Liouville Field, Modular Forms and Elliptic Genera

(Version published in Journal of High Energy Physics (2007) 0703:119

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Abstract

When we describe non-compact or singular Calabi-Yau manifolds by CFT, continuous as well as discrete representations appear in the theory. These representations mix in an intricate way under the modular transformations. In this article, we propose a method of combining discrete and continuous representations so that the resulting combinations have a simpler modular behavior and can be used as conformal blocks of the theory. We compute elliptic genera of ALE spaces and obtain results which agree with those suggested from the decompactification of K3 surface. Consistency of our approach is assured by some remarkable identity of theta functions whose proof, by D. Zagier, is included in an appendix.

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1 Introduction

Description of strings propagating on non-compact curved background is a challenging problem in particular when the space-time develops a singularity. A better grasp of underlying conformal field theory (CFT) should shed light on the physics of such space-time.

When a Calabi-Yau (CY) manifold is non-compact or singular, it is necessary to introduce a CFT possessing continuous as well as discrete representations in order to describe its geometry. These CFT's have a central charge above the "threshold", i.e. $c = 3$ for $\mathcal{N} = 2$ supersymmetric case, and are of non-minimal type. We may call these theories generically as Liouville type theories. Since continuous and discrete representations mix under modular transformations, representations of Liouville theories in general do not have good modular properties. Thus it is a non-trivial problem to construct suitable modular invariants describing the geometry of non-compact CY.

In this paper we present an attempt at constructing (holomorphic) modular invariants for some non-compact CY manifolds. In particular we propose the elliptic genera for the ALE spaces which are the degenerate limits of K3 surface. It turns out that the consistency of our approach hinges on the validity of some theta-function identities. Theses non-trivial identities have recently been proved by D.Zagier and the proof is presented in an appendix.

1.1 Bosonic Liouville theory

We start our discussions by reviewing the simple case of bosonic Liouville theory. Its stress tensor is given by

$$T(z) = -\frac{1}{2}(\partial\phi)^2 + \frac{\mathcal{Q}}{2}\partial^2\phi \tag{1.1}$$

where $\mathcal{Q}$ is the background charge. Central charge is given by

$$c = 1 + 3\mathcal{Q}^2. \tag{1.2}$$

If we parameterize $\mathcal{Q}$ as $\mathcal{Q} = \sqrt{2}(b + 1/b)$, the vertex operator

$$\exp(\sqrt{2}b\phi) \tag{1.3}$$

has a conformal dimension $h = 1$. Liouville theory is defined as a theory perturbed by this marginal operator (Liouville potential) from free fields.

Dynamics of boundary Liouville theory became clarified in late 1990's by the method of conformal bootstrap [1]. We first reintroduce the result of conformal bootstrap using representation theory and the modular properties of character formulas.

It is known that there are two types of representations in bosonic Liouville theory: continuous and identity representation. Their character formulas and their S-transformation are given by

continuous representations; $p > 0$

$$\chi_p(\tau) = \frac{q^{h-\frac{c}{24}}}{\prod_{n=1}(1-q^n)} = \frac{q^{\frac{p^2}{2}}}{\eta(\tau)}, \qquad h = \frac{p^2}{2} + \frac{\mathcal{Q}^2}{8}$$
$$\chi_p(-\frac{1}{\tau}) = 2\int_0^\infty dp' \cos(2\pi pp')\chi_{p'}(\tau), \tag{1.4}$$

identity representation; $h = 0$

$$\chi_{h=0}(\tau) = \frac{q^{-\frac{\mathcal{Q}^2}{8}}(1-q)}{\eta(\tau)},$$
$$\chi_{h=0}(-\frac{1}{\tau}) = 4\int_0^\infty dp \sinh(2\pi bp)\sinh(\frac{2\pi p}{b})\chi_p(\tau) \tag{1.5}$$

We identify the LHS of the above equations as describing the open string channel and RHS as the closed string channel. We then find that open and closed channels have different spectra:

open	closed
{ continuous rep. / identity rep. }	continuous rep.

Namely, there exist no identity representation in the closed string channel. This is consistent with the presence of mass gap and the decoupling of gravity in non-compact space-time. Indeed the conformal dimension of a vertex operator $e^{\alpha\phi}$ is given by

$$\begin{aligned} h(e^{\alpha\phi}) &= -\frac{\alpha^2}{2} + \frac{\alpha\mathcal{Q}}{2} = -\frac{(\alpha-\frac{1}{2}Q)^2}{2} + \frac{\mathcal{Q}^2}{8} \\ &= \frac{p^2}{2} + \frac{\mathcal{Q}^2}{8} \geq \frac{\mathcal{Q}^2}{8} \quad \text{for} \quad \alpha = ip + \frac{1}{2}\mathcal{Q} \end{aligned} \tag{1.6}$$

for continuous representations. Thus there is a gap of $\mathcal{Q}^2/8$ in the spectrum of continuous representations.

Let us next turn to the brane-interpretation of transformations (1.4), (1.5). We introduce ZZ and FZZT brane boundary states $|ZZ\rangle$, $|FZZT\rangle$ and identify the character functions as the inner product

$$\chi_0(-\frac{1}{\tau}) = \langle ZZ|e^{i\pi\tau H^{(c)}}|ZZ\rangle \tag{1.7}$$

$$\chi_p(-\frac{1}{\tau}) = \langle FZZT; p\,|e^{i\pi\tau H^{(c)}}|ZZ\rangle \tag{1.8}$$

where $H^{(c)} = L_0 + \bar{L}_0 - \frac{c}{12}$ is the closed string Hamiltonian. Using Ishibashi states $|p\,\rangle\rangle$ with momentum p which diagonalize the closed string Hamiltonian

$$\langle\langle p\,|e^{i\pi\tau H^{(c)}}|p'\rangle\rangle = \delta(p-p')\chi_p(\tau) \tag{1.9}$$

boundary states are expanded as

$$|ZZ\rangle = \int_0^\infty dp\,\Psi_0(p)\,|p\,\rangle\rangle \tag{1.10}$$

$$|FZZT; p\,\rangle = \int_0^\infty dp'\,\Psi_p(p')\,|p'\rangle\rangle. \tag{1.11}$$

We then have

$$|\Psi_0(p)|^2 = 4\sinh\sqrt{2}\pi pb\sinh\frac{\sqrt{2}\pi p}{b}\,, \tag{1.12}$$

$$\Psi_p(p')^*\Psi_0(p') = 2\cos 2\pi pp'. \tag{1.13}$$

Solving these relations one finds the boundary wave-functions

$$\Psi_0(p) = \frac{2\sqrt{2}\pi ip}{\Gamma(1+i\sqrt{2}pb)\Gamma(1+\frac{i\sqrt{2}p}{b})}, \tag{1.14}$$

$$\Psi_p(p') = \frac{-1}{\sqrt{2}\pi ip'}\Gamma(1-\sqrt{2}ibp')\Gamma(1-\frac{\sqrt{2}ip'}{b})\cos(2\pi pp'). \tag{1.15}$$

Up to phase factors the above results agree with those of conformal bootstrap [1].

1.2 $\mathcal{N}=2$ Liouville theory

For the sake of applications to string theory let us now consider $\mathcal{N}=2$ supersymmetric version of Liouville theory. In $\mathcal{N}=2$ system possesses two bosons, one of them coupled to background charge and the other one is a compact boson, and two free fermions. It is known that $\mathcal{N}=2$ Liouville theory is T-dual to $SL(2;\mathbb{R})/U(1)$ supercoset model which describes the space-time of the two-dimensional black hole [2]. In general $\mathcal{N}=2$ Liouville is geometrically interpreted as describing the radial direction of a complex cone.

