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Approximating the Number of Acyclic Orientations for a Class of Sparse Graphs

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The Tutte polynomial $T(G;x,y)$ of a graph evaluates to many interesting combinatorial quantities at various points in the (x,y) plane, including the number of spanning trees, number of forests, number of acyclic orientations, the reliability polynomial, the partition function of the Q-state Potts model of a graph, and the Jones polynomial of an alternating link. The exact computation of $T(G;x,y)$ has been shown by Vertigan and Welsh [8] to be #P-hard at all but a few special points and on two hyperbolae, even in the restricted class of planar bipartite graphs. Attention has therefore been focused on approximation schemes. To date, positive results have been restricted to the upper half plane $y>1$, and most results have relied on a condition of sufficient denseness in the graph. In this paper we present an approach that yields a fully polynomial randomized approximation scheme for $T(G;x,y)$ for $x>1,\ y=1$, and for $T(G;2,0)$, in a class of sparse graphs. This is the first positive result that includes the important point $(2,0)$.

1. Introduction

An acyclic orientation is an assignment of orientations to the edges of the graph G, such that the resulting directed graph contains no cycles. The associated counting problem is the evaluation of the Tutte polynomial at $T(G;2,0)$. Although this may seem a somewhat special problem, it turns out by a classic result of Zaslavsky [10] that this is related to hyperplane arrangements. We regard each edge (x,y) of a graph G on n vertices as the hyperplane $x=y$ in n-dimensional Euclidean space. The number of acyclic orientations of G equals the number of chambers in the arrangement of hyperplanes described in this way by G. For example the well-known braid arrangement of hyperplanes comes from the complete graph K_n (see Orlik and Terao [7]). The number of forests in a graph equals $T(G;2,1)$. These are two of a host of points in the (x,y) plane at which the evaluation

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of the Tutte polynomial is an interesting invariant of the underlying graph. The Tutte polynomial is a polynomial of two variables, which can be defined for a graph, matrix or a general matroid. It is defined recursively, and it follows that direct computation of the Tutte polynomial of a graph on m edges takes time exponential in m. A diverse collection of properties turns out to be determined by the Tutte polynomial. In addition to standard graph-theoretic properties, there are examples from many other fields including statistical physics, knot theory and network theory. The following well-known problems are specializations of the Tutte polynomial to particular points or lines in the (x, y) plane:

(i) the chromatic polynomial of a graph, along $y = 0$;
(ii) the flow polynomial of a graph, along $x = 0$;
(iii) the all terminal network reliability probability of a network, along $x = 1, y > 1$;
(iv) the partition function of the Ising and Q-state Potts model, on the hyperbola $(x - 1)(y - 1) = Q$;
(v) the Jones polynomial of an alternating knot, on the hyperbola $xy = 1$;
(vi) the weight enumerator of a linear code over $GF(q)$, on the hyperbola $(x - 1)(y - 1) = q$.

Exact computation of the Tutte polynomial has been shown to be #P-hard at all but a few points in the plane [8]. Indeed, $T(G; -2, 0)$ counts the number of proper three-colourings of the graph G. Since the associated decision problem is NP-complete, no approximation scheme that reliably differentiates zero from nonzero can exist for this point, unless NP $=$ RP. In other regions the decision problems are not NP-complete, indeed they are often trivial: every graph has an acyclic orientation. So at these points the assumption that NP $\neq$ RP does not immediately rule out a fully polynomial randomized approximation scheme (FPRAS).

A RAS for a quantity $\pi(G)$ is a randomized approximation algorithm such that, for any given $\epsilon > 0, \delta > 0$, with probability greater than $1 - \delta$ the output $\hat{\pi}(G, \epsilon, \delta)$ is within a relative error of $1 \pm \epsilon$,

$$\mathbf{Pr}[|\pi(G) - \hat{\pi}(G, \epsilon, \delta)| > \epsilon\pi(G)] < \delta.$$

A RAS is described as *fully polynomial* (FPRAS) if the running time is bounded by a polynomial in $|G|$, $1/\delta$ and $1/\epsilon$. The question of where in the (x, y) plane and for which classes of graphs there exists an FPRAS is wide open.

The positive results are few: Jerrum and Sinclair [4] presented an FPRAS along the hyperbola $(x - 1)(y - 1) = 2$ for all graphs, Annan [2] dealt with the case $y = 1, x \geqslant 1$ for dense graphs (those having minimum degree $\Omega(|V(G)|)$, Alon, Frieze and Welsh [1] showed that an FPRAS exists for $x \geqslant 1$, $y \geqslant 1$ for dense graphs, and for all $y \geqslant 1$ for strongly dense graphs (minimum degree $> |V(G)|/2$). Recently Karger [5] proved the existence of a similar scheme (for all $y > 1$) for graphs with no small cutset (edge connectivity at least $c \log |V(G)|$ for some c depending on x and y). Even though all these previous results except [4] have made use of a denseness or similar condition, Welsh conjectured [9] that there exists an FPRAS scheme for all graphs, in the region $x \geqslant 1$, $y \geqslant 1$.

A full survey on the Tutte polynomial can be found in [9]. Throughout, our graph G will have n vertices and m edges. A circuit C is a connected subgraph of G such that every vertex in C has degree two (a simple cycle). The length of a circuit is the number of edges it contains. The girth g of a graph is the length of the smallest circuit. The evaluations considered are trivial for graphs with no circuits, therefore we assume that G has at least one circuit, and hence that g is well defined for all G.

1.1. Results

In this paper we present an FPRAS for $T(G; x, y)$ along the half line $x > 1,\ y = 1$, and at the point $(2, 0)$, for a class of graphs with large girth (girth at least $c \log |V(G)|$ for some c depending on x). The evaluations of the Tutte polynomial at these points includes the number of acyclic orientations and the number of forests of the graph. To be precise we prove the following result.

