Amalgamations of factorizations of complete graphs

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Abstract

Let t be a positive integer and let $L = (l_1, \ldots, l_t)$ and $K = (k_1, \ldots, k_t)$ be collections of nonnegative integers. A (t, K, L)-factorization of a graph is a decomposition of the graph into factors $F_1, \ldots, F_t$ such that F_i is k_i-regular and at least l_i-edge-connected. In this paper we apply the technique of amalgamations of graphs to study (t, K, L)-factorizations of complete graphs. In particular, we describe precisely when it is possible to embed a factorization of K_m in a (t, K, L)-factorization of K_n.

keywords: factorizations, embeddings, amalgamations

1 Introduction

A factor of a graph is a subgraph with the same vertex set as the graph. A factorization of a graph is a set of factors with the property that the edge sets of the factors partition the edge set of the graph. In this paper we consider factorizations of complete graphs. Let t be a positive integer and let $K = (k_1, k_2, \ldots, k_t)$ and $L = (l_1, l_2, \ldots, l_t)$ be lists of nonnegative integers. We shall consider factorizations $F_1, \ldots, F_t$ of the complete graph K_n in which, for $1 \leq i \leq t$, F_i is a k_i-regular l_i-edge-connected graph. These are called (t, K, L)-factorizations. Johnstone [8] proved the following result that describes precisely when they exist.

Theorem 1 *A (t, K, L)-factorization of K_n exists if and only if*

(A1) $\displaystyle\sum_{i=1}^{t} k_i = n - 1$,

(A2) *if n is odd, then each k_i is even,*

(A3) *for $1 \leq i \leq t$, $l_i \leq k_i$, and*

(A4) *if $n \geq 3$, $l_i = 0$ if $k_i = 1$.*

Johnstone proved Theorem 1 by constructing the factorizations. At the end of the next section we shall give a proof using *amalgamations*. Many combinatorial problems have been solved using amalgamations; see, for example, [1, 2, 3, 4, 6, 7, 11]. Let us sketch how the technique is used on graph factorizations. Consider a partition of a graph G's vertex set into subsets $V_1, \ldots, V_r$. Then an amalgamation of G has vertex set $V_1, \ldots, V_r$, and for each edge in G joining a pair of vertices in V_i, $1 \leq i \leq r$, there is a loop on V_i in the amalgamation, and for each edge in G joining a vertex in V_i to a vertex in V_j, $1 \leq i < j \leq r$, there is an edge V_iV_j in the amalgamation. (We can think of the amalgamation as being obtained from G by merging vertices that belong to the same subset whilst retaining all edges.)

If G has a factorization, then we can represent it as an edge-colouring with the factors as the colour classes (in this paper we frequently use the equivalence of factorizations and edge-colourings). This colouring can be transferred to an amalgamation of G—each edge of the amalgamation has the same colour as the corresponding edge of G. Henceforth when we refer to an amalgamation we mean a graph with an edge-colouring. Suppose that $G = K_n$ and that it has a (t, K, L)-factorization. Then we can find some properties that an amalgamation of G must possess. For example we can find the number of loops on each vertex, the number of edges between each pair of vertices and the number of edges of each colour incident with each vertex. We call *any* edge-coloured graph that satisfies these properties an *outline* (t, K, L)-factorization of K_n. In Theorem 3 we prove that every outline (t, K, L)-factorization is an amalgamated graph. That is, given an outline graph G we find a (t, K, L)-factorization of K_n of which G is an amalgamation. This will allow us to give a simple proof of Theorem 1

This kind of outline/amalgamation result is a staple of papers on combinatorial amalgamations, but we were not able to apply the standard techniques (such as those used on problems on amalgamations of factorization of graphs in [4, 6, 7, 11]). An innovation of this paper is to show how a new technique for finding factorizations of graphs introduced by Hilton and Johnson [5] can be applied to amalgamations.

In the final section we use the outline/amalgamation result to solve the problem of embedding a factorization of K_m in a (t, K, L)-factorization of K_n. We describe briefly how this will be done. Suppose that we have a factorization (or an edge-colouring) of K_m. Add to it a vertex v. Join v to each vertex of K_m by $(n - m)$ edges and put $\binom{n-m}{2}$ loops on v to form a graph G. Complete the edge-colouring of G by colouring the edges incident with v. (Note that G can be seen to be K_n with $(n - m)$ vertices merged.) If G is an outline (t, K, L)-factorization of K_n, then there is a (t, K, L)-factorization of K_n in which the factorization of K_m is embedded; we can think of this factorization of K_n as being obtained from G by splitting v into

$(n-m)$ vertices. From the properties that define an outline factorization we can work back to find the properties that K_m must possess if it is to be embedded.

