Mass terms in the Skyrme model

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We consider various forms of the mass term that can be used in the Skyrme model and their implications on the properties of baryonic states. It is shown that modifications of the mass term, without changing the asymptotic behavior of the profile function at large r, can considerably alter the mass term contribution to the classical mass of the soliton. We find that, for some choices of this mass term, multibaryon configurations can be classically bound at large baryon numbers. It is pointed out that some restrictions on the possible form of the mass term can follow from studies of multimeson interactions.

I. INTRODUCTION

The Skyrme model has enjoyed a lot of interest ever since it was realized that, although it is a nonlinear theory of pions, it is also an effective theory of low-energy nucleon interactions. In fact, it may also provide a new approach to nuclear physics; as the lowest states of the model, corresponding to higher baryon numbers, are expected to provide a classical description of nuclei. In the Skyrme model approach the baryon number is identified with the soliton number.

Multi-Skyrmions are the stationary points of the static energy functional which, in natural units of the model, $3\pi^2 F_\pi/e$, is given by

$$E = \frac{1}{12\pi^2} \int_{R^3} \left\{ -\frac{1}{2} \text{Tr}(\partial_i U U^{-1})^2 \right.$$
$$\left. + \frac{1}{16} \text{Tr}[\partial_i U U^{-1}, \partial_j U U^{-1}]^2 \right\} d^3\vec{x}, \qquad (1)$$

where $U(\vec{x}) \in SU(2)$ and x is in units of $2/(F_\pi e)$.

Most of the phenomenological applications of the Skyrme model, especially to the study of the nucleon or hyperon properties, included also the pion (or kaon, or D-meson) mass term in the Lagrangian chosen in the simplest possible form (see e.g. Adkins and Nappi [1]). In particular, the kaon mass term has to be added to describe the mass splittings within the $SU(3)$ multiplets of baryons: octet, decuplet, antidecuplet, etc. [2]. However, the role of the mass terms in multi-Skyrmion configurations, especially at large baryon numbers, has not been investigated in much detail; the theoretical work performed so far has involved mostly the Skyrme model in which pions are massless (i.e. given by the Lagrangian above). It is only very recently that some attention has been paid also to the effects associated with the pion mass for large B

configurations [3,4]; one of the effects being the exponential localization of multi-Skyrmions. In particular, it was stressed that the contribution of the mass term can change the binding properties of large B classical configurations and, in particular, their decay properties into configurations with smaller B-numbers [4].

In most of these approaches the pion mass term has been introduced via the addition to (1) of the following term:

$$\frac{1}{12\pi^2} \int_{R^3} m^2 \text{Tr}(1 - U) d^3\vec{x}, \qquad (2)$$

where m is related to physical pion mass $\mu_\pi \simeq 138$ MeV by the relation $m = 2\mu_\pi/(F_\pi e)$, where $F_\pi \simeq 186$ MeV is the pion decay constant, taken usually from the experiment, and e is the Skyrme constant.[1] The appearance of the mass term in effective field theories was discussed, e.g. in [6]. Although the effects associated with the pion mass are small for small values of this mass, they increase if either the baryon number or the pion mass are larger. For massless pions all the known minimal energy multi-Skyrmion configurations have a shell-like structure. These field configurations were obtained in both numerical simulations and in studies involving the so-called "rational map ansatz." In the rational map ansatz, one approximates the full multi-Skyrmion field by assuming that its angular dependence is approximately described by a rational map between Riemann spheres. This approximation was, first of all, shown to be very good in a theory with massless pions

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[1]In [1], the masses of the nucleon and the $\Delta(1232)$ isobar were fitted using an $SU(2)$ quantization procedure and, as a result, the authors obtained $F_\pi = 108$ MeV, $e = 4.84$, but these values did not allow one to describe the mass splittings within $SU(3)$ multiplets of baryons. The approach of [1] has been revised and another set of parameters is widely accepted now. The baryon mass splittings are described with the experimental values of F_π, F_K and $e \simeq 4.1$. For these parameters the absolute values of the baryon masses are not fitted because they are controlled by the loop corrections, or the so-called Casimir energy which, for the baryon number 1, was estimated in [5].

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and it was later extended also to massive pions—where the agreement was again shown to be quite good.

Given these observations it is extremely important to have the right (correct) mass term. The problem, however, is that the mass term is nonunique and the expression (2) is only one of many that can be used. Indeed, the origin of the chiral symmetry conserving and chiral symmetry breaking, or the mass terms considered[2] in [6], may be very different in nature. Possible modifications of the mass term had been considered in [7], where consequences of these modifications were studied, for the case of baryon number $B = 1$. Here we have decided to reexamine this issue further and to look at mass terms other than (2), and to see what effects they have on the properties of multi-Skyrmion configurations with large values of B.

Initially introduced as a model to describe nucleons from pion fields, the Skyrme model has since then been shown [8] to be an effective theory for the low-energy regime of QCD. In that context one can view the Skyrme model as an effective theory to describe nucleons and try to modify the model to better fit experimental data. The addition of the standard mass term complements this picture and the introduction of a generalized mass term, as we do in the paper, constitutes a natural generalization of the Skyrme model. We would like to point out at this stage that the usual mass term not only gives the pion fields a mass, but it also introduces some extra interactions between the pions. This is obvious when one describes the Skyrme model using the S^4 valued pion field $\vec{\phi} = (\sigma, \pi_1, \pi_2, \pi_3)$ where π_1, π_2 and π_3 are the pion fields, σ is an auxiliary field, and $|\vec{\phi}|^2 = 1$. In that formulation, the standard mass term reads, up to an overall constant, $V_M = 1 - \sigma = 1 - (1 - \pi_1^2 + \pi_2^2 + \pi_3^2)^{1/2}$, while the generalized mass term is given, up to an overall constant, by $V_{M_g} = 1 - \sigma^p$ when $p = 2$ or $V_{M_g} = \pi_1^2 + \pi_2^2 + \pi_3^2$ (see next section). From a pion point of view, this mass term thus looks definitely more natural than the usual one [though the case of $p \leq 1$ might be more suitable for the phenomenological description of mass splittings within $SU(3)$ multiplets of baryons] and it is certainly worth studying the differences between these different models.

Although we present our discussion in terms of mesons and baryons, the results we have obtained may be of more general interest and can be applied not only for (3 + 1)-dimensional, but also for (2 + 1)-dimensional models. It is quite general and a first priority problem for any kind of soliton models to find configurations of lowest energy. Since the modified mass term discussed here can lead to a considerable decrease or increase of the classical configurations energy, depending on the type of mass term one chooses, especially for large topological (baryon) numbers,

they could find applications in different fields of physics, including condensed matter physics, astrophysics and cosmology. At the same time, the Skyrme model has received a lot of attention from a purely mathematical point of view [9], and our modified model has several properties that are definitely worth studying in detail.

In the particular case of baryons or nuclei, some restrictions on the possible form of the mass term may follow from the existing and nonexisting data on multipion interactions and also from symmetry considerations of effective chiral Lagrangians. This point is discussed in the next section where we discuss various choices of the mass term, pointing out what is fixed and what can be changed. In the following sections we look at some simple examples of such mass terms. Expressions for the static energy of Skyrmions and some definitions necessary for the description of multi-Skyrmion within the rational map approximation are presented in Secs. III and IV. Our numerical results are presented in Sec. V and the analytical discussion useful to establish asymptotic behavior is presented in Sec. VI. We finish with a short section discussing our conclusions and ideas for further work.

II. MASS TERMS

To consider the mass term, we first note that the pion fields $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ are given by $U = \sigma + i\vec{\pi} \cdot \vec{\tau}$, where $\vec{\tau}$ denotes the triplet of Pauli matrices and σ is determined by the constraint $\sigma^2 + \vec{\pi} \cdot \vec{\pi} = 1$.

Then the square of the pion mass is the coefficient of the expansion of the mass term in powers of $\vec{\pi} \cdot \vec{\pi}$; in fact it is the coefficient of the lowest term, i.e. $\vec{\pi} \cdot \vec{\pi}$, in this expansion. In the case above we have

$$m^2 \text{Tr}(1 - U) = m^2 2(1 - \sigma) \sim m^2 \vec{\pi} \cdot \vec{\pi} + \cdots, \quad (3)$$

where $+ \cdots$ stands for further powers of $\vec{\pi} \cdot \vec{\pi}$ to be interpreted as pion interaction terms shortly discussed below. So the mass of the pion field is proportional to m, since the canonical mass term in the Lagrangian is $-\mu_\pi^2 \vec{\pi} \cdot \vec{\pi}/2$.

However, (2) is not the only term we can use as the pion mass term. It is clear that we can multiply $(1 - U)$ in (2) by any function of U which in the limit $U \to 1$ reduces to 1. Thus we could multiply it by, say, $\frac{(U+1)}{2}$.

