

Rotating spacetimes with a cosmological constant

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ABSTRACT: We develop solution-generating techniques for stationary metrics with one angular momentum and axial symmetry, in the presence of a cosmological constant and in arbitrary spacetime dimension. In parallel we study the related lower dimensional Einstein-Maxwell-dilaton static spacetimes with a Liouville potential. For vanishing cosmological constant, we show that the field equations in more than four dimensions decouple into a four dimensional Papapetrou system and a Weyl system. We also show that given any four dimensional “seed” solution, one can construct an infinity of higher dimensional solutions parametrised by the Weyl potentials, associated to the extra dimensions. When the cosmological constant is non-zero, we discuss the symmetries of the field equations, and then extend the well known works of Papapetrou and Ernst (concerning the complex Ernst equation) in four-dimensional general relativity, to arbitrary dimensions. In particular, we demonstrate that the Papapetrou hypothesis generically reduces a stationary system to a static one even in the presence of a cosmological constant. We also give a particular class of solutions which are deformations of the (planar) adS soliton and the (planar) adS black hole. We give example solutions of these techniques and determine the four-dimensional seed solutions of the 5 dimensional black ring and the Myers-Perry black hole.

Contents

1. Introduction	1
2. An overview of $D = 4$ axially symmetric solutions of the vacuum	4
3. Rotating spacetimes and the Einstein-Maxwell-dilatonic (EMD) system	7
3.1 Set-up of the field equations and their symmetries	7
3.2 Ernst potentials with a cosmological constant	14
3.3 Extending the Papapetrou method	16
3.4 Set-up for uplifted spacetimes in D dimensions	19
4. Examples in $D = 4$ dimensions	22
5. Examples in $D = 5$ dimensions	25
6. Constructing solutions for $\Lambda = 0$	31
7. Conclusions	33
8. Acknowledgements	34

1. Introduction

Exact solutions in General Relativity are essential in order to gain insight on the nature of gravity, and for this reason much effort has been devoted to their systematic construction. In four-dimensional Einstein general relativity, numerous methods have been developed to obtain solutions, usually by assuming some symmetries for spacetime beforehand [1]. An important class of such solutions are spacetimes in vacuum which are axially symmetric and either static or stationary. In the former case, Weyl [2] showed that spacetime metrics can be generated from solutions of the Laplace equation in three-dimensional cylindrical coordinates, and hence that the field equations are essentially integrable¹. Many solutions of physical interest belong to this class: Rindler spacetime, the Schwarzschild black hole, as well as the C-metric

¹By essential here we mean that any solution can be expanded as an infinite series over a self-adjoint basis of orthonormal functions; see for example [3].

[4] describing in part an accelerating black hole, and multiple black hole solutions [5].

The work of Weyl was extended to stationary and axisymmetric spacetimes by Lewis [6] and Papapetrou [7, 8]. Typical examples of such spacetimes are the rotating black hole solution found in the 60's by Kerr [9], and the Taub-NUT (TN) solution [10] which has a new charge and non-trivial spacetime asymptotical behaviour. A great deal of work has also been devoted to developing and extending solution generating methods, and then to the analysis of the resulting new solutions: see [1] for a review of this vast subject and references therein.

In this paper we focus on the powerful methods first developed by Ernst [11]. Their extension enabled relativists (see [1], [12], [13] and references within) to demonstrate that, for vanishing cosmological constant, stationary and axisymmetric metrics are also essentially integrable. Although there have been an important number of papers on the subject, little is known when one includes the cosmological constant in Einstein's field equations. As we shall discuss in detail, the system is no longer integrable in this case², and methods such as those introduced by Papapetrou and Ernst, at first glance, seem to fail. In rather simple terms, integrability breaks down because equations which were homogeneous for $\Lambda = 0$ become inhomogeneous when $\Lambda \neq 0$. Examples of interesting stationary axisymmetric solutions with $\Lambda \neq 0$ are scarce: Carter, for example, found the extension of Kerr's solution with a cosmological constant (as well as a Taub-NUT parameter) by considering separable ansätze for Einstein's equations [14].

With the advent of modern theories of unification and in particular string theory, interest in solutions and solution generating methods in higher dimensional gravity has gradually developed. Myers and Perry [15] first gave the extension of Kerr's solution to higher dimensions, whereas extensions of Carter's solution were undertaken in [16]. In parallel, given the p -brane solutions of Horowitz and Strominger [17] and their importance in the understanding of string theory [18], much work has been devoted to Einstein-Maxwell-Dilaton (EMD) theories. When an EMD solution is Weyl symmetric (i.e. static and axisymmetric) it can, via an exact Kaluza-Klein mechanism and for certain values of the coupling constants appearing in the action, be uplifted to a higher dimensional axisymmetric and stationary vacuum solution. An example is the 4 dimensional Reissner-Nordstrom solution which can be mapped to a 5 dimensional rotating black 1-brane, and for which the black hole charge turns into the rotation potential and vice-versa. Rather less trivially, the work of Dowker et al. [19] in four dimensions, where the C-metric was upgraded to an EMD solution, allowed Emparan and Reall to discover the black ring solution³ in 5 dimensions [21] (see also [22] for a supersymmetric version). This solution represents a rotating black

²This is true even for Weyl's static case.

³See [20] for a full review on black ring type of solutions and a full list of references on the subject.

hole of given mass and angular-momentum, with a horizon of ring topology, $S^2 \times S^1$, thus making it different to the Myers-Perry solution. Indeed, the black ring is a typical higher dimensional solution preventing the extension of 4 dimensional uniqueness theorems [23]⁴. Regarding solution generating methods (see [25] for work on classification of higher dimensional solutions), Emparan and Reall [26] extended Weyl's work to higher dimensions while a cosmological constant was included in the analysis of [27]. Recently Harmark et al. [28, 29] analysed stationary and axisymmetric metrics for $\Lambda = 0$, giving the relevant mappings of solutions in multiple coordinate systems.

This paper aims to study solution generating methods for stationary and axisymmetric spacetimes in arbitrary dimension, and with non-vanishing cosmological constant $\Lambda \neq 0$. Apart from the interest in classical and higher dimensional general relativity, one must stress the importance of asymptotically adS solutions in string theory. Any such solution is a classical background with which to put the adS/CFT correspondence to the test [30]. Furthermore, recent exotic developments in cosmology, such as braneworlds, have brought particular attention to gravitating solutions of axial symmetry in adS. Indeed, an axially symmetric metric in 5 dimensions corresponds to a spherically symmetric geometry on the brane. A solution that describes a 4 dimensional black hole localised on a Randall-Sundrum (RS) braneworld [31] (see [32] for a clear explanation in lower dimensions, and also [33] and references within), if it exists would enter this category. One in particular, would seek a very particular metric of axial symmetry: the equivalent of a C-metric in 4 dimensions which describes, in part, an accelerating black hole. The reason for this is the following: an RS brane, embedded in a negatively curved spacetime, is charted in Poincaré coordinates, so that the brane induced metric is flat. This coordinate system from the bulk point of view is an accelerating patch covering a part of adS space. In rather loose terms this patch is similar for adS to the Rindler coordinates for Minkowski spacetime. Therefore a localised RS black hole has to be accelerating in order to keep up with the brane, meaning in turn that in the 5 dimensional bulk one wants a generalised C-metric: such a solution is yet unknown, even when $\Lambda = 0$ (see [27] for a recent discussion). More generally, axisymmetric solutions are important for theoretical, related in particular to the issue of stability of higher dimensional black hole solutions [34], and phenomenological reasons, particularly in the context of detecting extra dimensions. Furthermore, they are also related to solutions describing anisotropic Bianchi type cosmologies with perfect fluid sources (see [35]) or again to the gravitational field of sources such as the linear cosmic string [36]. It was found in [36] that the weak field approximation for the metric around a RS localised cosmic string differs from the 4 dimensional one [37]. These questions are even more intriguing since recent work [38], making use of the adS/CFT correspondance relating such

⁴However, see [24], where it is shown that a stationary rotating black hole must have an axial Killing vector.

bulk backgrounds with their brane-boundaries, can promote such classical solutions as probes of quantum effects on the braneworld.

In this paper we consider D -dimensional Einstein gravity with a cosmological constant term, and search for stationary and axisymmetric solutions. From a Kaluza-Klein perspective we also consider $d = D - 1$ dimensional EMD solutions (see, for example, [39]) with a Liouville potential for the dilaton (see, for example, [40]). For simplicity we consider a single angular momentum parameter throughout, thus postulating the existence of $D - 2$ Killing vectors of which only two are non orthogonal. We begin in section 2 by reviewing stationary and axisymmetric spacetimes with $\Lambda = 0$ in 4 dimensions, and also the well known solution generating methods of Ernst and Papapetrou. Then, in section 3, we introduce the cosmological constant and generalise the dimensionality of spacetime. The field equations are set up in a convenient form which resembles (but is not identical to) the original Lewis-Papapetrou 4 dimensional form, and this enables us to discuss their symmetries and extend 4 dimensional electromagnetic duality to include the presence of a cosmological constant. Furthermore it allows us to generalise the Ernst equation to arbitrary d and $\Lambda \neq 0$ (subsection 3.2); to extend Papapetrou's method (for arbitrary d and $\Lambda \neq 0$) and demonstrate that any Weyl solution gives a class of rotating solutions satisfying Papapetrou's hypothesis (subsection 3.3); to give a special class of solutions which describe deformations of the adS soliton and the planar adS black hole; and finally to present a method which allows for the direct construction of higher dimensional rotating metrics from lower dimensional ones (subsection 3.4). Finally, in sections 4, 5 and 6, we give some simple examples and put into practice the methods developed. Conclusions are given in section 7.

2. An overview of $D = 4$ axially symmetric solutions of the vacuum

In four dimensions, a static and axisymmetric metric can be written in the form

$$ds^2 = -e^{2\lambda} dt^2 + e^{-2\lambda} [\alpha^2 d\varphi^2 + e^{2\chi} (dR^2 + dZ^2)], \quad (2.1)$$

where α , λ and χ are functions of R and Z only. It follows from the vacuum Einstein equations $R_{ab} = 0$ that α is harmonic, $\Delta\alpha = (\partial_Z^2 + \partial_R^2)\alpha = 0$, and hence that one can always set $\alpha = r$ by a two dimensional conformal transformation in the (R, Z) plane. Without loss of generality, the metric then takes the well-known Weyl form [2]

$$ds^2 = -e^{2\lambda} dt^2 + e^{-2\lambda} [r^2 d\varphi^2 + e^{2\chi} (dr^2 + dz^2)], \quad (2.2)$$

and in this special coordinate system $\lambda(r, z)$ now satisfies

$$\left(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 \right) \lambda = 0. \quad (2.3)$$

Since this is just the three-dimensional flat Laplace equation in cylindrical coordinates, formally λ can be seen as the Newtonian potential generated by an axisymmetric Newtonian source [2]. Once a solution (or potential) λ is chosen in (2.3), the full metric is determined by solving the remaining Einstein's equations for χ :

$$\partial_r \chi = r [(\partial_r \lambda)^2 - (\partial_z \lambda)^2], \quad \partial_z \chi = 2 r \partial_r \lambda \partial_z \lambda, \quad (2.4)$$

which carry the full non-linearity of $R_{ab} = 0$. Since (2.3) is linear, one can superpose λ -potentials and then calculate the relevant χ field from (2.4). For instance, the Schwarzschild solution corresponds to the Newtonian potential of a rod placed at $z = 0$ and of finite length (per unit mass) in the z direction (see for example [5] or [27]); the Rindler spacetime corresponds to a semi-infinite rod; and their superposition gives rise to the Newtonian potential corresponding to the C-metric describing, in part, the spacetime of an accelerating black hole [4]. This is one intuitive way of obtaining solutions in the form (2.2). Alternatively it is useful to recall that since (2.3) is a linear second order equation one can solve it directly by separation of variables, find the relevant eigenfunctions for the separate Sturm-Liouville problems, and then expand in terms of the basis of functions (see [3]).

The choice of a coordinate system in which to undertake the task of writing down the metric solutions can be crucial. Although the Weyl canonical form is particularly helpful for the analysis of the system of equations at hand and for classifying the solutions, it is often useful to write specific solutions in coordinates differing from those in (2.2). A particularly appropriate coordinate system turns out to be the spheroidal coordinates discussed by Zipoy [41], which have ellipsoids and hyperboloids of revolution as coordinate surfaces. As we will see below, they are tailored to describe the Schwarzschild Weyl potential and were first introduced in order to express the exact Newtonian potential around the earth. Thus rather than Weyl coordinates (r, z) , consider polar-like coordinates (u, ψ) but with hyperbolae as radial functions, that is

$$\begin{aligned} z &= \cosh u \cos \psi, \\ r &= \sinh u \sin \psi, \end{aligned} \quad (2.5)$$

so that in the (r, z) plane $\psi = \text{const}$ curves are hyperboloids and $u = \text{const}$ are ellipsoids. On setting $x = \cosh u$ and $y = \cos \psi$, the coordinate system becomes symmetric in x and y : the 2 dimensional line element is given by

$$dr^2 + dz^2 = (x^2 - y^2) \left[\frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right] \quad (2.6)$$

and the Laplace equation (2.3) takes the form

$$\frac{1}{x^2 - y^2} \left\{ \frac{\partial}{\partial x} \left[(x^2 - 1) \frac{\partial \lambda}{\partial x} \right] + \frac{\partial}{\partial y} \left[(1 - y^2) \frac{\partial \lambda}{\partial y} \right] \right\} = 0. \quad (2.7)$$

As an example of these different coordinate shuffles and one in which spheroidal coordinates appear naturally, consider a Schwarzschild black hole: the standard metric

$$ds^2 = - \left(1 - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_{II}^2 \quad (2.8)$$

can be rewritten in Weyl coordinates (R, Z) of (2.1) where $r/2M = \cosh^2(R/2)$ and $\theta = Z$. The conformal transformation to (2.2) then gives $z = \cos Z \cosh R$ and $r = \sin Z \sinh R$ as in (2.5) (that is, $u = R$ and $\psi = Z$), and

$$e^{2\lambda} = \frac{x-1}{x+1}. \quad (2.9)$$

It can be easily checked that this Weyl potential λ is indeed a solution of (2.7). More generally, the solutions of (2.7) are separable and consist of products of Legendre polynomials [41]. Appropriate boundary conditions (as well as other coordinate systems) have been considered by different authors [3, 41]. Spheroidal coordinates are also very relevant for the analysis of *stationary* axisymmetric vacuum solutions, as we now discuss.