Theorem 1.1. *Let $\delta > 0, x > 1$ be fixed, and let $\mathcal{G}_{\delta,x}$ be the class of graphs with girth $g \geqslant (5 + \delta) \log_x(n)$. Then*
(i) *there is an FPRAS for $T(G; x, 1)$ for all $G \in \mathcal{G}_{\delta,x}$,*
(ii) *there is an FPRAS for $T(G; 2, 0)$ for all $G \in \mathcal{G}_{\delta,2}$.*

This is the first result that provides an approximation scheme outside the hyperbola $(x - 1)(y - 1) = 2$ for any class of sparse graphs, and also the first positive result that includes the point $(2, 0)$. We include a proof that exact evaluation is #P-hard even on this restricted class of graphs.

Although this class of sparse graphs (those with large girth) may look restrictive, in computing there is much interest in expander graphs, which make efficient networks using few edges. There are many well-known constructions of expanders with constant degree and girth of order $\log n$ [3].

Our approach is essentially a dualization (in the matroidal sense) of that of Karger [5], who presents an FPRAS for evaluating the reliability polynomial for the class of graphs with minimum edge cut-set of size $c \log_y n$. In Section 2 we bound the number of minimum circuits (those with length equal to the girth), and near-minimum circuits of graphs. In Section 3 we show that the near-minimum circuits can be enumerated in polynomial time. In Section 4 we introduce WASTE(p), the probability that a random subgraph of G contains a cycle. Using the previous results, we first bound WASTE(p) for graphs with girth greater than $c \log_{1/p} n$, and then by converting the problem into a DNF boolean formula we present an FPRAS for WASTE(p) for all graphs. In Section 5 we introduce the *efficiency probability* $\mathrm{Eff}(G, p)$, a dual to the well-known reliability probability and essentially given by the Tutte polynomial along the line $y = 1, x > 1$. We observe that

$$1 - \mathrm{WASTE}(G, p) = \mathrm{Eff}(G, p) = (p^{-1} - 1)^{m-n+1} p^m T(G; p^{-1}, 1)$$

and hence we construct an FPRAS for evaluating $T(G; x, 1)$ for $x > 1$ from the FPRAS for WASTE(p), whenever WASTE(p) can be bounded away from 1. Then, in Section 6, we extend the previous work to incorporate the point $(2, 0)$, by using an alternative reduction to DNF formulae. Section 7 contains a proof that even on this class of sparse graphs, exact evaluation of the Tutte polynomial is #P-hard in the region considered. Finally Section 8 presents an FPRAS at some additional points on the hyperbola $(x - 1)(y - 1) = -1$.

2. Counting paths and circuits

In this section we calculate bounds on the number of short circuits in simple graphs, via bounds on the number of paths of short length.

Lemma 2.1. *Let G be a simple graph on n vertices, with girth g. Then, for any $v \in V(G)$ there are at most $n-1$ distinct paths of length strictly less than $g/2$, that have v as one endpoint.*

Proof. Consider a fixed vertex $v \in V(G)$. Let P_1 and P_2 be any two distinct paths of length strictly less than $g/2$, with v as one endpoint. Let u_1 and u_2 be the other endpoints of P_1 and P_2 respectively. Then u_1 and u_2 are distinct, for otherwise G must contain a circuit of length strictly less than g (since $|P_i| < g/2$, for $i = 1, 2$). The number of distinct endpoints of paths starting at v is at most $n-1$, hence the number of distinct paths of length strictly less than $g/2$ with v as one endpoint is at most $n-1$. □

We now use this lemma to put a bound on the number of short circuits. To do this we regard a simple circuit as being made up of a collection of short paths, and establish bounds on the possible ways of combining short paths. The following result may be known, but I have been unable to find it in the literature.

Proposition 2.2. *Let G be a simple graph on n vertices, with girth g. Then, for any $h \geqslant g$, there are at most $n^{2\frac{h}{g-2}+1}$, circuits of length h.*

Proof. Let d be the largest integer less than $g/2$. In other words $d = g/2 - 1$ for g even, and $d = (g-1)/2$ for g odd. Any circuit C of length h can be expressed as the union of $\lfloor h/d \rfloor$ paths of length d, and one path of length $l = h - \lfloor h/d \rfloor d$ (if h is not an exact multiple of d). We shall label these paths $P_1, P_2, \ldots, P_t$.

Let us first count the number of circuits containing a fixed vertex v. These circuits can certainly be expressed as a union of paths such that P_1 has v as an endpoint. Hence by Lemma 2.1 there are at most $n-1$ choices for P_1, and since P_{i+1} must start at the endpoint of P_i, there are at most $n-1$ ways of choosing each of the subsequent paths (again by Lemma 2.1). Finally, the last path ($P_t = P_{\lceil h/d \rceil}$) is forced in order to make a circuit, since both endpoints are fixed and the existence of two short (length l) paths between them would imply the existence of a circuit of length less than g. Hence the total number of circuits containing v is at most $(n-1)^{\lfloor h/d \rfloor}$. Hence the total number of circuits of length h is at most

$$n(n-1)^{\lfloor h/d \rfloor} < n^{\lfloor h/d \rfloor + 1} \leqslant n^{2\frac{h}{(g-2)}+1}.$$ □

Note that, for $h = g \geqslant 6$, there are at most $n^{2\frac{g}{g-2}+1} \leqslant n^4$ minimum circuits, for $h = g < 6$ we can achieve the same bound by using a little more care. We now define an *α-small circuit*. A circuit C in a graph of girth g is called α-small if $|C| \leqslant \alpha g$.

Corollary 2.3. *Let G be a graph on n vertices, with girth $g \geqslant 6$. Then, for any $\alpha \geqslant 1$, G has at most $n^{3\alpha+2}$ α-small circuits.*

Proof. Since no circuit can have length greater than n, we may assume $\alpha g \leqslant n$. From Proposition 2.2, the number of α-small circuits in G is at most

$$\begin{aligned}\sum_{h \leqslant \alpha g} n^{2\frac{h}{g-2}+1} &< \alpha g n^{2\frac{\alpha g}{g-2}+1} \\ &\leqslant \alpha g n^{3\alpha+1} \\ &\leqslant n^{3\alpha+2}.\end{aligned}$$ □

3. Listing small circuits

In Section 2 we have bounded the number of α-small circuits in a graph G. When we come to constructing an FPRAS in Section 4, we will need a full list of the α-small circuits for some fixed α. However, the observations used in proving the theorems of Section 2 will enable us to create such a list by exhaustive search.

