Embedding Long Paths in *k*-Ary *n*-Cubes with Faulty Nodes and Links

Iain A. Stewart and Yonghong Xiang

Abstract—Let $k \geq 4$ be even and let $n \geq 2$. Consider a faulty k-ary n-cube Q_n^k in which the number of node faults f_v and the number of link faults f_e are such that $f_v + f_e \leq 2n - 2$. We prove that given any two healthy nodes s and e of Q_n^k, there is a path from s to e of length at least $k^n - 2f_v - 1$ (respectively, $k^n - 2f_v - 2$) if the nodes s and e have different (respectively, the same) parities (the parity of a node in Q_n^k is the sum modulo 2 of the elements in the n-tuple over $\{0, 1, \ldots, k-1\}$ representing the node). Our result is optimal in the sense that there are pairs of nodes and fault configurations for which these bounds cannot be improved, and it answers questions recently posed by Yang et al. [22] and by Fu [11]. Furthermore, we extend known results, obtained by Kim and Park [15], for the case when $n = 2$.

Index Terms—Interconnection networks, k-ary n-cubes, fault tolerance, embeddings, longest paths.

✦

1 INTRODUCTION

THE choice of how we connect the processors in a distributed-memory parallel machine is a fundamental design decision. There are numerous, often conflicting considerations to bear in mind. For instance, we would like our interconnection topology to be symmetric, have a small diameter, be recursively decomposable, be highly connected, be regular of low degree, support rapid and easy interprocessor communication, support the simulation of other machines based on other topologies, and so on. There does not exist an interconnection topology that is optimal on all accounts, and trade-offs generally have to be made.

An extremely popular interconnection topology is the hypercube. The hypercube has been used as the interconnection topology of a number of distributed-memory multiprocessors such as the Cosmic Cube [19], the Ametek S/14 [4], the iPSC [9], [10], the Ncube [5], [10], and the CM-200 [6], and the properties of hypercubes relevant to parallel computing have been well studied. One drawback of the hypercube is that as the dimension of a hypercube increases, so does the degree of its nodes. Consequently, given a collection of processors, if we wish to connect these processors in the topology of a hypercube, then we have no choice as to the degree of the nodes of the resulting network. The k-ary n-cube has been proposed as an alternative to the hypercube. The k-ary n-cube is very "hypercube-like" and has similar properties to the hypercube. Furthermore, the two parameters available, k and n, allow us to regulate the degree of the nodes yet still incorporate large numbers of processors, although usually at a cost to some other property such as the diameter or the connectivity. A number of distributed memory multiprocessors have been built with a k-ary n-cube forming the

underlying topology, such as the Mosaic [20], the iWARP [7], the J-machine [18], the Cray T3D [14], and the Cray T3E [3].

As more and more processors are incorporated into parallel machines, faults become more common, be it faults in the processors themselves or faults on the interprocessor connections. Given the significant cost of parallel machines, we would prefer to be able to tolerate small numbers of faults and still be able to use our parallel machine. A key property we would like our "faulty" machine to have is that a large number of the healthy processors should remain in a connected component and be able to undertake significant parallel computations. Numerous existing algorithms for k-ary n-cubes utilize the orderings of processors and involve the use of long paths and cycles (note that a path in a multiport bidirectional network results in a closed path containing the nodes of the path exactly twice), and we would wish to utilize such structures even in the presence of faults. Also, fundamental and abundant in parallel computing are linear arrays and rings of processors, and at the very least, our faulty machine should be able to simulate (the large number of) algorithms designed for machines whose processors are joined in the form of linear arrays or rings (see, for example, [2] and [16]). We remark that our situation is of a different nature to that where a faulty network is to simulate another (healthy) network but where this simulation comes about due to an embedding of the healthy network in the faulty network with low load, congestion, and/or dilation (such a scenario can be found in, for example, [1], [8], [12], and [17], where hypercubes, arrays, and butterflies are considered).

In this paper, we continue the study of the k-ary n-cube with regard to the existence of long paths and cycles in the presence of limited numbers of node and link faults. We are motivated by the work in four recent publications. In [15], Kim and Park study the existence of Hamiltonian paths in two-dimensional tori. They provide conditions when a two-dimensional torus with at most two faulty nodes is Hamiltonian, Hamiltonian-connected, and bi-Hamiltonian connected. In [11], Fu proves that an n-dimensional hypercube with $f \leq n - 2$ faulty nodes is such that there is a path of length at least $2^n - 2f - \epsilon$ between any two distinct healthy

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nodes, where $\epsilon = 1$ if the two nodes have different parities and $\epsilon = 2$ otherwise. In [13], Hsieh and Chang show that Fu's result holds even when $f \leq 2n - 5$ but only so long as every healthy node is adjacent to at least two healthy nodes (the so-called conditional fault assumption). In [22], Yang et al. proved that in a k-ary n-cube where k is odd, if the number of faulty nodes and links is at most $2n - 3$, then there is a Hamiltonian cycle, and if the number of faulty nodes and links is at most $2n - 2$, then there is a Hamiltonian path joining any two distinct healthy nodes. Note that Yang et al. prove no results when k is even beyond remarking that when k is even, the k-ary n-cube is bipartite, and so, if there is one faulty node, then there can be no Hamiltonian cycle, and there exists a pair of distinct healthy nodes not joined by a Hamiltonian path.

Our main result is given as follows: Let $k \geq 4$ be even and let $n \geq 2$. In a faulty k-ary n-cube Q_n^k in which the number of node faults f_n and the number of link faults f_e are such that $f_n + f_e \leq 2n - 2$, given any two healthy nodes s and e of Q_n^k, there is a path from s to e of length at least $k^n - 2f_n - 1$ (respectively, $k^n - 2f_n - 2$) if the nodes s and e have different (respectively, the same) parities. Our result resolves the situation in [22] when k is even, answers the questions posed by Yang et al. and by Fu, and extends known results, obtained by Kim and Park, for the case when $n = 2$. The rest of this paper is devoted to a proof by induction of our main theorem. Section 2 contains the basic definitions. In Section 3, we deal with the base case of the induction, and in Section 4, we deal with the inductive step. We present our conclusions in Section 5.

2 BASIC DEFINITIONS

The k-ary n-cube Q_n^k, for $k \geq 3$ and $n \geq 2$, has k^n nodes indexed by $\{0, 1, \ldots, k-1\}^n$, and there is a link $((u_1, u_2, \ldots, u_n), (v_1, v_2, \ldots, v_n))$ if and only if there exists $d \in \{1, 2, \ldots, n\}$ such that $min\{|u_d - v_d|, k - |u_d - v_d|\} = 1$ and $u_i = v_i$, for every $i \in \{1, 2, \ldots, n\} \setminus \{d\}$. Many structural properties of k-ary n-cubes are known, but of particular relevance for us is that a k-ary n-cube is node symmetric; that is, given any two distinct nodes $\mathbf{v}$ and $\mathbf{v}'$ of Q_n^k, there is an automorphism of Q_n^k mapping $\mathbf{v}$ to $\mathbf{v}'$. Throughout, we assume that the addition of tuple elements is modulo k.

An index $d \in \{1, 2, \ldots, n\}$ is often referred to as a *dimension*. We can *partition Q_n^k over dimension d* by fixing the dth element of any node tuple at some value v, for every $v \in \{0, 1, \ldots, k-1\}$. This results in k copies $Q_{d,0}, Q_{d,1}, \ldots, Q_{d,k-1}$ of Q_{n-1}^k (with $Q_{d,v}$ obtained to fix the dth element at v), with corresponding nodes in $Q_{d,0}, Q_{d,1}, \ldots, Q_{d,k-1}$ joined in a cycle of length k (in dimension d). Such a partition proves to be extremely useful (in proofs by induction, as we shall see).

The *parity* of a node $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ of Q_n^k is defined to be $\sum_{i=1}^n v_i$ modulo 2. We speak of a node as being *odd* or *even* according to whether its parity is odd or even. A pair of nodes $\{\mathbf{v}, \mathbf{v}'\}$ is *odd* (respectively, *even*) if $\mathbf{v}$ and $\mathbf{v}'$ have different (respectively, the same) parities.

We write paths in Q_n^k as sequences of incident links, and when k is even, paths necessarily consist of links joining, alternatively, odd and even nodes. We often refer to a path as $\rho(\mathbf{u}, \mathbf{v})$; the notation denotes that this is a path joining node $\mathbf{u}$ and node $\mathbf{v}$. On occasion, we might refer to a

link $(\mathbf{x}, \mathbf{y})$ as appearing on a path $\rho(\mathbf{u}, \mathbf{v})$ or, equivalently, the path $\rho(\mathbf{u}, \mathbf{v})$ as containing the link $(\mathbf{x}, \mathbf{y})$; when we do, the notation denotes that if we traverse the path $\rho(\mathbf{u}, \mathbf{v})$ starting at node $\mathbf{u}$, then we shall reach node $\mathbf{x}$ immediately before we reach node $\mathbf{y}$. If $\rho(\mathbf{u}, \mathbf{v})$ is a path and $\mathbf{x}$ and $\mathbf{y}$ are nodes on this path, then $\rho(\mathbf{x}, \mathbf{y})$ denotes the subpath of $\rho(\mathbf{u}, \mathbf{v})$ starting at $\mathbf{x}$ and ending at $\mathbf{y}$.

A *fault* in Q_n^k refers to a faulty node or a faulty link. If a node is faulty, then we imagine that the node and its incident links do not exist; if a link is faulty, then we imagine that this link does not exist. When we refer to a path in a faulty Q_n^k, we mean that all nodes and links on the path should be nonfaulty, that is, *healthy* (unless otherwise stated).

We repeatedly apply the following construction throughout. Suppose that we have partitioned a k-ary n-cube Q_n^k over some dimension d so as to obtain k-ary $(n-1)$-cubes $Q_{d,0}, Q_{d,1}, \ldots, Q_{d,k-1}$ and that we have a path $\rho(\mathbf{u}, \mathbf{v})$ in Q_n^k of length l. Suppose also that $(\mathbf{x}_i, \mathbf{y}_i)$ is a link of $\rho(\mathbf{u}, \mathbf{v})$, with $\mathbf{x}_i, \mathbf{y}_i \in Q_{d,i}$ and that we have another path $\rho'(\mathbf{x}_{i+1}, \mathbf{y}_{i+1})$ of length l' that shares no nodes in common with $\rho(\mathbf{u}, \mathbf{v})$, where $\mathbf{x}_{i+1}$ and $\mathbf{y}_{i+1}$ are the neighbors of $\mathbf{x}_i$ and $\mathbf{y}_i$, respectively, in $Q_{d,i+1}$. We refer to the path obtained by removing the link $(\mathbf{x}_i, \mathbf{y}_i)$ from $\rho(\mathbf{u}, \mathbf{v})$ and replacing it with the path $(\mathbf{x}_i, \mathbf{x}_{i+1})$, $\rho'(\mathbf{x}_{i+1}, \mathbf{y}_{i+1})$, $(\mathbf{y}_{i+1}, \mathbf{y}_i)$, so as to obtain a new path from $\mathbf{u}$ to $\mathbf{v}$ of length $l + l' + 1$, as the *join of* $\rho(\mathbf{u}, \mathbf{v})$ to $\rho'(\mathbf{x}_{i+1}, \mathbf{y}_{i+1})$ over $(\mathbf{x}_i, \mathbf{y}_i)$. We can equally well join two paths over a subpath rather than a link; with the above notation, we would remove a subpath $\rho(\mathbf{x}_i, \mathbf{y}_i)$ from $\rho(\mathbf{u}, \mathbf{v})$ and replace it with the path $(\mathbf{x}_i, \mathbf{x}_{i+1})$, $\rho'(\mathbf{x}_{i+1}, \mathbf{y}_{i+1})$, $(\mathbf{y}_{i+1}, \mathbf{y}_i)$. We have analogous constructions should we wish to join a cycle and a path to obtain a path or two cycles to obtain a cycle (when joining a cycle, we lose one edge from the cycle).

Henceforth, for reasons of clarity, we drop the use of bold type to denote the nodes of Q_n^k (hitherto, we have used bold type to emphasize the representation of nodes as tuples of elements).

3 THE BASE CASE

In this section, we deal with the base case of our forthcoming inductive proof of the main result, namely, when we have a k-ary 2-cube with no more than two faults. We begin with some notation specific to our constructions in this section.

We consider Q_2^k as a $k \times k$ grid with wraparound, and we think of a node $v_{i,j}$ as indexed by its *row i* and *column j*. Given two row indices $i, j \in \{0, 1, \ldots, k-1\}$, where $j \neq i$, we define the *row torus* $rt(i, j)$ to be the subgraph of Q_2^k induced by the nodes on rows $i, i+1, \ldots, j$ if $i < j$ or rows $i, i+1, \ldots, k-1, 0, \ldots, j$ if $j < i$ but with all column links between nodes on row j and nodes on row i removed if $i = j + 1$ or ($i = 0$ and $j = k - 1$). Throughout, we assume that the addition of row or column indices is modulo k.