Lewis and Papapetrou generalised the approach of Weyl to stationary and axisymmetric solutions in vacuum [6, 7, 8]. After a conformal transformation, the metric takes the Lewis-Papapetrou form

$$ds^2 = -e^{2\lambda} (dt + Ad\varphi)^2 + e^{-2\lambda} [r^2 d\varphi^2 + e^{2x}(dr^2 + dz^2)], \quad (2.10)$$

which differs from the static form by the additional component $A = A(r, z)$. Note that ∂_t is no longer a static but rather a stationary (locally) timelike Killing vector field, and that one cannot, via a coordinate transformation, remove the non-diagonal metric component whilst keeping the line-element ‘ t ’ independent. For the metric (2.10), Ernst [11] pointed out an interesting reformulation of Einstein’s equations for A and λ , which read respectively

$$\partial_r \left(\frac{e^{4\lambda}}{r} \partial_r A \right) + \partial_z \left(\frac{e^{4\lambda}}{r} \partial_z A \right) = 0, \quad \left(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 \right) \lambda = \frac{e^{4\lambda}}{2r^2} [(\partial_r A)^2 + (\partial_z A)^2]. \quad (2.11)$$

Indeed, on introducing an auxiliary field, ω , defined⁵ by

$$(-\partial_z \omega, \partial_r \omega) = \frac{e^{4\lambda}}{r} (\partial_r A, \partial_z A), \quad (2.12)$$

the complex function

$$\mathcal{E} = e^{2\lambda} + i\omega \quad (2.13)$$

⁵As we will see later on, this auxiliary field describes nothing but the passage from an electric to a magnetic potential and vice-versa.

then satisfies the complex differential equation

$$\frac{1}{r} \overrightarrow{\nabla} \cdot \left(r \overrightarrow{\nabla} \mathcal{E} \right) = \frac{(\overrightarrow{\nabla} \mathcal{E})^2}{\text{Re}(\mathcal{E})} \quad (2.14)$$

known as the Ernst equation. Its real and imaginary part are exactly (2.11).⁶ In this language, the Weyl potential λ is simply given by the real part of the Ernst potential \mathcal{E} , whereas rotation is embodied by a non-trivial ω .

Using the symmetries of complex functions, several methods have been proposed to obtain solutions of the Ernst equation (2.14) and hence to generate new metrics (see [1], [11], [13] and references within). An elegant application appeared in Ernst's original paper [11], namely a simple method to obtain the Kerr solution from the Schwarzschild solution. This example also underlines the importance of the choice of coordinates. Indeed, let

$$\mathcal{E} = \frac{\xi - 1}{\xi + 1}. \quad (2.15)$$

Then in spheroidal coordinates and for the Schwarzschild solution, it follows from (2.9) and (2.13) that $\xi = x$. Note that our new metric component ξ is now the 'radial' coordinate x , rather as in (2.2) where $\alpha = r$. We have adapted the coordinate system to the real part of the black hole Ernst potential. By symmetry, $\xi = y$ is also solution of (2.14), as is $\xi = x \sin \vartheta + iy \cos \vartheta$. It turns out that this is nothing other than the Ernst potential of the Kerr black hole, where $\sin \vartheta = a/M$ is the ratio between the angular momentum parameter and the mass of the black hole [11].

In a similar manner, Papapetrou noted that if one makes the hypothesis $\lambda = \lambda(\omega)$ then the system (2.11), with (2.12), is integrable [7]. Solutions obtained this way generally have non-trivial asymptotic properties and, in particular, Gautreau and Hoffman [42] showed that the above hypothesis reduces the stationary Papapetrou system to a Weyl static system. They also showed that starting from the Weyl potential of the Schwarzschild black hole one could easily construct the TN solution [10], which thus belongs to the Papapetrou class.

We now proceed to generalise the work of Ernst and Papapetrou to higher dimensions, including a non-vanishing cosmological constant.

3. Rotating spacetimes and the Einstein-Maxwell-dilatonic (EMD) system

3.1 Set-up of the field equations and their symmetries

We consider D -dimensional stationary axisymmetric metrics of the form

$$ds_D^2 = -e^{2W} (dt + A d\varphi)^2 + e^{2U_\varphi} d\varphi^2 + \sum_{i=1}^{D-4} e^{2U_i} dx_i^2 + e^{2V} (dr^2 + dz^2) \quad (3.1)$$

⁶Note that here, $\overrightarrow{\nabla} = (\partial_r, \partial_z)$, whereas in the literature $\overrightarrow{\nabla}$ is often used to denote a three-dimensional gradient in cylindrical coordinates.

where all the metric components are functions of r and z only, and we search for solutions of the vacuum Einstein equations with a cosmological constant

$$G_{AB} + \Lambda g_{AB} = 0. \quad (3.2)$$

The metric (3.1) possesses $(D - 2)$ Killing vector fields, of which ∂_t and ∂_φ are not orthogonal to each other, so that the spacetime is stationary rather than static. When $D = 4$, (3.1) is the most general stationary axisymmetric metric (which, when $\Lambda = 0$, can be written in the form (2.10)). For $D > 4$ multiple angular momenta are possible: here, however, we work with (3.1) which can be seen as the simplest generalization, through the addition of a single angular momentum A , of a static axisymmetric D -dimensional Weyl solution.

For the following analysis, it will be useful to recall (see for example [43]) that (3.1) can be dimensionally reduced to a $(D - 1)$ dimensional EMD system. Numerous higher-dimensional solutions have been obtained this way, [43], [44]. Indeed, Kaluza-Klein reduction of the metric (3.1) yields, putting aside the question of the signature for the moment, a $(D - 1)$ -dimensional metric together with a scalar field and a vector potential. More explicitly, if one starts from a D -dimensional metric \tilde{g}_{AB} , with dynamics governed by

$$S_D = \int d^D x \sqrt{-\tilde{g}} \left(\tilde{R} - 2\Lambda \right), \quad (3.3)$$

and decompose \tilde{g}_{AB} as

$$\tilde{d}s_D^2 = e^{-2a\phi} ds_{D-1}^2 + e^{2(D-3)a\phi} (dw + A_\nu dx^\nu)^2. \quad (3.4)$$

then the $(D - 1)$ -dimensional metric $g_{\mu\nu}$, the $(D - 1)$ form A_ν and the scalar field ϕ obey the system of equations derived from the action

$$S_{D-1} = \int d^{D-1} x \sqrt{-g} \left[R - (D - 2)(D - 3)a^2 (\partial\phi)^2 - \frac{1}{4} e^{-2(D-2)a\phi} F^2 - 2\Lambda e^{2a\phi} \right], \quad (3.5)$$

with field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (3.6)$$

Note that the dependence on the dilaton, ϕ , in (3.4) has been chosen so that the $(D - 1)$ dimensional action (3.5) corresponds to the Einstein frame. Notice also that since $\Lambda \neq 0$, the dilaton acquires an exponential potential. We now set

$$a = \pm \frac{1}{\sqrt{2(D - 2)(D - 3)}} \quad (3.7)$$

so that the kinetic term for ϕ is canonically normalised, and in turn the dilaton's potential and its coupling to the field strength are completely determined.

We now generalise one step further and, rather than (3.5), consider the action

$$S_d = \int d^d x \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{\gamma\phi}F^2 - 2\Lambda e^{-\delta\phi} \right], \quad (3.8)$$

where now the parameters γ and δ are *arbitrary*. Solutions to (3.8) have been studied in the past (see e.g. [45]) including spacetimes with non-trivial asymptotic behaviour [39]. Broadening the parameter space in this way will enable us to study the generic properties of the system of equations derived from (3.8), which are the subject of the remainder of this paper. Indeed, it is worth stressing that the black ring solution [21], which is a five-dimensional vacuum solution, was derived from a four-dimensional solution of an Einstein-Maxwell dilatonic system [19]. The solutions of (3.8) are, of course, solutions of the D -dimensional action (3.5) with⁷

$$d = D - 1, \quad (3.9)$$

if the coupling parameters take the specific values

$$\gamma = \pm \sqrt{\frac{2(D-2)}{(D-3)}}, \quad \delta = \pm \sqrt{\frac{2}{(D-2)(D-3)}} = 2a. \quad (3.10)$$

From (3.1), the d -dimensional metric $g_{\mu\nu}$ in (3.8) is fully diagonal and a Weyl metric.

We suppose here that the vector potential A_μ in (3.8) has only one non-zero component since we only consider a single angular momentum for the uplifted case. This non-zero component A_{μ_*} can be timelike ($\mu_* = 0$), in which case the vector potential A_μ is said to be electric, whereas if it is spacelike ($\mu_* \neq 0$), A_μ is magnetic. In both cases, we consider a diagonal d -dimensional metric of the form

$$ds_d^2 = -e^{2U_0} dx_0^2 + \sum_{i=1}^{d-3} e^{2U_i} dx_i^2 + e^{2V} (dr^2 + dz^2), \quad (3.11)$$

where the functions U_μ (with $\mu = 0, \dots, d-3$) and V only depend on r and z .

When the d -dimensional EMD solution is related to a $D = d + 1$ dimensional vacuum solution, i.e. when the coupling parameters satisfy (3.10), then an electric solution can be uplifted to a D -dimensional metric of the form (3.1) via a double Wick rotation of the metric (3.4)

$$w \rightarrow it \quad (3.12)$$

$$x_0 \rightarrow i\varphi. \quad (3.13)$$

In the magnetic case, one must not only use the double Wick rotation

$$w \rightarrow it \quad (3.14)$$

$$x_0 \rightarrow iy \quad (3.15)$$

⁷Throughout this paper we will note by D the dimension of the uplifted metrics, whereas d will denote the dimension of the EMD spacetime.

where y is one of the space-like coordinates x_i in (3.1), but also transform A_{μ^*} according to

$$A_{\mu^*} \rightarrow iA. \quad (3.16)$$

The case $d = 3$ is rather special since there is only one extra coordinate other than r and z . In the magnetic case, the extra coordinate, say x_1 , is necessarily spacelike which implies that the 3-dimensional metric (3.11) is *a priori* of Riemannian signature. One can then obtain a Lorentzian $D = 4$ metric of the form (3.1) via the transformations $w \rightarrow it$ and $A_{\mu^*} = A_1 \rightarrow iA$.

When $d = 4$ and $\Lambda = 0$ the electric and magnetic spacetimes are linked via the well-known electromagnetic duality relating strong to weak dilaton coupling, namely

$$\phi \rightarrow \bar{\phi} = -\phi, \quad F_{\mu\nu} \rightarrow \bar{F}_{\mu\nu} = \frac{1}{2} e^{\gamma\phi} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}. \quad (3.17)$$

We will discuss duality relations for $\Lambda \neq 0$ at the end of this section.

Given these well-known preliminaries and notation issues, we are now ready to analyse the equations of motion coming from (3.8) with $g_{\mu\nu}$ given in (3.11). It is useful to define

$$\alpha = \exp\left(\sum_{\mu=0}^{d-3} U_\mu\right), \quad \hat{U}_\mu = U_\mu - \frac{1}{d-2} \ln \alpha, \quad \chi = V + \frac{d-3}{2(d-2)} \ln \alpha, \quad (3.18)$$

so that the deviations, \hat{U}_μ , from the average, α , sum to zero:

$$\sum_{\mu=0}^{d-3} \hat{U}_\mu = 0. \quad (3.19)$$

In terms of these functions the metric (3.11) is given by

$$ds_d^2 = e^{2\chi} \alpha^{-\frac{d-3}{d-2}} (dr^2 + dz^2) + \alpha^{\frac{2}{d-2}} \sum_{\mu=0}^{d-3} \eta_{\mu\mu} e^{2\hat{U}_\mu} (dx^\mu)^2, \quad (3.20)$$

where $\eta_{\mu\nu}$ is the Minkowski metric. Let us also introduce the complex conjugate coordinates u and v such that

$$u = \frac{r - iz}{2}, \quad v = \frac{r + iz}{2}, \quad \text{and} \quad 4 dudv = dr^2 + dz^2, \quad r, z \in \mathbb{R}. \quad (3.21)$$

Then the equations of motion derived from (3.8) are

$$\Delta\alpha = -2\Lambda\alpha^{\frac{1}{d-2}} e^{2\chi - \delta\phi}, \quad (3.22)$$

$$0 = \bar{\nabla} \cdot \left(e^{\gamma\phi - 2\hat{U}_*} \alpha^{\frac{d-4}{d-2}} \bar{\nabla} A \right), \quad (3.23)$$

$$\frac{1}{\alpha} \bar{\nabla} \cdot \left(\alpha \bar{\nabla} \phi \right) = \frac{\gamma\epsilon}{2} e^{\gamma\phi} \alpha^{-\frac{2}{d-2}} e^{-2\hat{U}_*} \left(\bar{\nabla} A \right)^2 - 2\delta\Lambda\alpha^{-\frac{d-3}{d-2}} e^{2\chi - \delta\phi}, \quad (3.24)$$

$$\frac{1}{\alpha} \vec{\nabla} \cdot (\alpha \vec{\nabla} \hat{U}_*) = -\epsilon \frac{d-3}{2(d-2)} e^{\gamma\phi-2\hat{U}_*} \alpha^{-\frac{2}{d-2}} (\vec{\nabla} A)^2, \quad (3.25)$$

$$\frac{1}{\alpha} \vec{\nabla} \cdot (\alpha \vec{\nabla} \hat{U}_{\hat{\mu}}) = \frac{\epsilon}{2(d-2)} e^{\gamma\phi-2\hat{U}_*} \alpha^{-\frac{2}{d-2}} (\vec{\nabla} A)^2, \quad (\hat{\mu} \neq \mu_*) \quad (3.26)$$

$$2\chi_{,u} \frac{\alpha_u}{\alpha} - \frac{\alpha_{,uu}}{\alpha} = \hat{U}_{,u}^2 + \frac{1}{2} \phi_{,u}^2 + \frac{\epsilon}{2} e^{\gamma\phi} \alpha^{-\frac{2}{d-2}} e^{-2\hat{U}_*} (A_{,u})^2 \quad (u \leftrightarrow v), \quad (3.27)$$

where we have distinguished the component $\hat{U}_* \equiv \hat{U}_{\mu_*}$ (along the direction in which the potential A_μ is switched on) from the other components denoted by $\hat{U}_{\hat{\mu}}$. An extra equation exists for χ but it is just a Bianchi identity so we omit it. The parameter ϵ takes the value $\epsilon = -1$ when the potential is electric, and $\epsilon = 1$ when it is magnetic. Equation (3.23) is simply Maxwell's equation, whilst equation (3.24) is the equation of motion for the dilaton. Finally, the ordinary (complex) differential equation (3.27) and its complex conjugate, where we have set $\hat{U}_{,u}^2 = \sum_{\mu=0}^{d-3} U_{\mu,u}^2$, yield two real partial differential equations by restriction to their real and imaginary parts.