Theorem 3.1. *Let G be a simple graph on n vertices, with girth $g \geqslant 6$. Then, for $\alpha \geqslant 1$, a complete list of all α-small circuits can be constructed in running time $O(n^{3\alpha+3})$.*

Proof. First form a list of all the paths of length d, where d is the largest integer less than $g/2$, as in Theorem 2.2. This can be done in time $O(n^3)$, since there are at most n such paths starting at each vertex, and these can be found in time $O(n^2)$. Next, for each h, $g \leqslant h \leqslant \alpha g$, we can use the construction of Theorem 2.2 to exhaustively check all of $n^{\lfloor h/d \rfloor}$ possible circuits through each vertex v. So for each h there are at most $n^{3\alpha+1}$ circuits to be checked, and they can be checked in time $O(n^{3\alpha+2})$. This gives a total running time bounded by $O(n^{3\alpha+3})$. □

4. Approximating WASTE(G, p)

For a given graph G, let $\nu(G)$ be the cyclomatic number of G, in other words, for G with $\kappa(G)$ connected components,

$$\nu(G) = |E(G)| - |V(G)| + \kappa(G) = m - n + \kappa(G).$$

Note that $\nu(G) = 0$ if and only if G is a forest, *i.e.*, G has no cycles. For a fixed graph G, let G_p be a random subgraph of G obtained by deleting each edge of G independently with probability $(1-p)$. Let WASTE(G, p) be the probability that $\nu(G_p) > 0$. WASTE refers to the fact that G_p has at least one cycle, therefore 'wasted' edges that could be removed, while retaining the same connected components. In the remainder of the paper we shall write WASTE(p) for WASTE(G, p) where there is no confusion over the subject graph. We now use the results of Section 2 to bound WASTE(p), and then to present an FPRAS for WASTE(p).

4.1. Bounding WASTE(p)

Theorem 4.1. *For fixed $0 < p < 1$, let G be a graph with girth $g = \frac{(5+\delta)}{|\log(p)|}\log(n)$ for some $\delta > 0$. Let G_p be a random subgraph of G obtained by independently deleting each edge with probability $(1-p)$. Then:*

(1) *the probability that $\nu(G_p) > 0$ is at most $n^{-\delta}(1+\frac{5}{\delta})$,*

(2) *for $\alpha > 1$ the probability that some cycle of length at least αg is present in G_p is at most $n^{-\alpha\delta}(1+\frac{5}{\delta})$.*

Proof. Let all the circuits of G be listed in order of nondecreasing length $C_1, C_2, \ldots$. Let $p_i = p^{|C_i|}$. First note that $p^g = n^{-(5+\delta)}$. Also, from Corollary 2.3 there are fewer than $n^{3\alpha+2} < n^{5\alpha}$ circuits of length at most αg, so we have that $p_{n^{5\alpha}} < p^{\alpha g} = n^{-5\alpha(1+\delta/5)}$. Hence,

$$\begin{aligned}
\mathrm{WASTE}(p) &\leqslant \sum_{C:\text{ cycles in G}} p^{|C|} \\
&\leqslant \sum_{i=1}^{n^5} p^g + \sum_{i>n^5} p_i \\
&\leqslant n^5 n^{-(5+\delta)} + \sum_{i>n^5} i^{-(1+\delta/5)} \\
&< n^{-\delta} + \int_{n^5}^{\infty} x^{-(1+\delta/5)}\,\mathrm{d}x \\
&< n^{-\delta} + \left[-\frac{5}{\delta}x^{-\frac{\delta}{5}}\right]_{n^5}^{\infty} \\
&< n^{-\delta} + \frac{5}{\delta}n^{-\delta}.
\end{aligned}$$

This gives part (1). For part (2) we split the sum slightly differently to get the result:

$$\begin{aligned}
\mathbf{Pr}(\exists\ C \in G_p : |C| \geqslant \alpha g) &\leqslant \sum_{C:|C|\geqslant \alpha g} p^{|C|} \\
&\leqslant \sum_{i=1}^{n^{5\alpha}} p^{\alpha g} + \sum_{i>n^{5\alpha}} p_i \\
&\leqslant n^{5\alpha} n^{-(5\alpha+\alpha\delta)} + \sum_{i>n^{5\alpha}} i^{-(1+\delta/5)} \\
&< n^{-\alpha\delta} + \frac{5}{\delta}n^{-\alpha\delta}.
\end{aligned}$$

□

4.2. An FPRAS for WASTE(p)

We now present an FPRAS for WASTE(p). We begin with a technical lemma, which asserts that we can find an α which splits the circuits of G into α-small circuits, which we can deal with, and larger circuits which are unlikely to appear in G_p.

Lemma 4.2. *Given $\epsilon > 0$ and $0 < p < 1$. Let G be a graph. If $p^g \leqslant n^{-10}$, then for $\alpha = 2 - \frac{\ln(\epsilon/6)}{2\ln n}$ the probability that any circuit of length greater than αg is present in G_p is less than $(\epsilon/3)$ WASTE(p).*

Proof. Since $p^g < n^{-10} < n^{-5}$, we have that

$$p^g = n^{-(5+\delta)} \quad \text{some } \delta \geqslant 5, \qquad g = \frac{(5+\delta)\log n}{|\log p|}. \tag{4.1}$$

Hence g satisfies the conditions of Theorem 4.1. Using the observation that WASTE(p) $\geqslant$ p^g, we see that

$$\begin{aligned}
\alpha &= 2 - \frac{\ln(\epsilon/6)}{2\ln n}, \\
\alpha &> 1 + \frac{5}{\delta} - \frac{\ln(\epsilon/6)}{\delta \ln n}, \\
\alpha\delta \ln n &> (\delta+5)\ln n - \ln(\epsilon/6), \\
n^{-\alpha\delta} &< \frac{\epsilon}{6} n^{-(\delta+5)}, \\
2n^{-\alpha\delta} &< \epsilon/3 \ \mathrm{WASTE}(p), \\
(1+5/\delta)n^{-\alpha\delta} &< \epsilon/3 \ \mathrm{WASTE}(p).
\end{aligned}$$

The result follows by Theorem 4.1. □

Theorem 4.3. *Given fixed* $0 < p < 1$*, then for any graph* G *there is an FPRAS for* WASTE(p).