In the following we concentrate on the case when $\mathcal{N}=2$ Liouville has a central charge

$$\hat{c}=\frac{c}{3}=1+\frac{2}{N},$$

which we denote as L_N, for the sake of simplicity ($\mathcal{Q}=\sqrt{2/N}$). Here N is an arbitrary positive integer. This theory is T-dual to two-dimensional black hole with an asymptotic radius of the cigar $\sqrt{2N}$.

Unitary representations of $\mathcal{N}=2$ superconformal algebra with $\hat{c}=1+\frac{2}{N}$ are given by

$$\begin{cases} \text{identity rep.} & h=0, \quad j=0 \quad \text{vacuum} \\ \text{continuous reps.} & p>0, \quad j=\frac{1}{2}+i\frac{p}{\mathcal{Q}} \quad \text{non-BPS states} \\ \text{discrete reps.} & 1\leq s\leq N, \quad j=\frac{s}{2} \quad \text{BPS states, chiral primaries} \end{cases}$$

Here p and s label continuous and discrete representations of $\mathcal{N}=2$ Liouville theory, respectively. $\mathcal{N}=2$ representations are in one to one correspondence with those of level $k=N$ $SL(2;\mathbb{R})/U(1)$ coset theory with the value of spin j indicated as above.

In applications to string theory we consider the sum over spectral flows of each $\mathcal{N}=2$ representation and define an extended character [3][1]

$$\chi_*^{NS}(r;\tau,z)=\sum_{n\in r+N\mathbb{Z}} q^{\frac{\hat{c}}{2}n^2}e^{2\pi i\hat{c}zn}ch_*^{NS}(\tau;z+n\tau) \tag{1.16}$$

Here $ch_*^{NS}(\tau;z)$ denotes an irreducible character of $\mathcal{N}=2$ superconformal algebra (in NS

[1] Here the spectral flow is summed over modulo N for the sake of convenience. Idea of extended character has been introduced in [4] where the irreducible characters of $\mathcal{N}=4$ algebra are identified as extended characters of $\mathcal{N}=2$ algebra. For related works see [5, 6, 7].

sector). Extended characters carry some additional label

$$\begin{aligned}&\text{1. Identity representations :}\\&\chi^{NS}_{id}(r;\tau,z); \qquad r\in\mathbb{Z}_N,\end{aligned} \tag{1.17}$$

$$\begin{aligned}&\text{2. Continuous representations :}\\&\chi^{NS}_{cont}(p,\alpha;\tau,z);\end{aligned} \tag{1.18}$$

$$\begin{aligned}&\text{3. Discrete representations :}\\&\chi^{NS}_{dis}(s,s+2r;\tau,z); \qquad r\in\mathbb{Z}_N, 1\le s\le N\end{aligned} \tag{1.19}$$

Explicit form of these characters are presented in the Appendix A. We also present the form of modular transformation in the case of discrete representations. (Formulas for continuous and identity representations are given in [3]). Here we recall that the S transform of these functions has the following pattern

$$(\text{continuous rep}) \xrightarrow{S} (\text{continuous rep}) \tag{1.20}$$

$$(\text{identity rep}) \xrightarrow{S} (\text{discrete rep}) + (\text{continuous rep}) \tag{1.21}$$

$$(\text{discrete rep}) \xrightarrow{S} (\text{discrete rep}) + (\text{continuous rep}) \tag{1.22}$$

Namely, a continuous representation transforms into an integral over continuous representations while an identity and discrete representation transforms into a sum of discrete representation and an integral over continuous representations. Such a pattern was first observed in $\mathcal{N}=4$ representation theory [4].

1. As in the bosonic Liouville theory, there appear no identity representations in the RHS of above formulas.
2. While the identity representation disappears after a first S-transform, it comes back after a 2nd transform: this happens when one deforms the contour of momentum integration for the sake of convergence and picks up a pole in the complex plane corresponding to

the identity representation. It is further possible to check that $S^2 = C$ and $(ST)^3 = C$, where C is a charge conjugation matrix which acts as $C : (\tau, z) \to (\tau, -z)$. For details see [7, 8].

As compared with the case of minimal theories where only discrete representations exist which rotate into each other under the S-transform, the above transformation laws (1.20)-(1.22) are much more complex and in particular discrete representations mix with continuous representations. We can check that even under the transformation ST^2S^{-1} one can not eliminate the contribution of continuous representations in the transform of discrete representations (ST^2S^{-1} is a generator of $\Gamma(2)$ which is the subgroup of $SL(2;\mathbb{Z})$ keeping the spin-structure fixed). It seems not possible to eliminate the mixing of continuous representations under any subgroup of the modular group. This is what we mean by the extended characters of the $\mathcal{N} = 2$ algebra as a whole not having good modular properties.

We have three types of boundary states of $\mathcal{N} = 2$ theory corresponding to each representation. The boundary wave functions are again given by the elements of the modular S matrix. We can compare our expressions with known results of $SL(2;\mathbb{R})/U(1)$ theory obtained by semi-classical method using the geometry of 2d black hole and DBI action. It is found [3, 9] that $\mathcal{N} = 2$ theory reproduces essentially the correct wave functions of D-branes of 2d black hole [10]. Thus the representation theory seems quite consistent with the semi-classical analysis. However, the character formulas themselves do not have good modular properties and it is non-trivial problem to construct conformal blocks with good modular behaviors.

2 Geometry of $\mathcal{N} = 2$ Liouville Fields

Let us now consider models of the following type: tensor product of $\mathcal{N} = 2$ Liouville theory L_N (of $\hat{c} = 1 + \frac{2}{N}$) and $\mathcal{N} = 2$ minimal model M_k with level k [11]

$$L_N \otimes M_k. \tag{2.1}$$

If we choose

$$N = k + 2 \tag{2.2}$$

the central charge becomes integral

$$c_L + c_M = 3(1 + \frac{2}{N}) + 3(1 - \frac{2}{k+2}) = 6 \tag{2.3}$$

and the theory (after Z_N orbifolding) describes (complex) 2 dimensional CY manifolds. They are identified as the (A-type) ALE spaces which are obtained by blowing up A_{N-1} singularities [11]. At $N=1$ (without minimal model), we have $\hat{c}=3$ and the space-time of a conifold [12]. We may as well consider the tensor products of Liouville theories and minimal models. These describe other various singular geometries like A_{N-1} spaces fibered on P^1 etc. [13, 14, 15]

2.1 Elliptic genus and CY/LG correspondence

The elliptic genus is defined by taking the sum over all states in the left-moving sector of the theory while the right-moving sector is fixed at the Ramond ground states;

$$Z(\tau,z)=Tr_{R\otimes R}(-1)^{F_L+F_R}e^{2\pi i z J_o^L}q^{L_0-\frac{\hat{c}}{8}}\bar{q}^{\bar{L}_0-\frac{\hat{c}}{8}}. \tag{2.4}$$

Here J_0^L denotes the $U(1)_R$ charge in the left-moving sector. The trace is taken in the Ramond-Ramond sector. At specific values of z we have

$$\begin{aligned} &Z(\tau,z=0)=\chi, && \text{Euler number}\\ &Z(\tau,z=1/2)=\sigma+\mathcal{O}(q), && \text{Hirzebruch signature}\\ &Z(\tau,z=(\tau+1)/2)=\hat{A}q^{-1/4}+\mathcal{O}(q^{1/4}), && \hat{A}\ \text{genus}\end{aligned}$$

The elliptic genus is an invariant under smooth variations of the parameters of the theory and is useful, for instance, in counting the number of BPS states. We compute the elliptic genus of a non-compact CY manifolds by pairing the Liouville theory with $\mathcal{N}=2$ minimal models.