For the following analysis it is expedient to rewrite equations (3.22)-(3.27) in a form as close as possible to the original Papapetrou and Ernst formulation of the $D = 4$ equations of motion with $\Lambda = 0$ (section 2). To do so we follow the following strategy: decouple whenever possible the field equations between them; use (3.22) to absorb the cosmological constant Λ ; and finally render the field equations as independent of the dimension d as possible. Consider therefore the linear combinations

$$\Psi_{\mu_*} \equiv \Psi_* = \sqrt{\frac{d-3}{d-2}} \left[\sqrt{\frac{d-3}{d-2}} (\phi - \delta \ln \alpha) + \gamma \hat{U}_* \right], \quad (d > 3) \quad (3.28)$$

$$\Psi_{\hat{\mu}} = \hat{U}_{\hat{\mu}} + \frac{1}{d-3} \hat{U}_*, \quad (d > 3)$$

$$\Omega = \gamma(\phi - \delta \ln \alpha) - 2\hat{U}_*$$

$$2\nu = 2\chi - \delta\phi + \frac{\delta^2}{2} \ln \alpha,$$

and we take $\Psi_\mu = 0$ for $d = 3$. From (3.19), it follows that $\sum_{\hat{\mu} \neq \mu_*} \Psi_{\hat{\mu}} = 0$. On defining the positive constant

$$s \equiv \gamma^2 + 2 \frac{d-3}{d-2}, \quad (3.29)$$

the equations (3.22-3.27) simplify to

$$\Delta \alpha = -2\Lambda \alpha^{\frac{1}{d-2} - \frac{\delta^2}{2}} e^{2\nu}, \quad (3.30)$$

$$0 = \vec{\nabla} \cdot \left(e^{\Omega} \alpha^{\frac{d-4}{d-2} + \gamma\delta} \vec{\nabla} A \right), \quad (3.31)$$

$$\frac{1}{\alpha} \vec{\nabla} \cdot (\alpha \vec{\nabla} \Omega) = \frac{\epsilon s}{2} e^{\Omega} \alpha^{\gamma\delta - \frac{2}{d-2}} (\vec{\nabla} A)^2, \quad (3.32)$$

$$\vec{\nabla} \cdot (\alpha \vec{\nabla} \Psi_\mu) = 0, \quad (\mu = 0, \dots, d-3) \quad (3.33)$$

$$\begin{aligned}
2\nu_{,u} \frac{\alpha_u}{\alpha} - \frac{\alpha_{,uu}}{\alpha} &= \frac{1}{s} \left(\Psi_{*,u}^2 + \frac{1}{2} \Omega_{,u}^2 \right) + \frac{\epsilon}{2} e^{\Omega} \alpha^{\gamma\delta - \frac{2}{d-2}} (A_{,u})^2 + \\
&+ \sum_{\hat{\mu} \neq \mu^*} \Psi_{\hat{\mu},u}^2, \quad (u \leftrightarrow v).
\end{aligned} \tag{3.34}$$

These equations form the basis of the following analysis, and hence a few remarks are in order.

First suppose that $\Lambda = 0$. Then, given (3.30), α is harmonic and, as before, we can set $\alpha = r$ without loss of generality. Note then that equations (3.31)-(3.32) for (Ω, A) and (3.33) for the potentials Ψ_μ completely decouple. The former pair are analogous to the Weyl-Papapetrou equations of (2.11) with, however, γ and δ arbitrary, whereas (3.33) are just Weyl potential equations (2.3). The equations (3.34), which we shall call integrability conditions, relate all potentials together giving the function ν . Thus, we have shown that, when $\Lambda = 0$, the d -dimensional system decouples to a ‘‘Lewis-Papapetrou pair’’ on the one hand and $d - 2$ Weyl potentials on the other hand. Given the analysis of section 2, for $\Lambda = 0$ and d arbitrary, the system involving a single A -component is therefore (essentially) integrable. Note that this decoupling is a consequence of three facts: *i*) we have set $\Lambda = 0$; *ii*) there is only one non-zero angular momentum and *iii*) the choice of our metric components (3.28). Indeed, with the choice of (3.28) we can conveniently rewrite the matrix of potentials [28] so that they are all diagonal modulo the 2 by 2 matrix involving (Ω, A) .

When $\Lambda \neq 0$, α is no longer harmonic and, hence, an adapted coordinate system for α can no longer be chosen: this is the major difficulty with the addition of the cosmological constant. Now (3.30) gives ν in terms of α which can then be substituted in (3.34) at the expense of raising the order of the equation. Furthermore, the different potentials (3.31-3.33) are coupled through α . Despite this, a number of symmetries can presently be identified and extended from $\Lambda = 0$ to $\Lambda \neq 0$, as we will now see.

Consider in particular the possible generalisation of EM duality (3.17). Given the form of Maxwell’s equation (3.31), define a dual potential ω through

$$(-\partial_z \omega, \partial_r \omega) = e^{\Omega} \alpha^{\frac{d-4}{d-2} + \gamma\delta} (\partial_r A, \partial_z A). \tag{3.35}$$

This is simply the analogous of the second equation of (3.17), with ω the vector potential of the Hodge dual of $F_{\mu\nu}$. In terms of ω , the equations (3.31-3.32) and (3.34) take the rather similar form

$$0 = \vec{\nabla} \cdot \left(e^{-\Omega} \alpha^{-\frac{d-4}{d-2} - \gamma\delta} \vec{\nabla} \omega \right) \tag{3.36}$$

$$\frac{1}{\alpha} \vec{\nabla} \cdot \left(\alpha \vec{\nabla} \Omega \right) = \frac{\epsilon s}{2} e^{-\Omega} \alpha^{-\gamma\delta - \frac{2(d-3)}{d-2}} \left(\vec{\nabla} \omega \right)^2 \tag{3.37}$$

$$2\nu_{,u} \frac{\alpha_u}{\alpha} - \frac{\alpha_{,uu}}{\alpha} = \frac{1}{s} \left(\Psi_{*,u}^2 + \frac{1}{2} \Omega_{,u}^2 \right) - \frac{\epsilon}{2} e^{-\Omega} \alpha^{-\gamma\delta - \frac{2(d-3)}{d-2}} (\omega_{,u})^2 +$$

$$+ \sum_{\hat{\mu} \neq \mu_*} \Psi_{\hat{\mu}, u}^2, \quad (u \leftrightarrow v). \quad (3.38)$$

Comparing (3.31-3.34) with (3.36-3.38), it is clear that, for $\Lambda = 0$ ⁸ and $d = 4$, one can associate to every given solution of (3.31-3.34) a dual solution of these same equations through the map

$$\mathcal{E}\mathcal{M} = \begin{cases} \Omega \rightarrow -\Omega, \\ A \rightarrow \omega, \\ \epsilon \rightarrow -\epsilon \end{cases}. \quad (3.39)$$

This is just the EM duality. For $\Lambda \neq 0$ this duality no longer holds. Consider instead the map

$$\mathcal{A} = \begin{cases} \Omega \rightarrow \bar{\Omega} = -\Omega, \\ \omega \rightarrow \bar{A} = \omega, \\ \gamma \rightarrow \bar{\gamma} = \pm\gamma \\ \delta \rightarrow \bar{\delta} = \mp \left[\delta + \frac{2(d-4)}{(d-2)\gamma} \right] \\ \epsilon \rightarrow -\epsilon \end{cases}. \quad (3.40)$$

It is straightforward to observe that any solution (Ω, A, Ψ_μ) of (3.31-3.34) with given values of $\gamma\delta$ and ϵ gives rise, through (3.35), to a solution $(\Omega, \omega, \Psi_\mu)$ of the dual system (3.36-3.38) with the same values of $\gamma\delta$ and ϵ and that the latter can be mapped to a new solution of (3.31-3.34) through \mathcal{A} . Unfortunately, the map \mathcal{A} also alters (3.30) and therefore the symmetry is lost. There is one exception and it occurs if and only if $d = 4$: then \mathcal{A} simply changes the sign of γ or of δ leaving (3.30) unaffected. As we will see in section 5, a consequence of this is that given a dilatonic electric solution with $d = 4$ and $\gamma\delta = -1$, the map \mathcal{A} can be used to generate a $D = 5$ dimensional magnetic solution (and conversely). Finally, note that whilst

$$(\Omega, A, \Psi_\mu) \xrightarrow{(3.35)} (\Omega, \omega, \Psi_\mu) \xrightarrow{\mathcal{A}} (\bar{\Omega}, \bar{A}, \Psi_\mu) \quad (3.41)$$

is built as an extension of the EM duality to $\Lambda \neq 0$ for $d = 4$, it is *not* the EM duality (3.17). Indeed, the EM duality leaves unchanged the action parameter γ and exchanges solutions within the *same* theory, i.e. with the same dimension (here $d = 4$) and the same parameter γ . In contrast, the transformation (3.41) with $d = 4$ and $\Lambda \neq 0$ exchanges solutions corresponding to *different* theories, i.e. with different parameters γ and δ . In other words an uplifted $D = 5$ rotating solution will always be mapped to a $d = 4$ EMD solution and not to a new $D = 5$ solution.

⁸Note that the parameter δ is redundant in the absence of a cosmological constant and can be set to any value. Here we take $\delta = 0$.

3.2 Ernst potentials with a cosmological constant

We now proceed to generalise the method of Ernst to $\Lambda \neq 0$ and $d > 3$. In analogy with (2.13), let us define a complex potential

$$\mathcal{E}_- = e^{\frac{\Omega}{2}} \alpha^{\frac{\gamma\delta}{2} + \frac{d-3}{d-2}} + i \frac{\sqrt{s}}{2} \omega. \quad (3.42)$$

Then, *in the electric field case only*, $\epsilon = -1$, the Maxwell and scalar field equations (3.36) and (3.37) reduce to the single complex equation

$$\frac{1}{\alpha} \vec{\nabla} \cdot (\alpha \vec{\nabla} \mathcal{E}_-) = \frac{(\vec{\nabla} \mathcal{E}_-)^2}{\text{Re}(\mathcal{E}_-)} + \left(\frac{\gamma\delta}{2} + \frac{d-3}{d-2} \right) \text{Re}(\mathcal{E}_-) \frac{\Delta\alpha}{\alpha}. \quad (3.43)$$

For $\Lambda \neq 0$, α is not a harmonic function and therefore there is an extra term in the Ernst equation relative to its original form (2.14).

In the magnetic field case, $\epsilon = +1$, equations (3.36) and (3.37) can no longer be written in such an Ernst form. However, one can return to the system (3.31) and (3.32): *in the magnetic field case $\epsilon = +1$ only*, these may be derived from the potential

$$\mathcal{E}_+ = e^{-\frac{\Omega}{2}} \alpha^{-\frac{\gamma\delta}{2} + \frac{1}{d-2}} + i \frac{\sqrt{s}}{2} A \quad (3.44)$$

with corresponding equation

$$\frac{1}{\alpha} \vec{\nabla} \cdot (\alpha \vec{\nabla} \mathcal{E}_+) = \frac{(\vec{\nabla} \mathcal{E}_+)^2}{\text{Re}(\mathcal{E}_+)} + \left(-\frac{\gamma\delta}{2} + \frac{1}{d-2} \right) \text{Re}(\mathcal{E}_+) \frac{\Delta\alpha}{\alpha}. \quad (3.45)$$

Let us now consider the cases where the last term on the RHS of (3.43) or (3.45) vanishes. The electric or magnetic Ernst equation then reduces to the standard one (2.14), however with the difference that α is not harmonic. Furthermore, in these cases, we note that (3.33) or (3.45) can be derived from the two-dimensional action

$$S_2 = \int dr dz \alpha(r, z) \left[\frac{\vec{\nabla} \mathcal{E} \cdot \vec{\nabla} \mathcal{E}^*}{(\mathcal{E} + \mathcal{E}^*)^2} + \sum_{\mu=0}^{d-3} (d\Psi_\mu)^2 \right]. \quad (3.46)$$

where \mathcal{E} stands for either \mathcal{E}_- or \mathcal{E}_+ . Thus we have ended up with a non-linear σ -model, whose target space is spanned by the coordinates $(\mathcal{E}, \mathcal{E}^*, \Psi_\mu)$ and is endowed with the d -dimensional metric

$$G_d = \frac{d\mathcal{E} d\mathcal{E}^*}{(\mathcal{E} + \mathcal{E}^*)^2} + \sum_{\mu=0}^{d-3} (d\Psi_\mu)^2 = \frac{d\xi d\xi^*}{(1 - |\xi|)^2} + \sum_{\mu=0}^{d-3} (d\Psi_\mu)^2, \quad (3.47)$$

where, as in (2.15), we have set

$$\mathcal{E} \equiv \frac{\xi - 1}{\xi + 1}. \quad (3.48)$$

The target space is thus a d -dimensional Riemannian manifold which is locally isometric to $\mathbb{H}_2 \times \mathbb{R}^{d-2}$, where \mathbb{H}_2 is the hyperbolic plane.