In fact, a little thought shows that, instead of U in (2), we can take

$$\int_{-\infty}^{\infty} g(p)U^p dp \quad (4)$$

where

$$\int_{-\infty}^{\infty} g(p)dp = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} g(p)p^2 dp = 1. \quad (5)$$

The usual choice then corresponds to

$$g(p) = \delta(p - 1). \quad (6)$$

As the second condition in (5) can be eliminated by redefining the coefficient m^2 in (2), we see that a more general mass term is given by

$$\frac{1}{12\pi^2} Am^2 \int_{R^3} \text{Tr}\left[1 - \int_{-\infty}^{\infty} g(p) U^p dp \right] d^3\vec{x}, \quad (7)$$

where

$$A^{-1} = \int_{-\infty}^{\infty} g(p) p^2 dp. \quad (8)$$

For a single integer value of p, $A = 1/p^2$, and the mass term takes the simple form

$$E_M = \frac{m^2}{12\pi^2 p^2} \int_{R^3} \text{Tr}[1 - U^p] d^3\vec{x}. \quad (9)$$

In fact, it is the simplest and natural modification of the mass term which does not change the value of meson mass. When we consider applications of our model to elementary particle physics, then certain restrictions on the possible forms of the mass term can follow from the data on multipion interactions. Indeed, when the chiral field is small, and this corresponds to the case of large distances from the center of the soliton, then the expansion of the matrix U^p in (9) in powers of $\vec{\pi}^2$ can be made leading to the Lagrangian density

$$L_M = -\frac{m^2}{12\pi^2}\left[\vec{\pi}^2 - \frac{p^2}{12}\vec{\pi}^4 + \frac{p^4}{360}\vec{\pi}^6 - \cdots \right]. \quad (10)$$

Here we have made the substitution $f^2 \rightarrow \vec{\pi}^2$, introducing the quantized pion field, to obtain pion contributions to amplitudes of various physical processes (some numerical coefficients still have to be inserted but their values are not essential for the qualitative discussion here). The second term in this expansion, proportional to $p^2\vec{\pi}^4$, gives a contribution to the low-energy pion-pion scattering amplitude, usually extracted from the data on the reactions involving the two-pion production on nucleons, $\pi N \rightarrow 2\pi N$ [10], or from heavy meson decays to final states containing two or more pions, see e.g. [11]. The next term, proportional to $p^4\vec{\pi}^6$, gives a contribution to e.g. the process $2\pi \rightarrow 4\pi$, which could be studied in a similar reaction, e.g. $\pi N \rightarrow 4\pi N$, etc. The data on the pion-pion scattering are known not to be in contradiction with the usual value $p = 1$, although they have a considerable uncertainty, see e.g. [12] and references therein. The data on the reactions $2\pi \rightarrow 4\pi$ or $2\pi \rightarrow 6\pi$ do not seem to be available yet. A phenomenological analysis of the existing data aimed at providing restrictions on possible values of the power p would be of interest, although it is behind the scope of the present paper. Anyway, it is clear that the power p cannot be arbitrarily large to fit the data on multipion interactions.

Another class of restrictions can follow from the invariance properties imposed on the modified mass term. It is well known that the Lagrangian of the model depending on the chiral derivatives (1) is invariant under chiral trans-

formations, left L or right R: $U \rightarrow LUR^\dagger$ with $L \in SU(2)$ and $R \in SU(2)$ being arbitrary constant unitary matrices, see e.g. [1,6] and references therein. The mass term, in general, violates this invariance, but is invariant under such a transformation when $L = R$.

In the particle physics context, the mass term of the underlying QCD Lagrangian

$$L_M \sim \bar{\psi}_R \mathcal{M} \psi_L + \bar{\psi}_L \mathcal{M}^\dagger \psi_R \quad (11)$$

is invariant under simultaneous transformations $\psi_L \rightarrow L\psi_L$, $\psi_R \rightarrow R\psi_R$ and $\mathcal{M} \rightarrow L\mathcal{M}R^\dagger$, see e.g. [7]. According to this, the mass term in the Lagrangian density of the chiral soliton model, which can be written as

$$L_M \sim \text{Tr}(MU + U^\dagger M^\dagger - MU_0 - U_0^\dagger M^\dagger), \quad (12)$$

is invariant under the transformation $U \rightarrow LUR^\dagger$, $M \rightarrow RML^\dagger$, which we may call the UM transformation; here the meson mass matrix $M \sim \mathcal{M}$.

The commonly made assumption in particle physics [1,13] is that after such a transformation the vacuum value of the matrix U can be made equal to the unit matrix, $U_0 = 1$. Simultaneously, the mass matrix M takes a diagonal form and is proportional to the unit matrix $M^{diag} \sim \mu^2 \text{diag}(1, 1)$, since, phenomenologically, we can neglect the isospin violation in the mass matrix (i.e. the difference of masses of u and d quarks, or $\pi^\pm$ and π^0 mesons).

In our case of the generalized mass term we also assume that it can be written in an analogous way:

$$L_M \sim \text{Tr}[(M_p U)^p + (U^\dagger M_p^\dagger)^p - (M_p U_0)^p - (U_0^\dagger M_p^\dagger)^p] \quad (13)$$

and so is invariant under the transformations $U \rightarrow LUR^\dagger$, $M_p \rightarrow RM_pL^\dagger$. Clearly, the terms like (13) with $p = 2, 3 \ldots$ do arise when we consider second, third, etc. order contributions in the quark masses or meson masses squared; in this case $M_p \sim M$. For pions such terms are expected to be very small, but they become important for heavier quarks/mesons—the case we are studying here.

More interesting is a possibility to have the terms (13) in the lowest order in the quark/meson masses; i.e. instead of the usual mass term with $p = 1$, as we discussed above. In this case, by dimensional arguments, $(M_p)^p \sim M$. After an appropriate transformation we have $U_0 = 1$, $M_p^{diag} \sim \text{diag}(1, 1)$, with the relation $(M_p^{diag})^p = M^{diag}$, and we obtain

$$L_M \sim \text{Tr}(M_p^{diag})^p[U^p + (U^\dagger)^p - 2]$$
$$= 2\text{Tr}\,M^{diag}(U^p - 1), \quad (14)$$

and so we see that the mass term takes the form (9).

It would be important to determine the form of the mass term from the underlying QCD Lagrangian, or to show that (9) is possible, at least for some values of power p. Strictly speaking, this is an unresolved problem, which deserves

further study. In this context we can mention here the recent work of Battye, Krusch and Sutcliffe [14], who modified the model by changing its potential term, in analogy to our modifications of the potential term. Of course, in concrete physical applications one should consider also the Casimir energy contribution related to loop corrections, but one can look at their work as a study of a classical problem.

In this paper we shall consider several reasonable values of the parameter p, and study its effects, in spite of possible restrictions on its range in particle physics applications, bearing in mind applications in wider classes of models, not only connected to QCD or to effective chiral Lagrangians. One particular case of interest has already been studied in the so-called $(2 + 1)$-dimensional ''new baby Skyrme model'' [15–17] describing anisotropic systems, where the mass term corresponds to the value $p = 2$, see (9). As it was shown in [17], this variant of the model can be especially well approximated analytically.

III. $B = 1$ SKYRMION

Consider first the case of one Skyrmion. The single Skyrmion has the hedgehog form

$$U = \exp(if(r)\hat{\vec{r}} \cdot \vec{\tau}), \tag{15}$$

where $\hat{\vec{r}}$ is the unit vector in the $\vec{r}$ direction and $f(r)$ is the radial profile function which is required to satisfy the boundary conditions $f(0) = \pi$ and $f(\infty) = 0$.

Putting (15) into the energy functional we find that the energy of the field is given by

$$E = \frac{1}{3\pi} \int_0^\infty \left(r^2 \dot{f}^2 + 2(\dot{f}^2 + 1)\sin^2 f + \frac{\sin^4 f}{r^2} \right.$$
$$\left. + 2Am^2r^2 \left[1 - \int_{-\infty}^\infty g(k)\cos(kf)dk \right] \right) dr, \tag{16}$$

where A is given by (8).

Thus, for the minimal field, $f(r)$ satisfies the equation

$$\ddot{f}[r^2 + 2\sin^2 f] + 2r\dot{f} + 2(\dot{f}^2 - 1)\sin f \cos f$$
$$- \frac{2\sin^3 f \cos f}{r^2} - m^2r^2 \frac{\int_{-\infty}^\infty g(k)\sin(kf)kdk}{\int_{-\infty}^\infty g(k)k^2dk} = 0. \tag{17}$$

We have investigated several classes of such functions:
(i) $g(k) = \delta(k - p)$ for several values of p.
  The cases of even or odd integer values for p have also been investigated analytically. Moreover, it is easy to notice that the mass term for $p > 1$ is smaller than that for $p = 1$, since $(1 - \cos(pf))/p^2 = 2\sin^2(pf/2)/p^2$ and $\sin^2(pf/2)/p^2 < \sin^2(f/2)$ for $p > 1$.
(ii) g given by a Gaussian centered around $p = 1$, or around $p = 0$.
  In the latter two cases we have taken

$$g(p) = \frac{\sqrt{\lambda}}{\sqrt{\pi}} \exp(-\lambda(p - 1)^2) \quad \text{and}$$
$$g(p) = \frac{\sqrt{\lambda}}{\sqrt{\pi}} \exp(-\lambda p^2). \tag{18}$$

The corresponding expressions for the energy are given by (for the case with $p = 1$)

$$E = \frac{1}{3\pi} \int_0^\infty \left\{ r^2 \dot{f}^2 + 2(\dot{f}^2 + 1)\sin^2 f + \frac{\sin^4 f}{r^2} \right.$$
$$+ 2m^2r^2 \frac{1}{\frac{2}{\sqrt{\pi\lambda}} + 1 + \frac{1}{2\lambda}}$$
$$\left. \times \left[1 - \cos f \exp\left(-\frac{f^2}{4\lambda}\right) \right] \right\} dr, \tag{19}$$

with a similarly looking expression for the other case.