We define the following paths in the row torus $rt(0, 1)$ (of some Q_2^k). The names of these paths are derived from the shape of their pictorial representations (see the figures coming up). Also, if $i = 0$, then $\bar{i} = 1$, and if $i = 1$, then $\bar{\bar{i}} = 0$. The paths are given as follows:

$$C_m^+(v_{i,j}, v_{\bar{i},j}) = (v_{i,j}, v_{i,j+1}), (v_{i,j+1}, v_{i,j+2}), \ldots,$$
$$(v_{i,m-1}, v_{i,m}), (v_{i,m}, v_{\bar{i},m}), (v_{\bar{i},m}, v_{\bar{i},m-1}),$$
$$(v_{\bar{i},m-1}, v_{\bar{i},m-2}), \ldots, (v_{\bar{i},j+1}, v_{\bar{i},j}),$$
$$\text{where } 0 \le i \le 1, 0 \le j \le k-1,$$
$$0 \le m \le k-1, \text{ and } m \ne j.$$

$$C_m^-(v_{i,j}, v_{\bar{i},j}) = (v_{i,j}, v_{i,j-1}), (v_{i,j-1}, v_{i,j-2}), \ldots,$$
$$(v_{i,m+1}, v_{i,m}), (v_{i,m}, v_{\bar{i},m}), (v_{\bar{i},m}, v_{\bar{i},m+1}),$$
$$(v_{\bar{i},m+1}, v_{\bar{i},m+2}), \ldots, (v_{\bar{i},j-1}, v_{\bar{i},j}),$$
$$\text{where } 0 \le i \le 1, 0 \le j \le k-1,$$
$$0 \le m \le k-1, \text{ and } m \ne j.$$

$$N^+(v_{i,j}, v_{i,j'}) = (v_{i,j}, v_{\bar{i},j}), (v_{\bar{i},j}, v_{\bar{i},j+1}), (v_{\bar{i},j+1}, v_{i,j+1}),$$
$$(v_{i,j+1}, v_{i,j+2}), (v_{i,j+2}, v_{\bar{i},j+2}), (v_{\bar{i},j+2},$$
$$v_{\bar{i},j+3}), (v_{\bar{i},j+3}, v_{i,j+3}), (v_{i,j+3}, v_{i,j+4}),$$
$$\ldots, (v_{i,j'-1}, v_{i,j'}), \text{ where } 0 \le i \le 1,$$
$$0 \le j \ne j' \le k-1, \text{ and } |j - j'| \text{ is even.}$$

$$N^-(v_{i,j}, v_{i,j'}) = (v_{i,j}, v_{\bar{i},j}), (v_{\bar{i},j}, v_{\bar{i},j-1}), (v_{\bar{i},j-1}, v_{i,j-1}),$$
$$(v_{i,j-1}, v_{i,j-2}), (v_{i,j-2}, v_{\bar{i},j-2}), (v_{\bar{i},j-2},$$
$$v_{\bar{i},j-3}), (v_{\bar{i},j-3}, v_{i,j-3}), (v_{i,j-3}, v_{i,j-4}),$$
$$\ldots, (v_{i,j'+1}, v_{i,j'}), \text{ where } 0 \le i \le 1,$$
$$0 \le j' \ne j \le k-1, \text{ and } |j - j'| \text{ is even.}$$

$$Z^+(v_{i,j}, v_{i,j'}) = (v_{i,j}, v_{i,j+1}), (v_{i,j+1}, v_{\bar{i},j+1}),$$
$$(v_{\bar{i},j+1}, v_{\bar{i},j+2}), (v_{\bar{i},j+2}, v_{i,j+2}),$$
$$(v_{i,j+2}, v_{i,j+3}), (v_{i,j+3}, v_{\bar{i},j+3}),$$
$$(v_{\bar{i},j+3}, v_{\bar{i},j+4}), (v_{\bar{i},j+4}, v_{i,j+4}),$$
$$\ldots, (v_{\bar{i},j'}, v_{i,j'}), \text{ where } 0 \le i \le 1,$$
$$1 \le j \ne j' \le k-1, \text{ and } |j - j'| \text{ is even.}$$

$$Z^-(v_{i,j}, v_{i,j'}) = (v_{i,j}, v_{i,j-1}), (v_{i,j-1}, v_{\bar{i},j-1}),$$
$$(v_{\bar{i},j-1}, v_{\bar{i},j-2}), (v_{\bar{i},j-2}, v_{i,j-2}),$$
$$(v_{i,j-2}, v_{i,j-3}), (v_{i,j-3}, v_{\bar{i},j-3}),$$
$$(v_{\bar{i},j-3}, v_{\bar{i},j-4}), (v_{\bar{i},j-4}, v_{i,j-4}),$$
$$\ldots, (v_{\bar{i},j'}, v_{i,j'}), \text{ where } 0 \le i \le 1,$$
$$1 \le j' \ne j \le k-1, \text{ and } |j - j'| \text{ is even.}$$

In addition, we define $C_j^+(v_{i,j}, v_{\bar{i},j}) = C_j^-(v_{i,j}, v_{\bar{i},j}) = (v_{i,j}, v_{\bar{i},j})$. We also use the above notation to describe paths in other row tori of the form $rt(l, l+1)$ in Q_2^k. Furthermore, if we write, for example, $N^+(v_{i,j}, v_{i,j+1})$, $Z^-(v_{i,j}, v_{i,j})$ or some other illegal node pairing, then we regard the path so denoted as being the empty path.

We begin with two lemmas: the first concerning paths in a row torus $rt(0, 1)$ in which there is a faulty node and the second concerning paths in a row torus $rt(0, p-1)$ in which there are no faults. These two lemmas are used repeatedly in the proofs of the subsequent propositions, each of which deals with a specific configuration of faults relating to the base case.

Lemma 1. *Let $k \ge 4$ be even and consider the row torus $rt(0, 1)$ in Q_2^k, where one node of the row torus is faulty. If the pair of distinct healthy nodes $\{s, e\}$ of the row torus is odd (respectively, even), then there is a path $\rho(s, e)$ in the row torus joining s and e of length at least $2k - 3$ (respectively, $2k - 4$).*
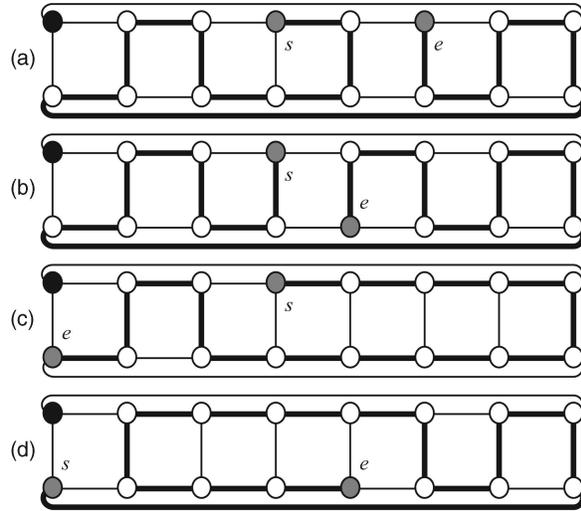
Fig. 1. Cases (a), (b), (c), and (d) when $k = 8$.

Proof. By the symmetric properties of the row torus $rt(0, 1)$, without loss of generality, we may assume that the fault is the node $v_{0,0}$.

Suppose that s and e are both odd. Without loss of generality, there are four cases. (Throughout, we proceed by a case-by-case analysis, eliminating some cases by applying automorphisms of $rt(0, 1)$ such as "reflections in the vertical bisecting plane" or "toroidal rotations.")

Case a. s and e both lie on row 0, with $s = v_{0,i}$, $e = v_{0,j}$, and $i < j$. Consider the path

$$C_{j-1}^+(v_{0,i}, v_{1,i}), Z^-(v_{1,i}, v_{1,1}), (v_{1,1}, v_{1,0}), (v_{1,0}, v_{1,k-1}),$$
$$N^-(v_{1,k-1}, v_{1,j}), (v_{1,j}, v_{0,j}).$$

This path has length $2k - 2$ and is as depicted in Fig. 1a.

Case b. s and e lie on different rows, with $s = v_{0,i}$, $e = v_{1,j}$, and $i < j$. Consider the path

$$C_{j-1}^+(v_{0,i}, v_{1,i}), Z^-(v_{1,i}, v_{1,1}), (v_{1,1}, v_{1,0}), (v_{1,0}, v_{1,k-1}),$$
$$N^-(v_{1,k-1}, v_{1,j+1}), (v_{1,j+1}, v_{0,j+1}), (v_{0,j+1}, v_{0,j}), (v_{0,j}, v_{1,j}).$$

This path has length $2k - 2$ and is as depicted in Fig. 1b.

Case c. s and e lie on different rows, with $s = v_{0,i}$ and $e = v_{1,0}$. Consider the path

$$C_{k-1}^+(v_{0,i}, v_{1,i}), Z^-(v_{1,i}, v_{1,1}), (v_{1,1}, v_{1,0}).$$

This path has length $2k - 2$ and is as depicted in Fig. 1c.

Case d. s and e both lie on row 1, with $s = v_{1,i}$, $e = v_{1,j}$, and $i < j$. Consider the path

$$N^-(v_{1,i}, v_{1,0}), (v_{1,0}, v_{1,k-1}), N^-(v_{1,k-1}, v_{1,j+1}),$$
$$(v_{1,j+1}, v_{0,j+1}), (v_{0,j+1}, v_{0,j}), C_{i+1}^-(v_{0,j}, v_{1,j}).$$

This path has length $2k - 2$ and is as depicted in Fig. 1d.

Suppose now that s and e are both even. Without loss of generality, there are three cases.

Case e. s and e both lie on row 0, with $s = v_{0,i}$, $e = v_{0,j}$, and $i < j$. Consider the path

$$C_{j-1}^+(v_{0,i}, v_{1,i}), Z^-(v_{1,i}, v_{1,2}), (v_{1,2}, v_{1,1}), (v_{1,1}, v_{1,0}),$$
$$(v_{1,0}, v_{1,k-1}), N^-(v_{1,k-1}, v_{1,j+1}), (v_{1,j+1}, v_{0,j+1}), (v_{0,j+1}, v_{0,j}).$$
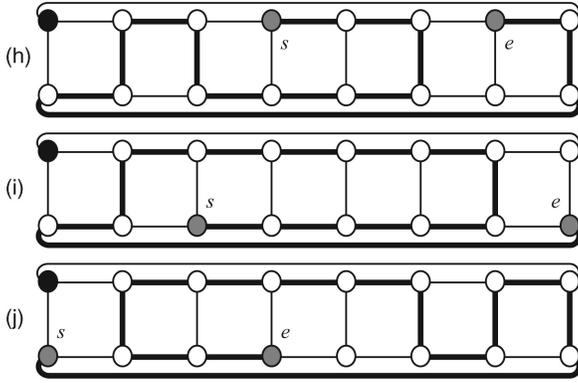
Fig. 2. Cases (h), (i), and (j) when $k = 8$.

This path has length $2k - 4$ and is similar to the path depicted in Fig. 1a.

Case f. s and e lie on different rows, with $s = v_{0,i}$, $e = v_{1,j}$, and $i < j$. Consider the path

$$C_{j-1}^+(v_{0,i}, v_{1,i}), Z^-(v_{1,i}, v_{1,2}), (v_{1,2}, v_{1,1}), (v_{1,1}, v_{1,0}),$$
$$(v_{1,0}, v_{1,k-1}), N^-(v_{1,k-1}, v_{1,j}).$$

This path has length $2k - 4$ and is similar to the path depicted in Fig. 1b.

Case g. s and e both lie on row 1, with $s = v_{1,i}$, $e = v_{1,j}$, and $i < j$. Consider the path

$$N^-(v_{1,i}, v_{1,1}), (v_{1,1}, v_{1,0}), (v_{1,0}, v_{1,k-1}), N^-(v_{1,k-1}, v_{1,j+2}),$$
$$(v_{1,j+2}, v_{0,j+2}), (v_{0,j+2}, v_{0,j+1}), (v_{0,j+1}, v_{0,j}), C_{i+1}^+(v_{0,j}, v_{1,j}).$$

This path has length $2k - 4$ and is similar to the path depicted in Fig. 1d.

Suppose now that one of s and e is odd and the other is even and, further, that s and e lie on the same row. Without loss of generality, there are three cases.

Case h. s and e both lie on row 0, with $s = v_{0,i}$ odd, $e = v_{0,j}$ even, and $i < j$. Consider the path

$$C_{j-1}^+(v_{0,i}, v_{1,i}), Z^-(v_{1,i}, v_{1,1}), (v_{1,1}, v_{1,0}), (v_{1,0}, v_{1,k-1}),$$
$$N^-(v_{1,k-1}, v_{1,j+1}), (v_{1,j+1}, v_{0,j+1}), (v_{0,j+1}, v_{0,j}).$$

This path has length $2k - 3$ and is as depicted in Fig. 2h.

Case i. s and e both lie on row 1, with $s = v_{1,i}$ odd, $e = v_{1,j}$ even, and $0 \neq i < j$. Consider the path

$$C_{j-1}^+(v_{1,i}, v_{0,i}), Z^-(v_{0,i}, v_{0,2}), (v_{0,2}, v_{0,1}), (v_{0,1}, v_{1,1}),$$
$$(v_{1,1}, v_{1,0}), (v_{1,0}, v_{1,k-1}), N^-(v_{1,k-1}, v_{1,j}).$$

This path has length $2k - 3$ and is as depicted in Fig. 2i.

Case j. s and e both lie on row 1, with $s = v_{1,0}$ and $e = v_{1,j}$ even. Consider the path

$$(v_{1,0}, v_{1,k-1}), N^-(v_{1,k-1}, v_{1,j+2}), (v_{1,j+2}, v_{0,j+2}),$$
$$(v_{0,j+2}, v_{0,j+1}), (v_{0,j+1}, v_{0,j}), C_1^-(v_{0,j}, v_{1,j}).$$

This path has length $2k - 3$ and is as depicted in Fig. 2j.

Suppose now that one of s and e is odd and the other is even and, further, that s and e lie on different rows. Without loss of generality, there are five cases.
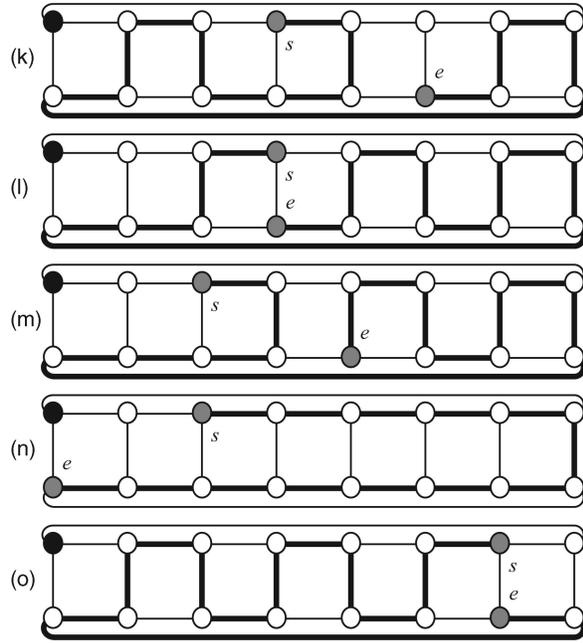
Fig. 3. Cases (k), (l), (m), (n), and (o) when $k = 8$.

Case k. s lies on row 0, and e lies on row 1, with $s = v_{0,i}$ odd, $e = v_{1,j}$ even, and $i < j$. Consider the path

$$C_{j-1}^+(v_{0,i}, v_{1,i}), Z^-(v_{1,i}, v_{1,1}), (v_{1,1}, v_{1,0}), (v_{1,0}, v_{1,k-1}),$$
$$N^-(v_{1,k-1}, v_{1,j}).$$

This path has length $2k - 3$ and is as depicted in Fig. 3k.