Before going into the computation of elliptic genera we first recall the results of CY/LG correspondence [16]. We consider a Landau-Ginzburg (LG) theory with a superpotential

$$W=g\,(X^{k+2}+Y^2+Z^2) \tag{2.5}$$

which in the infra-red limit acquires scale invariance and reproduces the $\mathcal{N}=2$ minimal theory with $\hat{c}=1-2/k$.

In the $\mathcal{N}=2$ minimal theory M_{N-2}, the contribution to elliptic genus comes from the Ramond ground states

$$Z_{\text{minimal}}(\tau,z)=\sum_{\ell=0}^{N-2}ch^{\tilde{R}}_{\ell,\ell+1}(\tau;z). \tag{2.6}$$

Here $ch^{\tilde{R}}_{\ell,\ell+1}(\tau;z)$ denotes the character of minimal model M_k associated to the Ramond ground state labeled by $\ell = 0, 1, \ldots, N-2$. See e.g. [17, 18] for their explicit expressions. $\tilde{R}$ denotes the Ramond sector with $(-1)^F$ insertion. On the other hand as the coupling parameter is turned off $g \to 0$, LG theory becomes a free theory of chiral field X with $U(1)_R$ charge $= 1/N$. Thus the theory possesses a free boson of charge $1/N$ and free fermion of charge $1/N - 1$. Combining these contributions one obtains [19]

$$Z_{LG}(\tau, z) = \frac{\theta_1(\tau, (1-\frac{1}{N})z)}{\theta_1(\tau, \frac{1}{N}z)}. \tag{2.7}$$

These two expressions (2.6),(2.7) in fact agree with each other

$$Z_{\text{minimal}} = Z_{LG}. \tag{2.8}$$

We would like to try a similar construction in Liouville sector as in the case of minimal models. Ramond ground states corresponds to the extended discrete characters;

$$\chi^{\tilde{R}}_{dis}(s, s-1; \tau, z), \qquad s = 1, \cdots, N \tag{2.9}$$

and the elliptic genus is expressed as the sum of them, which is explicitly evaluated in [20] as follows; [2]

$$\begin{aligned} Z_{\text{Liouville}} &= -\sum_{s=1}^{N} \chi^{\tilde{R}}_{dis}(s, s-1; \tau, z) \\ &= -\mathcal{K}_{2N}(\tau, \frac{z}{N}) \frac{i\theta_1(\tau, z)}{\eta(\tau)^3} \end{aligned} \tag{2.10}$$

Here we have introduced the notation of an Appell function $\mathcal{K}_k$ [21, 22]

$$\mathcal{K}_k(\tau, z) \equiv \sum_{n \in \mathbb{Z}} \frac{q^{\frac{k}{2}n^2} y^{kn}}{1 - yq^n}, \qquad y = e^{2\pi i z}. \tag{2.11}$$

We also use the anti-symmetrized version of Appell function defined as

$$\widehat{\mathcal{K}}_k(\tau, z) \equiv \frac{1}{2}\left(\mathcal{K}_k(\tau, z) - \mathcal{K}_k(\tau, -z)\right) \equiv \mathcal{K}_k(\tau, z) - \frac{1}{2}\Theta_{0,\frac{k}{2}}(\tau, 2z) \tag{2.12}$$

[2]Precisely speaking, in [20] we adopt a slightly different convention for the 'boundary contribution ($s = 1, N+1$)' of discrete representations, which yields the anti-symmetrized Appell function (2.12)

$$Z_{\text{Liouville}} = -\widehat{\mathcal{K}}_{2N}(\tau, z) \frac{i\theta_1(\tau, z)}{\eta(\tau)^3}$$

rather than (2.10). However, the difference drops off in the orbifold procedure (2.13) [20]. Namely, one may replace $\mathcal{K}_{2N}(\tau, z)$ with $\widehat{\mathcal{K}}_{2N}(\tau, z)$ in (2.13).

Unlike the theta functions of the minimal models, the Appell function in Liouville theory does not have a good modular transformation law [21]. Complication comes from the non-trivial denominator of the function (2.11) which arises due to existence of fermionic singular vectors in BPS (short) representations.

The Appell function is closely related to the function used by Miki in [6]: they are transformed to each other by spectral flow. The Appell function corresponds to an expression in $\tilde{R}$ sector while Miki's function in NS sector.

When we couple minimal and Liouville theory to compute elliptic genera of A_{N-1} spaces, we may use the orbifoldization procedure [23] and we find [20]

$$\begin{aligned} Z_{ALE(A_{N-1})}(\tau,z) &= \frac{1}{N}\sum_{a,b\in\mathbb{Z}_N} q^{a^2}e^{4\pi iaz}Z_{\text{minimal}}(\tau,z+a\tau+b)Z_{\text{Liouville}}(\tau,z+a\tau+b) \\ &= -\frac{1}{N}\sum_{a,b\in\mathbb{Z}_N} q^{\frac{a^2}{2}}e^{2\pi iaz}(-1)^{a+b}\frac{\theta_1(\tau,\frac{N-1}{N}(z+a\tau+b))}{\theta_1(\tau,\frac{1}{N}(z+a\tau+b))} \\ &\qquad\times\mathcal{K}_{2N}(\tau,\frac{1}{N}(z+a\tau+b))\frac{i\theta_1(\tau,z)}{\eta(\tau)^3} \end{aligned} \tag{2.13}$$

In the special case of $N=2$ we have ($y=e^{2\pi iz}$)

$$Z_{ALE(A_1)}(\tau,z) = -\sum_{n\in\mathbb{Z}}(-1)^n\frac{q^{\frac{1}{2}n(n+1)}y^{n+\frac{1}{2}}}{1-yq^n}\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\Big(\equiv -ch_0^{\tilde{R}}(I=0;\tau,z)\Big). \tag{2.14}$$

This formula coincides with a massless character of $\mathcal{N}=4$ algebra [4]. Unfortunately these formulas do not have well-behaved modular properties and we must make a suitable modification.

The elliptic genus is associated with a conformal field theory defined on the torus and hence it must be invariant under $SL(2;\mathbb{Z})$ or under one of its subgroups. Since we are dealing with superconformal field theory, it seems natural to demand invariance under the subgroup $\Gamma(2)$ which leave fixed the spin structures

$$\Gamma(2) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL(2;\mathbb{Z}),\ a=d=1, b=c=0 \mod 2\right\}$$

It is known that $\Gamma(2)$ is generated by T^2 and ST^2S^{-1}. In the following we construct elliptic genera which are invariant under $\Gamma(2)$.