IV. MULTI-SKYRMIONS

For multi-Skyrmion fields we use the rational map ansatz of Houghton *et al.* [18]. The ansatz involves the introduction of the spherical coordinates in $\mathbb{R}^3$, so that a point $\mathbf{x} \in \mathbb{R}^3$ is given by a pair (r, ξ), where $r = |\mathbf{x}|$ is the distance from the origin, and ξ is a Riemann sphere coordinate giving the point on the unit two-sphere which intersects the half line through the origin and the point $\mathbf{x}$, i.e., $\xi = \tan(\frac{\theta}{2})e^{i\varphi}$, where θ and φ are the usual spherical coordinates on the unit sphere.

Then one observes [19] that a general $SU(2)$ matrix, U, can always be written in the form

$$U = \exp(if(2P - I)) \tag{20}$$

where f is real and P is a 2×2 Hermitian projector i.e., $P = P^2 = P^\dagger$. The rational map ansatz assumes that the Skyrme field has the above form and, in addition, that f depends only on the radial coordinate, i.e., $f = f(r)$, and that the projector depends only on the angular coordinates, i.e., $P(\xi, \bar{\xi})$.

The projector is then taken in the form

$$P = \frac{\mathbf{f} \otimes \mathbf{f}^\dagger}{|\mathbf{f}|^2} \tag{21}$$

where $\mathbf{f}(\xi)$ is a 2-component vector, each entry of which is a degree k polynomial in ξ. Incidentally, given the projective nature of $\mathbf{f}$, one can also use the parametrization $\mathbf{f} = (1, R)^t$, where $R(\xi)$ is the ratio of $R = f_1/f_2$.

For $B = 1$ this ansatz reproduces the one Skyrmion field configuration discussed in the last section, while for $B > 1$ the ansatz (20) is not compatible with the equations which come from (1), so the ansatz cannot produce any exact multi-Skyrmion configurations. However, as was shown in many papers, see e.g. [18,20], it gives approximate field configurations which turn out to be very close to the

numerically computed minimal energy states. To do this, one selects a specific map **f** and puts it into the Skyrme energy functional (1). Performing the integration over the angular coordinates results in a one-dimensional energy functional for $f(r)$ which has then to be solved numerically.

Hence, if we write $\mathbf{f} = (1, R)^t$, then the Skyrme energy is

$$E = \frac{1}{3\pi} \int \left(r^2 f'^2 + 2B(f'^2 + 1)\sin^2 f + I\frac{\sin^4 f}{r^2} \right.$$
$$\left. + 2Am^2 r^2 \left[1 - \int_{-\infty}^{\infty} g(k)\cos(kf)dk \right] \right) dr, \quad (22)$$

where I denotes the integral

$$I = \frac{1}{4\pi} \int \left(\frac{1 + |\xi|^2}{1 + |R|^2} \left| \frac{dR}{d\xi} \right| \right)^4 \frac{2id\xi d\bar{\xi}}{(1 + |\xi|^2)^2}. \quad (23)$$

The values of I have already been calculated; in what follows we take our values from [20]. The equation for the profile function f is very similar to the equation (17) of the last section except that the coefficients of terms involving $\sin f \cos f$ are multiplied by B in the first term and I in the second.

V. NUMERICAL RESULTS

We have looked at the values of various quantities for different choices of p [taking $g(k) = \delta(k - p)$], and also at some Gaussians. The Gaussian cases were not particularly illuminating so here we discuss only the cases of fixed values of p.

Note that when $p \rightarrow 0$ we have a nontrivial contribution of the mass term. This involves taking the limit $p \rightarrow 0$ of the expression in (22) and then its last line becomes $m^2 r^2 f^2$. We can also consider the limit when $m \rightarrow 0$ which corresponds to the massless Skyrme model.

We present our numerical results in Figs. 1–3 in which we plot the normalized energy

$$En = \frac{E(B)}{BE(1)} \quad (24)$$

and the shell radius, as a function of B, for several values of the mass[3] and for p from 0 to 5.

The normalized energy (24) is a dimensionless energy which describes the binding of the configuration by comparing it to that of the $B = 1$ solution. Note that when $En > 1$ the B multi-Skyrmion configuration has an energy larger than the energy of B single Skyrmions thus showing that this configuration is unstable. The jagged curve near the origin is caused by the value of I which varies a lot

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[3]In our numerical calculations we take for pions $m = m_\pi = 0.361\,92$, for kaons $m_{st} = 1.299\,96$ and for the charmed mass scale $m_{ch} = 4.130\,964$. In what follows, in the text and in the captions, we refer to those values as $m = 0.362, 1.300$ and 4.131 respectively.
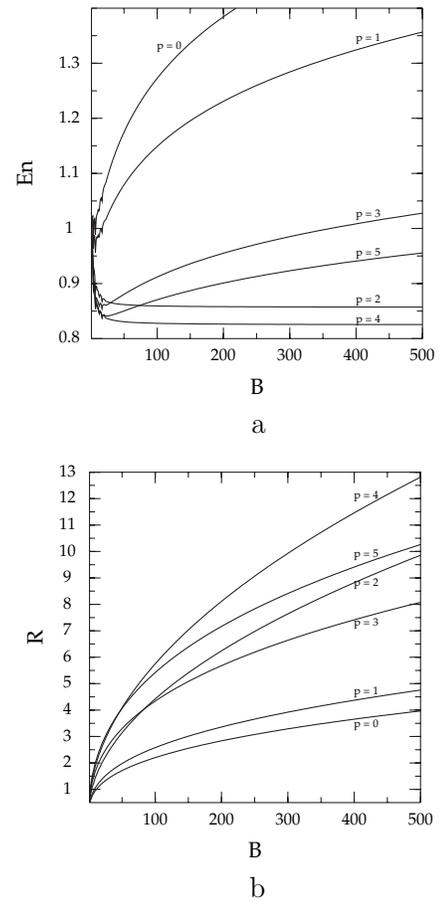

FIG. 1. Normalized energy (24) (a) and radius (25) (b) of multi-Skyrmion configurations for $m = 4.131$, $E(1) = 2.056$.

when B is small. When $B > 22$, we have taken $I = 1.28B^2$ (see [20]) and so the curves are smooth.

We also present the radius of the solutions defined as follows:

$$R = \frac{\int rE(r)r^2 dr}{\int E(r)r^2 dr}, \quad (25)$$

where $E(r)$ is the radial energy density.

Looking at the figures we see that, as B gets very large, the normalized energy converges to a finite value when p is even, but that it slowly diverges when p is odd. This was also observed by Battye and Sutcliffe [4] in the case of $p = 1$. When $m = 4.131$ (Fig. 1) the curve for $p = 3$ crosses the value of $En = 1$ at $B \approx 365$. For a smaller value of m, the curves cross this bound state threshold for much larger values of B. This is not surprising, as for odd p the mass term adds a nonvanishing contribution to the energy density from the region where $f = \pi$ and so to make a large shell configuration one needs to put an increasing amount of energy inside the shell. When B is large, this becomes energetically too expensive and the configuration is not a bound state anymore.
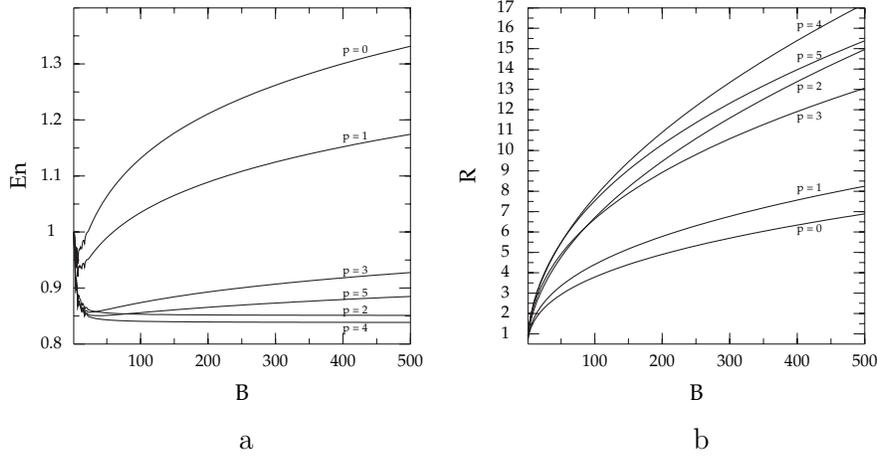
FIG. 2. Normalized energy (24) (a) and radius (25) (b) of multi-Skyrmion configurations for $m = 1.300$, $E(1) = 1.486$.
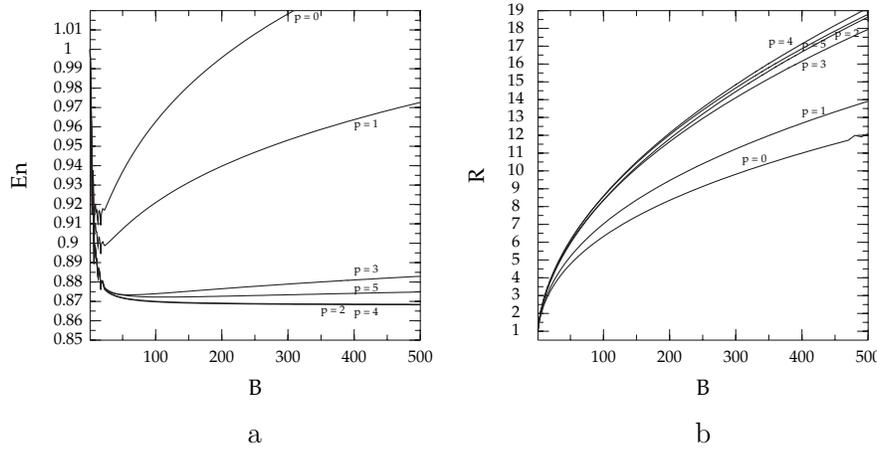

FIG. 3. Normalized energy (24) (a) and radius (25) (b) of multi-Skyrmion configurations for $m = 0.362$, $E(1) = 1.274$.