Case l. s and e lie on different rows, with $s = v_{0,i}$ odd, $e = v_{1,i}$ even, and $i \neq 1$. Consider the path

$$Z^-(v_{0,i}, v_{0,3}), (v_{0,3}, v_{0,2}), (v_{0,2}, v_{1,2}), (v_{1,2}, v_{1,1}), (v_{1,1}, v_{1,0}),$$
$$(v_{1,0}, v_{1,k-1}), N^-(v_{1,k-1}, v_{1,j}).$$

This path has length $2k - 3$ and is as depicted in Fig. 3l.

Case m. s and e lie on different rows, with $s = v_{0,i}$ even, $e = v_{1,i}$ odd, and $i < j$. Consider the path

$$C_{j-1}^+(v_{0,i}, v_{1,i}), Z^-(v_{1,i}, v_{1,2}), (v_{1,2}, v_{1,1}), (v_{1,1}, v_{1,0}),$$
$$(v_{1,0}, v_{1,k-1}), N^-(v_{1,k-1}, v_{1,j+1}), (v_{1,j+1}, v_{0,j+1}),$$
$$(v_{0,j+1}, v_{0,j}), (v_{0,j}, v_{1,j}).$$

This path has length $2k - 3$ and is as depicted in Fig. 3m.

Case n. s and e lie on different rows, with $s = v_{0,i}$ even and $e = v_{1,0}$. Consider the path

$$C_{j-1}^+(v_{0,i}, v_{1,i}), Z^-(v_{1,i}, v_{1,2}), (v_{1,2}, v_{1,1}), (v_{1,1}, v_{1,0}).$$

This path has length $2k - 3$ and is as depicted in Fig. 3n.

Case o. s and e lie on different rows, with $s = v_{0,i}$ even and $e = v_{1,i}$ odd. Consider the path

$$Z^-(v_{0,i}, v_{0,2}), (v_{0,2}, v_{0,1}), (v_{0,1}, v_{1,1}), (v_{1,1}, v_{1,0}),$$
$$(v_{1,0}, v_{1,k-1}), N^-(v_{1,k-1}, v_{1,j+1}), (v_{1,j+1}, v_{1,j}).$$

This path has length $2k - 3$ and is as depicted in Fig. 3o. The result follows.   □

The following lemma proves to be useful throughout.

Lemma 2. *Let $k \geq 4$ be even and consider the row torus $rt(0, p-1)$ in Q_2^k, where $2 \leq p \leq k$. If the pair of distinct nodes $\{s, e\}$ of the row torus is odd (respectively, even), then there is a path $\rho(s, e)$ in the row torus joining s and e of length $pk - 1$ (respectively, $pk - 2$).*

Proof. We proceed by induction on p. Suppose that $p = 2$ and consider the row torus $rt(0, 1)$. Without loss of generality, we may assume that $e = v_{0,0}$.

Suppose that $s = v_{0,i}$ is odd. The path

$$C_{k-1}^+(s, v_{1,i}), Z^-(v_{1,i}, v_{1,1}), (v_{1,1}, v_{1,0}), (v_{1,0}, e)$$

has length $2k - 1$.

Suppose that $s = v_{0,i}$ is even. The path

$$C_{k-1}^+(s, v_{1,i}), Z^-(v_{1,i-2}, v_{1,2}), (v_{1,2}, v_{1,1}), (v_{1,1}, v_{1,0}), (v_{1,0}, e)$$

has length $2k - 2$.

Suppose that $s = v_{1,i}$ is odd. The path

$$C_{k-1}^+(s, v_{0,i}), Z^-(v_{0,i}, e)$$

has length $2k - 1$.

Suppose that $s = v_{1,i}$ is even. The path

$$C_{k-1}^+(s, v_{0,i}), Z^-(v_{0,i}, v_{0,1}), (v_{0,1}, e)$$

has length $2k - 2$. Therefore, the result holds for $p = 2$.

Suppose, as our induction hypothesis, that the result holds for all p such that $1 \leq p < q$, where $1 < q \leq k - 1$. Consider $rt(0, q)$.

Case a. It is not the case that s lies on row 0 and e lies on row q, and it is not the case that s lies on row q and e lies on row 0.

Without loss of generality, assume that s and e lie in $rt(0, q-1)$. By the induction hypothesis, there is a path $\rho(s, e)$ in $rt(0, q-1)$ of length $qk - 1$ (respectively, $qk - 2$) if $\{s, e\}$ is odd (respectively, even). A simple counting argument yields that the path $\rho(s, e)$ must contain a link $(v_{q-1,i}, v_{q-1,i+1})$ lying on row $q - 1$. Consider the path

$$\rho(s, v_{q-1,i}), (v_{q-1,i}, v_{q,i}), (v_{q,i}, v_{q,i-1}), (v_{q,i-1}, v_{q,i-2}), \ldots,$$
$$(v_{q,i+2}, v_{q,i+1}), (v_{q,i+1}, v_{q-1,i+1}), \rho(v_{q-1,i+1}, e).$$

This path is as required (with reference to our construction as detailed at the beginning of this section, an alternative description of this path would be as that obtained by joining $\rho(s, e)$ to the cycle

$$(v_{q,0}, v_{q,1}), (v_{q,1}, v_{q,2}), \ldots, (v_{q,k-2}, v_{q,k-1}), (v_{q,k-1}, v_{q,0}),$$

over the links $(v_{q-1,i}, v_{q-1,i+1})$ and $(v_{q,i}, v_{q,i+1})$).

Case b. The node s lies on row 0, and the node e lies on row q.

If $e = v_{q,i}$, then define $e' = v_{q-1,i-1}$. Note that e is odd if and only if e' is odd. By the induction hypothesis, there is a path $\rho(s, e')$ in $rt(0, q-1)$ of length $qk - 1$ (respectively, $qk - 2$) if $\{s, e\}$ is odd (respectively, even). The path

$$\rho(s, e'), (e', v_{q,i-1}), (v_{q,i-1}, v_{q,i-2}), (v_{q,i-2}, v_{q,i-3}),$$
$$\ldots, (v_{q,i+1}, e)$$

is as required.

The result follows by induction. □
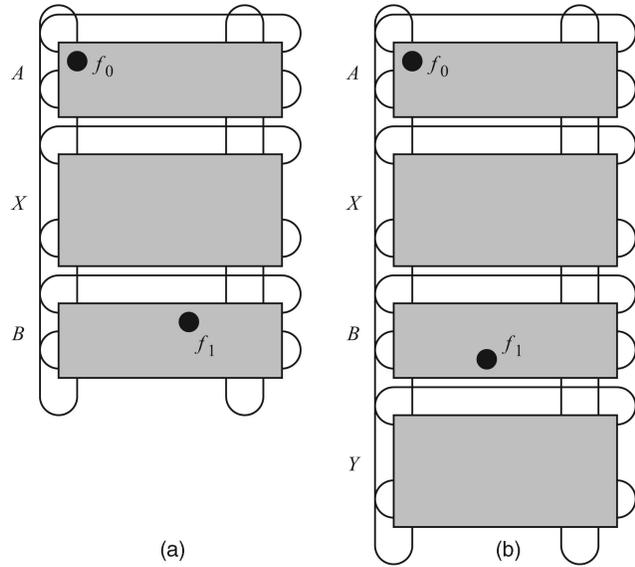
Fig. 4. Partitioned Q_2^k's.

We now deal with the first scenario in the base case.

Proposition 3. *Consider the k-ary 2-cube Q_2^k, where $k \geq 6$ is even and where two of the nodes are faulty. Let s and e be any two distinct nonfaulty nodes. There is a path of length at least $k^2 - 5$ (respectively, $k^2 - 6$) from s to e if $\{s, e\}$ is odd (respectively, even).*

Proof. Without loss of generality, suppose that the two faulty nodes are $f_0 = v_{0,0}$ and $f_1 = v_{p,p'}$, with $p \neq 0$. We begin by partitioning Q_2^k into three or four row tori. If $p \in \{1, 2, k-2, k-1\}$, then

- if $p = 1$ or $p = 2$, then we partition Q_2^k into $A = rt(k-1, 0)$, $B = rt(1, 2)$, and $X = rt(3, k-2)$, and
- if $p = k - 2$ or $p = k - 1$, then we partition Q_2^k into $A = rt(0, 1)$, $X = rt(2, k-3)$, and $B = rt(k-2, k-1)$.

If $p \notin \{1, 2, k-2, k-1\}$, then

- if $p \neq 3$ is odd, then we partition Q_2^k into $A = rt(0, 1)$, $X = rt(2, p-2)$, $B = rt(p-1, p)$, and $Y = rt(p+1, k-1)$,
- if $p = 3$, then we partition Q_2^k into $A = rt(k-1, 0)$, $X = rt(1, 2)$, $B = rt(3, 4)$, and $Y = rt(5, k-2)$,
- if p is even, then we partition Q_2^k into $A = rt(0, 1)$, $X = rt(2, p-1)$, $B = rt(p, p+1)$, and $Y = rt(p+2, k-1)$.

The outcome is that we have one of the two partitioned structures as in Fig. 4, where consecutive row tori are joined by column links. In particular, without loss of generality, we may assume that when the partition involves three row tori, we have the situation as in Fig. 4a, with $f_0 = v_{0,0} \in A = rt(0, 1)$, $X = rt(2, k-3)$ and $f_1 \in B = rt(k-2, k-1)$; and when the partition involves four row tori, we have the situation as in Fig. 4b, with $f_0 = v_{0,0} \in A = rt(0, 1)$, $X = rt(2, q-1)$, $f_1 \in B = rt(q, q+1)$, and $Y = rt(q+2, k-1)$, for some even q, where $4 \leq q \leq k - 4$.

Throughout the proof, $\epsilon = 1$ if $\{s, e\}$ is odd, and $\epsilon = 2$ if $\{s, e\}$ is even.

Case a. Q_2^k is partitioned into three row tori.

Subcase a.1. The nodes s and e both lie in A.

By Lemma 1, there exists a path $\rho_A(s, e)$ in A of length at least $2k - 2 - \epsilon$. A simple counting argument yields that there is at least one link of $\rho_A(s, e)$ lying on row 1; without loss of generality, let $(v_{1,i}, v_{1,i+1})$ be such a link (the case when the link is $(v_{1,i+1}, v_{1,i})$ is almost identical). By Lemma 2, there exists a path $\rho_X(v_{2,i}, v_{2,i+1})$ in X of length $k(k - 4) - 1$. Let $\rho(s, e)$ be obtained by joining $\rho_A(s, e)$ to $\rho_X(v_{2,i}, v_{2,i+1})$ over $(v_{1,i}, v_{1,i+1})$. Again, a simple counting argument yields that there are at least two nonincident links of $\rho(s, e)$ lying on row $k - 3$; without loss of generality, let $(v_{k-3,j}, v_{k-3,j+1})$ be such a link where $v_{k-2,j} \neq f_1 \neq v_{k-2,j+1}$. By Lemma 1, there exists a path $\rho_B(v_{k-2,j}, v_{k-2,j+1})$ in B of length at least $2k - 3$. The path obtained by joining $\rho(s, e)$ to $\rho_B(v_{k-2,j}, v_{k-2,j+1})$ over $(v_{k-3,j}, v_{k-3,j+1})$ has length at least $k^2 - 4 - \epsilon$.

Subcase a.2. The node s is in A, and the node e is in X.

Choose $v_{1,i}$ such that $v_{1,i}$ is odd if and only if s is even and $v_{2,i} \neq e$ (a simple counting argument yields that such a node $v_{1,i}$ exists). By Lemma 1, there exists a path $\rho_A(s, v_{1,i})$ in A of length at least $2k - 3$. By Lemma 2, there exists a path $\rho_X(v_{2,i}, e)$ in X of length $k(k - 4) - \epsilon$. Let $\rho(s, e)$ be the path

$$\rho_A(s, v_{1,i}), (v_{1,i}, v_{2,i}), \rho_X(v_{2,i}, e).$$

A simple counting argument yields that $\rho(s, e)$ contains at least two nonincident links on row $k - 3$; without loss of generality, let $(v_{k-3,j}, v_{k-3,j+1})$ be a link of $\rho(s, e)$ such that $v_{k-2,j} \neq f_1 \neq v_{k-2,j+1}$. By Lemma 1, there exists a path $\rho_B(v_{k-2,j}, v_{k-2,j+1})$ in B of length at least $2k - 3$. The path obtained by joining $\rho(s, e)$ to $\rho_B(v_{k-2,j}, v_{k-2,j+1})$ over $(v_{k-3,j}, v_{k-3,j+1})$ has length at least $k^2 - 4 - \epsilon$.

Subcase a.3. The node s is in A, and the node e is in B.

Choose $v_{1,i}$ such that $v_{1,i}$ is odd if and only if s is even and $v_{1,i} \neq s$. By Lemma 1, there exists a path $\rho_A(s, v_{1,i})$ in A of length at least $2k - 3$. Choose $v_{k-2,j}$ such that $v_{k-2,j}$ is odd if and only if e is even and $f_1 \neq v_{k-2,j}$. By Lemma 1, there exists a path $\rho_B(v_{k-2,j}, e)$ in B of length at least $2k - 3$. By Lemma 2, there exists a path $\rho_X(v_{2,i}, v_{k-3,j})$ in X of length $k(k - 4) - \epsilon$. The path

$$\rho_A(s, v_{1,i}), (v_{1,i}, v_{2,i}), \rho_X(v_{2,i}, v_{k-3,j}), (v_{k-3,j}, v_{k-2,j}),$$
$$\rho_B(v_{k-2,j}, e)$$

has length at least $k^2 - 4 - \epsilon$.

Subcase a.4. The nodes s and e both lie in X.