2.2 Elliptic genus of K3

A hint for our construction comes from the study of elliptic genus of K3 surface (we denote $\theta_i(\tau) \equiv \theta_i(\tau, 0)$)

$$Z_{K3}(\tau, z) = 8\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^2 + \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^2 + \left(\frac{\theta_2(\tau,z)}{\theta_2(\tau)}\right)^2\right]. \qquad (2.15)$$

This formula can be easily derived by orbifold calculation on $T^4/\mathbb{Z}_2$ [24] or by using LG theory and LG/CY correspondence. One can check $Z_{K3}(z=0) = 24$, $Z_{K3}(z=1/2) = 16+...$, $Z_{K3}(z=(\tau+1)/2) = -2q^{-1/4}+...$ and Z_{K3} reproduces classical topological invariants, χ=24, σ=16 and $\hat{A} = -2$.

In the case of K3 surface the manifold has a hyperKäler structure and the CFT possesses an $\mathcal{N} = 4$ symmetry. Thus one can use the representation theory of $\mathcal{N} = 4$ superconformal algebra [4].

At $\hat{c} = 2$ $\mathcal{N} = 4$ theory contains an $SU(2)$ current algebra at level 1. Unitary representations of $\mathcal{N} = 4$ algebra in the NS sector are given by

$$\text{massive rep.}: ch^{NS}(h, I=0; \tau, z) = q^{h-\frac{1}{8}} \frac{\theta_3(\tau,z)^2}{\eta(\tau)^3}, \qquad (2.16)$$

$$\text{massless rep.}: ch_0^{NS}(I=0; \tau, z), \quad ch_0^{NS}(I=1/2; \tau, z). \qquad (2.17)$$

Massive representations exist only for isospin $I = 0$ and are analogous to continuous representations of $\mathcal{N} = 2$. The $I = 0$ and $I = 1/2$ massless representations are analogues of identity and discrete representations. There exists a relation among them

$$ch_0^{NS}(I=0) + 2ch_0^{NS}(I=1/2) = ch^{NS}(h=0, I=0) \qquad (2.18)$$

which shows that the (non-BPS) massive representation becomes reducible as $h \to 0$ and splits into a sum of massless (BPS) representations.

There are various ways of writing the massless characters , however, particularly convenient expressions for our discussion are given by [24]

$$ch_0^{NS}(I=1/2, \tau, z) = -\left(\frac{\theta_1(\tau,z)}{\theta_3(\tau)}\right)^2 + h_3(\tau)\left(\frac{\theta_3(\tau,z)}{\eta(\tau)}\right)^2, \qquad (2.19)$$

$$= \left(\frac{\theta_2(\tau,z)}{\theta_4(\tau)}\right)^2 + h_4(\tau)\left(\frac{\theta_3(\tau,z)}{\eta(\tau)}\right)^2, \qquad (2.20)$$

$$= -\left(\frac{\theta_4(\tau,z)}{\theta_2(\tau)}\right)^2 + h_2(\tau)\left(\frac{\theta_3(\tau,z)}{\eta(\tau)}\right)^2, \qquad (2.21)$$

where the functions $h_i(\tau)$, $i=2,3,4$ are defined by

$$h_3(\tau) = \frac{1}{\eta(\tau)\theta_3(\tau)} \sum_{m\in\mathbb{Z}} \frac{q^{m^2/2-1/8}}{1+q^{m-1/2}}\,, \tag{2.22}$$

$$h_4(\tau) = \frac{1}{\eta(\tau)\theta_4(\tau)} \sum_{m\in\mathbb{Z}} \frac{q^{m^2/2-1/8}(-1)^m}{1-q^{m-1/2}}\,, \tag{2.23}$$

$$h_2(\tau) = \frac{1}{\eta(\tau)\theta_2(\tau)} \sum_{m\in\mathbb{Z}} \frac{q^{m^2/2+m/2}}{1+q^{m}}\,. \tag{2.24}$$

We note that h_i's obey identities [25]

$$h_3(\tau)-h_4(\tau) = \frac{1}{4}\left(\frac{\theta_2(\tau)}{\eta(\tau)}\right)^4, \quad h_2(\tau)-h_3(\tau) = \frac{1}{4}\left(\frac{\theta_4(\tau)}{\eta(\tau)}\right)^4, \quad h_2(\tau)-h_4(\tau) = \frac{1}{4}\left(\frac{\theta_3(\tau)}{\eta(\tau)}\right)^4. \tag{2.25}$$

Now using (2.19-2.21) we can rewrite K3 elliptic genus as

$$\begin{aligned} q^{\frac{1}{4}}y^{-1}Z_{K3}(\tau,z') &= 8\left[-\left(\frac{\theta_1(\tau,z)}{\theta_3(\tau)}\right)^2 + \left(\frac{\theta_2(\tau,z)}{\theta_4(\tau)}\right)^2 - \left(\frac{\theta_4(\tau,z)}{\theta_2(\tau)}\right)^2\right] \\ &= 24ch_0^{NS}(I=1/2;z) - 8\sum_{i=2,3,4} h_i(\tau)\frac{\theta_3(\tau,z)^2}{\eta(\tau)^2}, \\ &\qquad (\, z' \equiv z - (\tau/2+1/2)\,) \end{aligned} \tag{2.26}$$

If one considers the product of $\eta(\tau)$ times the sum of $h_i(\tau)$ functions

$$8\,\eta(\tau)\sum_{i=2,3,4} h_i(\tau) = q^{-1/8}\left[2 - \sum_{n=1}^{\infty} a_n q^n\right] \tag{2.27}$$

one finds that the coefficients a_n of q-expansion are positive integers. Then using the relation (2.18) we can rewrite Z_{K3} into a sum of irreducible characters

$$q^{\frac{1}{4}}y^{-1}Z_{K3}(\tau,z') = 20ch_0^{NS}(I=1/2;\tau,z) - 2ch_0^{NS}(I=0;\tau,z) + \sum_{n=1}^{\infty} a_n ch^{NS}(h=n;\tau,z). \tag{2.28}$$

Under the spectral flow from NS to R sector the $I=0$ and $1/2$ representations turn into the $I=1/2$ and 0 representations, respectively. Thus the coefficient -2 in front of $ch_0^{NS}(I=0)$ in the above formula comes from the multiplicity of the ground states of Ramond $I=1/2$ representation in the right-moving sector. Therefore the net multiplicity of $I=0$ massless representation is 1. Hence in the NS sector the theory contains

1 $\quad I=0 \quad$ rep.

20 $\quad I=1/2 \quad$ reps.

∞ $\quad$ of massive reps. $(h=1,2,\cdots)$

$I = 0$ NS representation corresponds to the gravity multiplet and $I = 1/2$ NS representation corresponds to matter multiplets (vector in IIA, tensor in IIB). This is the well-known field content in the supergravity description of string theory compactified on K3 [26]. Note that the values of the dimension h of massive representations are quantized at positive integers. This is consistent with the T-invariance of the elliptic genus.