We also see that, for a given parity of p, the energy at a given value of B decreases as we increase p by a multiple of 2, i.e. $En_p(B) > En_{p+2}(B)$.

The plots of the radius also show that when p is odd the shell is smaller, but otherwise, for a given parity of p, the radius increases with p. On the other hand, for fixed values of p and B, the radius decreases when the mass m increases. This is exactly what one expects for odd values of p as the energy density inside the shell is nonzero, but it is also true for even p, i.e. the radius of the shell decreases with the increase of the mass.

One property worth investigating for these low-energy configurations is their ability to decay into two or more shells of smaller baryon charges. To do this we have computed the derivative of the energy with respect to B and compared the obtained values with the energy per baryon of some small B configurations of low energy (typically $B = 2$, $B = 4$, $B = 7$ and $B = 17$). When the value of the derivative is larger than the energy per baryon of other configurations, it implies that the larger configuration can decay into two shells. In Fig. 4 we see that, when
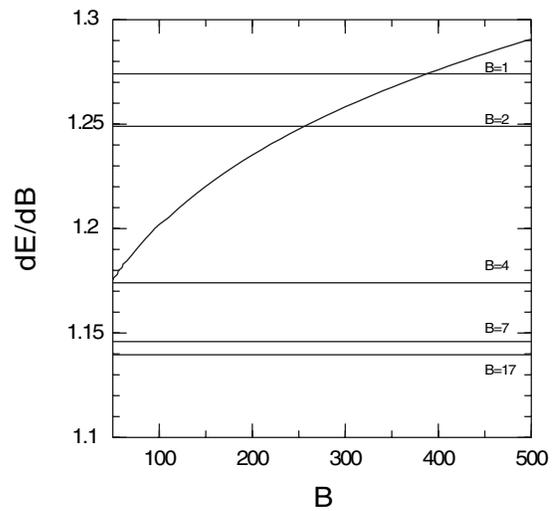

FIG. 4. Derivative of the normalized energy with respect to B and the normalized energy of $B = 2$, $B = 4$, $B = 7$ and $B = 17$ for $m = 0.362$ and $p = 1$.

TABLE I. Decay of low-energy configurations into subshells. Each column corresponds to a decay mode and the baryon charge given corresponds to the threshold from which the decay is always possible.

$m = 0.362$

p	$B = 1$	$B = 2$	$B = 4$	$B = 7$	$B = 17$
0	>95	>70	≥ 26	≥ 18	≥ 20
1	>380	>250	≥ 50	≥ 28	≥ 27
3	$\cdots$	$\cdots$	$\cdots$	>410	>100
5	$\cdots$	$\cdots$	$\cdots$	$\cdots$	>390

$m = 1.300$

p	$B = 1$	$B = 2$	$B = 4$	$B = 7$	$B = 17$
0	≥ 18	≥ 18	≥ 8	≥ 7	≥ 17
1	≥ 27	≥ 25	≥ 14	≥ 8	≥ 18
3	$\cdots$	>480	>120	≥ 45	≥ 34
5	$\cdots$	$\cdots$	>350	>110	≥ 33

$m = 4.131$

p	$B = 1$	$B = 2$	$B = 4$	$B = 7$	$B = 17$
0	≥ 18	≥ 18	≥ 8	≥ 8	≥ 18
1	≥ 18	≥ 18	≥ 8	≥ 8	≥ 18
3	>160	>110	≥ 38	≥ 24	≥ 24
5	>390	>240	>75	≥ 36	≥ 29

$m = 0.362$ and $p = 1$, all large shells ($B > 50$) can decay into shells with $B = 17$, $B = 7$ and $B = 4$ but can only decay into a $B = 2$ if $B > 250$ and $B = 1$ if $B > 380$. When p is even but nonzero, the normalized energy decreases as B increases implying that the binding energy of the configurations increases with B. This in turn implies that the configurations never decay into smaller shells.

We summarize our observation in Table I where we have given the threshold value corresponding to several decay modes. The threshold values are the values of B above which the decay is always possible, but sometimes some configurations with lower values of B (smaller than 32) can decay in such a mode, too (but only for very special values of B). The values given with a "$\geq$" refer to the values obtained by comparing the energy of the configuration directly (low B) instead of using the derivative of the energy.

In Fig. 5, we present the normalized energy (24) as a function of m for various values of B. We see that for $p = 0$ and $p = 1$ the energy increases rapidly with m and that very quickly the configurations become unstable. When $p > 1$, the normalized energy decreases for small m when m increases, then it reaches a minimum and finally it increases with m. The value of the mass for which the
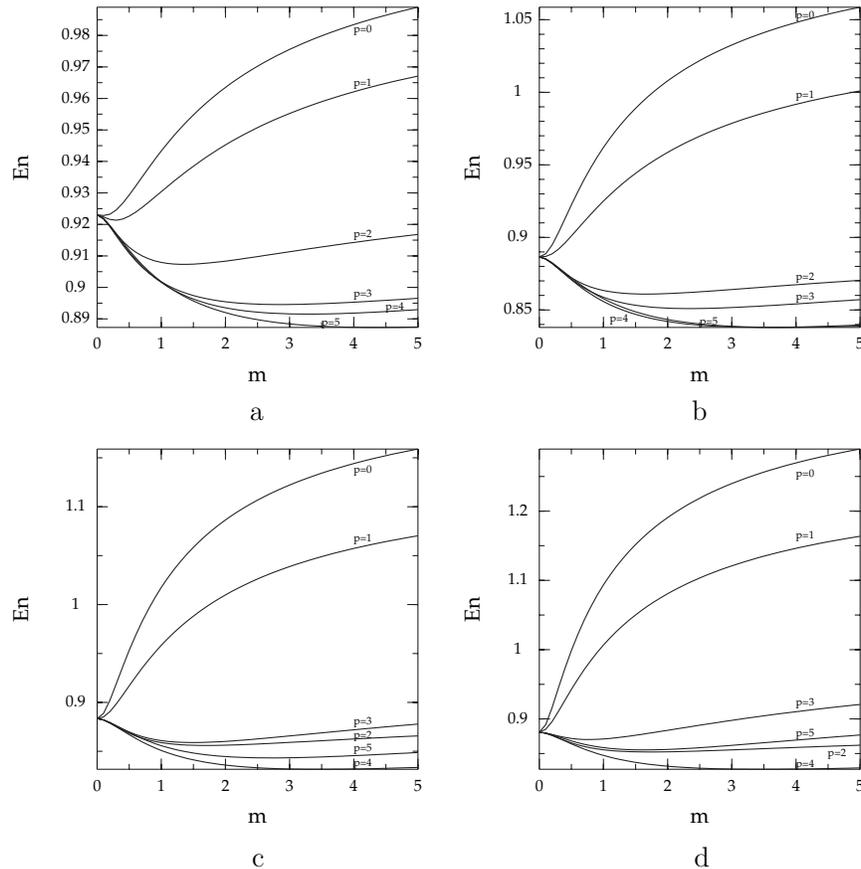

FIG. 5. Normalized energy En as a function of m for various values of p for $B = 4$ (a), $B = 17$ (b), $B = 40$ (c) and $B = 100$ (d).

solution is the most bound depends on p and on the baryon number.

In the next section we present an analytical description of the multi-Skyrmion configurations which will explain some of the features we have observed numerically.

VI. APPROXIMATE ANALYTICAL TREATMENT

It was shown in [3] that many properties of multi-Skyrmions, including their classical mass, spatial distribution, moments of inertia, etc., can be described with a good accuracy using a relatively simple power step (or "inclined" step) approximation of the profile function. A similar approach was also used successfully to describe "baby"-Skyrmions in the $2 + 1$ dimensional Skyrme model [15–17,21]. This approximation also turns out to be useful for the study of the asymptotics of our massive multi-Skyrmion field configurations for large values of the baryon number B.

Let us consider first the large r asymptotics of the profile function. Clearly, this asymptotic behavior is governed by the second derivative term and the mass term in the Lagrangian. The Euler-Lagrange equation then becomes asymptotically

$$(r^2 f')' = (2B + m^2 r^2) f, \qquad (26)$$

and so, if $m^2 r^2 \gg 2B$, we have $2r f' + r^2 f'' = m^2 r^2 f$, which has the asymptotics $f \sim \exp(-mr)$.

For the values of B in the region of r where $m^2 r^2 < 2B$, the profile f has a power behavior, and it is in this region that most of the mass and most of the baryon density of the multi-Skyrmion is concentrated [3], while the exponential tail of the profile function gives only a small correction to all such quantities and so can be neglected. Thus if we can neglect the $\sim m^2$ term on the right side of (26), we obtain the power law $f \sim r^{-\sqrt{2B}}$. As we shall see, the dominant range of r is always such that we can make this approximation, at least for pions and kaons.