Proof. We split the proof into two cases, depending on the number of vertices n of G, the girth g and the fixed probability p.

Case 1: $p^g \geqslant n^{-10}$.
If $p^g \geqslant n^{-10}$ then WASTE(p) $\geqslant n^{-10}$, since the probability that a given minimum circuit is present is at least n^{-10}. Hence we can use a simple Monte Carlo approximation as follows. For $j = 1$ to t we simulate G_p by deleting each edge with probability $(1-p)$, and set $X_j = 1$ if $\nu(G_p) > 0$ and $X_j = 0$ otherwise. Our estimate for WASTE(p) will be $X = \frac{1}{t}\sum_{j=1}^{t} X_j$. Clearly $\mathbf{E}[X] = \mathrm{WASTE}(p)$, and the variance of X is $\mathrm{WASTE}(p)(1-\mathrm{WASTE}(p))/t$. Hence, given $\delta, \epsilon > 0$, we set $t = \lceil \frac{n^{10}}{\epsilon^2\delta} \rceil$ and Chebyshev's inequality gives

$$\begin{aligned}
\mathbf{Pr}[|X - \mathrm{WASTE}(p)| \geqslant \epsilon \mathrm{WASTE}(p)] &\leqslant \frac{\mathbf{Var}(X)}{\epsilon^2 \mathrm{WASTE}(p)^2} \\
&\leqslant \frac{(1-\mathrm{WASTE}(p))}{\epsilon^2 \mathrm{WASTE}(p) t} \\
&< \frac{\epsilon^2 \delta}{\epsilon^2 \mathrm{WASTE}(p) n^{10}} \\
&< \delta.
\end{aligned}$$

Finally, since t is polynomial in $\delta^{-1}, \epsilon^{-1}$ and n, and also since each simulation takes time polynomial in n, we have an FPRAS for WASTE(p).

Case 2: $p^g < n^{-10}$.
As shown in Lemma 4.2, we can find an α which splits the set of circuits of G into large circuits, such that the probability of any one being present is less than $(\epsilon/3)$ WASTE(p),

and α-small circuits which we can list. Following the approach of Karger [5], once the α-small circuits have been listed, we can encode the information in a boolean formula in disjunctive normal form. We take a boolean variable x_e for each edge in G. We take x_e to be true if e is present, and false otherwise. Then, if we take a list of all the α-small circuits, $C_1, C_2, \ldots$ the clause corresponding to circuit C_i is $\hat{C}_i = \wedge_{e \in C_i} x_e$. The event that at least one α-small circuit is present is then $\hat{C} = \vee_i \hat{C}_i$. This is a formula of $|E(G)|$ variables, in disjunctive normal form, of length bounded by $\alpha g c$, where c is the number of small circuits. Karp, Luby and Madras [6] present an FPRAS for any $\epsilon > 0$ which approximates the probability that a randomly generated assignment to such a formula is satisfying, to within a relative error of $1 \pm \epsilon$ in $O(l/\epsilon^2)$ time, where l is the length of the formula. We use this FPRAS to approximate the probability that no α-small circuit is present, and hence WASTE(p).

To be precise, given any $\epsilon, \delta > 0$ we take $\alpha = 2 - \frac{\ln(\epsilon/6)}{2 \ln n}$, ignore the circuits of length greater than αg, and by Lemma 4.2 only incur an absolute error of $(\epsilon/3)$WASTE(p) by doing so. By Theorem 3.1, we can list all the circuits of length at most αg in time $O(n^{3\alpha+3}) = O(n^9 \epsilon^{-3/2})$. Hence, using Karp, Luby and Madras's FPRAS for DNF formulae, with input size $O(\alpha g n^{3\alpha+2}) = O(n^9 \epsilon^{-3/2})$, we can approximate the probability that a circuit of length up to αg is present in G_p to within a relative error of $\epsilon/3$, with high probability in polynomial time (*i.e.*, for any $\delta > 0$, this probability can be made greater than $1 - \delta$ in time polynomial in δ^{-1}). Hence we have an approximation for WASTE(p) to within a relative error of $\epsilon/3 + \epsilon/3 < \epsilon$ with high probability in polynomial time. $\square$

5. Approximating the Tutte polynomial

Alon, Frieze and Welsh [1] used an approximation scheme for the reliability polynomial (for dense graphs) and a manipulation of the Tutte polynomial, in order to get an approximation scheme for $T(G; x, y)$ for $x, y \geqslant 1$. We introduce the efficiency probability, a dual concept to the reliability probability. As before we have a fixed graph G, and a random subgraph G_p. Whereas Rel(G, p), gives the probability that G_p is connected, we define Eff(G, p) to be the probability that G_p is a forest. Thus Eff$(G, p) = 1 -$ WASTE(G, p).

Recall that $v(G_p) = |E(G_p)| - |V(G_p)| + \kappa(G_p) = |E(G_p)| - r(G_p)$, where $\kappa(G_p)$ is the number of connected components of G_p, $r(G_p)$ is the (matroid) rank of G_p. The Tutte polynomial can be evaluated by analysis of the probability distribution of $v(G_p)$, where $p = \frac{1}{x}$ (for $x \geqslant 1$).