This symmetry can be exploited only if it is respected by the integrability condition, (3.34) or (3.38). For both electric and magnetic cases, this is possible only if $s = 4$. Indeed, in this case, the integrability condition can be conveniently rewritten in terms of $(\mathcal{E}, \mathcal{E}^*, \Psi_\mu)$, as

$$2\nu_{,u} \frac{\alpha_{,u}}{\alpha} - \frac{\alpha_{,uu}}{\alpha} = 2 \frac{\mathcal{E}_{,u} \mathcal{E}^*_{,u}}{(\mathcal{E} + \mathcal{E}^*)^2} + \frac{1}{4} \Psi_{*,u}^2 + \sum_{i=1}^{d-3} \Psi_{i,u}^2 \quad (3.49)$$

where, up to a renormalisation of the Ψ_μ fields, we recognise on the RHS the target space metric G_d . Since the fields $(\mathcal{E}, \mathcal{E}^*, \Psi_\mu)$ only enter the field equations through G_d , each transformation of the target space isometry group leaves the field equations invariant. For example, the transformation

$$\forall \vartheta \in \mathbb{R}, \quad \xi \rightarrow e^{i\vartheta} \xi \quad (3.50)$$

is clearly such an isometry and for each constant phase ϑ will yield a different solution. Thus we can generate different solutions of the field equations through the action of the universal cover $SU(1, 1) \times E_{d-2}$ of the isometry group $SO(2, 1) \times E_{d-2}$ of the target space.

It is interesting to reflect on a geometric interpretation of the field equations (3.30-3.34), or (3.36-3.38). Note for a start the volume element $dr dz \alpha$ appearing in (3.46). For $\Lambda = 0$, in (3.46) the manifold over which integration takes place is the 3-dimensional flat cylindrical metric. When $\Lambda \neq 0$, on the other hand, the metric is still axially symmetric but is no longer flat

$$dr^2 + dz^2 + \alpha(r, z)^2 d\varphi^2. \quad (3.51)$$

It is intriguing to note that the scalar curvature of (3.51) is given by the component $e^{2\nu}$ via equation (3.30) and this, in turn, says that in the presence of the cosmological constant, (3.51) is a curved metric whose curvature depends on \mathcal{E} and Ψ_μ . Actually, the LHS differential operators acting on \mathcal{E} and Ψ_μ in (3.31-3.33), or (3.36-3.38), are the Laplace operators associated to the metric (3.51). In some sense, the integrability condition (3.34), or (3.38), can be seen to relate the 'geometry', on the LHS, to 'matter', on the RHS of the field equations. This geometric interpretation is another way to approach the field equations that deserves future study.

For the magnetic case, the two conditions on γ and δ discussed above are equivalent to (3.10), which corresponds to the case where the d -dimensional system can be uplifted to a D -dimensional solution. For the electric case, the two conditions on the couplings are

$$\gamma = \pm \sqrt{\frac{2(d-1)}{d-2}}, \quad \gamma\delta = -2 \frac{d-3}{d-2}, \quad (3.52)$$

where the first condition ensures that the integrability equation (3.38) can be written in the form (3.49) with, whereas the second condition ensures that the last term on the RHS of (3.43) vanishes. In the particular case $d = 4$, it is possible, using the map \mathcal{A} , to relate an electric EMD $d = 4$ -dimensional solution to a magnetic 5-dimensional solution. This is an interesting way to lift dilatonic electric solutions to 5 dimensions. To summarize, we have the following diagram

$$\begin{array}{ccc}
\gamma\delta = 1 & (g_5)_{\text{Magn.}} \xrightarrow{SU(1,1)} & (g'_5)_{\text{Magn.}} \\
& \downarrow \mathcal{A} & \downarrow \mathcal{A} \\
\gamma\delta = -1 & (g_4 \oplus A \oplus \Phi)_{\text{Elec.}} \xrightarrow{SU(1,1)} & (g'_4 \oplus A' \oplus \Phi')_{\text{Elec.}}
\end{array}$$

3.3 Extending the Papapetrou method

We now consider the generalisation of a construction technique of Papapetrou [7] which was originally carried out in 4 dimensions with $\Lambda = 0$ (see section 2). Here we consider the general case of a d -dimensional EMD system with $\Lambda \neq 0$. We will show that when the real potentials Ω and Ψ_μ are functionals of the EM potential A or ω , the d -dimensional EMD system reduces to a Weyl system with $\Lambda \neq 0$ *provided* certain constraints on the coupling constants γ and δ are satisfied.

We will consider simultaneously the two cases $\Omega = \Omega(A)$ and $\Omega = \Omega(\omega)$ and write generically $\Omega = \Omega(X)$ with $X = A, \omega$. In both cases, the equations (3.31-3.32) and (3.36-3.37) reduce to

$$\left[\Delta X + \frac{\vec{\nabla} \alpha}{\alpha} \cdot \vec{\nabla} X \right] + q\Omega'(\vec{\nabla} X)^2 = 0 \quad (3.53)$$

$$\Omega' \left[\Delta X + \frac{\vec{\nabla} \alpha}{\alpha} \cdot \vec{\nabla} X \right] + \left\{ \Omega'' - \frac{\epsilon s}{2} e^{q\Omega} \right\} (\vec{\nabla} X)^2 = 0. \quad (3.54)$$

where a prime denotes an ordinary derivative with respect to X , provided

$$\begin{cases} q = 1, & \gamma\delta = \frac{2}{d-2}, & \text{for } X = A \\ q = -1, & \gamma\delta = -2\frac{d-3}{d-2}, & \text{for } X = \omega. \end{cases} \quad (3.55)$$

The conditions on the couplings, which are the same as those encountered in the previous subsection, are necessary to get the same expression in the brackets on the left hand side of (3.53) and (3.54). Taking the difference we get the ordinary differential equation

$$\Omega'' - \frac{s\epsilon}{2} e^{q\Omega} - q(\Omega')^2 = 0 \quad (3.56)$$

with solution

$$e^{-q\Omega} = \left(-\frac{\epsilon q s}{4} X^2 + k_1 X + k_0 \right), \quad (3.57)$$

where k_1 and k_0 are some integration constants. The same trick can be used in (3.33) for each Ψ_μ , once we let $\Psi_\mu = \Psi_\mu(X)$. The solution reads

$$\Psi'_\mu = l_\mu e^{q\Omega}, \quad (3.58)$$

where the l_μ 's are again constants of integration.

It is now convenient to introduce the function

$$\varphi(X) = \sqrt{2\lambda} \int^X \frac{dx}{-(\epsilon qs/4)x^2 + k_1x + k_0}, \quad (3.59)$$

so that, using (3.57), $\varphi_u^2 = 2\lambda e^{2q\Omega}(X_u)^2$. λ is a free constant which we now fix by taking into account the last equations — the integrability conditions (3.34) and (3.38) — which become

$$2\nu_{,u} \frac{\alpha_u}{\alpha} - \frac{\alpha_{,uu}}{\alpha} = \left(\frac{\varphi_u^2}{2s\lambda} \right) \left\{ l_0^2 + \left[\frac{X^2}{8}(s^2 - 16) - \frac{2X\epsilon qk_1}{s}(s - 4) \right] + \left(\frac{k_1^2}{2} + \frac{8\epsilon qk_0}{s} \right) + \sum_{i=1}^{D-4} l_i^2 \right\}. \quad (3.60)$$

Requiring that the RHS of the above equation be independent of X yields $s = 4$ or, according to (3.29),

$$\gamma = \pm \sqrt{\frac{2(d-1)}{(d-2)}}. \quad (3.61)$$

In this case

$$\varphi(X) = -q\epsilon\sqrt{2\lambda} \begin{cases} \frac{1}{\sqrt{k_1^2 + 4q\epsilon k_0}} \ln \left(\frac{X - (q\epsilon k_1/2) - \sqrt{(k_1^2/4) + q\epsilon k_0}}{X - (q\epsilon k_1/2) + \sqrt{(k_1^2/4) + q\epsilon k_0}} \right) + c_0, & k_1^2 > -4q\epsilon k_0 \\ -\frac{1}{X - q\epsilon k_1/2} + c_1, & k_1^2 = -4q\epsilon k_0 \\ \frac{1}{\sqrt{-q\epsilon k_0 - k_1^2/4}} \arctan \left(\frac{X - q\epsilon k_1/2}{\sqrt{-q\epsilon k_0 - k_1^2/4}} \right) + c_2, & k_1^2 < -4q\epsilon k_0 \end{cases} \quad (3.62)$$

where c_0 , c_1 and c_2 are integration constants. Then on choosing

$$\lambda = \frac{1}{4} \left\{ l_0^2 + \left(\frac{k_1^2}{2} + 2\epsilon qk_0 \right) + \sum_{i=1}^{D-4} l_i^2 \right\}, \quad (3.63)$$

the integrability equations (3.34) or (3.38) reduce to

$$2\nu_{,u} \frac{\alpha_{,u}}{\alpha} - \frac{\alpha_{,uu}}{\alpha} = \frac{1}{2} \varphi_{,u}^2, \quad (u \leftrightarrow v). \quad (3.64)$$

When $\Omega = \Omega(A)$, the conditions on γ and δ , (3.55) and (3.61), are equivalent to (3.10), *i.e.* to an uplifted D dimensional rotating spacetime. Thus, as stated earlier,

under the hypothesis (3.55) and (3.61), each $D = d + 1$ -dimensional Weyl solution of Einstein's equations yields a family of D -dimensional stationary and axisymmetric solutions. Indeed, the field equations (3.31)-(3.34) reduce to

$$\Delta\alpha = -2\Lambda\alpha^{\frac{1}{D-2}}e^{2\nu}, \quad (3.65)$$

$$\vec{\nabla} \cdot (\alpha \vec{\nabla} \varphi) = 0, \quad (3.66)$$

$$2\nu_{,u} \frac{\alpha_{,u}}{\alpha} - \frac{\alpha_{,uu}}{\alpha} = \frac{1}{2}\varphi_{,u}^2, \quad (u \leftrightarrow v). \quad (3.67)$$

The Weyl metric element is (here we take $\epsilon = -1$)

$$ds^2 = e^{2\nu} \alpha^{-\frac{D-3}{D-2}} (dr^2 + dz^2) + \alpha^{\frac{2}{D-2}} \left[-e^{\sqrt{\frac{2(D-3)}{(D-2)}}\varphi} dt^2 + e^{-\sqrt{\frac{2}{(D-2)(D-3)}}\varphi} \sum_{i=1}^{D-3} (dx^i)^2 \right], \quad (3.68)$$

where φ is given in (3.59) with $q = 1$, $X = A$ and $s = 4$. Note that even if $D > 4$ we have only a single Weyl field φ in (3.65-3.67) since we have assumed a single angular momentum component A in (3.62). The metric solutions obtained this way have a very particular form. Indeed, using (3.57) and (3.59) (see also (3.76)), we find that a rotating spacetime metric reduces to

$$ds^2 = e^{2\nu} \alpha^{-\frac{D-3}{D-2}} (dr^2 + dz^2) + \alpha^{\frac{2}{D-2}} \left[\frac{e^{-\sqrt{\frac{D-4}{2(D-2)}}\Psi_*}}{\sqrt{A^2 + k_1 A + k_0}} (-dt^2 - 2Ad\varphi dt + (k_1 A + k_0)d\varphi^2) + e^{\sqrt{\frac{2}{(D-2)(D-4)}}\Psi_*} \sum_{i=1}^{D-4} e^{2\Psi_i} (dx^i)^2 \right], \quad (3.69)$$

with

$$\Psi_\mu = \frac{l_\mu}{\sqrt{2\lambda}} \varphi. \quad (3.70)$$

When, in turn, $\Omega = \Omega(\omega)$, then (3.55) and (3.61) give

$$\gamma = \pm \sqrt{\frac{2(d-1)}{(d-2)}} \quad \delta = \mp (d-3) \sqrt{\frac{2}{(d-2)(d-1)}}, \quad (3.71)$$

which is *not* equivalent to a D -dimensional system but is a particular EMD d dimensional system. The duality of the previous section, however, tells us that when $d = 4$ in particular we will be able to map any Papapetrou solution to a $D = 5$ rotating spacetime solution. For the dual system, the field equations reduce to

$$\Delta\alpha = -2\Lambda\alpha^{-\frac{(d-5)}{(d-1)}}e^{2\nu}, \quad (3.72)$$

$$\vec{\nabla} \cdot (\alpha \vec{\nabla} \varphi) = 0, \quad (3.73)$$

$$2\nu_{,u} \frac{\alpha_{,u}}{\alpha} - \frac{\alpha_{,uu}}{\alpha} = \frac{1}{2}\varphi_{,u}^2, \quad (u \leftrightarrow v) \quad (3.74)$$

where φ is given in (3.59) with $q = -1$, $X = \omega$, $s = 4$ (see in particular [40]). Note that for $d = 4$ equations (3.72-3.74) are identical to (3.65-3.67) for $D = 5$ in agreement with the duality map \mathcal{A} .

Indeed, as discussed in section 2, Papapetrou's construction was originally carried out for the dual system and then mapped in $D = 4$ dimensions via EM duality. In other words, one supposes rather that $\Omega = \Omega(\omega)$ and evaluates A independently from (3.35). In crude terms, this means that the rotation field A will generically depend on a different coordinate from Ω and the metric will not be of the specific form (3.69). In the absence of a cosmological constant we can apply the same method in arbitrary dimensions: when $\Lambda \neq 0$, however, we can only do so for $D = 5$.

In [27] it was shown that the system (3.65)-(3.67) is completely integrable if one makes the hypothesis that φ depends only on one of the two coordinates, say z . It then follows that the canonical components A, Ω, Ψ_μ must also depend the same variable z . Furthermore, from (3.66), α is separable:

$$\alpha = f(r)g(z), \quad g(z) = \frac{c}{\varphi_{,z}} \quad (3.75)$$

where c is a nonvanishing constant if $\varphi_{,z} \neq 0$. The remaining two equations (3.65) and (3.67) then give $f(r)$ and $g(z)$. As was discussed in [27], there are three classes of possible solutions: class I with $f_{,r} = 0$, class II with $g_{,z} = 0$, and class III with both $f_{,r}, g_{,z} \neq 0$. We will return to these in sections 4 and 5 where we discuss solutions in $D = 4, 5$ dimensions. Finally, note that the same method also gives a large class of solutions to the dual system given in (3.72-3.74).