Denoting $\phi = \cos f$ and taking for $g(k) = \delta(k - p)$ where p is an integer, the energy of the multi-Skyrmion can be written as

$$M = \frac{1}{3\pi} \int \left\{ \frac{1}{(1 - \phi^2)} [r^2 \phi'^2 + 2B(1 - \phi^2)^2] + \left[2B\phi'^2 \right. \right.$$
$$\left. \left. + I \frac{(1 - \phi^2)^2}{r^2} \right] + 2m^2 \Psi_p(\phi) r^2 \right\} dr, \qquad (27)$$

with ϕ changing from -1 at $r = 0$ to 1 at $r \to \infty$. The first part of (27) is the second order term contribution while the second term is due to the Skyrme term. Note that at fixed $r = r_0$ the fourth order term is exactly proportional to a one-dimensional domain wall energy widely discussed in the literature, see e.g. [22]. The function $\Psi_p(\phi) = (1 - \cos(pf))/p^2$ can be written explicitly for each p: $\Psi_1 = 1 - \phi$, $\Psi_3 = (1 - \phi)(1 + 2\phi)^2/9 \le \Psi_1$, $\Psi_2 = (1 - \phi^2)/2 \le \Psi_1$, $\Psi_4 = \phi^2(1 - \phi^2)/2 \le \Psi_2$, etc. Also

it can be shown that $\Psi_3 \le \Psi_2$, $\Psi_4 \le \Psi_3$, and it follows immediately, for any p, that $\Psi_{4p} \le \Psi_{3p} \le \Psi_{2p} \le \Psi_p$, etc. The functions Ψ_p and the whole mass term have different properties for odd and even p and so, for this reason, these two cases will be considered separately.

It is possible to rewrite the second order term contribution in (27) as

$$M^{(2)} = \frac{1}{3\pi} \int \left\{ \frac{r^2}{(1 - \phi^2)} [\phi' - \sqrt{2B}(1 - \phi^2)/r]^2 \right.$$
$$\left. + 2r\sqrt{2B}\phi' \right\} dr, \qquad (28)$$

and similarly for the fourth order Skyrme term. Next we observe that, if ϕ satisfies $\phi' = \sqrt{2B}(1 - \phi^2)/r$, a large part of the integrand in $M^{(2)}$ vanishes. Therefore, it is natural to consider a function ϕ which satisfies the following differential equation [3]:

$$\phi' = \frac{b}{2r}(1 - \phi^2), \qquad (29)$$

where b is a constant. A solution of this equation, which satisfies the boundary conditions $\phi(0) = -1$ and $\phi(\infty) = 1$, is given by

$$\phi(r, r_0, b) = \frac{(r/r_0)^b - 1}{(r/r_0)^b + 1} \qquad (30)$$

where r_0 is the distance from the origin to the point where $\phi = 0$ and at which the profile $f = \pi/2$. r_0 can be considered as the radius of the multi-Skyrmion. Both b and r_0 are arbitrary at this stage; they will be determined later by means of the mass minimization procedure. Note that the radii of distributions of baryon number and of the mass of the multi-Skyrmion are close to r_0. Let us point out that our parametrization (30) is very accurate as, in the Skyrme model with the usual mass term, as shown in [3], the masses and other characteristics of multi-Skyrmions are described by such a parametrization to within a few percent.

A. Odd powers, $p = 1, 3, \ldots$

Consider first the case of $p = 1$. Then, using (22) and (30) we find that the soliton mass is given by

$$M(B, b) = \frac{1}{3\pi} \int \left\{ \left(\frac{b^2}{4} + 2B \right)(1 - \phi^2) + \left(I + \frac{Bb^2}{2} \right) \right.$$
$$\left. \times \frac{(1 - \phi^2)^2}{r^2} + 2r^2 m^2 (1 - \phi) \right\} dr, \qquad (31)$$

where ϕ is given by (30) and where we should take $m = 0.362$ for the pion case and $m = 1.30$ for kaons, etc.

Given the form of ϕ the integration over r can now be performed using the well-known expressions for the Euler-type integrals, e.g.

$$\int_0^\infty \frac{dr}{1 + (r/r_0)^b} = \frac{\pi r_0}{b \sin(\pi/b)}, \quad (32)$$

if $b > 1$, and, more generally [3],

$$\int_0^\infty \frac{(r/r_0)^c dr}{\beta + (r/r_0)^b} = \beta^{(1+c-b)/b} \frac{\pi r_0}{b \sin[\pi(1 + c)/b]}, \quad (33)$$

with $\beta > 0$. $b > 1 + c$, $c > -1$. Differentiating with respect to β allows us to get the integrals with any power of $1 + (r/r_0)^b$ in the denominator. Thus we can derive the following expressions for the integrals of ϕ given by (30):

$$\int (1 - \phi^2) dr = \frac{4\pi r_0}{b^2 \sin(\pi/b)},$$
$$\int \frac{(1 - \phi^2)^2}{r^2} dr = \frac{8\pi(1 - 1/b^2)}{3r_0 b^2 \sin(\pi/b)}, \quad (34)$$

and other examples useful for the calculation of the mass term,

$$\int (1 - \phi) r^2 dr = \frac{2\pi r_0^3}{b \sin(3\pi/b)},$$
$$\int \phi^2(1 - \phi^2) r^2 dr = \frac{4\pi r_0^3(1 + 18/b^2)}{b^2 \sin(3\pi/b)}. \quad (35)$$

The expressions (34) and (35) allow us to obtain the mass of the multi-Skyrmion field in a simple analytical form as a function of the parameters b and r_0 (in units $3\pi^2 F_\pi/e$):

$$M(B, r_0, b) = \alpha(B, b) r_0 + \beta(B, b)/r_0 + \delta(b) r_o^3. \quad (36)$$

where

$$\alpha = (b^2 + 8B)/(3b^2 \sin(\pi/b)),$$
$$\beta = 4(Bb^2 + 2I)(1 - 1/b^2)/(9b^2 \sin(\pi/b)), \quad (37)$$
$$\delta = 4m^2/(3b \sin(3\pi/b)).$$

The mass term contribution is proportional to the volume of the multi-Skyrmion, $\sim r_0^3$, as expected on general grounds, multiplied by the corresponding flavor content of the Skyrmion.[4] Next we minimize (36) with respect to r_0 and obtain, in a simple form, the precise minimal value of the mass

$$M(B, b) = \frac{2r_0^{\min}}{3}(\sqrt{\alpha^2 + 12\delta\beta} + 2\alpha) \quad (38)$$

where the value of r_0 is given by

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[4]There is a difference of principle between the pion and kaon (or other flavored meson) fields included in the Lagrangian. The mass of flavored mesons enters into the classical soliton mass multiplied by a corresponding flavor content which is always smaller than 1 and even smaller than 0.5 for a rigid or soft rotator quantization scheme. So, when we take the masses $m \simeq 1.30$ for the strangeness or $m \simeq 4.13$ for the charm, we establish the scale of the mass for these flavors, not more than that.

$$r_0^{\min}(B, b) = \left[\frac{\sqrt{\alpha^2 + 12\beta\delta} - \alpha}{6\delta}\right]^{1/2}. \quad (39)$$

Equations (36) and (38) give the upper bounds for the mass of the multi-Skyrmion state, because they are calculated for the profile (30) which is different from the true profile to be obtained by the true minimization of the energy functional (16) with the mass term included. At large values of B the power b is also large, $b \sim \sqrt{B}$, as we shall see, and $\alpha \simeq (b + 8B/b)/(3\pi)$, $\beta \simeq 4(Bb + 2I/b)/(9\pi)$, $\delta \simeq 4m^2/(9\pi)$.

The structure of (36) remains the same for values of p different from 1, except for the case of even p which will be considered separately. For $p = 3$, 5, etc. one must perform the substitution $\delta \to \delta/p^2$, and the volume contribution is reduced by a factor $1/p^2$. The energy (38) can be simplified and analyzed in two different cases, small m or δ, when $12\delta\beta \ll \alpha^2$ (which we will call in what follows the small mass approximation, or SMA), and in the case of large m or large B when $12\delta\beta \gg \alpha^2$, which we will call the large mass approximation, or LMA. Note that, at large B-numbers, $\alpha \sim \sqrt{B}$ and $\beta \sim B\sqrt{B}$; therefore, when B is large enough, the latter inequality can always be satisfied: it reads then, approximately, $m^2\sqrt{B} \gg 1$.