2 Amalgamated factorizations

Before we formally define amalgamations we require another definition. Let D and G be graphs. D is a *detachment* of G if there is a bijection ρ: $E(D) \longrightarrow E(G)$ and a surjection σ: $V(D) \longrightarrow V(G)$ such that

- if e is a loop on v in D, then $\rho(e)$ is a loop on $\sigma(v)$ in G,
- if e is an edge joining v and w in D and $\sigma(v) = \sigma(w)$, then $\rho(e)$ is a loop on $\sigma(v)$ in G, and
- if e is an edge joining v and w in D and $\sigma(v) \neq \sigma(w)$, then $\rho(e)$ is an edge joining $\sigma(v)$ and $\sigma(w)$ in G.

We can think of D as being obtained from G by splitting vertices. A detachment is the opposite of an amalgamation, except that we define amalgamations on graphs which have an edge-colouring.

Let t be a positive integer. Let F and H be t-edge-coloured graphs. H is an *amalgamation* of F if there is a bijection ϕ: $E(F) \longrightarrow E(H)$ and a surjection ψ: $V(F) \longrightarrow V(H)$ such that

- if e is a loop coloured i on v in F, then $\phi(e)$ is a loop coloured i on $\psi(v)$ in H,
- if e is an edge coloured i joining v and w in F and $\psi(v) = \psi(w)$, then $\phi(e)$ is a loop coloured i on $\psi(v)$ in H, and
- if e is an edge coloured i joining v and w in F and $\psi(v) \neq \psi(w)$, then $\phi(e)$ is an edge coloured i joining $\psi(v)$ and $\psi(w)$ in H.

We can think of the set of vertices $\{u : u \in V(K_n), \psi(u) = v\}$ as being merged to form v.

Let F_i and H_i be the subgraphs of F and H induced by edges coloured i, $1 \le i \le t$. Then F_i is a detachment of H_i.

Let t, n, K and L be as defined in the Introduction. Suppose that $F = K_n$ is t-edge-coloured and that F_i is k_i-regular and l_i-edge-connected, $1 \le i \le t$, (that is, the edge-colouring gives a (t, K, L)-factorization of K_n). If H is an amalgamation of K_n, then define f: $V(H) \longrightarrow \mathbb{N}$ by

$$f(v) = |\{u : u \in V(K_n), \psi(u) = v\}|.$$

So f counts the vertices that are merged to form v. Together H and f form an *amalgamated* (t, K, L)*-factorization of* K_n.

Theorem 2 *Let H and f be an amalgamated (t, K, L)-factorization of K_n. Then*

(B1) *for all pairs of distinct vertices $v, w \in V(H)$, there are $f(v)f(w)$ edges joining v to w,*

(B2) *for all $v \in V(H)$, there are $\binom{f(v)}{2}$ loops on v,*

(B3) *for all $v \in V(H)$, for $1 \le i \le t$, v is incident with $k_i f(v)$ edges of colour i (counting loops twice),*

(B4) $\sum_{v \in V(H)} f(v) = n$, *and*

(B5) *for $1 \le i \le t$, H_i has an l_i-edge-connected k_i-regular detachment.*

Proof: We know that $f(v)$ vertices in K_n are merged to form v and $f(w)$ vertices are merged to form w. In K_n there are $f(v)f(w)$ edges between these two sets of vertices, and when the vertices are merged these edges join v to w. Hence we obtain (B1).

The subgraph of K_n induced by the $f(v)$ vertices merged to form v is $K_{f(v)}$ and contains $\binom{f(v)}{2}$ edges. When the vertices are merged these edges become loops on v. Hence we obtain (B2).

In F_i the $f(v)$ vertices merged to form v each have degree k_i. The sum of these degrees is the degree of v in H_i. Hence we obtain (B3).

As f counts the number of vertices merged to form each vertex of the amalgamation of K_n and as each vertex of K_n corresponds to exactly one of the vertices of the amalgamation, we obtain (B4).

As we noted before, F_i is an l_i-edge-connected k_i-regular detachment of H_i so (B5) is satisfied. □

A t-edge-coloured-coloured graph H and a function f: $V(H) \longrightarrow \mathbb{N}$ form an *outline* (t, K, L)*-factorization of* K_n if they satisfy (B1) to (B5). By Theorem 2, an amalgamated (t, K, L)-factorization of K_n is an outline (t, K, L)-factorization of K_n. We prove that the converse is true.

Theorem 3 *Let H and f be an outline (t, K, L)-factorization of K_n. Then H and f are an amalgamated (t, K, L)-factorization of K_n.*

Before we prove Theorem 3, we must introduce an important tool first used in [5]. Let a and b be vertices each of degree d in a multigraph G. Let u be a neighbour of a and v be a neighbour of b in G. To (a, b)*-swap* the vertices u and v means to form a new graph from G by deleting the edges au and bv, and adding the edges av and bu. Clearly this manoeuvre leaves the degrees of all the vertices unaltered.