Let $Q = (x-1)(y-1)$ and $p = x^{-1}$. Assuming G is connected then

$$\begin{aligned}
T(G; x, y) &= \sum_{A \subseteq E} (x-1)^{n-1-r(A)} (y-1)^{|A|-r(A)} \\
&= \sum_{A \subseteq E} [(x-1)(y-1)]^{v(A)} (x-1)^{n-1-|A|} \\
&= \sum_{A \subseteq E} Q^{v(A)} (x-1)^{n-m-1} x^m \left(\frac{1}{x}\right)^{|A|} \left(\frac{x-1}{x}\right)^{m-|A|} \\
&= (x-1)^{n-m-1} x^m \mathbf{E}\big[Q^{v(G_p)}\big].
\end{aligned}$$

Note that when $y = 1$, $Q = 0$, and hence $Q^{v(G_p)}$ is nonzero only when $v = 0$, so

$$T(G;x,1) = (x-1)^{n-m-1}x^m\mathbf{Pr}(v(G_p) = 0),$$

and when $p = 1/2$, we get the familiar

$$T(G;2,1) = 2^m\mathbf{Pr}(A \subseteq E \text{ is acyclic}) = \#\text{forests of G}.$$

We already have enough machinery to approximate $T(G;x,1)$ for $x > 1$. The probability that $v(G_p) = 0$ is precisely $\mathrm{Eff}(G,p) = 1 - \mathrm{WASTE}(p)$. We have presented in Section 4.2 an FPRAS for $\mathrm{WASTE}(p)$. So provided $\mathrm{WASTE}(p)$ is small enough, we can use our FPRAS to get an accurate approximation for $\mathrm{Eff}(G,p)$ and hence $T(G;x,1)$. We illustrate this first with a general lemma.

Lemma 5.1. *Let $0 \leqslant \psi \leqslant 1$ be a quantity dependent on an input of size n, such that there is an FPRAS for ψ. Let d be a fixed integer such that $\psi < (1 - \Omega(n^{-d}))$. Then $(1-\psi)$ admits an FPRAS.*

Proof. Since $\psi < (1 - \Omega(n^{-d}))$, there exist positive numbers c and N such that, for all $n \geqslant N$, $\psi < (1 - cn^{-d})$. To show the existence of an FPRAS for $(1-\psi)$ it is enough to show the existence of an FPRAS for all $n > N$. Given $\epsilon, \delta > 0$, use the FPRAS for ψ to get an estimate $\hat{\psi}$ for ψ such that

$$\mathbf{Pr}\left[|\psi - \hat{\psi}| \geqslant \epsilon\frac{cn^{-d}}{1-cn^{-d}}\psi\right] < \delta$$

in running time polynomial in n, ϵ^{-1} and δ^{-1}. We take $(1-\hat{\psi})$ to be our estimate for $(1-\psi)$. Note that for $n \geqslant N$, $cn^{-d} < (1-\psi)$, and $\frac{\psi}{(1-cn^{-d})} < 1$. Hence

$$\begin{aligned}\mathbf{Pr}[|(1-\psi)-(1-\hat{\psi})| \geqslant \epsilon(1-\psi)] &< \mathbf{Pr}[|\psi - \hat{\psi}| \geqslant \epsilon cn^{-d}]\\ &< \mathbf{Pr}\left[|\psi-\hat{\psi}| \geqslant \epsilon cn^{-d}\frac{\psi}{1-cn^{-d}}\right]\\ &< \delta.\end{aligned}$$

□

Now we use this lemma to prove the first part of the main theorem of the paper. We show the existence of an FPRAS for $T(G;x,1)$, whenever $\mathrm{WASTE}(p)$ can be bounded away from 1.

Theorem 5.2. *Let $\delta > 0$, $x > 1$ be fixed. For any graph G on n vertices, with girth g at least $(5+\delta)\log_x(n)$, there is an FPRAS for $T(G;x,1)$.*

Proof. We take $p = 1/x$. By Section 4.2, there exists an FPRAS for $\mathrm{WASTE}(p)$. By Theorem 4.1, $\mathrm{WASTE}(p) \leqslant n^{-\delta}(1+\frac{5}{\delta})$. Hence we may apply Lemma 5.1, to deduce that there is an FPRAS for $\mathrm{Eff}(G,p) = (1 - \mathrm{WASTE}(p))$. Our approximation for $T(G;x,1)$ is $(x-1)^{n-m-1}x^m\mathrm{Eff}(G,p)$, which differs from $\mathrm{Eff}(G,p)$ by an easily computable multiplicative factor. Hence the FPRAS for $\mathrm{Eff}(G,p)$ induces an FPRAS for $T(G;x,1)$. □

Note that the running time of the FPRAS depends upon δ in the following sense. Since $\mathrm{WASTE}(p) \leqslant n^{-\delta}(1+\frac{5}{\delta})$, we have that $\mathrm{WASTE}(p) < 0.99$ for $n > N = \left(\frac{1+5/\delta}{0.99}\right)^{1/\delta}$. This is enough to show the existence of an FPRAS, by Lemma 5.1, since for fixed δ, 2^N is a constant, and we can handle the graphs of size at most N by direct calculation (in time $O(2^{N^2})$). However, as δ shrinks to towards zero, this constant increases.

6. Acyclic orientations

We now extend the work of previous sections to evaluating $T(G;2,0)$. The evaluation of the Tutte polynomial at this point counts the number of acyclic orientations of a graph. An acyclic orientation is an assignment of orientations to the edges of a graph, such that the resulting directed graph contains no directed circuits. We will call a circuit *consistent* in an orientation if it forms a directed circuit. We will call an orientation *cyclic* if it is not an acyclic orientation (*i.e.*, some circuit is consistent). We will denote the probability that a random orientation of G is cyclic by $\mathrm{Cyc}(G)$. Our approach will be to first present an FPRAS for $\mathrm{Cyc}(G)$, and then to bound this strictly below 1 for graphs with large girth, so that we can obtain an FPRAS for 1-$\mathrm{Cyc}(G)$, just as for $\mathrm{WASTE}(G,p)$ and 1-$\mathrm{WASTE}(G,p)$.

We will first take an arbitrary base orientation σ, for example by an ordering of the vertices, and directing each edge upwards. Any other orientation will be thought of as a function $\tau : E \to \{0,1\}$, where for each edge $e \in E$, $\tau(e) = 0$ if e is oriented as in σ and $\tau(e) = 1$ otherwise. Consider a random orientation of the edges $\pi(E(G))$, where $\pi(e)$ is selected uniformly at random from $\{0,1\}$ independently for each edge. Thus

$$\mathrm{Cyc}(G) = \mathbf{Pr}[\pi(E(G)) \text{ is cyclic}].$$

6.1. Bounding Cyc(G)

Now using the observation that the probability that a given circuit C is consistent in $\pi(E(G))$ is simply $2^{-|C|+1}$, we can bound $\mathrm{Cyc}(G)$.