Let us consider first the latter case of large $\beta\delta$ (the LMA case). Now we can neglect the term $\sim \alpha^2$ in the square root of (38) and (39), and obtain

$$M(B, b) \simeq \frac{4}{3^{3/4}} (\beta^3 \delta)^{1/4} \left[1 + \frac{\alpha}{4}\left(\frac{3}{\beta\delta}\right)^{1/2}\right] \quad (40)$$

and

$$r_0^{\min} \simeq \left(\frac{\beta}{3\delta}\right)^{1/4} \left[1 - \frac{\alpha}{4}\left(\frac{1}{3\beta\delta}\right)^{1/2}\right]. \quad (41)$$

It is clear that the minimum value of the mass is reached at the minimum of β (δ does not depend on b when b is large, and the correction term in the square bracket has little influence on the position of the minimum), which is equal to $\beta^{\min} = 8\sqrt{2BI}/(9\pi)$ at $b = \sqrt{2I/B}$. Then

$$M(B) \simeq \frac{16}{9\pi}\left(\frac{2}{3}\right)^{3/4} m^{1/2}(2BI)^{3/8}\left[1 + \frac{3\sqrt{3}(I + 4B^2)}{8\sqrt{2}m(2IB)^{3/4}}\right] \quad (42)$$

Since $I \sim B^2$ (strictly, $I \geq B^2$ [18]), we establish the following scaling law at large B: $M(B) \sim B^{9/8}m^{1/2}$, $r(B) \sim B^{3/8}/m^{1/2}$. Numerically, we have for $p = 1$ ($I = 1.28B^2$ in these estimates)

$$\frac{M}{B}(p = 1) \simeq 0.593\,95\sqrt{m}B^{1/8}\left(1 + \frac{1.1982}{mB^{1/4}}\right). \quad (43)$$

For other odd p, dividing the volume contribution to the

mass term by p^2, we obtain

$$\frac{M}{B}(p) \simeq \frac{0.593\,95}{\sqrt{p}} \sqrt{m} B^{1/8} \left(1 + p\frac{1.1982}{mB^{1/4}}\right). \quad (44)$$

The absolute lower bound for the energy which follows from (44) obviously does not depend on p. These estimates can be improved further: the $O(\alpha^2)$ terms in the expansion of the square root in (38) can be included; the surface contributions to the mass term, besides the volumelike one, can be calculated (this may be important for higher p since the volume contribution decreases like $\sim 1/p^2$); the shift in the position of $b^{\min}$ could also be taken into account.

As the ratio $M(B)/B \sim B^{1/8}$, we conclude that, for large B, the shell configurations will not form a bound state. This confirms what we have observed numerically. The second order term of the initial Lagrangian makes a small contribution in the large mass regime; thus the Skyrme term and the mass term approximately balance each other, and the mass term gives $\sim 1/4$ of the total mass, by the Derrick theorem. The difference between the cases $p = 3, 5, \ldots$ etc. and $p = 1$ resides in the fact that for larger p this "large mass term regime" is reached at higher values of the baryon number. One of the properties of multi-Skyrmions in this regime is that the average energy density does not depend on B, $\rho_M \sim m^2$, or, in ordinary units, $\rho_M \sim \mu_\pi^2 F_\pi^2$ (which does not depend on the Skyrme parameter e). The energy density in the shell can be estimated as well; we get $\rho_{shell} \sim \sqrt{B} \mu_\pi^2 F_\pi^2$ which grows when the baryon number increases. And, as it has been previously discussed in the literature, the transition to other types of classical configurations, like the Skyrmion crystals, may become possible at high values of B.

When the mass m is small enough, as for the pion, the expansion in $12\beta\delta/\alpha^2$ can be made, and one obtains the reduction of the multi-Skyrmion size r_0:

$$r_0 \rightarrow r_0 - \frac{3\delta}{2\alpha}\left(\frac{\beta}{\alpha}\right)^{3/2} \simeq \sqrt{\frac{2}{3}} I^{1/4}\left[1 - \frac{2m^2}{3} I^{1/4}\eta_B\right], \quad (45)$$

and the increase of its mass

$$\delta M = M_{m=0}\frac{\beta\delta}{2\alpha^2}\left[1 - \frac{9\beta\delta}{8\alpha^2}\right]$$

$$\simeq M_{m=0}\frac{2m^2}{9} I^{1/4}\eta_B\left[1 - \frac{m^2}{2} I^{1/4}\eta_B\right], \quad (46)$$

$\eta_B = \sqrt{I}/(2B + \sqrt{I})$, $M_{m=0} \simeq 4B\sqrt{2/3}(2 + \xi_B)/(3\pi)$ [3], and at large B, $M_{m=0}/B \simeq 1.0851$, $\eta_B \simeq 0.3613$ if we take $\xi_B = \sqrt{I/B^2} \simeq 1.131\,37$—constant value, according to [20]. Here, we have used also the observation that, at large B, $b^{\min} = 2I^{1/4}$ [3] and

$$\alpha \simeq \frac{2}{3\pi I^{1/4}}(2B + \sqrt{I}) \sim \sqrt{B},$$

$$\beta \simeq \frac{4I^{1/4}}{9\pi}(2B + \sqrt{I}) \sim B^{3/2}.$$

As expected, the size of the multi-Skyrmion state decreases with increasing m while its mass increases, and these changes become very large for very large B and/or m.

For p different from 1 the substitution $m^2 \rightarrow m^2/p^2$ should be made in (45) and (46) and the following relation can then be obtained for any pair of odd p's, p_1 and p_2:

$$\frac{M_B(p_1) - M_B(p_2)}{M_B} \simeq \frac{2m^2}{9}\left(\frac{1}{p_1^2} - \frac{1}{p_2^2}\right)I^{1/4}\eta_B. \quad (47)$$

Numerically this works well for $p = 1$ and $p = 3$, see Tables II, III, and IV; for larger p's the agreement is less good but then, apparently, other contributions to the mass term, besides the volumelike one, should be included.

In Tables II, III, and IV, we present several values of the energy per baryon obtained from (46) (Table II) and (43) and (44) (Tables III and IV), and compare them with the values obtained numerically. We see that our analytical approximation works very well when the mass is small and B-numbers not too large (pions case, Table II), or when it is large, as for the charm, but it does not work so well for intermediate values of the mass. It also works better for $p = 1$ than for $p = 3$. The case $p = 3$, presented in Table III, is of special interest: the LMA improves when increasing the baryon number but is still not as good as for $p = 1$, whereas SMA becomes worst when B increases and also is not perfect at small values of B. To improve it, the following terms in the expansion (46) should be included.

For not very large values of m the structure of the multi-Skyrmion at large B remains the same: it is given by the chiral symmetry broken phase inside a spherical wall where (on this spherical shell) the main contribution to the mass and topological charge is concentrated [3,20]. The value of the mass density inside this wall is defined completely by the mass term with $1 - \phi = 2$ and decreases with increasing B while the mass density of the shell itself is constant [3]. The baryon number density

TABLE II. Energy per baryon for $m = m_\pi = 0.362$. The analytical calculations are made in the small mass approximation (SMA) according to (46).

B	$p = 1$ (num.)	$p = 1$ (SMA)	$p = 3$ (num.)	$p = 3$ (SMA)
1	1.2740	$\cdots$	1.2576	$\cdots$
40	1.1519	1.1497	1.0990	1.0935
100	1.1734	1.1760	1.0991	1.0982
200	1.1973	1.1956	1.1023	1.1034
300	1.2144	1.2037	1.1053	1.1073
400	1.2279	1.2057	1.1080	1.1106
500	1.2392	1.2038	1.1104	1.1133

TABLE III. Energy per baryon for $m = m_s = 1.300$. The analytical estimates are made according to (44) (LMA) and, for $p = 3$, also in SMA.

B	$p = 1$ (num.)	$p = 1$ (LMA)	$p = 3$ (num.)	$p = 3$ (LMA)	$p = 3$ (SMA)
1	1.4860	$\cdots$	1.3842	$\cdots$	$\cdots$
40	1.4525	1.4675	1.1893	1.3018	1.1701
100	1.5375	1.5553	1.2097	1.3032	1.1968
200	1.6181	1.6351	1.2360	1.3157	1.2056
300	1.6714	1.6875	1.2554	1.3276	1.1982
400	1.7120	1.7273	1.2710	1.3381	1.1820
500	1.7450	1.7596	1.2841	1.3474	1.1603

distribution is quite similar, the only difference being that inside the spherical wall it vanishes.

If, for some physical reasons, we would use a distribution over p in the Lagrangian, as discussed in Secs. II and III, the analytical expression for the multi-Skyrmion energy can be obtained quite analogously. Let us put $p = p_0 + \Delta p$, where $p_0 = 1, 3$, etc., and Δp is assumed to be small. Then, taking into account the changes in the volume contribution to the mass term, we get instead of (44):

$$\frac{M}{B}(p) \simeq \frac{0.593\,95}{\sqrt{p_0}} \sqrt{m} B^{1/8} \left[1 - \frac{\Delta p}{2 p_0} - \frac{(\Delta p)^2}{16} \left(\pi^2 - \frac{6}{p_0^2} \right) \right]$$
$$\times \left[1 + p_0 \left(1 + \frac{\Delta p}{p_0} + \frac{(\Delta p)^2 \pi^2}{8} \right) \frac{1.1982}{m B^{1/4}} \right], \quad (48)$$

and the averaging over any distribution $g(p)$, as suggested by (4), (5), and (18) can be easily performed.

It is also possible to consider, in a similar way, the case of small values of p, near $p = 0$. In this case we have $(1 - \cos(pf))/p^2 \simeq f^2(1 - p^2 f^2/12)/2$, and $f \simeq \pi$ inside the multi-Skyrmions. Evaluations similar to those at the beginning of this section show that the energy per unit B-number, in the large mass regime, is given by

$$\frac{M}{B} \simeq 0.744\,41 \sqrt{m} B^{1/8} \left(1 - \frac{p^2 \pi^2}{48} \right) \left[1 + \frac{0.7628}{m B^{1/4}} \right.$$
$$\times \left. \left(1 + \frac{p^2 \pi^2}{24} \right) \right]. \quad (49)$$

Obviously, at large enough value of the mass, this is some-

TABLE IV. Energy per baryon for $m = m_{ch} = 4.131$. The analytical estimates are made in the large mass approximation, (43) and (44).