We can find d neighbours of a in G by counting a vertex u as a neighbour of a as many times as there are edges au. An (a, b)*-swap-set* is a collection of d pairs of vertices such that each neighbour of a is the first element of exactly one pair and each neighbour of b is the second element of exactly one pair. We call the pairs (a, b)*-pairs*. The proof of the following lemma uses an argument from [5]

Lemma 4 *If a and b are vertices each of degree d in a l-edge-connected multigraph G, then there exists an (a,b)-swap-set S such that a graph obtained from G by (a,b)-swapping any number of (a,b)-pairs in the swap-set is at least l-edge-connected.*

Proof: First form S. In G we can find l edge-disjoint a–b paths $au_j \cdots v_j b$, $1 \leq j \leq l$. Let (u_j, v_j) be a pair in S. For any edges ab in G not already considered as one of the paths, let (b, a) be a pair in S. Complete S by pairing off the remaining neighbours of a and b arbitrarily.

Consider a graph obtained from G by (a,b)-swapping pairs in S. It contains l edge-disjoint a–b paths since, for $1 \leq j \leq l$, it contains either $au_j \cdots v_j b$ or $bu_j \cdots v_j a$. Now we use induction to prove the lemma. We know that G is l-edge-connected. Suppose that after some number of (a,b)-swaps we have obtained a graph H that is l-edge-connected, and then we (a,b)-swap a further (a,b)-pair (u,v) to obtain a graph J. That is, au and bv are deleted in H and replaced by av and bu to obtain J. If J is not l-edge connected, then we can find a minimal edge-cutset E such that $|E| < l$. We show that H has an edge-cutset of the same size as E, a contradiction. Let C_1 and C_2 be the two connected components of $J - E$. In J there are l edge-disjoint a–b paths so a and b must be in the same component of $J - E$, say C_1. If u and v are also both in C_1, then in $J - E$ we could reverse the (a,b)-swap of u and v to obtain $H - E$ which would also have two components. If u and v are both in C_2, then av and bu must both be in E. Thus $(E \setminus \{av, bu\}) \cup \{au, bv\}$ is an edge-cutset of H. Finally, suppose that u is in C_1 and v is in C_2. Then $av \in E$ and $bu \in C_1$. Let $E' = (E \setminus \{av\}) \cup \{bv\}$ and $C_1' = (C_1 - \{bu\}) \cup \{au\}$. Thus $H - E'$ has two connected components, C_1' and C_2. □

Proof of Theorem 3: Given an outline graph H and f, we find a (t, K, L)-factorization of K_n of which H and f are an amalgamation.

By (B5), for $1 \leq i \leq t$, H_i has an l_i-edge-connected k_i-regular detachment which we denote F_i. In this proof we refer to the subgraphs $H_1, \ldots, H_t$ as *colour classes* and to their detachments $F_1, \ldots, F_t$ as *factors*. Each of the factors is a graph with n vertices so let the vertex set of each factor be $V(K_n)$. Label the vertices of each factor so that for each $v \in H$ the set of vertices formed by the splitting of v when F_i is obtained from H_i is the same for each i, $1 \leq i \leq t$. Let U be a graph on $V(K_n)$ that contains each edge of each factor. Thus U has the same number of edges as K_n. To prove the theorem we show that we can alter the edge sets of some of the factors F_i in such a way that each of the graphs obtained is also an l_i-edge-connected k_i-regular detachment of H_i and the union of the new graphs is K_n.

(As we remarked in the Introduction, this method of proof differs from that used previously in outline/amalgamation theorems on graphs. In the standard proof (see, for example, [4, 6, 11]) the outline graph is "disentangled" by considering in turn each vertex v with $f(v) > 1$. A new graph is obtained by splitting v into two vertices v_1 and v_2 with $f(v_1) = f(v) - 1$ and $f(v_2) = 1$ in such a way that the new graph is also an outline graph. By repetition, an outline graph in which $f(v) = 1$ for every vertex v is obtained. Such a graph is the required factorization.)

Let $V(H) = \{v_1, v_2, \ldots, v_r\}$. Let $V(K_n) = V_1 \cup V_2 \cup \cdots \cup V_r$, where V_j, $1 \leq j \leq r$, is the set of vertices $\{u_{j1}, u_{j2}, \ldots, u_{jf(v_j)}\}$ that was formed by the splitting of the vertex v_j in each H_i. We call these smaller vertex sets *sets of split vertices*. Notice that two subgraphs of K_n are both detachments of the same colour class if and only if for each pair of sets of split vertices V_j and V_z, $1 \leq j \leq z \leq r$, the number of edges that join a vertex in V_j to a vertex in V_z is the same in each subgraph. From the definition of an outline factorization we find that

(B1′) for all pairs of distinct sets of split vertices V_j and V_z, in U there are $f(v_j)f(v_z)$ edges joining vertices in V_j to vertices in V_z, and

(B2′) for all sets of split vertices V_j, there are $\binom{f(v_j)}{2}$ edges in the subgraph of U induced by the vertices of V_j.