By Lemma 2, there exists a path $\rho_X(s, e)$ in X of length $k(k - 4) - \epsilon$. A simple counting argument yields that $\rho_X(s, e)$ always contains at least one link on row 2 and also that there are two nonincident links on row $k - 3$, unless we have the special situation where $k = 6$, s and e have a common neighbor on row $k - 3$ with this neighbor not lying on $\rho_X(s, e)$, and neither s nor e is adjacent on $\rho_X(s, e)$ to a node on row $k - 3$. Suppose that there are two nonincident links on row $k - 3$. Without loss of generality, let $(v_{k-3,j}, v_{k-3,j+1})$ and $(v_{2,i}, v_{2,i+1})$ be links of $\rho_X(s, e)$, where $v_{k-2,j} \neq f_1 \neq v_{k-2,j+1}$. By Lemma 1, there exists a path $\rho_B(v_{k-2,j}, v_{k-2,j+1})$ (respectively, $\rho_A(v_{1,i}, v_{1,i+1})$) in B

(respectively, A) of length at least $2k - 3$. Without loss of generality, suppose that the nodes $v_{k-3,j}$, $v_{k-3,j+1}$, $v_{2,i}$, and $v_{2,i+1}$ come in that order as we move along the path $\rho_X(s, e)$. The path

$$\rho_X(s, v_{k-3,j}), (v_{k-3,j}, v_{k-2,j}), \rho_B(v_{k-2,j}, v_{k-2,j+1}),$$
$$(v_{k-2,j+1}, v_{k-3,j+1}), \rho_X(v_{k-3,j+1}, v_{2,i}), (v_{2,i}, v_{1,i}),$$
$$\rho_A(v_{1,i}, v_{1,i+1}), (v_{1,i+1}, v_{2,i+1}), \rho_X(v_{2,i+1}, e)$$

has length at least $k^2 - 4 - \epsilon$.

Alternatively, suppose that we are in the special situation described above (and so, $k = 6$). Without loss of generality, suppose that $s = v_{3,0}$ and $e = v_{3,2}$, so the path $(v_{3,3}, v_{3,4}), (v_{3,4}, v_{3,5})$ is a subpath of $\rho_X(s, e)$. If $f_1 \neq v_{4,4}$, then we can find two links $(v_{3,j}, v_{3,j+1})$ and $(v_{2,i}, v_{2,i+1})$ of $\rho_X(s, e)$, as above, and so obtain our path as required. Therefore, suppose that $f_1 = v_{4,4}$. Let $\rho_B(v_{4,3}, v_{4,5})$ be the path

$$(v_{4,3}, v_{4,2}), (v_{4,2}, v_{4,1}), (v_{4,1}, v_{4,0}), (v_{4,0}, v_{4,5}),$$

and join $\rho_X(s, e)$ to $\rho_B(v_{4,3}, v_{4,5})$ over $(v_{3,3}, v_{3,4}), (v_{3,4}, v_{3,5})$ to obtain the path $\rho(s, e)$ of length $16 - \epsilon$. We can now join $\rho(s, e)$ to the cycle induced by the nodes on row 5 over two appropriate links and to an appropriate path $\rho_A(v_{1,i}, v_{1,i+1})$ in A of length at least 9, as we did above, to obtain our required path of length at least $32 - \epsilon$ (that is, $k^2 - 4 - \epsilon$).

The remaining subcases are essentially identical to those already considered.

Case b. Q_2^k is partitioned into four row tori.

If s and e lie in $A \cup X \cup B$, then by the analysis for case a, there is a path $\rho(s, e)$ in $A \cup X \cup B$ (and the connecting column links) of length at least $k(q + 2) - 4 - \epsilon$ (note that all paths constructed in case a actually lie in the row torus induced by $A \cup X \cup B$). A simple counting argument yields that there is at least one link of $\rho(s, e)$ on row $q + 1$ or on row 0; without loss of generality, suppose that it is row $q + 1$ and let $(v_{q+1,j}, v_{q+1,j+1})$ be such a link. By Lemma 2, there exists a path $\rho_Y(v_{q+2,j}, v_{q+2,j+1})$ in Y of length $k(k - 1 - q - 1) - 1$. Join $\rho(s, e)$ to $\rho_Y(v_{q+2,j}, v_{q+2,j+1})$ over $(v_{q+1,j}, v_{q+1,j+1})$ to obtain a path of length at least $k^2 - 4 - \epsilon$. A similar argument holds should s and e lie in $B \cup Y \cup A$.

Necessarily, the only remaining case is when s lies in X and e lies in Y. Let $v_{0,i}$ be such that s and e do not lie on column i and $v_{0,i}$ is odd if and only if e is odd. By Lemma 2, there exists a path $\rho_Y(v_{k-1,i}, e)$ in Y of length $k(k - 1 - q - 1) - 1$. Let $v_{1,j}$ be such that s does not lie on column j and $v_{1,j}$ is odd if and only if s is odd. By Lemma 2, there exists a path $\rho_X(s, v_{2,j})$ in X of length $k(q - 2) - 1$. By Lemma 1, there exists a path $\rho_A(v_{1,j}, v_{0,i})$ in A of length at least $2k - 2 - \epsilon$. Let $\rho(s, e)$ be the path

$$\rho_X(s, v_{2,j}), (v_{2,j}, v_{1,j}), \rho_A(v_{1,j}, v_{0,i}), (v_{0,i}, v_{k-1,i}), \rho_Y(v_{k-1,i}, e).$$

Necessarily, there are at least two nonincident links of $\rho_X(s, v_{2,j})$ on row $q - 1$; without loss of generality, let $(v_{q-1,m}, v_{q-1,m+1})$ be such a link with $v_{q,m} \neq f_1 \neq v_{q,m+1}$. By Lemma 1, there exists a path $\rho_B(v_{q,m}, v_{q,m+1})$ in B of length $2k - 3$. The path obtained by joining $\rho(s, e)$ to $\rho_B(v_{q,m}, v_{q,m+1})$ over $(v_{q-1,m}, v_{q-1,m+1})$ has length at least $k^2 - 4 - \epsilon$. The result follows. $\qquad \square$
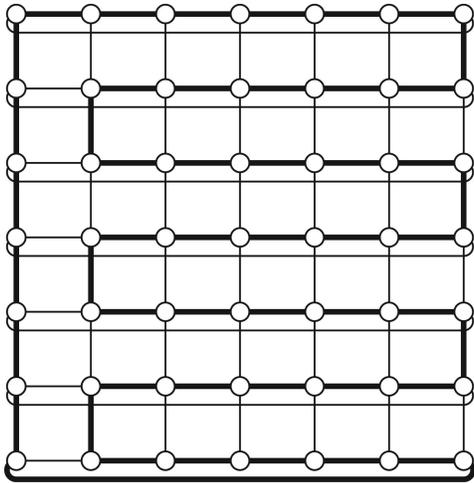
Fig. 5. The Hamiltonian cycle $E_{0,0}$ in $rt(0,6)$ in Q_2^7.

We deal with the case when $k = 4$ later (as we do also for subsequent propositions).

The next proposition deals with the next scenario in the base case.

Proposition 4. *Consider the k-ary 2-cube Q_2^k, where $k \geq 6$ is even and where one of the nodes is faulty. Let s and e be any two distinct nonfaulty nodes. There is a path of length at least $k^2 - 3$ (respectively, $k^2 - 4$) from s to e if $\{s, e\}$ is odd (respectively, even).*

Proof. The proof is a much simplified version of the proof of Proposition 3. Essentially, we partition Q_2^k into two row tori, $A = rt(0, 1)$ and $X = rt(2, k - 1)$, and follow the constructions in subcases a.1, a.2, and a.4 of Case a. The result follows. □

We now consider when there are only faulty links in Q_2^k, but first, we construct some basic Hamiltonian circuits on row tori. Consider the row torus $rt(0, p - 1)$ in Q_2^k, for some even p, where $2 \leq p \leq k - 1$. For every even $i \in \{0, 1, \ldots, p - 2\}$, build the following cycle C_i:

$$(v_{i,0}, v_{i,1}), (v_{i,1}, v_{i,2}), \ldots, (v_{i,k-2}, v_{i,k-1}), (v_{i,k-1}, v_{i+1,k-1}),$$
$$(v_{i+1,k-1}, v_{i+1,k-2}), \ldots, (v_{i+1,1}, v_{i+1,0}), (v_{i+1,0}, v_{i,0}).$$

Join the cycle C_0 to the cycle C_2 over the links $(v_{1,0}, v_{1,1})$ and $(v_{2,0}, v_{2,1})$ and denote the resulting cycle by $E_{0,0}$. Now, join $E_{0,0}$ to the cycle C_4 over the links $(v_{3,0}, v_{3,1})$ and $(v_{4,0}, v_{4,1})$ and denote the resulting cycle by $E_{0,0}$ also. Proceed in this way to obtain the Hamiltonian cycle $E_{0,0}$ of the row torus $rt(0, p - 1)$ rooted at $v_{0,0}$.

If $3 \leq p \leq k - 1$ is odd, then build the cycle $E_{0,0}$ in the row torus $rt(0, p - 2)$ and join it to the cycle induced by the nodes on row $p - 1$ over the links $(v_{p-2,0}, v_{p-2,1})$ and $(v_{p-1,0}, v_{p-1,1})$. Denote the resulting cycle as the cycle $E_{0,0}$ of $rt(0, p - 1)$ rooted at $v_{0,0}$. The Hamiltonian cycle $E_{0,0}$ in $rt(0, 6)$ in Q_2^7 can be visualized as in Fig. 5.

Note that we also have the Hamiltonian cycles $E_{0,i}$ of $rt(0, p - 1)$, for all $p \in \{2, 3, \ldots, k\}$ and $i \in \{1, 2, \ldots, k - 1\}$, obtained by starting the above process at the root node $v_{0,i}$ as opposed to node $v_{0,0}$.
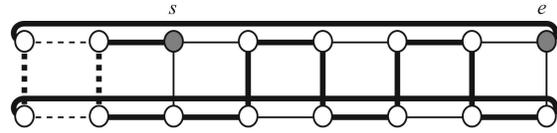
Fig. 6. Joining $\rho_0(s, e)$ to the amended cycle C.

Proposition 5. *Consider the k-ary 2-cube Q_2^k, where $k \geq 6$ is even and where there is one faulty link. Let s and e be any two distinct nodes in the row torus $rt(0, p - 1)$, where $2 \leq p \leq k$. There is a path in $rt(0, p - 1)$ from s to e of length $pk - 1$ (respectively, $pk - 2$) if $\{s, e\}$ is odd (respectively, even).*

Proof. By Lemma 2, we may assume that the faulty link lies in $rt(0, p - 1)$. Without loss of generality, we may assume that the faulty link is either $(v_{a,0}, v_{a+1,0})$ or $(v_{a,0}, v_{a,1})$, where $0 \leq a \leq p - 2$. As before, $\epsilon = 1$ if $\{s, e\}$ is odd, and $\epsilon = 2$ if $\{s, e\}$ is even.

Case a. $a = 0$, and the faulty link is $(v_{0,0}, v_{1,0})$.

Subcase a.1. s and e lie on row 0.

If $s = v_{0,i}$ and $e = v_{0,j}$, then without loss of generality, we may assume that $i < j$ and that it is not the case that $i = 0$ and $j = k - 1$.

Suppose that it is not the case that $i = 1$ and $j = k - 1$. Let $\rho_0(s, e)$ be the path

$$(s, v_{0,i-1}), (v_{0,i-1}, v_{0,i-2}), \ldots, (v_{0,j+1}, e).$$

Note that the length of $\rho_0(s, e)$ is odd if and only if $\{s, e\}$ is odd, so there are an even number of nodes on row 0 that are not on $\rho_0(s, e)$ if and only if $\{s, e\}$ is odd. Let C be the cycle induced by the nodes on row 1. Iteratively join C to appropriate links $(v_{0,l}, v_{0,l+1})$ over $(v_{1,l}, v_{1,l+1})$ so that the nodes used on row 0 do not already appear on $\rho_0(s, e)$. Links should be replaced (by paths) so that if $\{s, e\}$ is odd (respectively, even), then every node of $rt(0, 1)$ appears on the (amended) cycle C or on $\rho_0(s, e)$ (respectively, except one). Join $\rho_0(s, e)$ to C over two corresponding links (this is always possible) and denote the new path by $\rho_A(s, e)$. The path $\rho_A(s, e)$ has length $2k - \epsilon$. This construction can be visualized in Fig. 6, where the dashed links show how $\rho_0(s, e)$ is joined to the amended C.

Suppose that $i = 1$ and $j = k - 1$. Let $\rho_0(s, e)$ be the path

$$(s, v_{0,2}), (v_{0,2}, v_{0,3}), \ldots, (v_{0,k-2}, e).$$

Let C be the cycle induced by the nodes on row 1. Join $\rho_0(s, e)$ to C over $(v_{0,1}, v_{0,2})$ and $(v_{1,1}, v_{1,2})$ and denote the new path by $\rho_A(s, e)$. The path $\rho_A(s, e)$ has length $2k - 2$.

If $p = 2$, then we are done. If $p > 3$, then let D be the Hamiltonian cycle $E_{2,0}$ in the row torus $rt(2, p - 1)$, and if $p = 3$, then let D be the cycle induced by the nodes on row 2. Join $\rho_A(s, e)$ to D over two corresponding links, and the resulting path is as required.

Subcase a.2. s lies on row 0, and e lies on row 1.

Let $s = v_{0,i}$ and $e = v_{1,j}$; without loss of generality, we may assume that $i \neq k - 1$. If $i \neq 1$, then let e' be a neighbor of s on row 0 that does not lie in the same column as e. If $i = 1$ and $j \neq 2$, then let $e' = v_{0,2}$. Either

way, let $\rho_0(s, e')$ be a path on row 0 of length $k - 1$. If $i = 1$ and $j = 2$, then let $e' = v_{0,3}$ and let $\rho_0(s, e')$ be a path on row 0 of length $k - 2$.

Let s' be the neighbor of e' on row 1 and let $\rho_1(s', e)$ be a path on row 0 that contains the link $(v_{1,0}, v_{1,1})$. Define the path $\rho_A(s, e)$ as

$$\rho_0(s, e'), (e', s'), \rho_1(s', e).$$

Iteratively join $\rho_A(s, e)$ to appropriate links $(v_{1,l}, v_{1,l+1})$ over $(v_{0,l}, v_{0,l+1})$ so that the nodes used on row 1 do not already appear on $\rho_A(s, e)$. Links should be replaced (by paths) so that if $\{s, e\}$ is odd (respectively, even), then every node of $rt(0, 1)$ appears on (the amended) $\rho_A(s, e)$ (respectively, except one).

If $p = 2$, then we are done. If $p > 3$, then let D be the Hamiltonian cycle $E_{2,0}$ in the row torus $rt(2, p - 1)$, and if $p = 3$, then let D be the cycle induced by the nodes on row 2. Join $\rho_A(s, e)$ to D over the links $(v_{1,0}, v_{1,1})$ and $(v_{2,0}, v_{2,1})$. The resulting path is as required.

Note that if $p = 2$, then we have covered all cases, so henceforth, we assume that $p \geq 3$.

Subcase a.3. s lies on row 0, and e lies on rows $2, 3, \ldots, p - 1$.