3.4 Set-up for uplifted spacetimes in D dimensions

In this section, we focus on D -dimensional solutions which can be obtained from uplifting $d = D - 1$ dimensional EMD solutions. We start by summarizing our results in this specific case, corresponding to values of γ and δ given in (3.10). From (3.4), (3.11) and (3.28), the metric in the electric case (following a Wick rotation $x_0 \rightarrow i\varphi$ and $w \rightarrow it$) corresponds to a rotating metric,

$$ds^2 = e^{2\nu} \alpha^{-\frac{D-3}{D-2}} (dr^2 + dz^2) + \alpha^{\frac{2}{D-2}} \left[e^{-\sqrt{\frac{D-4}{2(D-2)}} \Psi_*} \left[-e^{\frac{\Omega}{2}} (dt + Ad\varphi)^2 + e^{\frac{-\Omega}{2}} d\varphi^2 \right] + e^{\sqrt{\frac{2}{(D-2)(D-4)}} \Psi_*} \sum_{i=1}^{D-4} e^{2\Psi_i} (dx^i)^2 \right]. \quad (3.76)$$

The pole at $D = 4$ is artificial since then the $\Psi_\mu = 0$. After an analytic continuation of the time coordinate $x_0 \rightarrow ix_{D-4}$, the magnetic spacetime is given by

$$ds^2 = e^{2\nu} \alpha^{-\frac{D-3}{D-2}} (dr^2 + dz^2) + \alpha^{\frac{2}{D-2}} \left[e^{-\sqrt{\frac{D-4}{2(D-2)}} \Psi_*} \left[e^{\frac{\Omega}{2}} (dw + Ad\varphi)^2 + e^{\frac{-\Omega}{2}} d\varphi^2 \right] + e^{\sqrt{\frac{2}{(D-2)(D-4)}} \Psi_*} \sum_{i=1}^{D-4} e^{2\Psi_i} (dx^i)^2 \right], \quad (3.77)$$

which is a purely Riemannian.

From (3.30)-(3.34), the field equations take the rather simplified form

$$\Delta\alpha = -2\Lambda\alpha^{\frac{1}{D-2}}e^{2\nu}, \quad (3.78)$$

$$0 = \vec{\nabla} \cdot (e^\Omega \alpha \vec{\nabla} A), \quad (3.79)$$

$$\frac{1}{\alpha} \vec{\nabla} \cdot (\alpha \vec{\nabla} \Omega) = 2\epsilon e^\Omega (\vec{\nabla} A)^2, \quad (3.80)$$

$$\vec{\nabla} \cdot (\alpha \vec{\nabla} \Psi_\mu) = 0, \quad \mu = 0 \dots d-3 \quad (3.81)$$

$$2\nu_{,u} \frac{\alpha_u}{\alpha} - \frac{\alpha_{,uu}}{\alpha} = \frac{1}{4} \left(\Psi_{*,u}^2 + \frac{1}{2} \Omega_{,u}^2 \right) + \frac{\epsilon}{2} e^\Omega (A_{,u})^2 + \sum_{i=1}^{D-4} \Psi_{i,u}^2, \quad (u \leftrightarrow v). \quad (3.82)$$

The electric Ernst potential (3.42) now reads

$$\mathcal{E}_- = e^{\frac{\Omega}{2}} \alpha + i\omega, \quad (3.83)$$

replacing (3.79-3.80) by

$$\frac{1}{\alpha} \vec{\nabla} \cdot (\alpha \vec{\nabla} \mathcal{E}_-) = \frac{(\vec{\nabla} \mathcal{E}_-)^2}{\text{Re}(\mathcal{E}_-)} + \text{Re}(\mathcal{E}_-) \frac{\Delta\alpha}{\alpha}. \quad (3.84)$$

In the magnetic case, $\epsilon = +1$, (3.44) becomes

$$\mathcal{E}_+ = e^{-\frac{\Omega}{2}} + iA \quad (3.85)$$

with corresponding equation

$$\frac{1}{\alpha} \vec{\nabla} \cdot (\alpha \vec{\nabla} \mathcal{E}_+) = \frac{(\vec{\nabla} \mathcal{E}_+)^2}{\text{Re}(\mathcal{E}_+)} \quad (3.86)$$

where, as we have already noted, the extra term of (3.45) drops out.

A particular class of solutions can be found taking advantage of the ‘decoupled’ form of the field equations (3.78-3.82). Indeed, suppose that α and ν only depend on r whereas Ω , Ψ_μ and A only depend on z . In that case the equations decouple into two separate systems of ODEs; one r -dependent for α and ν ; and one z -dependent for the remaining fields. Following the geometric interpretation of section 3 this amounts to splitting contributions from geometry and matter and treating them separately. The r -dependent system reads

$$\alpha'' = -2\Lambda\alpha^{\frac{1}{D-2}}e^{2\nu} \quad (3.87)$$

$$2\nu' \frac{\alpha'}{\alpha} = \frac{\alpha''}{\alpha}, \quad (3.88)$$

where a prime stands for a derivative with respect to the unique variable r . The system here is identical to the one appearing in [51, 52] and the solution reads

$$e^{2\nu} = \alpha' \quad (3.89)$$

$$\alpha' = -\frac{\mu}{(D-2)^2} - \frac{2(D-2)\Lambda}{(D-1)} \alpha^{\frac{D-1}{D-2}}, \quad (3.90)$$

where μ is some real integration constant. In [52] if z is a spacelike coordinate then the solutions of (3.90) can be coordinate transformed to an adS soliton [50]. On the other hand, if z is a timelike coordinate [51] one gets an adS planar black hole (see section 5). So we anticipate to recover these two solutions as a special case and furthermore to obtain continuous deformations of these. We will present these in detail in Section 5 for $D = 5$ dimensions.

Observe that all equations are independent of D except (3.78) when $\Lambda \neq 0$. The metrics (3.76) or (3.77), however, themselves depend on the dimension. Therefore, the form of the D -dimensional field equations dictates an important result: for $\Lambda = 0$ and given a D -dimensional solution, we can always construct a higher dimensional $D+n$ (n positive integer) dimensional solution. Indeed, recall first that when $\Lambda = 0$, α can be taken as the radial coordinate. Now, suppose one takes a known D dimensional solution (Ω, A, Ψ_μ) , where $\mu = 0, \dots, D-4$. Then, a new $D+n$ solution (Ω, A, Ψ_ν) , for $\nu = 0, \dots, D+n-4$, can be obtained from the D -dimensional solution simply by calculating the new Weyl potentials from (3.81), so that ν_{D+n} is given by direct integration of (3.82). That this is a new solution of the Einstein equations is due to the fact that (3.82) relates the different potentials together independently of the spacetime dimension. To summarise, taking an arbitrary stationary and axisymmetric solution in 4 dimensions, such as Kerr or TN say, we can construct higher dimensional solutions by adding n extra Weyl potentials. Unfortunately this property is spoiled once we switch on Λ , since from (3.78), the component ν becomes a D dependent quantity and α is no longer free.

Conversely, for $\Lambda = 0$, a higher dimensional stationary solution of axial symmetry with one angular momentum will always originate from a unique 4 dimensional seed solution *with the same Ernst potentials* \mathcal{E}_\pm . Consider, for example, a known $D+1$ dimensional solution and let us look for the D -dimensional seed solution. The only unknown metric component is ν_D which is immediately given from direct integration by (3.82),

$$2(\nu_{(D),u} - \nu_{(D+1),u}) \frac{\alpha_{,u}}{\alpha} = -\frac{1}{4} \Psi_{,u}^2, \quad (u \leftrightarrow v). \quad (3.91)$$

It will be useful for applications to define $\sigma = \nu_{(D),u} - \nu_{(D+1),u}$ and to rewrite the above equation in terms of r and z :

$$\begin{aligned} \sigma_{,z} &= \frac{\alpha}{8(\alpha_{,z}^2 + \alpha_{,r}^2)} [\alpha_{,z}(\Psi_{,r}^2 - \Psi_{,z}^2) - 2\alpha_{,r}\Psi_{,z}\Psi_{,r}] , \\ \sigma_{,r} &= -\frac{\alpha}{8(\alpha_{,z}^2 + \alpha_{,r}^2)} [\alpha_{,r}(\Psi_{,r}^2 - \Psi_{,z}^2) + 2\alpha_{,z}\Psi_{,z}\Psi_{,r}] . \end{aligned} \quad (3.92)$$

Note that these equations are particularly simple in Weyl coordinates. Simple examples of this and of previous methods will be given in the following sections.

4. Examples in $D = 4$ dimensions

The aim of this section is two fold. First, we make the connection between our general analysis of section 3 and the well known results of $D = 4$ and $\Lambda = 0$ general relativity (as summarised briefly in section 2) . Second, we give examples of Ernst potentials for well-known GR solutions, though now extended to the case of spacetimes with non-zero cosmological constant, $\Lambda \neq 0$.

Our general starting point is the electric EMD system (3.30-3.34) which reads for $d = 3$

$$\Delta\alpha = -2\Lambda\alpha^{1-\frac{\delta^2}{2}}e^{2\nu}, \quad (4.1)$$

$$\vec{\nabla} \cdot \left(\frac{e^\Omega}{\alpha^{1-\gamma\delta}} \vec{\nabla} A \right) = 0, \quad (4.2)$$

$$\frac{1}{\alpha} \vec{\nabla} \cdot (\alpha \vec{\nabla} \Omega) + \frac{\gamma^2}{2\alpha^{2-\gamma\delta}} e^\Omega (\vec{\nabla} A)^2 = 0, \quad (4.3)$$

$$\Delta\nu + \frac{1}{4\gamma^2} (\vec{\nabla} \Omega)^2 + \frac{1-\gamma\delta}{4\alpha^{2-\gamma\delta}} e^\Omega (\vec{\nabla} A)^2 = \frac{1}{2} \left(1 - \frac{\delta^2}{2} \right) \frac{\Delta\alpha}{\alpha}, \quad (4.4)$$

$$2\nu_{,u} \frac{\alpha_{,u}}{\alpha} - \frac{\alpha_{,uu}}{\alpha} = \frac{1}{2\gamma^2} (\Omega_{,u})^2 - \frac{1}{2\alpha^{2-\gamma\delta}} e^\Omega A_{,u}^2 \quad (u \leftrightarrow v) \quad (4.5)$$

From (3.10), for the special values of the coupling constants namely $\gamma = 2$ and $\delta = 1$, we can uplift to a $D = 4$ dimensional axisymmetric and stationary spacetime. Using (3.76), the metric in the above components reads

$$ds^2 = e^{2\nu} \alpha^{-1/2} (dr^2 + dz^2) + \alpha e^{-\frac{\Omega}{2}} d\varphi^2 - \alpha e^{\frac{\Omega}{2}} (dt + Ad\varphi)^2. \quad (4.6)$$

Note that the metric components differ from the original Weyl-Papapetrou ones (2.10). Indeed, the Weyl potential λ is now given by $e^{2\lambda} = \alpha e^{\frac{\Omega}{2}}$, although λ and Ω obey a similar differential equation (compare (2.11) and (4.3)). Furthermore, when $\Lambda = 0$ the component α is harmonic and is the radial coordinate r in (2.10). These slight differences are important, and result from having chosen variables which absorb the cosmological constant term in the field equations (4.1-4.5).

In the magnetic case, the 4 dimensional metric is of Euclidean signature and corresponds generically to a Euclidean instanton solution

$$ds^2 = e^{2\nu} \alpha^{-1/2} (dr^2 + dz^2) + \alpha e^{-\frac{\Omega}{2}} d\varphi^2 + \alpha e^{\frac{\Omega}{2}} (dw + Ad\varphi)^2. \quad (4.7)$$

As discussed in section 3, given the absence of an EM duality transformation (3.40) when $\Lambda \neq 0$ we can define two different Ernst potentials \mathcal{E}_\pm ; $\mathcal{E}_-(\omega)$ given in (3.83) for the electric spacetime (4.6), and $\mathcal{E}_+(A)$ given in (3.85) for a magnetic spacetime. The electric potential \mathcal{E}_- is identical to the original Ernst potential (2.14) for the metric (4.6). As was discussed in section 2, the electric Ernst potential and corresponding Ernst equation were used in [11, 13] to generate new solutions for

$\Lambda = 0$ such as, for example, Schwarzschild spacetime using spheroidal coordinates (2.5). Here, lacking a relevant coordinate system for $\Lambda \neq 0$, we merely construct the relevant potentials for some well known solutions.

Consider first Carter's metric [14] which describes a rotating Kerr black hole in an asymptotically adS spacetime:

$$ds_4^2 = -\frac{\Delta}{\rho^2} \left(dt - \frac{a \sin^2 \theta}{\Xi_a} d\varphi \right)^2 + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left(a dt - \frac{r^2 + a^2}{\Xi_a} d\varphi \right)^2 + \rho^2 \left(\frac{dr^2}{\Delta} + \frac{d\theta^2}{\Delta_\theta} \right), \quad (4.8)$$

where k is the curvature scale of adS, M is the black hole mass, a the angular momentum parameter and

$$\Delta = (r^2 + a^2)(1 + k^2 r^2) - 2Mr, \quad (4.9)$$

$$\Delta_\theta = 1 - a^2 k^2 \cos^2 \theta, \quad \Xi_a = 1 - a^2 k^2, \quad (4.10)$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Lambda = -3k^2. \quad (4.11)$$

As a general rule, metrics with a cosmological constant cannot be written explicitly in the coordinate system chosen in (4.6). However, this is not a problem since we can transit to the coordinate system of (4.8) by setting

$$\frac{dr^2}{\Delta} = dz^2, \quad \frac{d\theta^2}{\Delta_\theta} = dz^2, \quad (4.12)$$

meaning that z and r are implicitly given as functions of θ and r , respectively. Using (4.6), this is all we need to know in order to identify the different components:

$$\alpha = \frac{\sin \theta}{\Xi_a} \sqrt{\Delta \Delta_\theta}, \quad (4.13)$$

$$A = \frac{a \sin^2 \theta (\Delta - \Delta_\theta (r^2 + a^2))}{\Xi_a (a^2 \Delta_\theta \sin^2 \theta - \Delta)}, \quad (4.14)$$

$$e^\Omega = \frac{\Xi_a^2 (\Delta - a^2 \Delta_\theta \sin^2 \theta)^2}{\Delta \Delta_\theta \rho^4 \sin^2 \theta}, \quad (4.15)$$

$$e^{2\nu} = \rho^2 \alpha^{1/2}. \quad (4.16)$$

Using (3.83) and (3.35) one finds that the electric Ernst potential for Carter's solution is given by

$$\mathcal{E}_- = \frac{1}{\rho^2} (\Delta - a^2 \sin^2 \theta \Delta_\theta - 2ia \cos \theta (k^2 \rho^2 r + M)). \quad (4.17)$$