Now we alter the factors to obtain a factorization of K_n. Note that we shall refer to F_i before and after each alteration by the same name, and we shall also refer to the altered graphs as factors and define U in terms of the altered graphs. Our aim is to alter the factors so that $U = K_n$. The factors may have loops, and removing them is the first alteration we make. Suppose that there is a loop on a vertex a in F_i. Let V_z be the set of split vertices that contains a. By (B2′), $|V_z| \geq 2$ so there is a vertex $b \in V_z$, $a \neq b$. If there is also a loop on b, then we can delete the loops and replace them with two edges joining a to b. Clearly F_i is still k_i-regular and its edge-connectivity has not decreased. If there is no loop on b, then find l_i disjoint a–b paths (choosing edges ab if possible). We must have $l_i < k_i$ (else there could not be any loops), so we can find an edge bu that is not in one of these a–b paths and $u \neq a$ (as there is a loop on a, and a and b have the same degree, b must be adjacent to a vertex other than a). Delete the loop on a and bu and add edges ab and au. The new graph is k_i-regular and has one fewer loop than the original graph. We must check that the new graph is l_i-edge-connected. If it is not, then there is a minimal edge-cutset E, $|E| < l_i$. Note that E cannot separate a and b since they are joined by l_i disjoint paths. Thus $ab \notin E$. If $au \notin E$, then E is a cutset of the original graph, and if $au \in E$, then $(E \setminus \{au\}) \cup bu$ is a cutset of the original graph; a contradiction since the original graph was l_i-edge-connected.

By repetition we obtain a set of loopless factors. Note that in each case the new factor is still a detachment of the corresponding colour class.

By Lemma 4 for $1 \leq i \leq t$, if a and b are vertices in F_i, then we can find a set $S_i(a,b)$ that is a collection of k_i (a,b)-pairs such that

- each neighbour of a in F_i is the first element of exactly one pair and each neighbour of b is the second element of exactly one pair,
- there are l_i pairs (u_j, v_j) such that there exist in F_i edge-disjoint paths $au_j \cdots v_j b$, $1 \leq j \leq l_i$, and
- for each edge ab in F_i, there is an (a,b)-pair (b,a).

Note that if a and b are in the same set of split vertices V_j, then any graph obtained from a factor F_i by (a, b)-swapping a pair (u, v) in $S_i(a, b)$ is also a detachment of the corresponding colour class H_i since we delete an edge, au, that joins u to a vertex in V_j and replace it with another edge, bu, that also joins u to a vertex in V_j. Similarly for v. Also by Lemma 4, any graph obtained from F_i by (a, b)-swapping pairs in $S_i(a, b)$ is l_i-edge-connected. So if we alter the factors using only (a, b)-swaps for pairs of vertices a and b in the same set of split vertices and we obtain a (t, K, L)-factorization of K_n, then H and f will be an amalgamation of this factorization. We show how this is done.

There are two further stages to the proof. We will often say informally that two disjoint sets of vertices V and V' are joined by the *correct* number of edges if they are joined by $|V||V'|$ edges, that is, the number of edges between them in K_n. In the next stage of the proof we alter the factors so that each vertex is joined the correct number of times to each set of split vertices. That is, we alter the factors so that they satisfy

(C1) in U, for $1 \leq j \leq r$, $1 \leq h \leq f(v_j)$, u_{jh} is joined by $f(v_j) - 1$ edges to vertices of V_j and, for $1 \leq z \leq r$, $z \neq j$, by $f(v_z)$ edges to vertices in V_z.

We then complete the proof by further altering the edge sets of the factors so that

(C2) in U each pair of distinct vertices is joined by exactly one edge.

In other words, $U = K_n$.

First we alter the factors so that (C1) is satisfied. For any vertex $a \in V$, for $1 \leq j \leq r$,

- let $p(a, V_j)$ be the number of edges in U that join a to a vertex in the set of split vertices V_j,
- let $q(a, V_j) = f(v_j)$ if $a \notin V_j$, let $q(a, V_j) = f(v_j) - 1$ if $a \in V_j$.

That is, $q(a, V_j)$ is the number of edges that will join a to vertices in V_j in U when $U = K_n$. Thus to satisfy (C1) we must alter the factors so that for each vertex $a \in V(K_n)$, for $1 \leq j \leq r$, $p(a, V_j) = q(a, V_j)$. Let the *set-discrepancy* δ_s be defined by

$$\delta_s = \sum_{a \in V(K_n)} \sum_{j=1}^{r} |p(a, V_j) - q(a, V_j)|.$$

(C1) is satisfied when the set-discrepancy is zero. We describe a method that will reduce the set-discrepancy if it is greater than zero. By applying it repeatedly we obtain a set of factors that satisfies (C1).