Suppose that $s = v_{0,i}$. If $i \neq 1$, then define $e' = v_{0,i-1}$, and if $i = 1$, then define $e' = v_{0,i+1}$. Define the path $\rho_0(s, e')$ to be the path on row 0 of length $k - 1$. Let e'' be the neighbor of e' on row 1, and let e''' be a neighbor of e'' on row 1 that does not lie in the same column as e. Define the path $\rho_1(e'', e''')$ as the path of length $k - 1$ on row 1. Define the path $\rho_A(s, e''')$ as

$$\rho_0(s, e'), (e', e''), \rho_1(e'', e''').$$

The path $\rho_A(s, e''')$ has length $2k - 1$.

Let s' be the neighbor of e''' on row 2. If $p \geq 4$, then by Lemma 2, there is a path $\rho_X(s', e)$ in $rt(2, p - 1)$ of length $k(p - 2) - \epsilon$, and the path

$$\rho_A(s, e'''), (e''', s'), \rho_X(s', e)$$

is as required. If $p = 3$, then define the path $\rho_X(s', e)$ to be a path on row 2 and let $\rho(s, e)$ be the path

$$\rho_A(s, e'''), (e''', s'), \rho_X(s', e).$$

Iteratively join $\rho(s, e)$ to appropriate links $(v_{2,l}, v_{2,l+1})$ over $(v_{1,l}, v_{1,l+1})$ so that the nodes used on row 2 do not already appear on $\rho(s, e)$. Links should be replaced (by paths) so that if $\{s, e\}$ is odd (respectively, even), then every node of row 2 appears on the amended path (respectively, except one). The resulting path is as required.

Subcase a.4. s and e lie on row 1.

Proceed as in subcase a.1 to build a path (analogous to) $\rho_A(s, e)$. The path $\rho_A(s, e)$ is such that it contains a link on row 1. Join $\rho_A(s, e)$ to the cycle D, as constructed in subcase a.1 and over corresponding links, to obtain a required path.

Subcase a.5. s lies on rows $1, 2, \ldots, p - 1$, and e lies on rows $2, 3, \ldots, p - 1$.

By Lemma 2, there exists a path $\rho(s, e)$ in $rt(1, p - 1)$ of length $(p - 1)k - \epsilon$. There is at least one link of $\rho(s, e)$ on

row 1 that is not incident with $v_{1,0}$. Join $\rho(s, e)$ to the cycle induced by the nodes on row 0 over two corresponding links to obtain a required path.

Case b. $0 \neq a \neq p - 2$, and the faulty link is $(v_{a,0}, v_{a+1,0})$.

Subcase b.1. s and e lie on rows $0, 1, \ldots, a$.

By Lemma 2, there is a path $\rho_A(s, e)$ in $rt(0, a)$ of length $(a + 1)k - \epsilon$. Either there exist two disjoint links of $\rho_A(s, e)$ on row a, and so, we have a link of $\rho_A(s, e)$ on row a that is not incident with $v_{a,0}$, or $k = 6$, and the nodes $v_{a,2}, v_{a,3}, v_{a,4}$ constitute s, e, and a node not on $\rho_A(s, e)$. However, in this latter case, let $E_{0,0}$ be the Hamiltonian cycle in $rt(0, a)$ but with the subpath from s to e involving (some of) the nodes $v_{a,2}, v_{a,3}, v_{a,4}$ removed (therefore, the length of this subpath is one if $\{s, e\}$ is odd and two if $\{s, e\}$ is even). Either way, we obtain a path, call it $\rho_A(s, e)$, in $rt(0, a)$ of length $(a + 1)k - \epsilon$ with the property that there is a link of $\rho_A(s, e)$ on row a that is not incident with $v_{a,0}$.

Join $\rho_A(s, e)$ to the Hamiltonian cycle $E_{a+1,0}$ of $rt(a + 1, p - 1)$ over some appropriate links, and the path obtained is as required.

Subcase b.2. s lies on rows $0, 1, \ldots, a$, and e lies on rows $a + 1, a + 2, \ldots, p - 1$.

Suppose that we can choose e' on row a such that $v_{a,0} \neq e' \neq s$, e and e' are not adjacent, and $\{s, e'\} = \{s, e\}$. If so, then by Lemma 2, there is a path $\rho_A(s, e')$ in $rt(0, a)$ of length $(a + 1)k - \epsilon$ so that e is not adjacent to e'. Define s' to be the neighbor of e' on row $a + 1$. By Lemma 2, there is a path $\rho_X(s', e)$ in $rt(a + 1, p - 1)$ of length $(p - a - 1)k - 1$. The path

$$\rho_A(s, e'), (e', s'), \rho_X(s', e)$$

is as required.

Alternatively, suppose that e' does not exist. This only happens when $k = 6$, and $(s = v_{a,2}$ and $e = v_{a+1,4})$ or $(s = v_{a,4}$ and $e = v_{a+1,2})$. Define $e' = v_{a,3}$ and let $E_{0,0}$ be the Hamiltonian cycle in $rt(0, a)$ with the link (s, e') removed; call this path $\rho_A(s, e')$. By Lemma 2, there is a path $\rho_X(v_{a+1,3}, e)$ in $rt(a + 1, p - 1)$ of length $(p - a - 1)k - 1$. The path

$$\rho_A(s, e'), (e', v_{a+1,3}), \rho_X(v_{a+1,3}, e)$$

is as required.

Case c. $a = 0$, and the faulty link is $(v_{0,0}, v_{0,1})$.

Subcase c.1. s and e lie on row 0.

Let $\rho_0(s, e)$ be the path on row 0 that contains the faulty link $(v_{0,0}, v_{0,1})$ and let C be the cycle induced by the nodes on row 1. Join $\rho_0(s, e)$ to C over the links $(v_{0,0}, v_{0,1})$ and $(v_{1,0}, v_{1,1})$ and denote the resulting path by $\rho(s, e)$. Iteratively join $\rho(s, e)$ to appropriate links $(v_{0,l}, v_{0,l+1})$ over $(v_{1,l}, v_{1,l+1})$ so that the nodes used on row 0 do not already appear on $\rho(s, e)$. Links should be replaced (by paths) so that if $\{s, e\}$ is odd (respectively, even), then every node of row 0 appears on the amended path (respectively, except one). Denote the amended path by $\rho(s, e)$ also.

If $p > 3$, then let D be the Hamiltonian cycle $E_{2,0}$ in $rt(2, p - 1)$, and if $p = 3$, then let D be the cycle induced by the nodes of row 2. Joining $\rho(s, e)$ to D over two corresponding links yields a path as required.

Subcase c.2. s lies on row 0, and e lies on row 1.

Suppose that $s = v_{0,i}$ and $e = v_{1,j}$. Without loss of generality, we may assume that i is odd.

If $\{s, e\}$ is odd and $1 \leq j < i$, then define $\rho(s, e)$ as

$$C_0^+(s, v_{1,i}), Z^-(v_{1,i}, v_{1,j+2}), (v_{1,j+2}, v_{1,j+1}), (v_{1,j+1}, v_{0,j+1}),$$
$$(v_{0,j+1}, v_{0,j}), C_1^-(v_{0,j}, e).$$

If $\{s, e\}$ is odd and $i < j \leq k - 1$, then define $\rho(s, e)$ as

$$C_1^-(s, v_{1,i}), Z^+(v_{1,i}, v_{1,j-2}), (v_{1,j-2}, v_{1,j-1}), (v_{1,j-1}, v_{0,j-1}),$$
$$(v_{0,j-1}, v_{0,j}), C_{k-1}^+(v_{0,j}, e).$$

If $\{s, e\}$ is odd and $i = j$, then define $\rho(s, e)$ as $C_0^+(s, e)$, and if $i \neq 1$, then define C as the cycle

$$C_1^-(v_{0,i-1}, v_{1,i-1}), (v_{1,i-1}, v_{0,i-1}).$$

If $\{s, e\}$ is even and $2 \leq j < i$, then define $\rho(s, e)$ as

$$C_0^+(s, v_{1,i}), Z^-(v_{1,i}, v_{1,j+3}), (v_{1,j+3}, v_{1,j+2}), (v_{1,j+2}, v_{0,j+2}),$$
$$(v_{0,j+2}, v_{0,j+1}), (v_{0,j+1}, v_{0,j}), C_0^-(v_{0,j}, e).$$

If $\{s, e\}$ is even and $j = 0$, then define $\rho(s, e)$ as

$$C_1^-(s, v_{1,i}), Z^+(v_{1,i}, v_{1,k-1}), (v_{1,k-1}, e).$$

If $\{s, e\}$ is even and $i < j \leq k - 1$, then define $\rho(s, e)$ as

$$C_1^-(s, v_{1,i}), Z^+(v_{1,i}, v_{1,j-3}), (v_{1,j-3}, v_{1,j-2}), (v_{1,j-2}, v_{1,j-1}),$$
$$(v_{1,j-1}, v_{0,j-1}), (v_{0,j-1}, v_{0,j}), C_0^+(v_{0,j}, e).$$

If $p > 3$, then let D be the Hamiltonian cycle $E_{2,0}$ of $rt(2, p - 1)$, and if $p = 3$, then let D be the cycle induced by the nodes on row 2. If there is a cycle C, then join C and D over two corresponding links and denote the new cycle by D also. Now, join $\rho(s, e)$ to the cycle D, and the path obtained is as required.

Subcase c.3. s lies on row 0, and e lies on rows $2, 3, \ldots, p - 1$.

Suppose that $p > 3$. If $\{s, e\}$ is even, then let the node e' on row 1 be such that e' and s have a common neighbor on row 0 and also such that e' does not lie on the same column as e. If $\{s, e\}$ is odd, then let e' be the neighbor of s on row 1. By the construction in subcase c.2, there is a path $\rho_A(s, e')$ in $rt(0, 1)$ of length $2k - \epsilon$.

Let s' be the neighbor of e' on row 2 (note that $s' \neq e$ and that $\{s', e\}$ is odd). By Lemma 2, there is a path $\rho_X(s', e)$ in $rt(2, p - 1)$ of length $(p - 2)k - 1$. The path

$$\rho_A(s, e'), (e', s'), \rho_X(s', e)$$

is as required.

Suppose that $p = 3$. Let s' be a neighbor of e on row 2 so that s' does not lie on the same column as s and let e' be the neighbor of s' on row 2. By the construction in subcase c.2, there is a path $\rho_A(s, e')$ in $rt(0, 1)$ of length $2k - \epsilon$. Let $\rho_X(s', e)$ be the path on row 2 of length $k - 1$. The path

$$\rho_A(s, e'), (e', s'), \rho_X(s', e)$$

is as required.

Subcase c.4. s and e lie on row 1.

Let $s = v_{1,i}$ and $e = v_{1,j}$; without loss of generality, we may assume that $i < j$. Let $\rho_1(s, e)$ be the path on row 1 containing the link $(v_{1,0}, v_{1,1})$. Join $\rho_1(s, e)$ to the cycle induced by the nodes on row 0 over the links $(v_{1,0}, v_{1,1})$ and $(v_{0,0}, v_{0,1})$ and denote the resulting path by $\rho_A(s, e)$. Iteratively join $\rho_A(s, e)$ to appropriate links $(v_{1,l}, v_{1,l+1})$ over $(v_{0,l}, v_{0,l+1})$ so that the nodes used on row 1 do not already appear on $\rho_A(s, e)$. Links should be replaced (by paths) so that if $\{s, e\}$ is odd (respectively, even), then every node of row 1 appears on the amended path (respectively, except one). Denote the amended path by $\rho(s, e)$.

If $p \geq 4$, then let D be the Hamiltonian cycle $E_{2,1}$ of $rt(2, p - 1)$, and if $p = 3$, then let D be the cycle induced by the nodes on row 2. Join $\rho(s, e)$ to D over two corresponding links, and the resulting path is as required.

Subcase c.5. s lies on row 1, and e lies on rows $2, 3, \ldots, p - 1$.

Suppose that $p \geq 4$. Let e' be a neighbor of s on row 1 such that e does not lie on the same column as e'. We now define a path $\rho_A(s, e')$ in $rt(0, 1)$. If $s = v_{1,1}$ and $e' = v_{1,0}$, then define $\rho_A(s, e')$ as

$$N^+(s, v_{1,k-1}), (v_{1,k-1}, v_{0,k-1}), (v_{0,k-1}, v_{0,0}), (v_{0,0}, e'),$$

and if $s = v_{1,0}$ and $e' = v_{1,1}$, then define $\rho_A(s, e')$ as

$$N^-(s, v_{1,2}), (v_{1,2}, v_{0,2}), (v_{0,2}, v_{0,1}), (v_{0,1}, e').$$

Otherwise, let $\rho_1(s, e')$ be the path on row 1 containing the link $(v_{1,0}, v_{1,1})$ and join $\rho_1(s, e')$ to the cycle induced by the nodes on row 0 (which contains the faulty link) over the links $(v_{1,0}, v_{1,1})$ and $(v_{0,0}, v_{0,1})$, denoting the resulting path by $\rho_A(s, e')$ (joining as we do results in the path $\rho_A(s, e')$ being fault free).

Let s' be the neighbor of e' on row 2. By Lemma 2, there is a path $\rho_X(s', e)$ in $rt(2, p - 1)$ of length $(p - 2)k - \epsilon$. The path

$$\rho_A(s, e'), (e', s'), \rho_X(s', e)$$

is as required.

Suppose that $p = 3$. Let e' be a node on row 1 such that $s \neq e'$ and e' is in a column adjacent to the column on which e lies. Clearly, $\{s, e\}$ is odd if and only if $\{s, e'\}$ is odd. We now build a path $\rho_A(s, e')$ in $rt(0, 1)$; without loss of generality, we may assume that $s = v_{1,i}$, $e' = v_{i,j}$, and $i < j$, with $i \neq 0$ (as usual, we can apply automorphisms of $rt(0, 1)$ if necessary). If $\{s, e\}$ is odd and $i \neq 1$, then define $\rho_A(s, e')$ as

$$C_1^-(s, v_{0,i}), Z^+(v_{0,i}, v_{0,j-1}), (v_{0,j-1}, v_{0,j}), C_0^+(v_{0,j}, e').$$

If $\{s, e\}$ is odd and $i = 1$, then define $\rho_A(s, e')$ as

$$N^+(s, v_{1,j-1}), (v_{1,j-1}, v_{0,j-1}), (v_{0,j-1}, v_{0,j}), C_0^+(v_{0,j}, e').$$

If $\{s, e\}$ is even and $s \neq 1$, then define $\rho_A(s, e')$ as

$$C_1^-(s, v_{0,i}), Z^+(v_{0,i}, v_{0,j-2}), (v_{0,j-2}, v_{0,j-1}), (v_{0,j-1}, v_{0,j}),$$
$$C_0^+(v_{0,j}, e').$$

If $\{s, e\}$ is even and $s = 1$, then define $\rho_A(s, e')$ as

$$N^+(s, v_{1,j-2}), (v_{1,j-2}, v_{0,j-2}), (v_{0,j-2}, v_{0,j-1}), (v_{0,j-1}, v_{0,j}),$$
$$C_0^+(v_{0,j}, e').$$

Let s' be the neighbor of e' on row 2 and let $\rho_X(s', e)$ be the path on row 2 of length $k - 1$. The path

$$\rho_A(s, e'), (e', s'), \rho_X(s', e)$$

is as required.