Theorem 6.1. *Let G be a graph with girth $g = (5+\delta)\log_2 n$, for some $\delta > 0$. Then:*

(1) $\mathrm{Cyc}(G)$ *is at most* $n^{-\delta}(2+\frac{10}{\delta})$,

(2) *for $\alpha > 1$ the probability that some cycle of length at least αg is consistently orientated in $\pi(E(G))$ is at most $n^{-\alpha\delta}(2+\frac{10}{\delta})$.*

Proof. We use similar arguments to those used in the proof of Theorem 4.1. Let all the circuits of G be listed in order of nondecreasing length $C_1, C_2, \ldots$. Let $p_i = 2^{-|C_i|+1}$. First note that $2^{-g+1} = 2n^{-(5+\delta)}$. Also, from Corollary 2.3 there are fewer than $n^{3\alpha+2} < n^{5\alpha}$ circuits of length at most αg, so we have $p_{n^{5\alpha}} < 2^{-\alpha g+1} = 2n^{-5\alpha(1+\delta/5)}$. Hence

$$\begin{aligned}\mathrm{Cyc}(G) &\leqslant \sum_{C:\ \text{cycles in G}} 2^{-|C|+1} \\ &\leqslant \sum_{i=1}^{n^5} 2^{-g+1} + \sum_{i>n^5} p_i\end{aligned}$$

$$
\begin{aligned}
&\leqslant n^5 2n^{-(5+\delta)} + \sum_{i>n^5} 2i^{-(1+\delta/5)} \\
&< 2n^{-\delta} + 2\int_{n^5}^{\infty} x^{-(1+\delta/5)}\,\mathrm{d}x \\
&< 2n^{-\delta} + 2\left[-\frac{5}{\delta}x^{-\frac{\delta}{5}}\right]_{n^5}^{\infty} \\
&< 2n^{-\delta} + 2\frac{5}{\delta}n^{-\delta}.
\end{aligned}
$$

This gives (1). For (2) we split the sum slightly differently to get the result:

$$
\begin{aligned}
\mathbf{Pr}(\exists \text{ consistent } C \in \pi(E(G)) : |C| \geqslant \alpha g) &\leqslant \sum_{C:|C|\geqslant \alpha g} 2^{-|C|+1} \\
&\leqslant \sum_{i=1}^{n^{5\alpha}} 2^{-\alpha g+1} + \sum_{i>n^{5\alpha}} p_i \\
&\leqslant n^{5\alpha} 2n^{-(5\alpha+\alpha\delta)} + \sum_{i>n^{5\alpha}} 2i^{-(1+\delta/5)} \\
&< 2n^{-\alpha\delta} + 2\frac{5}{\delta}n^{-\alpha\delta}. \qquad \square
\end{aligned}
$$

6.2. An FPRAS for Cyc(G)

We now follow the same route that we took for constructing an FPRAS for WASTE(G, p). First, we prove a second technical lemma, similar to Lemma 4.2.

Lemma 6.2. *Given $\epsilon > 0$, and G a graph. If $2^{-g} \leqslant n^{-10}$ then for $\alpha = 2 - \frac{\ln(\epsilon/12)}{2\ln n}$ the probability that any circuit of length greater than αg is consistent in $\pi(E(G))$ is less than $(\epsilon/3)$ Cyc(G).*

Proof. Since $2^{-g} < n^{-10} < n^{-5}$, we have

$$2^{-g} = n^{-(5+\delta)} \quad \text{some } \delta \geqslant 5, \qquad g = (5+\delta)\log_2 \; n. \tag{6.1}$$

Hence g satisfies the conditions of Theorem 6.1. Observing that Cyc$(G) \geqslant 2^{-g}$, we see that

$$
\begin{aligned}
\alpha &= 2 - \frac{\ln(\epsilon/12)}{2\ln n} \\
\alpha &> 1 + \frac{5}{\delta} - \frac{\ln(\epsilon/12)}{\delta \ln n} \\
\alpha\delta \ln n &> (\delta+5)\ln n - \ln \\
n^{-\alpha\delta} &< \frac{\epsilon}{12} n^{-(\delta+5)}. \\
4n^{-\alpha\delta} &< \epsilon/3 \; \mathrm{Cyc}(G) \\
(2 + 10/\delta)n^{-\alpha\delta} &< \epsilon/3 \; \mathrm{Cyc}(G).
\end{aligned}
$$

The result follows by Theorem 6.1. $\square$

Theorem 6.3. *For any graph G there is an FPRAS for* $\mathrm{Cyc}(G)$, *the probability that a random orientation of the edges is cyclic.*

Proof. As in Theorem 4.3, we split the proof into two cases. Let n be the number of vertices of G, and g its girth. If $2^{-g} \geqslant n^{-10}$, we use simple Monte Carlo approximation, whereas if $2^{-g} < n^{-10}$ we use an efficient reduction to DNF counting.

Case 1: $2^{-g} \geqslant n^{-10}$.
If $2^{-g} \geqslant n^{-10}$ then $\mathrm{Cyc}(G) \geqslant n^{-10}$, since the probability that a given minimum circuit is consistent is at least n^{-10}. Hence we can use a simple Monte Carlo approximation as follows. For $j = 1$ to t we simulate $\pi(G)$ by orienting each edge randomly, and set $X_j = 1$ if $\pi(E(G))$ is cyclic and $X_j = 0$ otherwise. Our estimate for $\mathrm{Cyc}(G)$ will be $X = \frac{1}{t}\sum_{j=1}^{t} X_j$. Clearly $\mathbf{E}[X] = \mathrm{Cyc}(G)$, and the variance of X is $\mathrm{Cyc}(G)(1 - \mathrm{Cyc}(G))/t$. Hence, given $\delta, \epsilon > 0$ we set $t = \lceil \frac{n^{10}}{\epsilon^2\delta} \rceil$ and Chebyshev's inequality gives:

$$
\begin{aligned}
\mathbf{Pr}[|X - \mathrm{Cyc}(G)| \geqslant \epsilon\mathrm{Cyc}(G)] &\leqslant \frac{\mathbf{Var}(X)}{\epsilon^2\mathrm{Cyc}(G)^2} \\
&\leqslant \frac{(1 - \mathrm{Cyc}(G))}{\epsilon^2\mathrm{Cyc}(G)t} \\
&< \frac{\epsilon^2\delta}{\epsilon^2\mathrm{Cyc}(G)n^{10}} \\
&< \delta.
\end{aligned}
$$

Finally, since t is polynomial in $\delta^{-1}, \epsilon^{-1}$ and n, and also since each simulation takes time polynomial in n, we have an FPRAS for $\mathrm{Cyc}(G)$.