Let j and z be fixed. By (B1′) and (B2′) each pair of sets of split vertices is joined by the correct number of edges. Thus

$$\sum_{a \in V_z} p(a, V_j) = \sum_{a \in V_z} q(a, V_j). \tag{1}$$

If the set-discrepancy is greater than zero, then for some vertex a and some z_1, $p(a, V_{z_1}) \neq q(a, V_{z_1})$. We can assume that

$$p(a, V_{z_1}) > q(a, V_{z_1}), \tag{2}$$

since by (1) this implies, and is implied by, the existence of a vertex b in the same set of split vertices as a such that

$$p(b, V_{z_1}) < q(b, V_{z_1}). \tag{3}$$

Using the sets $S_i(a, b)$, $1 \leq i \leq t$, we create a further set, $S(a, b)$. For $1 \leq i \leq t$, if $(c, d) \in S_i(a, b)$, then $(i, c, d) \in S(a, b)$. So $S(a, b)$ contains ordered triples (i, c, d) where c is a neighbour of a and d is a neighbour of b in F_i. Note that there is an obvious one-to-one relationship between the triples of $S(a, b)$ and the neighbours, over all the factors, of a, and also between the triples of $S(a, b)$ and the neighbours, over all the factors, of b.

Claim 5 *There is a sequence of sets of split vertices*

$$\Gamma = V_{z_1}, V_{z_2}, \ldots, V_{z_m}$$

such that

(D1) $V_{z_\alpha} \neq V_{z_\beta}$ *if* $\alpha \neq \beta$,

(D2) *either* $p(a, V_{z_m}) < q(a, V_{z_m})$ *or* $p(b, V_{z_m}) > q(b, V_{z_m})$, *and*

(D3) *for* $2 \leq j \leq m$, *there is a triple* $(i_j, c_j, d_j) \in S(a, b)$ *where* $c_j \in V_{z_{j-1}}$ *and* $d_j \in V_{z_j}$.

The claim is proved below. First we use it to reduce δ_s. For $2 \leq j \leq m$, we (a, b)-swap c_j and d_j in F_{i_j}: the edges ac_j and bd_j are deleted and replaced with the edges ad_j and bc_j. Each new factor F_i obtained is clearly k_i-regular and, by Lemma 4, it is l_i-edge-connected. It is also a detachment of the corresponding colour class H_i since the number of edges in the factor between each pair of sets of split vertices does not change.

For $2 \leq j \leq m - 1$, an edge from a to a vertex, c_{j+1}, that is in V_{z_j}, has been deleted and an edge from a to a vertex, d_j, that is in V_{z_j} has been added. Thus $p(a, V_{z_j})$ is unchanged. Similarly $p(b, V_{z_j})$, $2 \leq j \leq m - 1$, is unchanged.

The only neighbour of a in V_{z_1} involved in an (a, b)-swap is c_2. The edge ac_2 is deleted so $p(a, V_{z_1})$ is reduced by 1. Hence, by (2), δ_s is also reduced by 1. The addition of bc_2 causes $p(b, V_{z_1})$ to increase by 1, so by (3), δ_s decreases further by 1.

The only neighbour of b in V_{z_m} involved in an (a, b)-swap is d_m. Consider (D2). If $p(a, V_{z_m}) < q(a, V_{z_m})$, then the addition of ad_m causes $p(a, V_{z_m})$ to increase by 1, and δ_s is reduced further by 1. The deletion of bd_m may cause δ_s to increase by 1, but at worst δ_s is reduced by 2 overall. The only other possibility is that $p(b, V_{z_m}) > q(b, V_{z_m})$, and by a similar argument δ_s is reduced overall by at least 2 in this case also.

We show that the factors remain loopless. A loop is put on a only if one of the triples is (i_j, c_j, d_j) with $d_j = a$. That is, (c_j, a) is a pair in $S_{i_j}(a, b)$. Recall that a is the second element of a pair in $S_i(a, b)$ only if b is the first element. But if $c_j = b$, then c_j and d_j are in the same set of split vertices, a contradiction by (D1) and (D3). By a similar argument b also remains loopless.

Proof of Claim 5: In fact we shall prove that there is a sequence of sets of split vertices

$$\Delta = V_{g_1}, V_{g_2}, \ldots, V_{g_{m'}}$$

such that

(E1) $V_{g_1} = V_{z_1}$,

(E2) $V_{g_\alpha} \neq V_{g_\beta}$ if $\alpha \neq \beta$,

(E3) either $p(a, V_{g_{m'}}) < q(a, V_{g_{m'}})$ or $p(b, V_{g_{m'}}) > q(b, V_{g_{m'}})$, and

(E4) for $2 \leq j \leq m'$, there is a triple $(i_j, c_j, d_j) \in S(a, b)$ where $c_j \in V_{g_h}$ for some $h \in \{1, 2, \ldots, j-1\}$ and $d_j \in V_{g_j}$.