Subcase c.6. s and e lie on rows $2, 3, \ldots, p - 1$.

Suppose that $p \geq 4$. By Lemma 2, there is a path $\rho_X(s, e)$ in $rt(2, p - 1)$ of length $(p - 2)k - \epsilon$. Let C be the cycle

$$C_1^- (v_{1,0}, v_{0,0}), (v_{0,0}, v_{1,0}).$$

Joining $\rho_X(s, e)$ to C over two corresponding links yields a required path.

Suppose that $p = 3$. If $(s = v_{2,0}$ and $e = v_{2,1})$ or $(e = v_{2,0}$ and $s = v_{2,1})$, then let $\rho_X(s, e)$ be the path on row 2 of length $k - 1$; otherwise, let $\rho_X(s, e)$ be the path on row 2 not containing the link $(v_{2,0}, v_{2,1})$. Join $\rho_X(s, e)$ to C over two corresponding links and denote the resulting path by $\rho(s, e)$.

If $(s = v_{2,0}$ and $e = v_{2,1})$ or $(e = v_{2,0}$ and $s = v_{2,1})$, then $\rho(s, e)$ is as required. Otherwise, iteratively join $\rho(s, e)$ to appropriate links $(v_{2,l}, v_{2,l+1})$ over $(v_{1,l}, v_{1,l+1})$ so that the nodes used on row 2 do not already appear on $\rho(s, e)$. Links should be replaced (by paths) so that if $\{s, e\}$ is odd (respectively, even), then every node of row 2 appears on the amended path (respectively, except one). The path so obtained is as required.

Case d. The faulty link is $(v_{a,0}, v_{a+1,0})$, where $1 \leq a \leq p - 3$.

Subcase d.1. s and e lie on rows $0, 1, \ldots, a + 1$.

By case c, there is a path $\rho_A(s, e)$ in $rt(0, a + 1)$ of length $(a + 2)k - \epsilon$. If $a \neq p - 3$, then let C be the Hamiltonian cycle $E_{a+2,0}$ of $rt(a + 2, p - 1)$, and if $a = p - 3$, then let C be the cycle induced by the nodes on row $p - 1$. Joining $\rho_A(s, e)$ and C over two corresponding links yields a path as required.

Subcase d.2. s lies on rows $0, 1, \ldots, a + 1$, and e lies on rows $a + 2, a + 3, \ldots, p - 1$.

Suppose that $a \neq p - 3$. Let the node e' on row $a + 1$ be such that $s \neq e'$ and $\{s, e\} = \{s, e'\}$. By case c, there is a path $\rho(s, e')$ in $rt(0, a + 1)$ of length $(a + 2)k - \epsilon$. Let s' be the node on row $a + 2$ adjacent to e'. By Lemma 2, there is a path $\rho_X(s', e)$ in $rt(a + 2, p - 1)$ of length $(p - a - 2)k - 1$. The path

$$\rho_A(s, e'), (e', s'), \rho_X(s', e)$$

is as required.

Suppose that $a = p - 3$. Let the node e' on row $a + 1$ be such that $e' \neq s$ and e' lies on a column adjacent to the column on which e lies. By case c, there is a path $\rho(s, e')$ in $rt(0, p - 2)$ of length $(p - 1)k - \epsilon$. Let s' be the neighbor of e' on row $p - 1$ and let $\rho_X(s', e)$ be the path of length $k - 1$ on row $p - 1$. The path

$$\rho_A(s, e'), (e', s'), \rho_X(s', e)$$

is as required.   □

Proposition 6. *Consider the k-ary 2-cube Q_2^k, where $k \geq 6$ is even and where two of the links are faulty. Let s and e be*

any two distinct nodes. There is a path of length $k^2 - 1$ (respectively, $k^2 - 2$) from s to e if $\{s, e\}$ is odd (respectively, even).

Proof. Without loss of generality, we may assume that $(v_{0,0}, v_{1,0})$ is a faulty link. Partition Q_2^k into $rt(k - 1, 0)$ and $rt(1, k - 2)$. As usual, $\epsilon = 1$ if $\{s, e\}$ is odd, and $\epsilon = 2$ if $\{s, e\}$ is even.

Case a. Both s and e lie in $rt(k - 1, 0)$.

By Proposition 5, there is a path $\rho_A(s, e)$ in $rt(k - 1, 0)$ of length $2k - \epsilon$. Either there is a link of $\rho_A(s, e)$ on row $k - 1$ that is not incident with any faulty link or there is a link of $\rho_A(s, e)$ on row 0 that is not incident with any faulty link; without loss of generality, suppose that $(v_{k-1,i}, v_{k-1,i+1})$ is a link of $\rho_A(s, e)$ such that neither $(v_{k-1,i}, v_{k-2,i})$ nor $(v_{k-1,i+1}, v_{k-2,i+1})$ is faulty (the alternative case is similar). By Proposition 5, there is a path $\rho_X(v_{k-2,i}, v_{k-2,i+1})$ in $rt(1, k - 2)$ of length $(k - 2)k - 1$. The path obtained by joining $\rho_A(s, e)$ to $\rho_X(v_{k-2,i}, v_{k-2,i+1})$ over $(v_{k-1,i}, v_{k-1,i+1})$ is as required.

Case b. s lies in $rt(k - 1, 0)$ and e lies in $rt(1, k - 2)$.

Let $(v_{k-1,i}, v_{k-2,i})$ be a healthy link such that $s \neq v_{k-1,i}$, $e \neq v_{k-2,i}$, and $\{s, v_{k-1,i}\} = \{s, e\}$. By Proposition 5, there is a path $\rho_A(s, v_{k-1,i})$ in $rt(k - 1, 0)$ of length $2k - \epsilon$, and there is a path $\rho_X(v_{k-2,i}, e)$ in $rt(1, k - 2)$ of length $(k - 2)k - 1$. The path

$$\rho_A(s, v_{k-1,i}), (v_{k-1,i}, v_{k-2,i}), \rho_X(v_{k-2,i}, e)$$

is as required.   □

Finally, we deal with the case when there is one faulty node and one faulty link.

Proposition 7. *Consider the k-ary 2-cube Q_2^k, where $k \geq 6$ is even and where there is a faulty node and a faulty link. Let s and e be any two distinct nonfaulty nodes. There is a path of length at least $k^2 - 3$ (respectively, $k^2 - 4$) from s to e if $\{s, e\}$ is odd (respectively, even).*

Proof. Without loss of generality, we may assume that the faulty node is $v_{0,0}$. Moreover, we may assume that either the faulty link does not lie in $rt(0, 1)$ or the faulty link is $(v_{0,0}, v_{0,1})$ (again, by applying the usual automorphisms). However, if the faulty link is $(v_{0,0}, v_{0,1})$, then we can assume that there are no faulty links as the fact that $v_{0,0}$ is a faulty node means that the link $(v_{0,0}, v_{0,1})$ is never used. Thus, we can assume that the faulty link does not lie in $rt(0, 1)$. As usual, $\epsilon = 1$ if $\{s, e\}$ is odd, and $\epsilon = 2$ if $\{s, e\}$ is even.

Case a. Both s and e lie in $rt(0, 1)$.

By Lemma 1, there is a path $\rho_A(s, e)$ in $rt(0, 1)$ of length at least $2k - 2 - \epsilon$. Either there is a link of $\rho_A(s, e)$ on row 0 that is not incident with $v_{0,0}$ nor a faulty link or there is a link of $\rho_A(s, e)$ on row 1 that is not incident with a faulty link. Without loss of generality, suppose that $(v_{1,i}, v_{1,i+1})$ is a link of $\rho_A(s, e)$ that is not incident with a faulty link (the alternative case is similar). By Proposition 5, there is a path $\rho_X(v_{2,i}, v_{2,i+1})$ in $rt(2, k - 1)$ of length $(k - 2)k - 1$. The path obtained by joining $\rho_A(s, e)$ to $\rho_X(v_{2,i}, v_{2,i+1})$ over $(v_{1,i}, v_{1,i+1})$ is as required.

Case b. s lies in $rt(0, 1)$, and e lies in $rt(2, k - 1)$.

Let $v_{1,i}$ be such that $s \neq v_{1,i}$, $(v_{1,i}, v_{2,i})$ is healthy and $\{s, v_{1,i}\} = \{s, e\}$. By Lemma 1, there is a path $\rho_A(s, v_{1,i})$ in

$rt(0, 1)$ of length at least $2k - 2 - \epsilon$. By Proposition 5, there is a path $\rho_X(v_{2,i}, e)$ in $rt(2, k - 1)$ of length $(k - 2)k - 1$. The path

$$\rho_X(s, v_{1,i}), (v_{1,i}, v_{2,i}), \rho_X(v_{2,i}, e)$$

is as required. □

From Propositions 3, 4, 6, and 7, we obtain the base case for our main result so long as $k \geq 6$. However, when $k = 4$, a simple computer program (implementing an exhaustive search) verifies that Propositions 3, 4, 6, and 7 all still hold (we leave this verification as an exercise). Hence, we have the following result.

Theorem 8. *Let $k \geq 4$ be even. In a faulty k-ary 2-cube Q_2^k in which the number of node faults f_v and the number of link faults f_e are such that $f_v + f_e \leq 2$, given any two healthy nodes s and e of Q_2^k, there is a path from s to e of length at least $k^2 - 2f_v - 1$ (respectively, $k^2 - 2f_v - 2$) if the nodes s and e have different (respectively, the same) parities.*

4 THE INDUCTIVE STEP

In this section, we complete the proof by induction of our main theorem. The following lemma simplifies the situation considerably.

Lemma 9. *Let Q_n^k have $2n - 2$ faulty nodes and links, where $n \geq 4$. There exists a dimension d such that when we partition Q_n^k over dimension d, the resulting k-ary $(n - 1)$-cubes $Q_{d,0}, Q_{d,1}, \ldots, Q_{d,k-1}$ each contain at most $2n - 4$ faulty nodes and links.*

Proof. Suppose, as our induction hypothesis, that $n \geq 5$ and that the result holds for Q_{n-1}^k (with $2n - 4$ faults). Let Q_n^k have $2n - 2$ faults. Partition Q_n^k over dimension 1; if the resulting k-ary $(n - 1)$-cubes $Q_{1,0}, Q_{1,1}, \ldots, Q_{1,k-1}$ are such that each contains at most $2n - 4$ faults, then we are done. Therefore, without loss of generality, suppose that $Q_{1,0}$ contains $2n - 2$ or $2n - 3$ faults.

Suppose that $Q_{1,0}$ contains $2n - 3$ faults, and so, there is exactly one fault not in $Q_{1,0}$. Temporarily regard some fault, w, say, of $Q_{1,0}$ as healthy and apply the induction hypothesis to $Q_{1,0}$ (note that w might be a node or a link). Thus, there is a dimension d such that when we partition $Q_{1,0}$ over dimension d, the resulting k-ary $(n - 2)$-cubes each contain at most $2n - 6$ faults. Consequently, when we partition Q_n^k over dimension d, each of the resulting k-ary $(n - 1)$-cubes contains at most $2n - 4$ faults (the "temporarily healthy fault" w needs to be recast as faulty, and there is one other fault not in $Q_{1,0}$ to consider).

Suppose that $Q_{1,0}$ contains $2n - 2$ faults, and so, there are no faults outside $Q_{1,0}$. Temporarily regard two faults, w and w', say, of $Q_{1,0}$ as healthy and apply the induction hypothesis to $Q_{1,0}$. Thus, there is a dimension d such that when we partition $Q_{1,0}$ over dimension d, the resulting k-ary $(n - 2)$-cubes each contain at most $2n - 6$ faults. Consequently, when we partition Q_n^k over dimension d, each of the resulting k-ary $(n - 1)$-cubes contains at most $2n - 4$ faults (the two "temporarily healthy faults" w and w' need to be recast as faulty).

In order for the result to follow by induction, all we need to do is to verify the statement of the lemma for when $n = 4$. Let the faults of Q_4^k be w_i, for $i = 1, 2, \ldots, 6$.

Partition Q_4^k over dimension 1. Either each resulting k-ary 3-cube contains at most four faults, and we are done, or the nodes involved in at least five of $\{w_i : i = 1, 2, \ldots, 6\}$ have identical first components (if w_i is a link, then the nodes involved in w_i are the nodes of the link, and if w_i is a node, then the node involved in w_i is the node itself). We may assume that it is the latter and that the five faults whose first components (of the nodes involved) are identical are w_1, w_2, w_3, w_4, and w_5.

Partition Q_4^k over dimension 2. Either each resulting k-ary 3-cube contains at most four faults, and we are done, or one of the resulting k-ary 3-cubes contains either five or six faults. We may assume that the second components of w_1, w_2, w_3, and w_4 are identical.

Partition Q_4^k over dimension 3. Either each resulting k-ary 3-cube contains at most four faults, and we are done, or one of the resulting k-ary 3-cubes contains either five or six faults. We may assume that the third components of w_1, w_2, and w_3 are identical.

Partition Q_4^k over dimension 4. Either each resulting k-ary 3-cube contains at most four faults, and we are done, or one of the resulting k-ary 3-cubes contains either five or six faults. We may assume that the fourth components of w_1 and w_2 are identical. This yields a contradiction as either w_1 and w_2 are nodes, and $w_1 \neq w_2$, or w_1 or w_2 is a link joining a node to itself. The result follows. □

Let us reexamine the proof of Lemma 9. Ideally, we would like Lemma 9 to apply when $n = 3$, but the argument in the proof fails. However, we can classify exactly the fault configurations leading to failure.