If there is no rotation, $a = 0$, the Ernst potential is real and corresponds to Kottler's black hole [48]. For $M = 0$ we have pure adS but the potential is still complex since the metric has non-zero angular momentum⁹. If $\Lambda = 0$, \mathcal{E}_- is the usual Ernst

⁹This is quite unlike the situation for Kerr's solution at asymptotic infinity.

potential in the coordinates of (4.8). Considering $\alpha = ia$ and $t = -iw$ we obtain the magnetic Carter instanton (see [19]). The corresponding magnetic Ernst potential is, according to (3.85),

$$\mathcal{E}_+ = \frac{\sin \theta}{\Xi_\alpha(\Delta + \alpha^2 \Delta_\theta \sin^2 \theta)} \left(\sqrt{\Delta \Delta_\theta} \rho^2 - i\alpha \sin \theta (\Delta - \Delta_\theta (r^2 - \alpha^2)) \right). \quad (4.18)$$

Another interesting example is Taub-NUT spacetime with a Λ term [10, 14]

$$ds^2 = -F(r)(dt + Ad\varphi)^2 + \frac{dr^2}{F(r)} + (r^2 + n^2)d\Omega_{II}^2 \quad (4.19)$$

with

$$A = 2n \cos \theta, \\ F(r) = \frac{1}{l^2(n^2 + r^2)} [r^4 + (l^2 + 6n^2)r^2 - 2Mr l^2 - n^2(l^2 - 3n^2)]. \quad (4.20)$$

The constants M , l and n are the mass, the length scale $l = 1/k$ and the Taub-NUT parameter, respectively. The electric Ernst potential is simply

$$\mathcal{E}_- = F(r) + i\omega(r) \quad (4.21)$$

with

$$\omega = -\frac{2n}{l^2(n^2 + r^2)}(r^3 - rl^2 + Ml^2) - \frac{6n^2}{l^2} \arctan(r/n), \quad (4.22)$$

whereas the magnetic potential is given by

$$\mathcal{E}_+ = \frac{\sin \theta \sqrt{r^2 - n^2}}{\sqrt{F(r)}} + 2in \cos \theta, \quad (4.23)$$

where we have taken $n = in$ to obtain a Riemannian metric. Switching off the Taub-NUT parameter yields the relevant static Kottler potential, and in the limit $l \rightarrow \infty$ we obtain the usual $\Lambda = 0$ potential. Indeed, in this $\Lambda = 0$ case, the TN solution was demonstrated in [42] (see also section 2) to be of the Papapetrou class: given ω in (4.22), it is possible to show that the relevant Weyl potential — $F(r)$ in (4.20) — is also a function of ω (note already that ω and F only depend on r unlike A). Furthermore, this result ties in with the fact that, quite generically, Papapetrou type solutions have non-trivial asymptotic properties. Indeed, note that the θ dependent A potential in (4.19) is non-vanishing in the large r limit.

When $\Lambda \neq 0$, we can longer do this trick since the Papapetrou ansatz works only for $\Omega = \Omega(A)$ and there is no duality relation to take us to $\Omega = \Omega(\omega)$. This fact, following the integrable cases of [27], limits the solutions to be in one of three classes (see the discussion at the end of subsection 3.3). For class I solutions in particular, there is an extra Killing vector field and the solutions in question are stationary and

cylindrically symmetric. When the extra Killing vector is null we obtain pp -wave solutions [1]. Such solutions can be obtained directly combining the results of the previous section with [27]. To illustrate the method we restrict ourselves here to a simple example. Start with the Weyl spacetime [27, 49]

$$ds^2 = (\cosh(kx))^2 \left[-y^2 V dt^2 + \frac{dy^2}{V k^2 y^2} + y^2 dz^2 \right] + dx^2, \quad (4.24)$$

where k is the adS curvature, and the potential V given by

$$V(y) = 1 - \frac{M}{y^2}. \quad (4.25)$$

The solution is regular at the adS horizon and there is an event horizon at $V = 0$. It describes a 3-dimensional planar BTZ black hole embedded in a locally 4-dimensional adS spacetime. Furthermore the metric (4.24) is a solution of the Weyl system (3.65-3.67) with

$$\alpha = (\cosh(kx))^2 y^2 \sqrt{V}, \quad e^{2\varphi} = V, \quad e^{2\nu} = \alpha^{1/2} (\cosh(kx))^2. \quad (4.26)$$

According to [27] it is a Class III solution since α is a function of x and y . It is now straightforward to calculate A and Ω (3.57) for the stationary version (3.76). Here, for simplicity, we take $k_0 = 0$ obtaining

$$A = \frac{k_1 V}{1 - V}, \quad e^\Omega = \frac{k_1^2 V}{(1 - V)^2}. \quad (4.27)$$

Thus metric (3.76) reads

$$ds^2 = \cosh^2(kx) \left(\frac{1}{(y^2 - M)k^2} dy^2 - dt^2 - \frac{2}{\sqrt{M}} (y^2 - M) dt d\phi + (y^2 - M) d\phi^2 \right) + dx^2. \quad (4.28)$$

It is also possible to construct a deformed adS soliton (or planar black hole) as we will explicitly show for $D = 5$ in the next section.

5. Examples in $D = 5$ dimensions

As we stressed earlier, the EMD $d = 4$ system and the uplifted $D = 5$ system have unique properties: in particular, the duality relation (3.40) applies even in the presence of a cosmological constant, and it can be used to bring Ernst's equation into its usual $\Lambda = 0$ form. The duality can also be used in relation to Papapetrou's method. Last but not least, for $\Lambda = 0$ we can construct an infinity of solutions seeded from given $D = 4$ stationary and axisymmetric solutions. We examine these properties one by one giving examples as we go along to illustrate them.

Our general starting point is again the electric EMD system (3.30-3.34), which reads for $d = 4$ and for arbitrary couplings γ and δ :

$$\Delta\alpha = -2\Lambda\alpha^{\frac{1}{2}-\frac{\delta^2}{2}}e^{2\nu}, \quad (5.1)$$

$$\vec{\nabla} \cdot \left(e^\Omega \alpha^{\gamma\delta} \vec{\nabla} A \right) = 0, \quad (5.2)$$

$$\frac{1}{\alpha} \vec{\nabla} \cdot (\alpha \nabla \Omega) + \frac{1+\gamma^2}{2\alpha^{1-\gamma\delta}} e^\Omega (\nabla A)^2 = 0, \quad (5.3)$$

$$\nabla \cdot \left(\alpha \vec{\nabla} \Psi_* \right) = 0, \quad (5.4)$$

$$2\nu_{,u} \frac{\alpha_u}{\alpha} - \frac{\alpha_{,uu}}{\alpha} = \frac{1}{\gamma^2+1} \left(2\Psi_{0,u}^2 + \frac{1}{2}\Omega_{,u}^2 \right) + \frac{\epsilon}{2} e^\Omega \alpha^{\gamma\delta-1} (A_{,u})^2, \quad (u \leftrightarrow v) \quad (5.5)$$

A solution is thus given by a set of functions $(\alpha, A, \Omega, \Psi_*)$, such that the dilatonic metric for arbitrary γ and δ reads

$$ds^2 = (dr^2 + dz^2) e^{2\nu} e^{\frac{2\delta\Psi_*}{\gamma^2+1}} e^{\frac{\gamma\delta\Omega}{\gamma^2+1}} \alpha^{\frac{\delta^2-1}{2}} + e^{\frac{(\Omega-2\gamma\Psi_*)}{\gamma^2+1}} \alpha \left(e^{\frac{2(2\gamma\Psi_*-\Omega)}{\gamma^2+1}} d\varphi^2 + d\psi^2 \right) \quad (5.6)$$

with dilaton $\phi = \frac{\gamma\Omega + \sqrt{2}\Psi_*}{1+\gamma^2} + \delta \ln \alpha$ and potential A . According to (3.4) and for specific values $\gamma = 1/\delta = \sqrt{3}$, this corresponds to a $D = 5$ dimensional stationary spacetime

$$ds^2 = (dr^2 + dz^2) e^{2\nu} \alpha^{-2/3} + \alpha^{2/3} \left\{ e^{-\frac{\Psi_*}{\sqrt{6}}} \left[-e^{\frac{\Omega}{2}} (dt + Ad\varphi)^2 + e^{\frac{-\Omega}{2}} d\varphi^2 \right] + e^{\frac{\sqrt{2}\Psi_*}{\sqrt{3}}} d\psi^2 \right\}. \quad (5.7)$$

Let us dwell on the duality map (3.40). First, it is important to note the γ and δ dependence of the field equations when spacetime is stationary and $\Lambda \neq 0$. Since the duality takes us from a $\gamma\delta = 1$ spacetime to $\gamma\delta = -1$ spacetime it cannot be used to map between 5 dimensional solutions. A $D = 5$ dimensional stationary and axisymmetric spacetime will be transformed into a $d = 4$ static and axisymmetric solution with scalar and magnetic/electric charge.

Suppose, however, that we have instead a $D = 5$ *static* spacetime i.e. $A = 0$. This corresponds to some Weyl solution with cosmological constant [27]. In that case the map (3.40) indeed takes us from a $D = 5$ dimensional to a $D = 5$ solution. This is obvious from the form of the action (3.8). A sign change of δ can always be compensated by a sign change of the scalar field ϕ . For the metric, consider $\gamma = -1/\delta = \sqrt{3}$ whereupon the $D = 5$ solution now reads

$$ds^2 = (dr^2 + dz^2) e^{2\nu} \alpha^{-2/3} + \alpha^{2/3} \left\{ e^{-\frac{\Psi_*}{\sqrt{6}}} \left(-e^{\frac{-\Omega}{2}} dt^2 + e^{\frac{\Omega}{2}} d\psi^2 \right) + e^{\frac{\sqrt{2}\Psi_*}{\sqrt{3}}} d\varphi^2 \right\}. \quad (5.8)$$

The duality \mathcal{A} which takes us back to the static version of (5.7) is simply a double Wick rotation. As an example, the $D = 5$ adS Schwarzschild solution,

$$ds^2 = r^2 \left(\frac{dr^2}{r^2 V(r)} + d\theta^2 \right) - V(r) dt^2 + r^2 \cos^2 \theta d\varphi^2 + r^2 \sin^2 \theta d\psi^2 \quad (5.9)$$

with $V(r) = 1 - \frac{\Lambda}{3}r^2 - \frac{\mu}{r^2}$, is transformed by \mathcal{A} into

$$ds^2 = r^2 \left(\frac{dr^2}{r^2 V(r)} + d\theta^2 \right) + V(r) d\psi^2 - r^2 \cos^2 \theta dt^2 + r^2 \sin^2 \theta d\varphi^2 \quad (5.10)$$

which is nothing but the adS soliton [50].

A stationary rather than static example is the 5 dimensional Λ -Kerr solution of Hawking et. al [16] with a single angular momentum. The metric reads

$$ds_5^2 = -\frac{\Delta}{\rho^2} \left(dt - \frac{a \sin^2 \theta}{\Xi_a} d\varphi \right)^2 + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left(a dt - \frac{r^2 + a^2}{\Xi_a} d\varphi \right)^2 + \rho^2 \left(\frac{dr^2}{\Delta} + \frac{d\theta^2}{\Delta_\theta} \right) + r^2 \cos^2 \theta d\psi^2 \quad (5.11)$$

with a , M , k the angular momentum parameter, the mass, and adS curvature scale respectively and

$$\Delta = (r^2 + a^2)(1 + k^2 r^2) - 2M, \quad (5.12)$$

$$\Delta_\theta = 1 - a^2 k^2 \cos^2 \theta, \quad \Xi_a = 1 - a^2 k^2, \quad (5.13)$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Lambda = -6k^2. \quad (5.14)$$

Using (5.7) and applying the same trick as in (4.8) it is straightforward to identify the components,

$$\begin{aligned} \alpha &= \frac{r \cos \theta \sin \theta}{\Xi_a} \sqrt{\Delta \Delta_\theta}, \quad (5.15) \\ A &= \frac{a \sin^2 \theta (\Delta - \Delta_\theta (r^2 + a^2))}{\Xi_a (a^2 \Delta_\theta \sin^2 \theta - \Delta)}, \\ e^\Omega &= \frac{\Xi_a^2 (\Delta - a^2 \Delta_\theta \sin^2 \theta)^2}{\Delta \Delta_\theta \rho^4 \sin^2 \theta}, \\ e^{2\nu} &= \rho^2 \alpha^{2/3}, \\ e^{-\frac{\sqrt{3}\Psi}{\sqrt{2}}} &= \frac{\tan \theta}{\Xi_a r^2 \cos \theta} \sqrt{\Delta \Delta_\theta}, \end{aligned}$$

where implicitly we perform the coordinate transformation $dr = dr/\sqrt{\Delta}$ and $dz = d\theta/\sqrt{\Delta_\theta}$. In order to use the duality we need to evaluate the dual potential (3.35), ω , defined by $(-\partial_z \omega, \partial_r \omega) = e^\Omega \alpha (\partial_r A, \partial_z A)$ (recall that in 5D, $\gamma\delta = 1$). We obtain

$$\omega = -\frac{a \cos^2 \theta}{\rho^2} (\mu + k^2 r^2 \rho^2), \quad (5.16)$$

The duality map (3.40) then takes us to a $d = 4$ EMD solution with $\gamma\delta = -1$:

$$ds^2 = \frac{\sqrt{\Delta \Delta_\theta} \rho \sin \theta}{\sqrt{\Delta - a^2 \Delta_\theta \sin^2 \theta}} \left[\frac{\rho^2}{\Xi} \left(\frac{dr^2}{\Delta} + \frac{d\theta^2}{\Delta_\theta} \right) + \frac{\Delta - a^2 \Delta_\theta \sin^2 \theta}{\rho^2} d\psi^2 + r^2 \cos^2 \theta d\varphi^2 \right] \quad (5.17)$$

with scalar field

$$e^{\frac{2\phi}{\sqrt{3}}} = \frac{\Xi^2(\Delta - a^2\Delta_\theta \sin^2 \theta)}{\Delta\Delta_\theta\rho^2 \sin^2 \theta}. \quad (5.18)$$

and potential (5.16). The (electric) Ernst potential (3.83) for the rotating black hole (5.11) is

$$\mathcal{E}_- = \frac{r \cos \theta (\Delta - a^2 \Delta_\theta \sin^2 \theta)}{\rho^2} - i \frac{a \cos^2 \theta}{\rho^2} (2M + k^2 r^2 \rho^2) \quad (5.19)$$

and shares a rather similar form to its 4 dimensional rotating counterpart (4.17).