It is easy to see that Δ has a subsequence that has $V_{g_1} = V_{z_1}$ as the first term and satisfies (D1), (D2) and (D3). (Let $V_{g_{m'}}$ be the final term and work backwards. If V_{g_α} is the last term reached, then if $\alpha = 1$ the subsequence is found. Otherwise there is a triple $(i_\alpha, c_\alpha, d_\alpha)$. Let the previous term of the sequence be the set of split vertices V_{g_β} that contains c_α. As $\beta < \alpha$ we must eventually get back to V_{g_1}.)

We find Δ. The first term $V_{g_1} = V_{z_1}$ was found before the claim was stated. Suppose that we have found the first ω terms, and that this sequence of ω terms satisfies (E1), (E2) and (E4) with $m' = \omega$. If for any $\alpha \in \{1, 2, \ldots, \omega\}$

$$\begin{aligned} p(a, V_{g_\alpha}) &< q(a, V_{g_\alpha}), \text{ or} \\ p(b, V_{g_\alpha}) &> q(b, V_{g_\alpha}), \end{aligned}$$

then we pick the smallest such α and let $\Delta = V_{g_1}, V_{g_2}, \ldots, V_{g_\alpha}$ as this also satisfies (E3). Otherwise, for $1 \leq j \leq \omega$,

$$p(a, V_{g_j}) \geq q(a, V_{g_j}), \tag{4}$$

$$p(b, V_{g_j}) \leq q(b, V_{g_j}). \tag{5}$$

Let $W = V_{g_1} \cup V_{g_2} \cup \cdots \cup V_{g_\omega}$. As a and b are in the same set of split vertices, $q(a, V_j) = q(b, V_j)$, $1 \leq j \leq r$. Therefore, by (2) and (3), a has more neighbours than b in V_{g_1} and, by (4) and (5), a has at least as many neighbours as b in V_{g_j}, $2 \leq j \leq \omega$. Therefore over all the factors a has more neighbours than b in W. Recall that in $S(a, b)$ there is a triple corresponding to each neighbour of a in each factor; similarly there is a triple corresponding to each neighbour of b. So there is a triple $(i_{\omega+1}, c_{\omega+1}, d_{\omega+1}) \in S(a, b)$, such that $c_{\omega+1} \in W$ and $d_{\omega+1} \notin W$. Let the set of split vertices containing $d_{\omega+1}$ be $V_{g_{\omega+1}}$. Then $V_{g_{\omega+1}} \neq V_{g_j}$, $1 \leq j \leq \omega$, since $V_{g_{\omega+1}} \not\subset W$.

We must eventually find a set of split vertices that satisfies (E3): note that

$$\sum_{j=1}^{r} p(a, V_j) = \sum_{j=1}^{r} q(a, V_j), \tag{6}$$

since both sums are equal to $n - 1$, the sum of the degrees of a taken over all the factors. As $p(a, V_{z_1}) > q(a, V_{z_1})$, there is at least one set of split vertices V_z such that $p(a, V_z) < q(a, V_z)$ and therefore V_z, at least, satisfies (E3). This completes the proof of Claim 5. □

We must now show that when (C1) is satisfied we can further alter the factors so that (C2) is also satisfied. For a pair of distinct vertices a and c, let $p(a, c)$ be the number of edges in U from a to c. Note that $p(a, c) = p(c, a)$.

Let the *vertex-discrepancy* δ_v be defined by

$$\delta_v = \sum_{ac \in E(K_n)} |p(a, c) - 1|.$$

If (C2) is satisfied, then for all pairs of distinct vertices a and c, $p(a, c) = 1$, and the vertex-discrepancy is zero. We describe a method that will reduce the vertex-discrepancy if it is greater than zero. By applying it repeatedly we shall obtain a set of factors that satisfies (C2).

We can see that if c is the only vertex in a set of split vertices V_z, then $p(a, c) = 1$: let a be some other vertex; as (C1) is satisfied, $p(a, V_z) = q(a, V_z) = f(v_z) = 1$, and as $p(a, c) = p(a, V_z)$, we already have $p(a, c) = 1$.