Suppose that Q_3^k has four faulty nodes. Following through the argument in the proof of Lemma 9 yields that up to isomorphism, the situations where the argument fails is when the four faults are of the form $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$, for some a, b, and c all different from zero.

Suppose that Q_3^k has three faulty nodes and one faulty link. Without loss of generality, suppose that the faulty link lies in dimension 1. Following the argument in Lemma 9 yields that up to isomorphism, the situations where the argument fails is when the three faulty nodes are of the form $(0, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$, for some b and c different from zero, and the faulty link is of the form $((a, 0, 0), (a + 1, 0, 0))$, for some a.

Suppose that Q_3^k has two faulty nodes and two faulty links. Without loss of generality, suppose that one of the faulty links lies in dimension 1 with the other in dimension 2 (the two links cannot lie in the same dimension, as otherwise, we could partition over this dimension and be done). Following the argument in Lemma 9 yields that up to isomorphism, the situations where the argument fails is when the two faulty nodes are of the form $(0, 0, 0)$ and $(0, 0, c)$, for some c different from zero, and the faulty links are of the form $((a, 0, 0), (a + 1, 0, 0))$ and $((0, b, 0), (0, b + 1, 0))$, for some a and b.

Suppose that Q_3^k has one faulty node and three faulty links. Without loss of generality, suppose that one of the faulty links lies in dimension 1, one in dimension 2, and one in dimension 3. Following the argument in Lemma 9

yields that up to isomorphism, the situations where the argument fails is when the faulty node is of the form $(0, 0, 0)$ and the faulty links are of the form $((a, 0, 0), (a+1, 0, 0))$, $((0, b, 0), (0, b+1, 0))$, and $((0, 0, c), (0, 0, c+1))$, for some a, b, and c.

Suppose that Q_3^k has four faulty links. In this case, Lemma 9 holds, as at least two faulty links lie in the same dimension, and we can partition over this dimension. We shall use these observations in the proof of the following theorem.

Throughout the rest of the paper, we adopt the following notation. Suppose that we partition Q_n^k over some dimension d to get the k-ary $(n-1)$-cubes $Q_{d,0}, Q_{d,1}, \ldots, Q_{d,k-1}$. Let x be a node of $Q_{d,i}$, say. Then, we refer to the node in $Q_{d,j}$ corresponding to x (that is, the node of $Q_{d,j}$ whose name is identical to that of x except that its dth component is j as opposed to i) as x_j. We also refer to the node x as x_i.

Theorem 10. *Let Q_n^k be a k-ary n-cube, for some $n \geq 2$ and some even $k \geq 4$, with f_v faulty nodes and f_e faulty links, where $0 \leq f_v + f_e \leq 2n - 2$. If s and e are distinct healthy nodes and $\{s, e\}$ is odd (respectively, even), then there exists a path from s to e of length at least $k^n - 2f_v - 1$ (respectively, $k^n - 2f_v - 2$).*

Proof. We proceed by induction on n. The base case of the induction is handled by Theorem 8. Suppose, as our induction hypothesis, that the result holds for Q_m^k, where $n \geq 3$, and for all $m < n$. Let Q_n^k be a k-ary n-cube as in the statement of the theorem. Throughout, $\epsilon = 1$ if $\{s, e\}$ is odd, and $\epsilon = 2$ if $\{s, e\}$ is even.

Suppose that $n \geq 4$. By Lemma 9, we may assume that when we partition Q_n^k over dimension 1, the resulting k-ary $(n-1)$-cubes $Q_{1,0}, Q_{1,1}, \ldots, Q_{1,k-1}$ each contain at most $2n - 4$ faults. Suppose that the number of faulty nodes in $Q_{1,i}$ is f_i, for $i = 0, 1, \ldots, k-1$.

Case a. s and e lie in $Q_{1,0}$.

By the induction hypothesis, there is a path $\rho_0(s, e)$ in $Q_{1,0}$ of length at least $k^{n-1} - 2f_0 - \epsilon$. Let (w_0, z_0) be a link of $\rho_0(s, e)$ for which w_1 and z_1 are healthy nodes (of $Q_{1,1}$) and (w_0, w_1) and (z_0, z_1) are healthy links (a simple counting argument shows the existence of such a link). By the induction hypothesis, there is a path $\rho_1(w_1, z_1)$ in $Q_{1,1}$ of length at least $k^{n-1} - 2f_1 - 1$. Let $\rho(s, e)$ be the join of $\rho_0(s, e)$ to $\rho_1(w_1, z_1)$ over (w_0, z_0). The path $\rho(s, e)$ has length at least $2k^{n-1} - 2(f_0 + f_1) - \epsilon$. Proceeding similarly and iteratively with appropriate paths in $Q_{1,2}, Q_{1,3}, \ldots, Q_{1,k-1}$ yields a path from s to e of the required length.

Case b. s lies in $Q_{1,0}$, and e lies in $Q_{1,a}$, for $a \neq 0$.

A simple counting argument yields that there exists a healthy node $w_0 \in Q_{1,0} \setminus \{e_0\}$ such that $\{s, w_0\}$ is odd, w_i is healthy, for all $i = 0, 1, \ldots, k-1$, and all links of $\{(w_i, w_{i+1}) : i = 0, 1, \ldots, k-2\} \cup \{(w_{k-1}, w_0)\}$ are healthy. By the induction hypothesis, there exists a path $\rho_0(s, w_0)$ in $Q_{1,0}$ of length at least $k^{n-1} - 2f_0 - 1$.

Suppose that $a \neq 1$. A simple counting argument yields that there exists a healthy node $z_1 \in Q_{1,1} \setminus \{e_1\}$ such that $\{w_1, z_1\}$ is odd, z_i is healthy, for all $i = 0, 1, \ldots, k-1$, and all links of $\{(z_i, z_{i+1}) : i = 0, 1, \ldots, k-2\} \cup \{(z_{k-1}, z_0)\}$ are healthy. By the induction hypothesis, there exists a path $\rho_1(w_1, z_1)$ in $Q_{1,1}$ of length at least $k^{n-1} - 2f_1 - 1$. Denote the path

$$\rho_0(s, w_0), (w_0, w_1), \rho_1(w_1, z_1)$$

by $\rho(s, z_1)$.
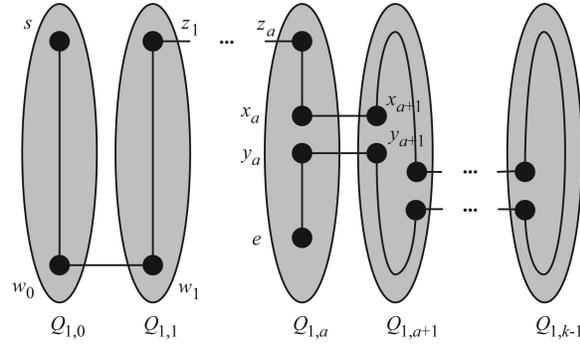
Fig. 7. The construction in case b.

Suppose that $a \neq 2$. By the induction hypothesis, there exists a path $\rho_2(z_2, w_2)$ in $Q_{1,2}$ of length at least $k^{n-1} - 2f_2 - 1$. Denote the path

$$\rho(s, z_1), (z_1, z_2), \rho_2(z_2, w_2)$$

by $\rho(s, w_2)$.

Proceeding iteratively in this way yields a path $\rho(s, z_{a-1})$ or $\rho(s, w_{a-1})$, depending upon whether $a - 1$ is odd or even, respectively, of length at least $ak^{n-1} - 2(f_0 + f_1 + \ldots + f_{a-1}) - 1$. Without loss of generality, suppose that the path is $\rho(s, z_{a-1})$ (the other case is similar). The node z_a is odd if and only if the node s is odd; hence, $\{s, e\} = \{z_a, e\}$.

By the induction hypothesis, there exists a path $\rho_a(z_a, e)$ in $Q_{1,a}$ of length at least $k^{n-1} - 2f_a - \epsilon$. Denote the path

$$\rho(s, z_{a-1}), (z_{a-1}, z_a), \rho_a(z_a, e)$$

by $\rho'(s, e)$. The path $\rho'(s, e)$ has length at least $(a+1)k^{n-1} - 2(f_0 + f_1 + \ldots + f_a) - \epsilon$.

A simple counting argument yields that there is a link (x_a, y_a) of $\rho_a(z_a, e)$ such that x_{a+1} and y_{a+1} are both healthy nodes and (x_a, x_{a+1}) and (y_a, y_{a+1}) are both healthy links (to see this, note that $\rho_a(z_a, e)$ has length at least $k^{n-1} - 2f_a - \epsilon \geq 2^{2n-2} - 2(2n-4) - 2 = 2^{2n-2} - 4n + 6$, and so, there are at least $2^{2n-3} - 2n + 3$ mutually disjoint links on $\rho_a(z_a, e)$; as there are at most $2n - 2$ faulty links in our Q_n^k and $2^{2n-3} - 2n + 3 > 2n - 2$, when $n \geq 3$, at least one such link (x_a, y_a) of $\rho_a(z_a, e)$ must be as required). By the induction hypothesis, there is a path $\rho_{a+1}(x_{a+1}, y_{a+1})$ in $Q_{1,a+1}$ of length at least $k^n - 2f_{a+1} - 1$. Form the path obtained by joining $\rho'(s, e)$ to $\rho_{a+1}(x_{a+1}, y_{a+1})$ over (x_a, y_a) and denote this path by $\rho''(s, e)$. The path $\rho''(s, e)$ has length at least $(a+2)k^{n-1} - 2(f_0 + f_1 + \ldots + f_{a+1}) - \epsilon$. Proceeding similarly and iteratively in $Q_{1,a+2}, Q_{1,a+3}, \ldots, Q_{1,k-1}$ results in a path from s to e of the required length (the construction can be visualized as in Fig. 7).

Now, suppose that $n = 3$ and suppose further that we have no faulty links (we deal with when there are faulty links later). From the observation following Lemma 9, we may assume that we have four faulty nodes and that these nodes are $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$, for some a, b, and c, all different from zero; otherwise, the construction above in cases 1 and 2 can be used to build our path. Partition Q_3^k over dimension 1 to obtain the k-ary 2-cubes $Q_{1,0}, Q_{1,1}, \ldots, Q_{1,k-1}$; note that $(0, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$ lie in $Q_{1,0}$.

Fig. 8. The construction in case c.

Fig. 9. The constructions in case d when $e \in Q_{1,1}$.
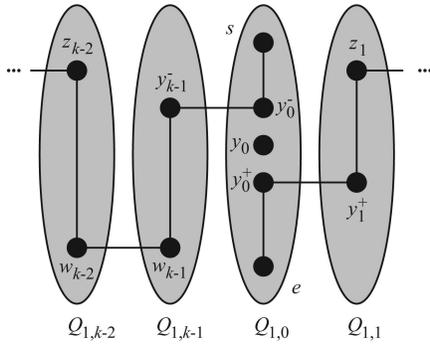
Case c. s and e lie in $Q_{1,0}$.

Temporarily suppose that $(0, 0, 0)$ is healthy. By Theorem 8, there is a path $\rho_0(s, e)$ in $Q_{1,0}$ of length at least $k^2 - 4 - \epsilon$ but upon which $(0, 0, 0)$ may lie. If $(0, 0, 0)$ lies on $\rho_0(s, e)$, then choose $y_0 = (0, 0, 0)$; otherwise, choose y_0 to be any node of $\rho_0(s, e)$ different from s and e.

Let y_0^- and y_0^+ be the nodes immediately before and after y_0, respectively, on $\rho_0(s, e)$. Without loss of generality, we may suppose that y_{k-1}^- and y_1^+ are healthy nodes (and that (y_0^-, y_{k-1}^-) and (y_0^+, y_1^+) are healthy links; recall that there is one faulty node outside $Q_{1,0}$). A simple counting argument yields that there exists a healthy node $w_{k-1} \in Q_{k-1} \setminus \{y_{k-1}^-\}$ such that $\{y_{k-1}^-, w_{k-1}\}$ is odd and w_i is healthy, for all $i = 1, 2, \ldots, k - 1$ (and the links of $\{(w_i, w_{i+1}) : i = 0, 1, \ldots, k - 2\}$ are healthy; to see this, note that there are at least $\lfloor (k^2 - 1)/2 \rfloor$ healthy nodes w_{k-1} for which $\{y_{k-1}^-, w_{k-1}\}$ is odd, and this number is greater than zero). By Theorem 8, there exists a path $\rho_{k-1}(y_{k-1}^-, w_{k-1})$ in $Q_{1,k-1}$ of length at least $k^2 - 2f_{k-1} - 1$.

A simple counting argument yields that there exists a healthy node $z_{k-2} \in Q_{k-2} \setminus \{y_{k-2}^+, w_{k-2}\}$ such that $\{w_{k-2}, z_{k-2}\}$ is odd and z_i is healthy, for all $i = 1, 2, \ldots, k - 1$ (and the links of $\{(z_i, z_{i+1}) : i = 0, 1, \ldots, k - 3\}$ are healthy). By Theorem 8, there exists a path $\rho_{k-2}(w_{k-2}, z_{k-2})$ in Q_{k-2} of length at least $k^2 - 2f_{k-2} - 1$.

Proceeding iteratively in this way yields a path $\rho'(s, z_1)$ defined as

$$\rho(s, y_0^-), (y_0^-, y_{k-1}^-), \rho_{k-1}(y_{k-1}^-, w_{k-1}), (w_{k-1}, w_{k-2}),$$
$$\rho_{k-2}(w_{k-2}, z_{k-2}), (z_{k-2}, z_{k-3}), \ldots, (z_2, z_1).$$

By Theorem 8, there is a path $\rho_1(z_1, y_1^+)$ in $Q_{1,1}$ of length at least $k^2 - 2f_1 - 2$. Consider the path $\rho''(s, e)$ defined as

$$\rho'(s, z_1), \rho_1(z_1, y_1^+), (y_1^+, y_0^+), \rho_0(y_0^+, e).$$

The length of this path is $k^3 - 2\Sigma_{i=1}^{k-1} f_i - 6 - \epsilon = k^3 - 8 - \epsilon$. Hence, the path $\rho''(s, e)$ is as required (the construction can be visualized as in Fig. 8).

Case d. s lies in $Q_{1,0}$, and e does not lie in $Q_{1,0}$.