Case 2: $2^{-g} < n^{-10}$.
Proceeding as before, by Lemma 6.2 we can find an α which splits the set of circuits of G into large circuits, such that the probability of any one being consistent is less than $(\epsilon/3)\,\mathrm{Cyc}(G)$, and α-small circuits which we can list. We can again encode the information in a boolean formula in disjunctive normal form; however, we take a different formula to the one used for $\mathrm{WASTE}(G, p)$. We take a boolean variable x_e for each edge in G. We take x_e to be true if $\pi(e) = 0$, and false otherwise. We form a list of all the α-small circuits, $C_1, C_2, \ldots$. We form two clauses for each circuit C_i as follows. Let $\tau_i(G)$ be some orientation such that C_i is consistent in $\tau_i(G)$; then the clauses $\hat{C}_i, \bar{C}_i$ corresponding to circuit C_i are $\hat{C}_i = \wedge_{e\in C_i}\tilde{x}_e$, and $\bar{C}_i = \wedge_{e\in C_i}\bar{\tilde{x}}_e$, where $\tilde{x}_e = x_e$ if $\tau_i(e) = 0$, and $\tilde{x}_e = \bar{x}_e$ if $\tau_i(e) = 1$. The event that C_i is consistent is $(\hat{C}_i \vee \bar{C}_i)$. Hence the event that at least one α-small circuit is consistent is then $F = \vee_i(\hat{C}_i \vee \bar{C}_i)$. This is a formula of $|E(G)|$ variables, in disjunctive normal form, of length bounded by $2\alpha gc$, where c is the number of small circuits. We again use the FPRAS of Karp, Luby and Madras [6] to approximate the probability that no α-small circuit is consistent, and hence $\mathrm{Cyc}(G)$.

So, given any $\epsilon, \delta > 0$, we take $\alpha = 2 - \frac{\ln(\epsilon/12)}{2\ln n}$, ignore the circuits of length greater than αg, and by Lemma 6.2 only incur an absolute error of $(\epsilon/3)\mathrm{Cyc}(G)$ by doing so. By Theorem 3.1, we can list all the circuits of length at most αg in time $O(n^{3\alpha+3}) = O(n^9\epsilon^{-3/2})$. Hence using Karp, Luby and Madras's FPRAS for DNF formulae, with input size $O(\alpha g n^{3\alpha+2}) = O(n^9\epsilon^{-3/2})$, we can approximate the probability that a circuit of length up to αg is consistent in $\pi(E(G))$ to within a relative error of $\epsilon/3$, with high probability in polynomial time. Hence we have an approximation for $\mathrm{Cyc}(G)$ to within a relative error of $\epsilon/3 + \epsilon/3 < \epsilon$ with high probability in polynomial time. □

6.3. Approximating $T(G;2,0)$

Theorem 6.4. *Let $\delta > 0$ be fixed. For any graph G with girth g at least $(5+\delta)\log_2(n)$, there is an FPRAS for $T(G;2,0)$.*

Proof. In order to approximate $T(G;2,0)$ it remains to observe

$$\begin{aligned} T(G;2,0) &= 2^m \mathbf{Pr}[\pi(E(G)) \text{ is acyclic}] \\ &= 2^m(1 - \mathbf{Pr}[\pi(E(G)) \text{ is cyclic}]) \\ &= 2^m(1 - \mathrm{Cyc}(G)). \end{aligned}$$

Hence, an FPRAS for $(1 - \mathrm{Cyc}(G))$ can be used to give an FPRAS for $T(G;2,0)$. By Section 6.2, there exists an FPRAS for $\mathrm{Cyc}(G)$. By Theorem 6.1, $\mathrm{Cyc}(G) \leqslant n^{-\delta}(2 + \frac{10}{\delta})$. Hence we may apply Lemma 5.1, to deduce that there is an FPRAS for $(1 - \mathrm{Cyc}(G))$. Our approximation for $T(G;2,0)$ is $2^m(1 - \mathrm{Cyc}(G))$. Hence the FPRAS for $(1 - \mathrm{Cyc}(G))$ induces an FPRAS for $T(G;2,0)$. □

Note that the running time of the FPRAS depends on δ as in Theorem 5.2. Theorems 5.2 and 6.4 together give Theorem 1.1.

7. Exact evaluation is #P-hard

We have shown the existence of approximation schemes for the Tutte polynomial at specific points in a certain class of graphs. We now show that exact evaluation of the Tutte polynomial is indeed #P-hard for this class of graphs.