Claim 6 *Suppose that a and b are vertices in the same set of split vertices, that $c_1 \notin \{a, b\}$ and that*

$$p(a, c_1) > 1, \quad (7)$$
$$p(b, c_1) < 1. \quad (8)$$

Let $S(a, b)$ be defined as before. There is a sequence of vertices $c_1, c_2, \ldots, c_m$ such that

(F1) *$c_j \notin \{a, b\}$, $2 \leq j \leq m$,*

(F2) *$c_\alpha \neq c_\beta$ if $\alpha \neq \beta$,*

(F3) *either $p(a, c_m) < 1$ or $p(b, c_m) > 1$, and*

(F4) *for $1 \leq j \leq m-1$ there is a triple $(i_j, c_j, c_{j+1}) \in S(a, b)$.*

Proof: The first term of the sequence is known by the hypothesis. Suppose that we have found the first ω terms and that this sequence satisfies (F1), (F2) and (F4) with $m = \omega$. If for some $h \in \{1, 2, \ldots, \omega\}$

$$p(a, c_h) < 1, \text{ or}$$
$$p(b, c_h) > 1,$$

then choose the smallest such h and let the complete sequence be $c_1, c_2, \ldots, c_h$ since this also satisfies (F3) with $m = h$. Otherwise, for $1 \leq j \leq \omega$,

$$p(a, c_j) \geq 1,$$
$$p(b, c_j) \leq 1.$$

As $p(a, c_\omega) \geq 1$ we can find a triple $(i_\omega, c_\omega, c_{\omega+1}) \in S(a, b)$. As there are no loops and $c_{\omega+1}$ is a neighbour of b, $c_{\omega+1} \neq b$. By (F1), $c_\omega \neq b$ and a is the second element of a pair in $S_{i_\omega}(a, b)$ only if b is the first element, so $c_{\omega+1} \neq a$. By (8), $p(b, c_1) = 0$, so $c_{\omega+1} \neq c_1$. As $p(b, c_j) \leq 1$, $2 \leq j \leq \omega$, there is at most one triple in $S(a, b)$ with c_j as the third element and we have already found one such triple (namely (i_{j-1}, c_{j-1}, c_j)). Therefore $c_{\omega+1} \neq c_j$, $2 \leq j \leq \omega$.

The sequence must terminate: there is a finite number of vertices and it is easily seen that $p(a, c_1) > 1$ implies that for some vertex c, $p(a, c) < 1$ (that is, if a vertex a is joined too many times to one vertex, then it must be joined too few times to some other vertex). This completes the proof of Claim 6. □

We describe how to use the claim to reduce the vertex discrepancy. First choose a set of split vertices V_z such that

(C1a) for every vertex $c \notin V_z$, $p(c, V_j) = q(c, V_j)$, $1 \leq j \leq r$.

Note that (C1) implies (C1a) so initially we can choose any set of split vertices as V_z. If possible choose a pair of vertices $a \in V_z$, $c_1 \notin V_z$ that satisfy (7). By (C1a) there is a vertex $b \in V_z$, $a \neq b$, that satisfies (8). Therefore, by Claim 6, there is a sequence of vertices $c_1, c_2, \ldots, c_m$ that satisfies (F1) to (F4). For $1 \leq j \leq m-1$, (a, b)-swap (c_j, c_{j+1}) in F_{i_j}. For $2 \leq j \leq m-1$, we add ac_j to $F_{i_{j-1}}$ and delete ac_j from F_{i_j}, so $p(a, c_j)$ is unchanged. Similarly $p(b, c_j)$ is unchanged, $2 \leq j \leq m-1$. By (7), the deletion of ac_1 reduces δ_v by 1, and, by (8), the addition of bc_1 reduces δ_v further by 1. By (F3), the addition of ac_m and the deletion of bc_m at worst has no net effect on δ_v. So overall δ_v is reduced by at least 2. As $c_j \notin \{a, b\}$, $1 \leq j \leq m$, no loops are created.

Consider the effect of these (a, b)-swaps on the set discrepancy. Let V_{z_j} be the set of split vertices that contains c_j, $1 \leq j \leq m$. For $2 \leq j \leq m-1$, $p(a, c_j)$ and $p(b, c_j)$ were unchanged so $p(a, V_{z_j})$ and $p(b, V_{z_j})$ are unchanged. Note that

$$p(a, V_{z_1}) \text{ and } p(b, V_{z_m}) \text{ are reduced by 1, and} \tag{9}$$

$$p(a, V_{z_m}) \text{ and } p(b, V_{z_1}) \text{ are increased by 1.} \tag{10}$$

As $a, b \in V_z$, (C1a) remains satisfied. So we can look for further pairs $a \in V_z$, $c_1 \notin V_z$ that satisfy (7) and repeat the procedure. When no such pairs remain we have $p(a,c) = 1$ for every $a \in V_z$, $c \notin V_z$. For $1 \le j \le r$, $j \ne z$, $p(a, V_j) = \sum_{c \in V_j} p(a,c) = |V_j|$. Thus $p(a,V_j) = q(a,V_j)$, $1 \le j \le r$, $j \ne z$. By (6), this implies that $p(a, V_z) = q(a, V_z)$ also. Thus

(C1b) for every vertex $a \in V_z$, $p(a,V_j) = q(a,V_j)$, $1 \le j \le r$.