Now let us throw away the gravity multiplet so that we can decompactify K3 into a sum of ALE spaces; it is known that K3 may be decomposed into a sum of 16 A_1 spaces [27]. Decompactification corresponds to dropping $I = 0$ massless representation. $I = 0$ representation comes from $q^{-1/8}$ piece in (2.27) which in turn originates from the $(\theta_2(\tau, z)/\theta_2(\tau))^2$ term in (2.15). This suggests

$$Z_{K3,\text{decompactified}} = 8\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^2 + \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^2\right]. \tag{2.29}$$

2.3 Elliptic genera of ALE spaces

We now propose the following formula for the elliptic genus of the A_1 space

$$Z_{A_1}(\tau, z) = \frac{1}{2}\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^2 + \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^2\right]. \tag{2.30}$$

Note that in the NS sector we have the decomposition

$$\begin{aligned} q^{\frac{1}{4}} y^{-1} Z_{A_1}(\tau, z') &= \frac{1}{2}\left[\left(\frac{\theta_2(\tau,z)}{\theta_4(\tau)}\right)^2 - \left(\frac{\theta_1(\tau,z)}{\theta_3(\tau)}\right)^2\right] \\ &= ch_0^{NS}(I = 1/2; \tau, z) - \frac{1}{2}\eta(\tau)\left(h_3(\tau) + h_4(\tau)\right)\frac{\theta_3(\tau,z)^2}{\eta(\tau)^3} \\ &\equiv ch_0^{NS}(I = 1/2; \tau, z) + \sum_{n=1}^{\infty} b_n\, ch^{NS}(h = n; \tau, z), \end{aligned} \tag{2.31}$$

where again the expansion

$$\frac{1}{2}\eta(\tau)\sum_{i=3,4} h_i(\tau) = -\sum_{n=1} b_n q^{n-1/8} \tag{2.32}$$

has positive integer coefficients b_n.

We also propose that elliptic genera of A_{N-1} spaces are simply $(N-1)$ times that of A_1

$$Z_{A_{N-1}}(\tau,z)=(N-1)\frac{1}{2}\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^2+\left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^2\right]. \tag{2.33}$$

Above construction (2.31) of Z_{A_1} suggests that instead of using the irreducible character $ch_0^{NS}(I=1/2)$ by itself we should use its combination with (an infinity of) massive representations defined by the R.H.S. of (2.31), which has a good modular property and is in fact invariant under $\Gamma(2)$. We call this combination as the $\Gamma(2)$-invariant completion of the massless representation and consider it as a conformal block in non-compact CFT.

2.4 Theta-function identity

It is a non-trivial problem to show that for a given representation of a superconformal algebra, it is always possible to define its $\Gamma(2)$-invariant completion uniquely by adding a suitable amount of non-BPS representations. According to our analysis this seems possible when we impose suitable additional conditions: massive contributions have only integer powers in Ramond sector and their conformal dimensions are above the gap, i.e. $h=n$ with $n=1,2,\cdots$.

The $\Gamma(2)$-invariant completion is effectively selecting a topological part of massless representations; this may be easily seen from the formula in the $\tilde{R}$ sector. For instance we consider

$$ch_0^{\tilde{R}}(I=0,\tau,z) = -\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^2-h_3(\tau)\left(\frac{\theta_1(\tau,z)}{\eta(\tau)}\right)^2, \tag{2.34}$$

$$= -\left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^2-h_4(\tau)\left(\frac{\theta_1(\tau,z)}{\eta(\tau)}\right)^2, \tag{2.35}$$

We see at $z=0$, the 2nd terms of (2.34),(2.35) vanish while the 1st term gives the Witten index$=-1$. Our prescription is to identify

$$\left[ch_0^{\tilde{R}}(I=0;\tau,z)\right]_{inv}=-\frac{1}{2}\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^2+\left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^2\right]. \tag{2.36}$$

Here we take an average of $(\theta_3(\tau,z)/\theta_3(\tau))^2$ and $(\theta_4(\tau,z)/\theta_4(\tau))^2$ since in Ramond sector q-expansion is integer-powered. We do not adopt $(\theta_2(\tau,z)/\theta_2(\tau))^2$ for the invariant completion since in this case massive representations start from $h=0$, i.e. below the threshold.

One of the most interesting examples of our analysis will be the Appell function: It turns out that the desired completion is given by

$$[\mathcal{K}_{2N}(\tau,z)]_{inv}\equiv\frac{1}{4}\frac{i\eta(\tau)^3\theta_1(\tau,2z)}{\theta_1(\tau,z)^2}\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^{2(N-1)}+\left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^{2(N-1)}\right]. \tag{2.37}$$

Derivation will be given in Appendix B.

Then we can plug (2.37) into the orbifold formula (2.13) and we can represent the elliptic genera for A_{N-1} spaces as

$$
\begin{aligned}
Z_{A_{N-1}}(\tau,z) &= \frac{1}{4N}\sum_{a,b\in\mathbb{Z}_N} q^{\frac{a^2}{2}}e^{2\pi iaz}(-1)^{a+b}\frac{\theta_1(\tau,\frac{N-1}{N}z_{a,b})\theta_1(\tau,\frac{2}{N}z_{a,b})\theta_1(\tau,z)}{\theta_1(\tau,\frac{1}{N}z_{a,b})^3} \\
&\quad\times\left[\left(\frac{\theta_3(\tau,\frac{1}{N}z_{a,b})}{\theta_3(\tau)}\right)^{2(N-1)}+\left(\frac{\theta_4(\tau,\frac{1}{N}z_{a,b})}{\theta_4(\tau)}\right)^{2(N-1)}\right]
\end{aligned}
\tag{2.38}
$$

where $z_{a,b}=z+a\tau+b$.

It turns out that somewhat strikingly the above formula agrees exactly with our proposed expression for $Z_{A_{N-1}}$

$$
\text{RHS of (2.38)}=\frac{N-1}{2}\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^2+\left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^2\right]. \tag{2.39}
$$

We have proved the above identity for $N=2$ using the addition theorem of theta functions and have checked its validity by Maple for lower values of N.

Thus our approach seems altogether consistent: we have arrived at the same expression (2.39) starting either from the decompactification of K3 or the pairing of $\mathcal{N}=2$ minimal and Liouville theories. We have managed to construct holomorphic modular ($\Gamma(2)$) invariant for a class of non-compact CY manifolds.

Actually the above identity (2.39) is a special case of identities of theta functions

$$
\begin{aligned}
&\frac{1}{2N}\sum_{a,b\in\mathbb{Z}_N} q^{\frac{a^2}{2}}e^{2\pi iaz}(-1)^{a+b}\frac{\theta_1(\tau,\frac{N-1}{N}z_{a,b})\theta_1(\tau,\frac{2}{N}z_{a,b})\theta_1(\tau,z)}{\theta_1(\tau,\frac{1}{N}z_{a,b})^3}\left(\frac{\theta_i(\tau,\frac{1}{N}z_{a,b})}{\theta_i(\tau)}\right)^{2(N-1)} \\
&\qquad = (N-1)\left(\frac{\theta_i(\tau,z)}{\theta_i(\tau)}\right)^2, \qquad\qquad i=2,3,4
\end{aligned}
\tag{2.40}
$$

We note that the above identities (2.40) for $i=2,3,4$ transform into each other under S and T transformations (more precisely under $SL(2;\mathbb{Z})/\Gamma(2)=S_3$).

We are informed of a mathematical proof of these identities (2.40) from D.Zagier [28]. We present his elegant proof using residue integrals in Appendix C.

3 Summary

When we consider a string theory on non-compact CY manifolds it is described by a CFT possessing continuous as well as discrete representations. Characters of representations of such CFT transform in a peculiar manner under S transformation as

$$\text{discrete} \underset{S}{\rightarrow} \sum \text{discrete} + \int \text{continuous}$$

$$\text{continuous} \underset{S}{\rightarrow} \int \text{continuous}$$

Mathematical nature of such transformation is currently not well understood. We have found an empirical method of constructing conformal blocks which have good modular behavior and obtained elliptic genera of some non-compact CY manifolds. Our method of construction of conformal blocks, however, is still provisional and needs further studies.