B	$p = 1$ (num.)	$p = 1$ (LMA)	$p = 3$ (num.)	$p = 3$ (LMA)
1	2.0558	$\cdots$	1.7370	$\cdots$
40	2.1778	2.1352	1.5160	1.4877
100	2.3619	2.3436	1.5839	1.5805
200	2.5304	2.5216	1.6589	1.6643
300	2.6398	2.6344	1.7111	1.7191
400	2.7222	2.7185	1.7516	1.7607
500	2.7888	2.7861	1.7850	1.7945

what greater than the energy given by (43). The expression (48) can be compared with our numerical results for $p = 0$ presented in Figs. 1 and 2. We note the agreement to within an accuracy of about (5–8)% for the largest values of m and B. The integration over any distribution in p, $g(p)$, near $p = 0$, as presented in (4) and (5), can also be easily made.

B. Even power $p = 2, 4, \ldots$

In the case of even p i.e., $p = 2, 4 \ldots$ the volume contribution to the energy density is reduced because $1 - \cos(pf) \simeq 0$ inside the multi-Skyrmion where the profile $f \simeq \pi$. Because of the dependence of the mass term on the parameter b and due to the connection between r_0 and b, this case is very different from the case of $p = 1, 3, \ldots$. However, we can still write

$$M(B, r_0, b) \simeq \alpha(B, b) r_0 + \frac{\beta(B, b)}{r_0} + \frac{\delta'(b) r_0^3}{b}. \quad (50)$$

This expression coincides with (36), except for the mass term where an additional factor $1/b$ appears and δ' is different from δ. For $p = 2$ we have $\delta'(p = 2) = 4m^2/(b \sin(3\pi/b)) \simeq 4m^2/(3\pi)$ at large b while for $p = 4$ an additional small factor appears as $\delta'(p = 4) = 4m^2(1 + 18/b^2)/(3b \sin(3\pi/b)) \simeq 4m^2/(9\pi)$. It is not easy to find the general expression for larger p; i.e., the expression for δ' which decreases with increasing p [see the discussion after (27)].

In general, we proceed as for odd values of p and after minimizing with respect to r_0 we obtain

$$M(B, b) = \frac{2 r_0^{\min}}{3} \left(\sqrt{\alpha^2 + 12\delta'\beta/b} + 2\alpha \right) \quad (51)$$

and the value of r_0

$$r_0^{\min}(B, b) = \left[\frac{\sqrt{\alpha^2 + 12\delta'\beta/b} - \alpha}{6\delta'/b} \right]^{1/2}. \quad (52)$$

The main difference from the previous case is that, at large values of B, the quantities $\alpha^2 \sim B$ and $\delta'\beta/b \sim B$, i.e., they are of the same order of magnitude, since $b \sim \sqrt{B}$. At large enough b or B we have $12\beta\delta'/(b\alpha^2) \simeq 0.35$ for pions, 4.5 for kaons and ~ 46 for charm.

TABLE V. Asymptotic values of the energy per baryon for $p = 2$ and $p = 4$. The analytical calculations correspond to (59), SMA, and (56) and (57) in LMA.

m	$p = 2$ (num. $B = 500$)	$p = 2$ (approx.)	$p = 4$ (num. $B = 500$)	$p = 4$ (approx.)
0.362	1.0879	1.1016 (SMA)	1.0881	1.0908 (SMA)
1.300	1.2186	1.2048 (SMA)	1.1362	1.1475 (SMA)
		1.2781 (LMA)		1.2596 (LMA)
4.131	1.6236	1.6448 (LMA)	1.3519	1.4116 (LMA)

Let us discuss first the large mass case when we can take

$$\sqrt{\alpha^2 + 12\delta'\beta/b} \simeq 2\sqrt{3\delta'\beta/b} \tag{53}$$

and

$$M(B, b) \simeq \frac{4}{3^{3/4}}\left(\frac{\beta^3\delta'}{b}\right)^{1/4}\left[1 + \frac{\alpha}{4}\left(\frac{3b}{\beta\delta'}\right)^{1/2}\right]. \tag{54}$$

The minimum is reached at $b \simeq 2\sqrt{I/B}$ and we have, recalling that at large B and $p = 2$, $\delta' \simeq 4m^2/(3\pi)$,

$$M(B, p = 2) \simeq Bm^{1/2}\frac{16\xi_B^{1/2}}{3\sqrt{3}\pi 6^{1/4}}\left[1 + \frac{3\sqrt{3}(\xi_B + 2/\xi_B)}{4\sqrt{2}m}\right], \tag{55}$$

at $r_0 \sim \sqrt{B/m}$. Note that $\xi_B = \sqrt{I/B^2}$ and, at large B, it is constant within the rational map approximation.

Numerically, for $p = 2$, we obtain from (55)

$$\frac{M}{B}(p = 2) \simeq 0.666\,12\sqrt{m}\left(1 + \frac{0.8877}{m}\right) \tag{56}$$

and for $p = 4$

$$\frac{M}{B}(p = 4) \simeq 0.506\,14\sqrt{m}\left(1 + \frac{1.5375}{m}\right). \tag{57}$$

In Table V, we present a few values of the asymptotic energy obtained from (59), SMA, and from (56) and (57) in LMA and compare them with the values obtained numerically for $B = 500$. In our calculations, for large B, we have again used the value $\xi_B = 1.131\,37$, [20], and $\delta' \simeq 4m^2/(9\pi)$ for $p = 4$. There is a good agreement between our numerical results and our analytical approximation values when the mass is small (pions), or large (charm scale), and not so good for the intermediate value $m = m_{st}$ where we give both the SMA and LMA results. Our approximation also works better for $p = 2$ than for $p = 4$. Nevertheless, analytical approximations work better for odd p at large m and B than for the even ones. It is possible to improve the analytical estimates although, for even p, the estimate of the preasymptotic contributions to the energy appears to be technically harder to obtain than for odd p.

So, for $p = 2, 4, \ldots$ etc. and for large meson masses the multi-Skyrmion mass is proportional to the baryon number, and the average (volume) mass density decreases as $\sim 1/\sqrt{B}$. At large B the thickness or width of the shell is given by $W \sim 1/\sqrt{m}$—i.e., it does not depend on the B-number and the mass density in the shell is constant, $\rho_{shell} \sim \mu_\pi^2 F_\pi^2$, in contradistinction to the case of odd p where it grows with B.[5]

When the meson mass is small, as for pions, we can perform the following expansion:

$$\sqrt{\alpha^2 + 12\delta'\beta/b} \simeq \alpha + 6\delta'\beta/(b\alpha) + \cdots. \tag{58}$$

The main contribution to the mass is then $M_0 = 2\sqrt{\alpha\beta}$ at $r_0 = \sqrt{\beta/\alpha}$, and the minimum is reached at $b^{min} \simeq 2I^{1/4}$, as in the massless case [3].

As a result we obtain the following expression for the mass of the multi-Skyrmion, given the mass term in the Lagrangian, to first order in δ':

$$M(B) \simeq M_{m=0}\left(1 + \frac{\delta'\beta}{2b\alpha^2}\right)$$

$$\simeq \frac{4B}{3\pi}\sqrt{\frac{2}{3}}(2 + \sqrt{I/B^2})$$

$$\times\left[1 + \frac{\pi\delta'}{4}\eta_B\left(1 - \frac{9\pi\delta'}{16}\eta_B\right)\right]. \tag{59}$$

At large values of B the relative contribution of the m^2 correction is constant (since δ' is constant at large B and $I/B^2 \to$ const, $\eta_B \to$ const $= 0.3613$), in contradistinction to the case of odd values of p, and the value of $M(B)/B$ from (59) is independent of B. Note that the difference of the multi-Skyrmion masses, between the $p = 2$ and $p = 4$ cases, is given by

$$\frac{M(B, p = 2) - M(B, p = 4)}{M(B, m = 0)} \simeq \frac{2m^2}{9}\eta_B(1 - m^2\eta_B). \tag{60}$$

For the pion mass $m^2 \simeq 0.13$, and (60) gives the value ~ 0.01, as shown in Table VI, in agreement with the numerical data, for greater m the agreement is not so

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[5]In this respect there is a direct analogy with the $(2 + 1) - D$ model [16,17,21], where the surface energy density of the rings, representing states of lowest energy, and their width, do not depend on the topological number when this number is large.

TABLE VI. $(M(B, p = 2) - M(B, p = 4))/M(B, m = 0)$ for numerical solutions (column 2 and 3) and according to SMA (59) and LMA (56) and (57), column 4.

m	$B = 100$	$B = 400$	(approx.)
0.362	0.009 89	0.009 87	0.0100 (SMA)
1.300	0.076 02	0.076 15	0.0528 (SMA)
4.131	0.250 53	0.250 89	0.2149 (LMA)

good, since the case of $p = 4$ is more difficult to describe analytically.

For any even p the mass term gives a contribution to the multi-Skyrmion mass which is constant at large baryon numbers (relatively), and which decreases with increasing p, as then δ' decreases. This is in agreement with the numerical results presented in the previous section.