As discussed in section 3.1, a convenient way to generate solutions using the Ernst potential [11] is to set

$$\mathcal{E}_+ = \frac{\xi - 1}{\xi + 1}, \quad (5.20)$$

where ξ is a complex field depending on (r, z) . In terms of ξ , equation (3.45) now reads

$$\frac{1}{\alpha} \overline{\nabla} \cdot (\alpha \overline{\nabla} \xi) = \frac{2\xi^* (\overline{\nabla} \xi)^2}{|\xi|^2 - 1}, \quad (5.21)$$

where a star denotes complex conjugation. In this representation of the potential, (5.21) is invariant under the complex transformation (3.50). Therefore a simple trick is to start with a real Ernst potential say, \mathcal{E}_+ , for $D = 5$ in other words a Weyl solution. Let us take an adS/Sch solution (5.9) as an example. We have

$$\mathcal{E}_+ = e^{-\Omega/2} = \sqrt{\frac{V}{r \sin \theta}} \quad (5.22)$$

and therefore, from (5.20),

$$\xi = \frac{\sqrt{V} + r \sin \theta}{r \sin \theta - \sqrt{V}}. \quad (5.23)$$

Then we can apply (3.50) for a convenient phase say $\vartheta = \pi/2$ in order to obtain an imaginary ξ . The newly generated Ernst potential from (5.20) is now complex and we find

$$(e^{-\Omega/2}, A) = \left(\frac{2\sqrt{V}r \sin \theta}{V + r^2 \sin^2 \theta}, -\frac{V - r^2 \sin^2 \theta}{V + r^2 \sin^2 \theta} \right). \quad (5.24)$$

Inserting this Ernst pair (Ω, A) , together with α , ν and Ψ_* from (5.9), into (5.7) gives a rotating solution.

Let us now briefly present an example solution following the Papapetrou method. Our starting point this time is a Class II solution of [27]

$$ds^2 = e^{\frac{2}{3}r} \left(-e^{-\frac{2\sqrt{2}z}{3}} dt^2 + e^{\frac{2}{3\sqrt{2}z}} (dx_1^2 + dx_2^2) \right) + \frac{1}{-2\Lambda} (dr^2 + dz^2) \quad (5.25)$$

which is static and axially symmetric. Setting for simplicity $k_0 = 0$ and $l_0 = 0$ we obtain $\Psi = 0$ and from (3.68) and (3.62)

$$\alpha = e^r, \quad \varphi = \sqrt{\frac{2}{3}}z, \quad A = \frac{k_1 e^{\frac{-2\sqrt{6}z}{3}}}{1 - e^{\frac{-2\sqrt{6}}{3}z}} \quad (5.26)$$

upon which using (3.69) gives a solution in rotating coordinates.

As we mentioned in Section 3.3, a special class of solutions can be found by supposing that α and ν only depend on r whereas Ω , Ψ and A only depend on z . The r -dependent part is given by (3.90) and is the same as in [51, 52]¹⁰. From (3.79-3.82), we deduce the second subsystem for the z dependent part

$$\left(e^{\Omega} \dot{A} \right)' = 0 \quad (5.27)$$

$$\ddot{\Omega} + 2e^{\Omega} \dot{A}^2 = 0 \quad (5.28)$$

$$\ddot{\Psi} = 0 \quad (5.29)$$

$$2\dot{\Psi}^2 + \dot{\Omega}^2 = 4e^{\Omega} \dot{A}^2, \quad (5.30)$$

where a dot now stands for a derivative with respect to z . From (5.29), we deduce

$$\Psi(z) = \frac{\beta z}{\sqrt{2}}, \quad (5.31)$$

where β is some real integration constant and we have taken $\Psi(0) = 0$ as a choice for the origin of the z coordinate. Now, from (5.27)

$$\dot{A} = \lambda e^{-\Omega}, \quad (5.32)$$

where λ is a real integration constant. Substituting (5.31) and (5.32) into (5.30), we get

$$\frac{1}{2}\dot{\Omega}^2 + \beta^2 = 4\lambda^2 e^{-\Omega}. \quad (5.33)$$

When $\lambda = 0$ and $\beta = 0$ then A , Ω are constant and $\Psi = 0$. As we anticipated the metric reduces to

$$ds^2 = \left(-\frac{\mu}{r^2} + k^2 r^2 \right) dz^2 + \frac{dr^2}{-\frac{\mu}{r^2} + k^2 r^2} + r^2 (-dt^2 + d\phi^2 + d\psi^2) \quad (5.34)$$

which is nothing but the planar adS soliton¹¹ [50]. When $\beta \neq 0$ and $\lambda \neq 0$ on the other hand we obtain a non-trivial deformation of this solution. We get

$$e^{\Omega} = \frac{2\lambda^2}{\beta^2} (1 \pm \sin(\beta z)), \quad A(z) = \mp \frac{\beta}{2\lambda} \frac{\cos(\beta z)}{1 \pm \sin(\beta z)} \quad (5.35)$$

¹⁰though there the fields $\Omega = A = \Psi = 0$ since the $D - 2$ dimensional subspaces are of maximal symmetry.

¹¹By a suitable double Wick rotation one can get a planar black hole with a compact Euclidean horizon.

and the five dimensional metric reads

$$\begin{aligned}
ds^2 = & -r^2 e^{\frac{\beta z}{2\sqrt{3}}} \sqrt{1 \pm \sin(\beta z)} (dt + A(z)d\phi)^2 + \frac{dr^2}{-\frac{\mu}{r^2} + k^2 r^2} \\
& + \left(-\frac{\mu}{r^2} + k^2 r^2\right) dz^2 + r^2 e^{-\frac{\beta z}{\sqrt{3}}} d\psi^2 + \frac{\beta^2}{2l^2} r^2 \frac{e^{\frac{\beta z}{2\sqrt{3}}}}{\sqrt{1 \pm \sin(\beta z)}} d\phi^2 \quad (5.36)
\end{aligned}$$

We can go one step further by absorbing $\lambda/2\beta$ in ϕ , renaming $\beta \rightarrow 2\beta$ and considering the translation $z \rightarrow z + \frac{\pi}{2\beta}$. We get

$$\begin{aligned}
ds^2 = & \frac{dr^2}{-\frac{\mu}{r^2} + k^2 r^2} + \left(-\frac{\mu}{r^2} + k^2 r^2\right) dz^2 + \\
& + r^2 \left[e^{\frac{\beta z}{\sqrt{3}}} |\cos(\beta z)| [-dt^2 + d\phi^2 + 2 \tan(\beta z) dt d\phi] + e^{-\frac{2\beta z}{\sqrt{3}}} d\psi^2 \right] \quad (5.37)
\end{aligned}$$

This solution is clearly a continuous deformation of the adS soliton which is obtained for $\beta = 0$. The metric is not however everywhere C^2 ; for every $z = n\pi + \frac{\pi}{2\beta}$ there is a discontinuity in the first derivative with respect to z which indicates the presence of δ sources to account for these jumps. The parameter $1/\beta$ indicates the distance between the singularities. Also we can easily show that for $r = \text{constant}$ the induced 4-dimensional metric is a vacuum solution to the 4 dimensional Einstein equations. Surprisingly the deformed solution (5.37) and the adS soliton have the same Kretschmann scalar indicating that the deformed solution is again regular. Note that the z coordinate varies throughout the real line because of the exponential warp factors, which for z negative and large effectively reduce the $t - \phi$ dimensions, whereas for z positive and large, reduce the ψ dimension. This solution has no 4 dimensional counterpart since the extra Weyl direction has to be switched on (5.31).

When $\beta = 0$ but $\lambda \neq 0$ we get

$$A(z) = -\frac{1}{\lambda^2 z} \quad (5.38)$$

and the five dimensional metric reads

$$ds^2 = \frac{dr^2}{-\frac{\mu}{r^2} + k^2 r^2} + \left(-\frac{\mu}{r^2} + k^2 r^2\right) dz^2 + r^2 (-\lambda z dt^2 + 2dt d\phi + d\psi^2). \quad (5.39)$$

Notice that, unlike the previous case, this solution can be Wick rotated to a non-static black hole. Indeed, let us take

$$r \rightarrow ir \quad (5.40)$$

$$\psi \rightarrow i\psi \quad (5.41)$$

$$t \rightarrow \theta \quad (5.42)$$

$$z \rightarrow t, \quad (5.43)$$

to get

$$ds^2 = - \left(-\frac{\mu}{r^2} + k^2 r^2 \right) dt^2 + \frac{dr^2}{-\frac{\mu}{r^2} + k^2 r^2} + r^2 (\lambda t d\theta^2 + 2d\theta d\phi + d\psi^2). \quad (5.44)$$

Although this metric has similar structure as the planar adS black hole the horizon surface here has a non-trivial curved embedding depending on the coordinate time t . This solution is not a continuous deformation of the black hole solution and ∂_ϕ is a null Killing vector. This solution can also be written in $D = 4$ by simply taking $\psi = \text{constant}$ and using instead the 4 dimensional black hole potential.

6. Constructing solutions for $\Lambda = 0$

As we pointed out in Section 2 we can construct solutions in a $D + n$ dimensional spacetime starting from a known seed solution in D dimensions. This is possible as long as $\Lambda = 0$. Say we start from some 4 dimensional solution which can even be flat spacetime. Then for each Weyl potential Ψ solution of (3.81) one can construct a new $D = 5$ dimensional solution finding the relevant σ component from (3.92). Schematically for each $D = 4$ dimensional solution there is an infinity of $D = 4 + n$ dimensional solutions that can be constructed, parametrised by the Weyl potentials $\Psi_i, i = 1, \dots, n$. A general analysis of this method is best done in Weyl coordinates starting from lower to higher dimension. Then, $\alpha = r$ and we keep the same coordinate system from lower to higher dimension. The Weyl potentials can be constrained in order to guarantee asymptotic flatness for the higher dimensional solution. This we leave for later study. In this section we will do the converse. Starting from two 5 dimensional examples, the Myers-Perry black hole and the black ring, we will go down to 4 dimensions.

Start with the Myers-Perry solution describing a rotating black hole with a single angular momentum in the coordinates (5.11). We set $\Lambda = 0$, i.e. $k = 0$, in the metric components (5.15) and we use (3.91) to obtain

$$e^{2\sigma} = \frac{(\rho^2 - 2M \sin^2 \theta)^{3/4}}{\Delta^{1/12} r^{2/3} \cos^{2/3} \theta \sin^{1/6} \theta}. \quad (6.1)$$

The corresponding four-dimensional metric, solution of Einstein's equations, is given by substituting the expressions for α, Ω, A into (5.15) and (6.1) in

$$ds^2 = e^{2\sigma} \alpha^{1/6} \rho^2 \left(\frac{dr^2}{\Delta} + d\theta^2 \right) + \alpha e^{-\frac{\Omega}{2}} d\phi^2 - \alpha e^{\frac{\Omega}{2}} (dt + Ad\phi)^2. \quad (6.2)$$

After the dust settles we obtain the following 4-dimensional metric¹²

$$ds^2 = \frac{\rho^2}{\sqrt{r \cos(\theta)}} [r^2 + (a^2 - 2M) \sin^2 \theta]^{3/4} \left(d\theta^2 + \frac{dr^2}{\Delta} \right) -$$

¹²A word of warning on notation. Here $\Delta = r^2 + a^2 - 2M$ stands for the 5-dimensional potential and thus the coordinate r is not the one appearing in (4.8).

$$-\frac{r \cos(\theta)(\rho^2 - 2M)}{\rho^2} \left(dt + \frac{2Ma \sin^2 \theta}{\rho^2 - 2M} d\phi \right)^2 + \frac{\rho^2 r \Delta \cos \theta \sin^2 \theta}{\rho^2 - 2M} d\phi^2.$$

Note that the resulting metric does not describe the Kerr geometry and is not asymptotically flat (actually, even for $M = a = 0$ this solution is not flat).

Now we work out the seed solution for the black ring solution [21] in $D = 4$ dimensions. The black ring is described in C-metric type coordinates by the line element,

$$ds^2 = -\frac{F(y)}{F(x)} \left(dt + C(\nu, \lambda) R \frac{1+y}{F(y)} d\phi \right)^2 + \frac{R^2 F(x)}{(x-y)^2} \left(-\frac{G(y)}{F(y)} d\phi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\psi^2 \right) \quad (6.3)$$

where $F(\xi) = 1 + \lambda\xi$, $G(\xi) = (1 - \xi^2)(1 + \nu\xi)$, R is a constant giving roughly the rings radius and

$$C(\nu, \lambda) = \sqrt{(\lambda - \nu)\lambda \frac{1 + \lambda}{1 - \lambda}}. \quad (6.4)$$

The radial and angular coordinates are respectively $y \in]-\infty, -1]$ and $x \in [-1, 1]$. A regular black ring without conical singularity is obtained when the rotation cancels out the gravitational attraction of the ring, for

$$\lambda = \frac{2\nu}{1 + \nu^2} \quad (6.5)$$

In all other cases a conical singularity naturally appears at $x = 1$ holding the black ring together and avoiding its collapse. Static black rings are obtained when $\lambda = \nu$. This solution presents a lot of interesting properties which are discussed in [21], [28] and [47].