It will be interesting to see if we can associate a simple free field interpretation to the Liouville contribution to the elliptic genus

$$Z_{\text{Liouville}} = \frac{1}{4}\frac{\theta_1(\tau, \frac{2}{N}z)\theta_1(\tau, z)}{\theta_1(\tau, \frac{1}{N}z)^2}\left[\left(\frac{\theta_3(\tau, \frac{1}{N}z)}{\theta_3(\tau)}\right)^{2(N-1)} + \left(\frac{\theta_4(\tau, \frac{1}{N}z)}{\theta_4(\tau)}\right)^{2(N-1)}\right] \tag{3.1}$$

as in the case of minimal theories. Above formula does not seem to fit to a LG-type description with a superpotential of a chiral field with some negative power.

Acknowledgment

We would like to thank E.Witten and Y.Tachikawa for discussions. We are particularly indebted to D.Zagier for informing us of his proof of theta-function identities and allowing us to present his results in one of the Appendices.

T.E. is grateful to Institute for Advanced Study for its kind hospitality and the Monell Foundation for financial supports. Research of T.E. and Y.S. is supported in part by Grant in Aid from the Japan Ministry of Education, Culture, Sports, Science and Technology. A.T. would like to thank Univ. of Tokyo for its hospitality at various stages of this work, and the LMS for a Collaborative Small Grant. Research of A.T. was partially supported by the TMR programme HPRN-CT-2002-00225.

Appendix:A

We first list the irreducible characters of $\mathcal{N}=2$ theory:

(1) continuous representations:

$$ch^{NS}(h,Q;\tau,z)=q^{h-\frac{\hat{c}-1}{8}}y^{Q}\frac{\theta_3(\tau,z)}{\eta(\tau)^3},\quad (h>\frac{|Q|}{2}) \tag{A.1}$$

(2) discrete representations:

$$ch^{NS}_{dis}(Q;\tau,z)=q^{\frac{|Q|}{2}-\frac{\hat{c}-1}{8}}y^{Q}\frac{1}{1+y^{sgn(Q)}q^{1/2}}\frac{\theta_3(\tau,z)}{\eta(\tau)^3} \tag{A.2}$$

(3) identity representation:

$$ch^{NS}_{id}(\tau,z)=q^{-\frac{\hat{c}-1}{8}}\frac{1-q}{(1+yq^{1/2})(1+y^{-1}q^{1/2})}\frac{\theta_3(\tau,z)}{\eta(\tau)^3} \tag{A.3}$$

Here $y=e^{2\pi iz}$ and Q denotes the $U(1)$ charge of $\mathcal{N}=2$ algebra.

Extended characters are given by the sum over spectral flow of irreducible characters (1.16):

(1) continuous representations:

$$\chi^{NS}_{cont}(p,\alpha;\tau,z) \;=\; q^{\frac{p^2}{2}}\Theta_{\alpha,N}(\tau,\frac{2z}{N})\frac{\theta_3(\tau,z)}{\eta(\tau)^3},\quad (h=\frac{p^2}{2}+\frac{4\alpha^2+1}{4N}) \tag{A.4}$$

(2) discrete representations:

$$\chi^{NS}_{dis}(s,s+2r;\tau,z) \;=\; \sum_{m\in\mathbb{Z}}\frac{\left(yq^{N(m+\frac{2r+1}{2N})}\right)^{\frac{s-1}{N}}}{1+yq^{N(m+\frac{2r+1}{2N})}}y^{2(m+\frac{2r+1}{2N})}q^{N(m+\frac{2r+1}{2N})^2}\frac{\theta_3(\tau,z)}{\eta(\tau)^3} \tag{A.5}$$

(3) identity representations:

$$\chi^{NS}_{id}(r;\tau,z) \;=\; q^{-\frac{1}{4N}}\sum_{m\in\mathbb{Z}}q^{N(m+\frac{r}{N})^2+N(m+\frac{2r-1}{2N})}y^{2(m+\frac{r}{N})+1} \times\frac{1-q}{\left(1+yq^{N(m+\frac{2r+1}{2N})}\right)\left(1+yq^{N(m+\frac{2r-1}{2N})}\right)}\frac{\theta_3(\tau,z)}{\eta(\tau)^3} \tag{A.6}$$

Here $\Theta_{k,N}(\tau,z)$ is the theta function

$$\Theta_{k,N}(\tau,z) = \sum_{m\in\mathbb{Z}} q^{N(m+\frac{k}{2N})^2} y^{N(m+\frac{k}{2N})} \tag{A.7}$$

Range of parameters r, s are

$$r \in \mathbb{Z}_N\,, \quad 1 \le s \le N, \quad (s \in \mathbb{Z}) \tag{A.8}$$

If we go to the Ramond sector with $(-1)^F$ insertion, one has

$$\chi^{\tilde{R}}_{dis}(s, s+2r;\tau,z) = \sum_{m\in\mathbb{Z}} \frac{\left(yq^{N(m+\frac{2r+1}{2N})}\right)^{\frac{s-1}{N}}}{1-yq^{N(m+\frac{2r+1}{2N})}} y^{2(m+\frac{2r+1}{2N})} q^{N(m+\frac{2r+1}{2N})^2} \frac{i\theta_1(\tau,z)}{\eta(\tau)^3} \tag{A.9}$$

r now takes half-integer values. We find discrete representations $1 \le s \le N$ with $r = -1/2$ carry a non-zero Witten index

$$\chi^{\tilde{R}}_{dis}(s, s-1;\tau, z=0) = -1 \tag{A.10}$$

Now we discuss S-transformation of extended characters. S-transform of continuous representations remains essentially the same as the Fourier transformation. For the sake of brevity we only list the S-transformation of discrete representations.

$$\begin{aligned}
\chi^{NS}_{dis}\left(s, m; -\frac{1}{\tau}, \frac{z}{\tau}\right) &= e^{i\pi\hat{c}\frac{z^2}{\tau}} \left[\frac{1}{\sqrt{2N}} \sum_{m'\in\mathbb{Z}_{2N}} e^{-2\pi i \frac{mm'}{2N}}\right.\\
&\times \int_0^\infty dp' \, \frac{\cosh\left(2\pi\frac{N-(s-1)}{\sqrt{2N}}p'\right) + e^{2\pi i\frac{m'}{2}} \cosh\left(2\pi\frac{s-1}{\sqrt{2N}}p'\right)}{2\left|\cosh\pi\left(\sqrt{\frac{N}{2}}p' + i\frac{m'}{2}\right)\right|^2} \chi^{NS}_{cont}(p', m';\tau,z)\\
&+ \frac{i}{N}\sum_{s'=1}^{N}\sum_{m'\in\mathbb{Z}_{2N}} e^{2\pi i\frac{(s-1)(s'-1)-mm'}{2N}} \chi^{NS}_{dis}(s', m';\tau,z)\\
&\left.- \frac{i}{2N}\sum_{m'\in\mathbb{Z}_{2N}} e^{-2\pi i\frac{mm'}{2N}} \chi^{NS}_{cont}(p'=0, m';\tau,z)\right],
\end{aligned} \tag{A.11}$$

where $m = s + 2r$. Transformation of identity representation is similar. We refer the reader to [3, 20] for the complete argument.