For the moment, regard the node $x_0 = (0, 0, 0)$ as healthy. By Theorem 8, there is a path $\rho_0(s, x_0)$ in $Q_{1,0}$ of length at least $k^2 - 5$ if $\{s, x_0\}$ is odd and $k^2 - 6$ if $\{s, x_0\}$ is even. Let w_0 be the node of $\rho_0(s, x_0)$ adjacent to x_0. Without loss of generality, we may assume that w_1 and
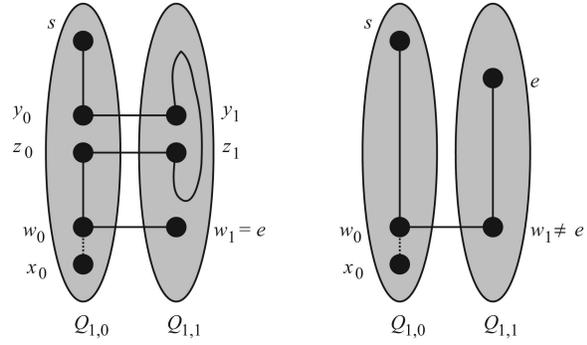
(w_0, w_1) are healthy. There are two possibilities: either $e \in Q_{1,1}$ or $e \in Q_{1,m}$, where $0 \neq m \neq 1$.

Suppose that $e \in Q_{1,1}$ and $w_1 = e$. A simple counting argument yields that there exists a link (y_0, z_0) of $\rho_0(s, w_0)$ such that $y_0 \neq w_0 \neq z_0$ and $y_1, z_1, (y_0, y_1)$, and (z_0, z_1) are healthy. By Theorem 8, there is a path $\rho_1(y_1, z_1)$ in $Q_{1,1}$ that avoids e and is of length at least $k^2 - 2(f_1 + 1) - 1$. Let $\rho(s, e)$ be the path obtained by joining

$$\rho_0(s, w_0), (w_0, e)$$

to $\rho_1(y_1, z_1)$ over the link (y_0, z_0). As $\{s, x_0\} = \{s, e\}$, the length of $\rho(s, e)$ is at least $2k^2 - 2f_1 - 6 - \epsilon$.

Suppose that $e \in Q_{1,1}$ and $w_1 \neq e$. By Theorem 8, there is a path $\rho_1(w_1, e)$ in $Q_{1,1}$ of length at least $k^2 - 2f_1 - 1$ if $\{w_1, e\}$ is odd and $k^{n-1} - 2f_1 - 2$ if $\{w_1, e\}$ is even. Define the path $\rho(s, e)$ as

$$\rho_0(s, w_0), (w_0, w_1), \rho_1(w_1, e).$$

If $\{s, e\}$ is odd, then $\{s, x_0\} = \{s, w_1\} \neq \{w_0, e\}$, and the length of $\rho(s, e)$ is at least $2k^2 - 2f_1 - 7$. If $\{s, e\}$ is even, then $\{s, x_0\} = \{s, w_1\} = \{w_0, e\}$, and the length of $\rho(s, e)$ is at least $2k^2 - 2f_1 - 8$.

Hence, if $e \in Q_{1,1}$, then we have a path $\rho(s, e)$ in $Q_{1,0} \cup Q_{1,1}$ of length at least $2k^2 - 2f_1 - 6 - \epsilon$ (the constructions can be visualized as in Fig. 9).

A simple counting argument yields that there is a link (u_1, v_1) of $\rho(s, e)$ such that (u_1, u_2) and (v_1, v_2) are both healthy. By Theorem 8, there is a path $\rho_2(u_2, v_2)$ in $Q_{1,2}$ of length at least $k^2 - 2f_2 - 1$. Join the path $\rho(s, e)$ to the path $\rho_2(u_2, v_2)$ over the link (u_1, v_1) and denote the resulting path by $\rho(s, e)$ also. Proceeding iteratively in this way in $Q_{1,3}, Q_{1,4}, \ldots, Q_{1,k-1}$ yields a path $\rho(s, e)$ whose length is at least $k^3 - 2\Sigma_{i=1}^{k-1} f_i - 6 - \epsilon = k^3 - 8 - \epsilon$. Hence, the path $\rho(s, e)$ is as required.

Alternatively, suppose that $e \in Q_{1,m}$, where $0 \neq m \neq 1$. Let $y_1 \in Q_{1,1}$ be such that $\{s, y_1\}$ is odd, $y_m \neq e$, and y_i is healthy, for $i = 1, 2, \ldots, k - 1$ (and the links of $\{(y_i, y_{i+1}) : i = 1, 2, \ldots, k - 2\}$ are healthy). By the construction above, there is a path $\rho'(s, y_1)$ in $Q_{1,0} \cup Q_{1,1}$ of length $2k^2 - 2f_1 - 7$.

Suppose that $m \neq 2$. Let $z_2 \in Q_{1,2}$ be such that $\{z_2, y_2\}$ is odd, $z_a \neq e$, and z_i is healthy, for $i = 1, 2, \ldots, k - 1$ (and the links of $\{(z_i, z_{i+1}) : i = 2, 3, \ldots, k - 2\}$ are healthy). By Theorem 8, there is a path $\rho_2(y_2, z_2)$ in $Q_{1,2}$ of length $k^2 - 2f_2 - 1$.

Suppose that $m \neq 3$. By Theorem 8, there is a path $\rho_3(z_3, y_3)$ in $Q_{1,3}$ of length $k^2 - 2f_3 - 1$. Proceeding in this

way, we obtain paths $\rho_2(y_2, z_2), \rho_3(z_3, y_3), \ldots$, and so on, until $\rho_{m-1}(y_{m-1}, z_{m-1})$ if m is odd or $\rho_{m-1}(z_{m-1}, y_{m-1})$ if m is even. Applying Theorem 8 again yields a path $\rho_m(z_m, e)$ or $\rho_m(y_m, e)$ in $Q_{1,m}$, depending upon whether m is odd or even, respectively. If m is odd (respectively, even), then $\rho_m(z_m, e)$ (respectively, $\rho_m(y_m, e)$) has length at least $k^2 - 2f_m - 1$ if $\{z_m, e\}$ (respectively, $\{y_m, e\}$) is odd and $k^2 - 2f_m - 2$ if $\{z_m, e\}$ (respectively, $\{y_m, e\}$) is even.

If m is odd, then let $\rho(s, e)$ be defined as

$$\rho'(s, y_1), (y_1, y_2), \rho_2(y_2, z_2), (z_2, z_3), \rho_3(z_3, y_3), \ldots,$$
$$(z_{m-1}, z_m), \rho_m(z_m, e),$$

and if m is even, then let $\rho(s, e)$ be defined as

$$\rho'(s, y_1), (y_1, y_2), \rho_2(y_2, z_2), (z_2, z_3), \rho_3(z_3, y_3), \ldots,$$
$$(y_{m-1}, y_m), \rho_m(y_m, e).$$

It can easily be verified that if m is odd, then $\{s, e\} = \{z_m, e\}$, and if m is even, then $\{s, e\} = \{y_m, e\}$. Thus, the length of the path $\rho(s, e)$ is at least $(m+1)k^2 - 2\Sigma_{i=1}^{m} f_i - 6 - \epsilon$. If $m \neq k - 1$, then the path $\rho(s, e)$ can be iteratively joined to a path in $Q_{1,i}$ of length $k^2 - 2f_i - 1$, for $i = m+1, m+2, \ldots, k-1$, just as we did above, to obtain a path, also denoted $\rho(s, e)$, of length at least $k^3 - 2\Sigma_{i=1}^{k-1} f_i - 6 - \epsilon$. Hence, our path $\rho(s, e)$ is as required.

Case e. s and e lie in $Q_{1,p}$ and $Q_{1,m}$, respectively, where $m \neq 0 \neq p \neq m$.

Without loss of generality, suppose that $p > m$. Let $s' \in Q_{1,0}$ be such that s', s'_{k-1} and (s'_{k-1}, s') are healthy and $\{s', s\}$ is odd. By the construction in Case d above, there is a path $\rho'(s', e)$ in $Q_{1,0} \cup Q_{1,1} \cup \ldots \cup Q_{1,m}$ of length at least $(m+1)k^2 - 2\Sigma_{i=0}^{a} f_i - 7$.

Let w_p be a node of Q_p such that $\{s, w_p\}$ is odd, $w_0 \neq s'$, and w_i is healthy, for $i = p, p+1, \ldots, k-1$ (and the links of $\{(w_i, w_{i+1}) : i = p, p+1, \ldots, k-2\}$ are healthy). By Theorem 8, there is a path $\rho_p(s, w_p)$ in $Q_{1,p}$ of length at least $k^2 - 2f_p - 1$.

Let y_{p+1} be a node of $Q_{1,p+1}$ such that $\{w_{p+1}, y_{p+1}\}$ is odd, $y_0 \neq s'$, and y_i is healthy, for $i = p+1, p+2, \ldots, k-1$ (and the links of $\{(y_i, y_{i+1}) : i = p+1, p+2, \ldots, k-2\}$ are healthy). By Theorem 8, there is a path $\rho_{p+1}(w_{p+1}, y_{p+1})$ in Q_{p+1} of length at least $k^2 - 2f_{p+1} - 1$.

Again, by Theorem 8, there are paths $\rho_{p+2}(y_{p+2}, w_{p+2})$, $\rho_{p+3}(w_{p+3}, y_{p+3})$, and so on, up to $\rho_{k-2}(y_{k-2}, w_{k-2})$ if p is even and $\rho_{k-2}(w_{k-2}, y_{k-2})$ if p is odd that are of lengths $k^2 - 2f_{p+2} - 1, k^2 - 2f_{p+3} - 1, \ldots, k^2 - 2f_{k-2} - 1$, respectively; note that $\{s, e\} = \{w_{k-1}, s'_{k-1}\}$ if p is odd (respectively, $\{s, e\} = \{y_{k-1}, s'_{k-1}\}$ if p is even). Yet again, by Theorem 8, there is a path $\rho_{k-1}(w_{k-1}, s'_{k-1})$ (respectively, $\rho_{k-1}(y_{k-1}, s'_{k-1})$) in Q_{k-1} of length at least $k^2 - 2f_{k-1} - \epsilon$ if p is even (respectively, odd). Let $\rho(s, e)$ be the path

$$\rho_p(s, w_p), (w_p, w_{p+1}), \rho_{p+1}(w_{p+1}, y_{p+1}), (y_{p+1}, y_{p+2}),$$
$$\rho_{p+2}(y_{p+2}, w_{p+2}), \ldots \ldots, (s'_{k-1}, s'), \rho'(s', e).$$

The path $\rho(s, e)$ has length at least

$$(k - p + m - 1)k^2 - 2\Sigma_{i=0}^{m} f_i - 2\Sigma_{i=p}^{k-1} f_i - 2 - \epsilon.$$

If $p \neq m+1$, then the path $\rho(s, e)$ can be iteratively joined to a path in $Q_{1,i}$ of length $k^2 - 2f_i - 1$, for $i = m+1, m+2, \ldots, p-1$, just as we did in Case d, to obtain a path, also denoted $\rho(s, e)$, of length at least $k^3 - 2\Sigma_{i=1}^{k-1} f_i - 6 - \epsilon$. Hence, our path $\rho(s, e)$ is as required.

Case f. s and e lie in $Q_{1,m}$, where $m \neq 0$.

By Theorem 8, there is a path $\rho_m(s, e)$ in $Q_{1,m}$ of length at least $k^2 - 2f_m - \epsilon$. There exists a link (w_m, y_m) of $\rho_m(s, e)$ such that $w_{m+1}, y_{m+1}, (w_m, w_{m+1})$, and (y_m, y_{m+1}) are healthy. By Theorem 8, there exists a path $\rho_{m+1}(w_{m+1}, y_{m+1})$ in $Q_{1,m+1}$ of length at least $k^2 - 2f_{m+1} - 1$. Join $\rho_m(s, e)$ to $\rho_{m+1}(w_{m+1}, y_{m+1})$ over (w_m, y_m) and denote this path by $\rho(s, e)$ also. The path $\rho(s, e)$ can be iteratively joined to a path in $Q_{1,i}$ of length $k^2 - 2f_i - 1$, for $i = m+2, m+3, \ldots, m-1$, to obtain a path of length at least $k^3 - 8 - \epsilon$ as required.

Now, suppose that we have one faulty link. Partition over the dimension containing this faulty link, and if each resulting k-ary 2-cube $Q_{1,0}, Q_{1,1}, \ldots, Q_{1,k-1}$ contains at most two faults, then apply the construction as in Cases a and b to build our path. Hence, we may assume that $Q_{1,0}$ contains three faulty nodes. However, if we follow exactly the constructions in each of Cases c, d, e, and f, then these constructions still apply, and we obtain a path of the required length. Exactly the same can be said of the scenarios when we have two and three faulty links. The result now follows. □

We note that given Q_n^k, where $k \geq 4$ is even, and f_n and f_e, where $f_n + f_e \leq 2n - 2$, there are configurations of f_n faulty nodes, f_e faulty links, and pairs of distinct healthy nodes so that the longest path joining the two nodes has length exactly $k^n - 2f_n - 1$ (respectively, $k^n - 2f_n - 2$) if the parities of the two nodes are different (respectively, the same). Hence, in this sense, our result can be viewed as optimal.

Also, there are configurations of $2n - 1$ faulty nodes in Q_n^k and pairs of healthy nodes such that the longest path joining the two nodes has length one; take healthy adjacent nodes x and y where all other neighbors of x are faulty. Hence, the total number of faults in Theorem 10 cannot be increased.

5 CONCLUSIONS

Theorem 10, allied with the result in [22], fully resolves the situation regarding the existence of longest cycles in k-ary n-cubes where the total number of faults (nodes and links) is at most $2n - 2$ and where the faults are configured in a "worst case" scenario with respect to the pair of nodes in question. Of course, there are configurations of, for example, $2n - 2$ faulty nodes in Q_n^k where certain pairs of nodes have paths joining them of lengths strictly greater than the bounds stated in Theorem 10. It would be interesting to build the longest paths joining pairs of nodes but taking into account the configuration of faults (though this would appear to be a demanding task).

We expect that if we assume the conditional fault assumption, then we should be able to tolerate more faults yet still prove a result analogous to Theorem 10. It would be worthwhile to investigate this scenario, and we conjecture that the path lengths will be exactly as in Theorem 10.

The existence of paths and cycles in (faulty) interconnection networks does not guarantee that we can efficiently construct these paths and cycles using a distributed algorithm implemented on the underlying topology (see

[21] regarding the issues involved with the distributed embedding of a Hamiltonian cycle in a faulty k-ary n-cube). The existence of an efficient distributed algorithm that "implements" Theorem 10 should be investigated.

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