Theorem 7.1. *For fixed $\delta > 0$, exact evaluation of $T(G;2,0)$ for the class of graphs with girth at least $(5+\delta)\log(n)$ is #P-hard.*

Proof. It is well known that evaluating the Tutte polynomial at $T(G;2,0)$ is #P-hard for general graphs [8]. Suppose that we can exactly evaluate $T(G;2,0)$ in time polynomial in n, the number of vertices of the graph, for graphs of girth at least $(5+\delta)\log(n)$. Let G be a general graph; let its girth be g. We define $s_k(G)$ to be the k-stretch of G: that is to say, we replace each edge (u,v) of G by a path of length k, joining u to v. Note that the girth, g', of $s_k(G)$ is kg, and the number of vertices, n', in $s_k(G)$ is $n + (k-1)m$. Since the girth is increasing by a multiplicative factor of k, but the log of the number of vertices is increasing by (approximately) an additive factor of $\log(k)$, we have $g' > (5+\delta)\log(n')$ for $k > K$, where K depends on G and δ, but is certainly at most $(5+\delta)n$ (for n at least 6). Hence we can evaluate $T(s_k(G);2,0)$ exactly for $k > K$, in time polynomial in n whenever k, and hence the number of vertices in $s_k(G)$, is only polynomially large. The

Tutte polynomial of a graph and its stretch are related by the following formula [8]:

$$T(s_k(G); x, y) = \left(\frac{x^k - 1}{x - 1}\right)^{\alpha} T(G; X, Y),$$

where

$$X = x^k \qquad Y = \frac{y + x + x^2 + \cdots + x^{k-1}}{1 + x + x^2 + \cdots + x^{k-1}}$$

and α is known and easily computable.

Note that $(x-1)(y-1) = (X-1)(Y-1)$. Since we are able to evaluate $T(s_k(G); 2, 0)$, we can evaluate $T(G; x, y)$ at a point (depending on k) on the hyperbola $(x-1)(y-1) = -1$. We now form $s_k(G)$ for sufficiently many $k > K$, so that we can obtain the univariate polynomial of the restriction of $T(G; x, y)$ to this hyperbola by Lagrange interpolation. This can be done in time polynomial in n, since the maximum degree of the Tutte polynomial, and therefore the number of points we will need to evaluate, and the time taken for each evaluation are both polynomial in n. We use this to recover $T(G; 2, 0)$. Since this is known to be #P-hard to evaluate exactly, we conclude that even on the class of graphs of girth at least $(5 + \delta)\log(n)$, it is #P-hard to evaluate $T(G; 2, 0)$ exactly. □

The same proof can be used to show that for $x > 1$, it is #P-hard to evaluate $T(G; x, 1)$ exactly even on the class of graphs of girth at least $(5 + \delta)\log_x(n)$.

8. Extension to other points in the Tutte plane

The technique of stretching graphs introduced in Section 7 can be used to provide an extension to the earlier work. We use stretching to get an FPRAS for some additional points on the hyperbola $(x-1)(y-1) = -1$.

Theorem 8.1. *Let $\delta > 0$ and k a positive integer, be fixed. For any graph G on n vertices with girth g at least $(5 + \delta)\log_2(n)$, there is an FPRAS for $T(G; x, y)$ whenever $x = 2^k$, $y = \frac{2 + 2^2 + \cdots + 2^{k-1}}{1 + 2 + 2^2 + \cdots + 2^{k-1}}$.*

Proof. Let $s_k(G)$ be the k-stretch of G as in Section 7. Then the girth, g', of $s_k(G)$ is kg, and the number of vertices of $s_k(G)$, n', is $n + (k-1)m$.

Case 1: $k > 2$.
Let $l = \lceil \log_n k \rceil$. Then

$$\begin{aligned} \log_2 n' &= \log_2(n + (k-1)m) \\ &\leqslant \log_2(n + (n^l - 1)n^2) \\ &\leqslant \log_2(n^{l+2}) \\ &\leqslant (l+2)\log_2 n. \end{aligned}$$

Note that for $k > n$ we have $k \geqslant n^{l-1} \geqslant (l+2)$ (for n at least 3), and for $k \leqslant n$ we have $l = 1$, so $k \geqslant 3 = (l+2)$. So

$$g' = kg \geqslant (l+2)(5+\delta)\log_2 n \geqslant (5+\delta)\log_2 n'.$$

Case 2: $k = 2$.
If $k = 2$, then $g' = 2g$, and $n' = n + m \leqslant n + n(n-1)/2 \leqslant n^2$. So

$$g' = 2g \geqslant 2(5+\delta)\log_2 n \geqslant (5+\delta)\log_2 n^2 \geqslant (5+\delta)\log_2 n'.$$

In either case $s_k(G)$ satisfies the conditions of Theorem 6.4, and we may evaluate $T(s_k(G);2,0)$, using the FPRAS of Theorem 6.4, in running time polynomial in the size of G, since the blow-up to $s_k(G)$ is polynomial (as k is fixed). Finally, $T(G;x,y) = (2^k-1)^{-\alpha}T(s_k(G);2,0)$ where α is known and easily computable, we have an FPRAS for $T(G;x,y)$. □

All previous results giving an FPRAS for some region of the Tutte plane have included all points along a given branch of a hyperbola (or none at all). Here we have presented an FPRAS for a large number of points along the hyperbola $(x-1)(y-1) = -1$, though these are restricted to $y \geqslant 0$. However, it suggests that, for this class of sparse graphs, approximation along the hyperbola $(x-1)(y-1) = -1$ should be possible.

Conjecture. *Let $\delta > 0$, $x > 1$ be fixed. For any graph G on n vertices, with girth g at least $(5+\delta)\log_x(n)$, there is an FPRAS for $T(G;x,\frac{x-2}{x-1})$.*

9. Conclusion

We have bounded the number of minimum and near-minimum circuits in all graphs. This has enabled us to show the existence of fully polynomial randomized approximation schemes for WASTE(G,p) and Cyc(G), using Monte Carlo simulation for large probabilities, and an efficient reduction to DNF counting for small probabilities. Interpreting the Tutte polynomial in the terms of the efficiency probability enabled us to use the FPRAS for WASTE(p) to approximate $T(G;x,1)$, $x > 1$ for graphs with large girth. We also used the FPRAS for Cyc(G) to approximate $T(G;2,0)$, the number of acyclic orientations.

A natural area for further research is to try to use the techniques introduced in this paper to prove the existence of FPRAS schemes for the same class of graphs in different regions of the Tutte plane. There are now known to be FPRAS schemes for $T(G;x,y)$, $x > 1, y = 1$ for dense graphs [1] and the class of sparse graph defined here. It is an interesting conjecture that there is an FPRAS for all graphs in this region of the Tutte plane (indeed in the entire region $x \geqslant 1$, $y \geqslant 1$ [9]).

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