Appendix:B

Let us consider the representation of $\mathcal{N} = 4$ theory at general values of central charge $c = 6k$ where k is an arbitrary positive integer. This theory possesses an affine $SU(2)$ current of level

k which is given by a diagonal sum of level $k-1$ bosonic $SU(2)$ current and level 1 current made of fermion bilinears. When we try to generalize the formula (2.34), (2.35) for a general level, we expect an expansion of the form

$$ch_0^{\tilde{R}}(I=0,\tau,z) = (-1)^k \left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^{2k} + (-1)^k \sum_{j=0}^{k-1} A_{3,j}(\tau)\chi_j^{k-1}(\tau,z)\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}. \quad \text{(B.1)}$$

where χ_j^k denotes the $SU(2)_k$ character for spin $j/2$ representation. It turns out that the expansion coefficients $A_{3,j}$ are given by

$$A_{3,j}(\tau) = \widehat{H}_{j+1}^{(2(k+1))}(\tau) + a_{j+1}^{(k+1)}(\tau), \quad (j=0,1,\cdots,k-1) \quad \text{(B.2)}$$

where

$$\widehat{H}_s^{(k)}(\tau) \equiv \frac{H_s^{(k)}(\tau) - H_{k-s}^{(k)}(\tau)}{\Theta_{s,\frac{k}{2}}(\tau,\tau)},$$
$$H_s^{(k)}(\tau) \equiv \sum_{n\in\mathbb{Z}} \frac{q^{\frac{k}{2}n(n+1)+(n+\frac{1}{2})s}}{1-q^{k(n+\frac{1}{2})}} \quad \text{(B.3)}$$

and $a_{j+1}^{(k+1)}$ is expressed in terms of values of theta functions and $SU(2)$ characters at special points $z = r/2(k+1)$, $(r=1,\cdots,2k+1)$. For details we refer to [8].

Similarly one has the expansion

$$ch_0^{\tilde{R}}(I=0,\tau,z) = (-1)^k \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^{2k} + (-1)^k \sum_{j=0}^{k-1} A_{4,j}(\tau)\chi_j^{k-1}(\tau,z)\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}. \quad \text{(B.4)}$$

We again take the average of (B.1) and (B.4) to achieve the q-expansion with integral powers, and obtain the $\Gamma(2)$-invariant completion

$$\left[ch_0^{\tilde{R}}(I=0,\tau,z)\right]_{inv} = \frac{(-1)^k}{2}\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^{2k} + \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^{2k}\right]. \quad \text{(B.5)}$$

If one recall the relation

$$q^{\frac{k}{4}}y^k ch_0^{NS}(I=\frac{k}{2};\tau,z+\frac{(\tau+1)}{2}) = ch_0^{\tilde{R}}(k,I=0;\tau,z) \quad \text{(B.6)}$$

one finds

$$\left[ch_0^{NS}(I=\frac{k}{2},\tau,z)\right]_{inv} = \frac{1}{2}\left[(-1)^k\left(\frac{\theta_1(\tau,z)}{\theta_3(\tau)}\right)^{2k} + \left(\frac{\theta_2(\tau,z)}{\theta_4(\tau)}\right)^{2k}\right]. \quad \text{(B.7)}$$

Taking the half spectral flow $z \mapsto z - \frac{\tau}{2} - \frac{1}{2}$, we can rewrite $\frac{1}{2}((B.1)+(B.4))$ as

$$\frac{1}{2}\left[(-1)^k\left(\frac{\theta_1(\tau,z)}{\theta_3(\tau)}\right)^{2k}+\left(\frac{\theta_2(\tau,z)}{\theta_4(\tau)}\right)^{2k}\right] = ch_0^{NS}(I=k/2;\tau,z)+\sum_{j=0}^{k-1}\sum_{n=0}^{\infty} b_{j,n}\, ch^{NS}(h_j^{(0)}+n,j;\tau,z),$$

$$h_j^{(0)} \equiv \frac{k}{2}\ (\mathrm{mod}\ \frac{1}{2})\ , \quad h_j^{(0)} \geq \frac{j(j+2)}{4(k+1)}+\frac{k^2}{4(k+1)} \quad \text{('above threshold')}, \tag{B.8}$$

where the coefficients $b_{j,n}$ are defined by the q-expansion

$$q^{h_j^{(0)}-\frac{j(j+2)}{4(k+1)}-\frac{k^2}{4(k+1)}}\sum_{n=0}^{\infty} b_{j,n}q^n = \frac{(-1)^{k-j}}{2}\left(A_{3,k-1-j}(\tau)+A_{4,k-1-j}(\tau)\right), \tag{B.9}$$

and the $\mathcal{N}=4$ massive character is given as

$$ch^{NS}(h,j;\tau,z) \equiv q^{\frac{p^2}{2}}\chi_j^{k-1}(\tau,z)\frac{\theta_3(\tau,z)^2}{\eta(\tau)^3}\ , \quad h \equiv \frac{p^2}{2}+\frac{j(j+2)}{4(k+1)}+\frac{k^2}{4(k+1)}. \tag{B.10}$$

It is remarkable that $b_{j,n}$ are always non-negative integers as in (2.31). We have explicitly checked this for the cases $k=2,3,4$ by Maple.

Let us next study the relation of $\mathcal{N}=4$ character and Appell function. Explicit form of $\mathcal{N}=4$ massless character in Ramond sector is given by [4]

$$ch_0^{\tilde{R}}(k,I=0;\tau,z) = (-1)^{k+1}\frac{i\theta_1(\tau,z)^2}{\eta(\tau)^3\theta_1(\tau,2z)}\sum_{n\in\mathbb{Z}}\frac{1+yq^n}{1-yq^n}q^{(k+1)n^2}y^{2(k+1)n}. \tag{B.11}$$

Thus

$$\begin{aligned} ch_0^{\tilde{R}}(k,I=0;\tau,z) &= (-1)^{k+1}\frac{(i\theta_1(\tau,z))^2}{\eta(\tau)^3}\frac{1}{i\theta_1(\tau,2z)}\left(\mathcal{K}_{2(k+1)}(\tau,z)-\mathcal{K}_{2(k+1)}(\tau,-z)\right) \\ &= (-1)^{k+1}2\frac{i\theta_1(\tau,z)^2}{\eta(\tau)^3\theta_1(\tau,2z)}\widehat{\mathcal{K}}_{2(k+1)}(\tau,z). \end{aligned} \tag{B.12}$$

By comparing (B.12) with (B.5) we obtain the invariant completion of Appell function

$$\left[\widehat{\mathcal{K}}_{2N}(\tau,z)\right]_{inv}(\equiv[\mathcal{K}_{2N}(\tau,z)]_{inv}) \equiv \frac{1}{4}\frac{i\eta(\tau)^3\theta_1(\tau,2z)}{\theta_1(\tau,z)^2}\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^{2(N-1)}+\left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^{2(N-1)}\right]. \tag{B.13}$$

We add a comment: For the special case $N=1$, we find the identity [21]

$$\widehat{\mathcal{K}}_2(\tau,z) = \left[\widehat{\mathcal{K}}_2(\tau,z)\right]_{inv} = \frac{1}{2}\frac{i\eta(\tau)^3\theta_1(\tau,2z)}{\theta_1(\tau,z)^2}. \tag{B.14}$$

Thus $\widehat{\mathcal{K}}_2(\tau, z)$ is already $\Gamma(2)$-invariant without adding any massive term (it further possess the invariance under full modular group). It would be also worthwhile to note the simple relation to the elliptic genus for the conifold as shown in [20];

$$Z_{\text{conifold}}(\tau, z) = -\widehat{\mathcal{K}}_2(\tau, z)\frac{i\theta(\tau, z)}{\eta(\tau)^3} = \frac{1}{2}\frac{\theta_1(\tau, 2z)}{\theta_1(\tau, z)}. \tag{B.15}$$

This is a Jacobi form of $\hat{c} = 3$.