The first thing we need to do is identify the components from (5.7). We get,

$$\begin{aligned} \alpha &= \sqrt{-G(y)G(x)} \frac{R^2}{(x-y)^2}, & e^\Omega &= \left(\frac{F(y)(x-y)}{F(x)R} \right)^2 \frac{1}{-G(y)} \\ e^{\frac{\sqrt{3}\Psi_*}{\sqrt{2}}} &= \frac{RG(x)}{(x-y)\sqrt{-G(y)}}, & A &= C(\nu, \lambda) R \frac{(1+y)}{F(y)} \\ e^{2\nu_{(5)}} &= \alpha^{2/3} \frac{R^2}{(x-y)^2} F(x) \end{aligned} \quad (6.6)$$

To construct the relevant $D = 4$ solution from the above we keep the same components A , Ω and α and we evaluate the component $\sigma = \nu_{(4)} - \nu_{(5)}$ using (3.91) given the components Ψ and $\nu_{(5)}$ from (6.6). It is then straightforward to note that the $D = 4$ metric (4.6) takes the form,

$$ds^2 = e^{2\sigma} \alpha^{1/6} \frac{R^2}{(x-y)^2} F(x) \left(\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)} \right) + \alpha e^{-\frac{\Omega}{2}} d\phi^2 - \alpha e^{\frac{\Omega}{2}} (dt + Ad\phi)^2 \quad (6.7)$$

and σ , in the above coordinate system, is given by two first order ODE's (3.91)

$$\begin{aligned}\sigma_{,x} &= -\frac{\alpha}{8(\alpha_{,x}^2 G(x) - \alpha_{,y}^2 G(y))} [\alpha_{,x} \{\Psi_{,x}^2 G(x) + \Psi_{,y}^2 G(y)\} - 2\Psi_{,x}\alpha_{,y}\Psi_{,y}G(y)] \\ \sigma_{,y} &= \frac{\alpha}{8(\alpha_{,x}^2 G(x) - \alpha_{,y}^2 G(y))} [\alpha_{,y} \{\Psi_{,x}^2 G(x) + \Psi_{,y}^2 G(y)\} - 2\Psi_{,x}\alpha_{,x}\Psi_{,y}G(x)]\end{aligned}\quad (6.8)$$

This can be integrated explicitly and we obtain

$$e^{2\sigma} = \frac{(x-y)^{1/12}(W(x,y))^{3/4}}{(-G(y))^{1/12}(G(x))^{1/3}}\quad (6.9)$$

where

$$W(x,y) = [y+x+\nu(1+xy)][\nu^2(xy-1)^2 - [2+\nu(x+y)]^2].\quad (6.10)$$

7. Conclusions

In this paper we have extensively analysed solution generating methods for Einstein's equations in D dimensions with a cosmological constant. In particular, we studied stationary spacetimes of axial symmetry, restricting our attention to the case of a single rotation parameter. Our analysis was also shown to apply, by a simple KK reduction, to an EMD (Einstein-Maxwell-dilaton) system with a Liouville potential. Our approach has been threefold. Firstly, to make the connection with the classical works of general relativity in $D = 4$ and $\Lambda = 0$ such as those of Papapetrou and Ernst, and also to connect with the relatively few recent studies in higher dimensions for $\Lambda = 0$ [28]. Our aim was to analyse the symmetries of the field equations including possible dualities, to classify and characterise the methods and solutions, and to give typical examples without necessarily writing out all the possible metrics.

Our analysis of the field equations has brought out a new solution generating method valid for $\Lambda = 0$. According to this recipe, for each 4 dimensional stationary and axisymmetric solution, one can generate an infinity of higher dimensional solutions, parametrised by a Weyl potential, for each extra dimension. In this way, even a flat 4 dimensional solution can generate an infinite number of higher dimensional solutions. As examples, we showed that the 5 dimensional black ring and the 5 dimensional Myers-Perry solution do not originate from Kerr's solution, the only stationary and axisymmetric black hole solution in $D = 4$. We have seen that this method does not generically preserve asymptotic flatness. A more systematic analysis of this method, in particular making use of the Weyl coordinates (2.2), will be undertaken in the future. For $\Lambda \neq 0$, we have found solutions which can be interpreted as deformations of the adS soliton and planar black holes. These solutions are of non-trivial topological charge characterised by an extra integration parameter.

We have demonstrated that classical methods such as those of Ernst and Papapetrou can be extended to spacetimes admitting a cosmological constant. We

generalised the results of Papapetrou mapping a certain class of stationary solutions to static ones and have found the extension of Ernst's equation in the presence of a cosmological constant. We have seen that one can interpret the field equations in a geometric way with respect to a three dimensional background manifold. Whereas when $\Lambda = 0$ the manifold in question is flat, the presence of Λ makes the manifold curved and the choice of an adequate coordinate system difficult. Our actual analysis leaves open the question of finding a suitable coordinate system for asymptotically dS or adS spaces, such as those available for $\Lambda = 0$; namely that of spheroidal coordinates [41] or Weyl coordinates [2]. A coordinate system adapted to the profile of the solution in question would be able to stretch the methods we have developed to their full potential. For example, we would expect to be able to generate Carter's solution [14] from Kottler's solution by a method similar to that exposed by Ernst for the $\Lambda = 0$ case [11]. The presence of the cosmological constant has been shown here not to burden the solution generating methods themselves, but rather to emphasise the the adequate choice of a coordinate system with which to apply these methods. This is of crucial importance in order to tackle solutions such as the adS black ring, the black ring solution in higher dimensions, or exact braneworld gravity solutions such as the black hole on the brane (see for example [38]) or that of a cosmic string [36].

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References

- [1] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, , Cambridge University Press, 1984.
- [2] H. Weyl, *Annalen Physik* **54**, 117 (1917).
- [3] H. Quevedo, *Phys. Rev. D* **33**, 324 (1986).
- [4] W. Kinnersley and M. Walker, *Phys. Rev. D* **2**, 1359 (1970).
- [5] W. Israel and K. A. Khan, *Nuovo Cim.* **33**, 331 (1964). R. C. Myers, *Phys. Rev. D* **35**, 455 (1987). H. F. Dowker and S. N. Thambayahpillai, *Class. Quant. Grav.* **20**, 127 (2003) [arXiv:gr-qc/0105044].
- [6] T.Lewis, *Proc Roy Soc Lond. A* **136**, 176 (1932).

- [7] A. Papapetrou, *Ann. Phys.* **12** (1953) 309.
- [8] A. Papapetrou, *Ann. Inst. H. Poincaré* **A4** (1966) 83.
- [9] R. P. Kerr, *Phys. Rev. Lett.* **11**, 237 (1963).
- [10] A. H. Taub, *Annals Math.* **53**, 472 (1951).
E. Newman, L. Tamubirino and T. Unti, *J. Math. Phys.* **4**, 915 (1963).
- [11] F.J. Ernst, *Phys. Rev.* **167** (1968) 1175. F.J. Ernst, *Phys. Rev.* **168** (1968) 1415
- [12] V.A Belinskii and V.E Zakharov, *Sov.Phys. JETP* **77**, 3.
C. Hoenselaers, W. Kinnersley, B.C. Xanthopoulos (Montana State U.), *J.Math.Phys* **20** , 2530-2536, 1979.
- [13] P. T. Chrusciel, G. M. Greuel, R. Meinel and S. J. Szybka, *Class. Quant. Grav.* **23**, 4399 (2006) [arXiv:gr-qc/0603041]. G. Neugebauer and R. Meinel, *J. Math. Phys.* **44**, 3407 (2003) [arXiv:gr-qc/0304086]. R. Meinel and G. Neugebauer, *Class. Quant. Grav.* **12**, 2045 (1995) [arXiv:gr-qc/0302114].
- [14] B. Carter, *Commun. Math. Phys.* **10**, 280 (1968).
- [15] R. C. Myers and M. J. Perry, *Annals Phys.* **172**, 304 (1986).
- [16] S.W. Hawking, C.J. Hunter and M.M. Taylor-Robinson, , hep-th/9811056
G. W. Gibbons, H. Lu, D. N. Page and C. N. Pope, *Phys. Rev. Lett.* **93**, 171102 (2004) [arXiv:hep-th/0409155]. G. W. Gibbons, H. Lu, D. N. Page and C. N. Pope, *J. Geom. Phys.* **53**, 49 (2005) [arXiv:hep-th/0404008].
- [17] G. T. Horowitz and A. Strominger, *Nucl. Phys. B* **360**, 197 (1991).
- [18] J. Polchinski, *Phys. Rev. Lett.* **75**, 4724 (1995) [arXiv:hep-th/9510017].
J. Polchinski, “String theory. Vol. 2: Superstring theory and beyond,” J. Polchinski, “String theory. Vol. 1: An introduction to the bosonic string,”
- [19] F. Dowker, J.P. Gauntlett, D.A. Kastor and J. Traschen, , hep-th/9309075 F.
Dowker, J.P. Gauntlett, G.W. Gibbons and G.T. Horowitz, hep-th/9507143
- [20] R. Emparan and H. S. Reall, arXiv:hep-th/0608012.
- [21] R. Emparan and H. S. Reall, *Phys. Rev. Lett.* **88**, 101101 (2002) [arXiv:hep-th/0110260].
- [22] H. Elvang, R. Emparan, D. Mateos and H. S. Reall, *Phys. Rev. Lett.* **93**, 211302 (2004) [arXiv:hep-th/0407065].
- [23] B. Carter, *Phys. Rev. Lett.* **26**, 331 (1971).
- [24] S. Hollands, A. Ishibashi and R. M. Wald, arXiv:gr-qc/0605106.

- [25] A. Coley and N. Pelavas, *Gen. Rel. Grav.* **38**, 445 (2006) [arXiv:gr-qc/0510064].
A. Coley, R. Milson, V. Pravda and A. Pravdova, *Class. Quant. Grav.* **21**, L35 (2004) [arXiv:gr-qc/0401008].
- [26] R. Emparan and H.S. Reall, *Phys. Rev.* **D65** (2002) 084025, hep-th/0110258.
- [27] C. Charmousis and R. Gregory, *Class. Quant. Grav.* **21**, 527 (2004) [arXiv:gr-qc/0306069].
- [28] T. Harmark, hep-th/0408141. *Phys. Rev. D* **70**, 124002 (2004) [arXiv:hep-th/0408141].
- [29] T. Harmark and P. Olesen, *Phys. Rev. D* **72**, 124017 (2005) [arXiv:hep-th/0508208].
- [30] J. M. Maldacena, "supergravity," *Adv. Theor. Math. Phys.* **2**, 231 (1998) [*Int. J. Theor. Phys.* **38**, 1113 (1999)] [arXiv:hep-th/9711200].
- [31] L. Randall and R. Sundrum, *Phys. Rev. Lett.* **83** (1999) 4690 [arXiv:hep-th/9906064].
- [32] R. Emparan, G. T. Horowitz and R. C. Myers, *JHEP* **0001**, 007 (2000) [arXiv:hep-th/9911043].
- [33] S. Creek, R. Gregory, P. Kanti and B. Mistry, arXiv:hep-th/0606006.
- [34] R. Gregory and R. Laflamme, *Phys. Rev. Lett.* **70**, 2837 (1993) [arXiv:hep-th/9301052].
R. Gregory and R. Laflamme, *Nucl. Phys. B* **428**, 399 (1994) [arXiv:hep-th/9404071].
R. Gregory, *Class. Quant. Grav.* **17**, L125 (2000) [arXiv:hep-th/0004101].
- [35] A. Fabbri, D. Langlois, D. A. Steer and R. Zegers, *JHEP* **0409**, 025 (2004) [arXiv:hep-th/0407262].
- [36] S. C. Davis, *Phys. Lett. B* **499**, 179 (2001) [arXiv:hep-ph/0010255].
- [37] A. Vilenkin, *Phys. Rev. D* **23**, 852 (1981).
- [38] R. Emparan, A. Fabbri and N. Kaloper, *JHEP* **0208**, 043 (2002) [arXiv:hep-th/0206155].
- [39] K. C. K. Chan, J. H. Horne and R. B. Mann, *Nucl. Phys. B* **447**, 441 (1995) [arXiv:gr-qc/9502042]. C. Leygnac, arXiv:gr-qc/0409040. G. Clement and C. Leygnac, *Phys. Rev. D* **70**, 084018 (2004) [arXiv:gr-qc/0405034]. G. Clement, D. Gal'tsov, C. Leygnac and D. Orlov, *Phys. Rev. D* **73**, 045018 (2006) [arXiv:hep-th/0512013]. G. Clement, D. Gal'tsov and C. Leygnac, *Phys. Rev. D* **71**, 084014 (2005) [arXiv:hep-th/0412321]. S. S. Yazadjiev, *Class. Quant. Grav.* **22**, 3875 (2005) [arXiv:gr-qc/0502024]. S. S. Yazadjiev, S. S. Yazadjiev, *Phys. Rev. D* **73**, 064008 (2006) [arXiv:gr-qc/0511114]. [arXiv:hep-th/0511016].
- [40] C. Charmousis, *Class. Quant. Grav.* **19**, 83 (2002) [arXiv:hep-th/0107126].

- [41] D.M. Zipoy, *J. Maths. Phys* **7** (1966) 1137.
- [42] R. Gautreau and R. B. Hoffman, *Nuovo Cim. B* **61**, 411 (1969).
- [43] G. W. Gibbons and D. L. Wiltshire, *Annals Phys.* **167**, 201 (1986) [Erratum-ibid. **176**, 393 (1987)]. G. T. Horowitz, arXiv:hep-th/9210119. D. Youm, *Phys. Rept.* **316**, 1 (1999) [arXiv:hep-th/9710046].
- [44] M. Cvetič and D. Youm, *Nucl. Phys. B* **476**, 118 (1996) [arXiv:hep-th/9603100].
M. Cvetič and C. M. Hull, *Nucl. Phys. B* **480**, 296 (1996) [arXiv:hep-th/9606193].
- [45] G. W. Gibbons, R. Guven and C. N. Pope, *Phys. Lett. B* **595**, 498 (2004) [arXiv:hep-th/0307238]. Y. Aghababaie *et al.*, *JHEP* **0309**, 037 (2003) [arXiv:hep-th/0308064]. C. P. Burgess, F. Quevedo, G. Tasinato and I. Zavala, *JHEP* **0411**, 069 (2004) [arXiv:hep-th/0408109].
- [46] R. Emparan, G. T. Horowitz and R. C. Myers, *JHEP* **0001**, 021 (2000) [arXiv:hep-th/9912135].
- [47] R. Emparan, rings,” *JHEP* **0403**, 064 (2004) [arXiv:hep-th/0402149].
- [48] F. Kottler, *Ann. Phys.* **56** 401.
- [49] R. Emparan, G. T. Horowitz and R. C. Myers, *JHEP* **0001**, 021 (2000) [arXiv:hep-th/9912135].
- [50] G. T. Horowitz and R. C. Myers, *Phys. Rev. D* **59**, 026005 (1999) [arXiv:hep-th/9808079].
- [51] P. Bowcock, C. Charmousis and R. Gregory, *Class. Quant. Grav.* **17**, 4745 (2000) [arXiv:hep-th/0007177].
- [52] R. Gregory and A. Padilla, *Class. Quant. Grav.* **19** (2002) 279 [arXiv:hep-th/0107108].