The radius of the multi-Skyrmion state, to first order in the mass term, can also be rewritten as

$$r_0 \simeq r_{0,m=0}\left(1 - \delta'\frac{3\beta}{2\alpha^2 b}\right) \simeq r_{0,m=0}\left(1 - \frac{3\pi}{4}\delta'\eta_B\right). \quad (61)$$

Since δ' decreases with increasing p, the radius of the multi-Skyrmion increases, in good agreement with the numerical results of the previous section. From (59) and (61) we have also that

$$\frac{r_B(p_2) - r_B(p_1)}{r_B(p_1)} \simeq -3\frac{M_B(p_2) - M_B(p_1)}{M_B(p_1)} \quad (62)$$

which is verified to a good accuracy for B larger than ~ 10.

To summarize, in the case of even p, i.e., $p = 2, 4 \ldots$, the multi-Skyrmions have the structure of empty shells; the mass and B-number densities are concentrated in the envelopes of these shells and the energy per unit B decreases with increasing B, asymptotically approaching a constant value.

C. Even p; "the inclined step approximation"

A natural question then arises: to what extent the structure of multi-Skyrmions and their properties depend on the parametrization we have used. Of course, we have to satisfy the boundary conditions on the profile function: $f(0) = \pi$ and $f(\infty) = 0$ and the function should minimize the value of the mass (27). However, the profile f could have been decreasing according to a law which is different from (30), thus giving us different mass and B-number distributions. But it is just the property of the Lagrangian (27) that produces the above-mentioned bubble structure as this structure leads to a low value of the mass. Another, perhaps the simplest, example of a description that we can make is provided by the "toy" model of "the inclined step" type [3]. Such an approximation is cruder than "the power step" considered previously. However, it has the advantage that the calculations can be made for arbitrary p.

Hence, we shall mention it here and compare its results with what we have obtained before.

Let W be the width of the step, and r_0—the radius of the multi-Skyrmion state, defined by the value of r at which the profile $f = \pi/2$. Then we can approximate the profile function by $f = \pi/2 - (r - r_o)\pi/W$ for $r_o - W/2 \leq r \leq r_o + W/2$. This approximation describes the usual domain wall energy (see, e.g., [22]) to within an accuracy of $\sim 9.5\%$.

Next, we write the energy in terms of W, r_0 (recall that $W \sim r_0/b$ in terms of the previous parametrization) and minimize it with respect to both these parameters thus finding the approximate value of the energy. The case of $p = 1$ was considered previously [3], and since the case of other odd p is similar, we restrict our attention here to the case of even p's.

Thus for an arbitrary even p we have

$$\frac{1}{p^2}\int_{r_0-W/2}^{r_0+W/2}(1 - \cos(pf))r^2 dr = \frac{r_0^2}{p^2}W + \frac{W^3}{12p^2} + \frac{2W^3}{p^4\pi^2}. \quad (63)$$

The volume term $\sim r_0^3$ is absent, and since $W \ll r_0$ at large B, we retain the term $\sim r_0^2 W$ on the right-hand side of (63) and omit other terms. Then, for the classical mass of the multi-Skyrmion, we have (the second and fourth order terms were presented in [3])

$$M(B, r_0, W) \simeq \frac{1}{3\pi}\left[\frac{\pi^2}{W}(r_0^2 + B) + W\left(B + \frac{3I}{8r_0^2}\right) + m^2\frac{2r_0^2 W}{p^2}\right]. \quad (64)$$

The minimization with respect of r_0 is straightforward and it gives us

$$M(B, W) \simeq \frac{1}{3\pi}\left[\sqrt{3I}\left(\frac{m^2 W^2}{p^2} + \frac{\pi^2}{2}\right)^{1/2} + B\left(\frac{\pi^2}{W} + W\right)\right] \quad (65)$$

while $(r_0^{\min})^2 = p\sqrt{3I}/[4m\sqrt{1 + p^2\pi^2/(2m^2 W^2)}]$.

We can now consider the two opposite cases. In the case of a large mass, when $2m^2 W^2 \gg p^2\pi^2$, we can expand $(m^2 W^2/p^2 + \pi^2/2)^{1/2} \simeq mW/p + \pi^2 p/(4mW)$ and obtain

$$M(B, W) \simeq \frac{1}{3\pi}\left[W(B + \sqrt{3I}m/p) + \frac{\pi^2}{W}\left(B + \frac{p\sqrt{3I}}{4m}\right)\right]. \quad (66)$$

This gives us

$$M(B) \simeq \frac{2B}{3}\left(1 + \sqrt{3}\xi_B\frac{m}{p}\right)^{1/2}\left(1 + \frac{\sqrt{3}p\xi_B}{4m}\right)^{1/2} \quad (67)$$

for $W^{\min} = \pi[(1 + \sqrt{3}\xi_B p/(4m))/(1 + m\sqrt{3}\xi_B/(2p))]^{1/2}$ where $\xi_B = \sqrt{I/B^2}$. At large m the simple formula (67) provides an asymptotic (at large B) value of the static

energy per unit B which, as is easily seen, is in a good agreement with our numerical results, i.e., for $m \simeq 4.13$ it gives $M/B \simeq 1.666$ for $p = 2$ and $M/B \simeq 1.408$ for $p = 4$ which agree with the numbers in Table V to within (3–4)%.

On the other hand, the large mass limit cannot be assumed when p is large and so we can consider only the small mass limit: $2m^2 W^2 \ll p^2 \pi^2$. In this case we can consider the m^2 dependent term in (65) as a perturbation, as it was done in [3] and in the section above. Now $W^{\min} = \pi$ and

$$M(B) \simeq \frac{1}{3}\left(2B + \sqrt{3\mathcal{I}/2} + \frac{m^2}{p^2}\sqrt{3\mathcal{I}/2}\right). \qquad (68)$$

The radius r_0 is now given by [3]

$$r_0^2 \simeq (3\mathcal{I}/8)^{1/2} \qquad (69)$$

and so, $r_0 \sim \sqrt{B}$, and the corrections to r_0 can be easily found, following the steps similar to those of [3]. The difference of masses of the $p = 2$ and $p = 4$ cases is reproduced well; however, this approximation is too crude at higher values of p.

To conclude, we see that the results obtained within "the inclined step" approximation reproduce well the results of the preceding subsection for even p and describe well the transition to higher values of p where the small mass limit can be applied. Further refinements and improvements of this analytical discussion are possible, e.g. subasymptotics of the B-number dependence can be calculated, but we shall not do this here since our results are already in a good agreement with the numerical data of Sec. V, and as the asymptotic behavior of the solitons mass also is well understood, (43), (44), and (56), etc.

VII. CONCLUSIONS

We have investigated different possible forms of the mass term in the $SU(2)$ Skyrme model, concentrating our attention on a class of terms involving a parameter p. We have found that the case of even p, but $p \neq 0$, in the parametrization of the mass term, $\sim(1 - \cos(pf))/p^2$ is of special interest. In this case the contribution to the static mass proportional to the volume of multi-Skyrmion is absent, and the multi-Skyrmion states, at any value of the chiral meson mass m in the Lagrangian, are the spherical bubbles empty inside, their energy and baryon number being concentrated on the surface of the bubble. The energy per B-number decreases with increasing B and approaches a constant value, which is proportional to $\sqrt{m}$ when m is large. Thus as B increases the shell configurations become more bound.

For odd values of p and $p = 0$, there is a volumelike contribution to the static mass of multi-Skyrmions, within the rational map approximation, which grows faster than B. Thus it is responsible for the asymptotic behavior of the multi-Skyrmion mass $\sim B^{9/8}\sqrt{m}$ and makes the multi-Skyrmions unstable with respect to the decay into Skyrmions of smaller values of B. These unbound configurations have recently been observed by Battye and Sutcliffe [4], for $p = 1$, for relatively low values of B even for the pion mass. We have also observed them for the other odd values of p: $p = 3$ and $p = 5$. When the mass m is small, the configurations decay is into a $B = 17$ or $B = 7$ shell only for very large values of B. As the mass increases, the decay becomes possible for smaller values of B. When p is even but non-null, the configuration cannot decay into smaller shells. Thus if we want our states to be bound states we should consider p even, or a combination of terms with p even.

Note that the fact that some rational map configurations do not form classically bound states suggests that the states of minimal energy for those values of p and B can be of a different form. They might correspond to embedded shells, or not have any shell structure at all. This needs to be investigated further but this can be done only by solving the full equations of the model.

The configurations we obtained for $p > 1$ have lower energy than those considered within the standard mass term and, for this reason, they have a good chance to find realization in nature, not only in elementary particle and nuclear physics, but also in astrophysics and cosmology. Of course, what really happens for physical nuclei is unclear as our results are classical; i.e., to compare them with physical nuclei we would have had to compute quantum corrections, and this has not yet been done for nonzero modes.

Investigations of multi-Skyrmions could be extended also to variants of the model where the higher derivative terms (sixth order, eighth order, etc.) are included into the effective Lagrangian. The studies performed in [23] for the case of the sixth order term and recently in [24] for some generalizations of the Skyrme model including an eighth order term in the chiral derivatives, have shown that topological structures of minimal energy configurations are the same for these model extensions as in the original variant of the model, for values of baryon numbers not too large. Therefore, most probably, the observations of bound large B states made in the present paper for certain modifications of the mass term will be confirmed in such generalizations of the model, although this requires detailed studies.

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