Appendix:C

In this appendix we present a proof of the identity (2.40) due to D. Zagier [28].

Throughout this appendix we fix $\tau \in \mathbb{H}$ (i.e. $\operatorname{Im}\tau > 0$) and $N \geq 2$. Also, for convenience we abbreviate $\theta(z) \equiv \theta_1(\tau, z)/\theta_1'(\tau, 0)$ and $f_i(z) \equiv \theta_i(\tau, z)/\theta_i(\tau, 0)$ $(i = 2, 3, 4)$. We have $\theta(z + a\tau + b) = (-1)^{a+b} q^{-a^2/2} y^{-a}\theta(z)$ and similarly for $f_i(z)$, but with $(-1)^{a+b}$ replaced by $(-1)^b$, 1 or $(-1)^a$ for $i = 2$, 3 or 4, respectively. The identity (2.40) can therefore be rewritten

$$\frac{1}{2N}\sum_{w}\frac{\theta((N-1)w)\,\theta(2w)}{\theta(Nw)\,\theta(w)^3}\, f_i(w)^{2N-2} \quad = \quad (N-1)\,\frac{f_i(z)^2}{\theta(z)^2} \qquad (i = 2,\, 3,\, 4), \tag{C.1}$$

where the sum is over $w = (z + a\tau + b)/N$ with a, $b \in \mathbb{Z}_N$, or more invariantly over $w \in E_\tau = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ with $Nw = z$.

The proposed identity (C.1) is a special case of the more general ones:

$$\frac{1}{2N}\sum_{Nw=z}\frac{\theta((N-1)w)\;\theta(2w)}{\theta(Nw)}\,\theta(w)^{2a-3} f_2(w)^{2b} f_3(w)^{2c} f_4(w)^{2d} = \begin{cases} \dfrac{b\, f_2(z)^2 + c\, f_3(z)^2 + d\, f_4(z)^2}{\theta(z)^2} & \text{if } a = 0 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } a \geq 2 \end{cases} \tag{C.2}$$

for any a, b, c, $d \geq 0$ with $a + b + c + d = N - 1$.

If we write $\wp(z)$ for $\wp(z;\tau)$ and observe that $f_i(z)^2/\theta(z)^2 = \wp(z) - e_i$ where $e_2 \equiv \wp(1/2)$, $e_3 \equiv \wp((\tau+1)/2)$ and $e_4 \equiv \wp(\tau/2)$, then we find that this identity (C.2) follows from (and is in fact equivalent to) the following proposition:

Proposition

For $N \geq 1$, let F_N be the even elliptic function

$$F_N(w) \;=\; \frac{\theta((N-1)w)\,\theta(2w)\,\theta(w)^{2N-5}}{\theta(Nw)}\,,$$

and $P(X) = c_0 X^{N-1} + c_1 X^{N-2} + \mathrm{O}(X^{N-3})$ be a polynomial of degree $\leq N-1$. Then

$$\frac{1}{2N}\sum_{Nw\,=\,z} F_N(w)\,P(\wp(w)) \;=\; (N-1)\,c_0\,\wp(z) \;+\; c_1\,. \tag{C.3}$$

[Proof]

Set $\zeta(z) \equiv \theta'(z)/\theta(z)$. This function satisfies $\zeta(z + a\tau + b) = \zeta(z) - 2\pi i a$ for $a,\, b \in \mathbb{Z}$. If we write the beginning of the Taylor expansion of $\theta(z)$ at 0 as $\theta(z) = z + Az^3 + \mathrm{O}(z^5)$ with $A = A(\tau)$ (A is a multiple of $E_2(\tau)$), then we have $\zeta(z) = z^{-1} + 2Az + \mathrm{O}(z^3)$ and $\zeta'(z) = -z^{-2} + 2A + \mathrm{O}(z^2) = -\wp(z) + 2A$. Fix $z \in \mathbb{C}$ (with $Nz \neq 0$ in E_τ) and define a function $t(w)$ by

$$t(w) \equiv \frac{1}{2}\,\{\zeta(z+Nw) - \zeta(z-Nw)\} \,-\, \zeta((N-1)w) \,-\, \zeta(w)\,.$$

From the transformation law of ζ we find that $t(w)$ is elliptic. Therefore, by the residue theorem, we have

$$\sum_{\alpha\in E_\tau} \mathrm{Res}_{w=\alpha}\big(F_N(w)\,P(\wp(w))\,t(w)\,dw\big) \;=\; 0\,,$$

where the sum is over all singularities $\alpha \in \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ of $F_N(w)P(\wp(w))t(w)$. These singularities occur only at $Nw = \pm z$ or $w = 0$. (The function $F_N(w)$ has further simple poles at $Nw = 0,\ w \neq 0$, but $t(w)$ vanishes at these points, and the function $t(w)$ has simple poles at $(N-1)w = 0,\ w \neq 0$, but $F_N(w)$ vanishes at these points.) Since the residue of $t(w)$ at a point w with $Nw = \pm z$ is $1/2N$ and $F_N(w)P(\wp(w))$ is even, the identity above becomes

$$\frac{1}{N}\sum_{Nw\,=\,z} F_N(w)\,P(\wp(w)) \,+\, \mathrm{Res}_{w=0}\big(F_N(w)\,P(\wp(w))\,t(w)\,dw\big) \;=\; 0\,.$$

But for $w \to 0$ we have

$$\begin{aligned}
F_N(w) \;&=\; \frac{2(N-1)}{N}\,w^{2N-4}\big(1 \,+\, A\,[(N-1)^2 + 2^2 + 2N - 5 - N^2]\,w^2 + \mathrm{O}(w^4)\big)\\
&=\; \frac{N-1}{N}\,w^{2N-4} \,+\, \mathrm{O}(w^{2N})\,,\\
P(\wp(w)) \;&=\; c_0\,\big(\frac{1}{w^2} + \mathrm{O}(w^2)\big)^{N-1} \,+\, c_1\,\big(\frac{1}{w^2} + \mathrm{O}(w^2)\big)^{N-2} \,+\, \mathrm{O}\big(\frac{1}{w^2}\big)^{N-3}\\
&=\; \frac{c_0}{w^{2N-2}} \,+\, \frac{c_1}{w^{2N-4}} \,+\, \mathrm{O}\big(\frac{1}{w^{2N-6}}\big)\,,\\
t(w) \;&=\; N\zeta'(z)\,w \,-\, \frac{1}{(N-1)w} \,-\, 2A(N-1)w \,-\, \frac{1}{w} \,-\, 2Aw \,+\, \mathrm{O}(w^2)\\
&=\; -\,\frac{N}{N-1}\,w^{-1} \,-\, N\wp(z)\,w \,+\, \mathrm{O}(w^2),
\end{aligned}$$

and hence $\mathrm{Res}_{w=0}(F_N(w)\,P(\wp(w))\,t(w)\,dw) = -2(N-1)c_0\wp(z) - 2c_1\,.$ $\square$

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