Note that (C1a) and (C1b) together imply (C1).

Now if possible choose a pair $a \in V_z$, $c \in V_z$ that satisfies (7). By (C1b), there is a vertex $b \in V_z$ that satisfies (8), so we can use the claim and the method of (a,b)-swapping just described to reduce δ_v. Note that $V_{z_1} = V_z$ (since V_{z_1} is the set that contains c_1) and that $V_{z_m} = V_z$ (since V_{z_m} is the set that contains c_m, c_m satisfies (F3) and we know that $p(a,c) = 1$ for all $a \in V_z$, $c \notin V_z$). Thus (9) and (10) cancel each other out and (C1a) and (C1b) remain satisfied. Look for further pairs $a, c_1 \in V_z$ that satisfy (7) and reduce δ_v further. When no such pairs remain since (C1a) and (C1b) are satisfied, (C1) is satisfied and we can begin the process again with another choice of V_z. Eventually δ_v is reduced to zero and (C2) is satisfied. This completes the proof of Theorem 3 □

Proof of Theorem 1: The following four sentences prove the necessity of (A1) to (A4). The sum, taken over all the factors, of the degrees of a vertex is equal to its degree in K_n. By the handshaking lemma, a regular graph with an odd number of vertices has even degree. As the set of all edges incident with a vertex forms an edge-cutset of a graph, the edge-connectivity of a graph is at most its minimum degree. A 1-regular graph is a set of independent edges and is not connected if it has more than 2 vertices.

By Theorem 2, we can show that the conditions are sufficient by finding an outline (t, K, L)-factorization, H and f. Let $V(H) = \{v\}$. Let there be $\binom{n}{2}$ loops on v, and let $nk_i/2$ of the loops be coloured i, $1 \le i \le t$. Let $f(v) = n$. It is easy to check that H and f satisfy (B1) to (B5). □

3 Embedding factorizations

Here we answer this question: when can a factorization $G_1, \ldots, G_t$ of K_m be embedded in a (t, K, L)-factorization $F_1, \ldots F_t$ of K_n. By embed we mean that the vertices of K_m are identified with m of the vertices of K_n in such a way that G_i is a subgraph of F_i, $1 \leq i \leq t$. Note that we can think of $G_1, \ldots, G_t$ as the colour classes of a t-edge-colouring of K_m. In some cases a solution to the embedding problem is known. When each $l_i = 0$, that is, when there is no constraint on the connectivity of the factors, the solution was found by Andersen and Hilton [2] (and independently by Rodger and Wantland [11]). A solution in the case where each $l_i = 1$ was found by Hilton, Johnson, Rodger and Wantland [6]. Solutions when each $l_i = 2$ are also known: Hilton [4] solved the subcase where each $k_i = 2$, and this was generalized by Rodger and Wantland [11] (it also follows from a result of Nash-Williams [9]). Below in Theorem 8 we solve the general case where t, K and L are required to satisfy (G1) to (G4).

First we need a result about detachments. Recall that, by (B5), if H is a (t, K, L)-outline factorization, then each H_i (the subgraph induced by edges coloured i) has an l_i-edge-connected k_i-regular detachment. Theorem 7 is a result of Nash-Williams that characterizes those graphs that have such detachments (in fact, this is a specialization of a much more general result). We need some definitions first. Let G be a graph of which we seek to find a detachment. We define three functions f, c, e: $\mathcal{P}(V(G)) \longrightarrow \mathbb{Z}$, ($\mathcal{P}(V(G))$ is the power set of $V(G)$). For each set of vertices $V \subseteq V(G)$, let $f(V)$ be the total number of vertices we wish to split the vertices of V into, let $c(V)$ be the number of components in $G - V$, and let $e(V)$ be the number of edges (including loops) that are incident with at least one vertex in V (loops and edges incident twice with vertices in V are only counted once).

Theorem 7 [10] *Let k and l be nonnegative integers. Let G be a graph in which the degree of each vertex is a multiple of k. Then G has an l-edge-connected k-regular detachment if and only if*

(X1) *G is l-edge-connected,*

(X2) *if $l = 1$, then for all $V \subseteq V(G)$, $f(V) + c(V) \leq e(V) + 1$,*

(X3) *if l is odd and $l = k$, then G has no cutvertex with degree $2l$, and*

(X4) *if l is odd and $l = k$, then G is not a loopless graph that contains exactly two vertices each with degree $2l$.* □

We need some more definitions before we state the embedding result. Let ω_i be the number of connected components of G_i and let these components be $C_{i,1}, C_{i,2}, \ldots, C_{i,\omega_i}$. Let $\varepsilon_{i,j} = \sum_{v \in V(C_{i,j})} k_i - d_{G_i}(v)$, and let $\varepsilon_i = \sum_{j=1}^{\omega_i} \varepsilon_{i,j}$. Let $r_{i,j}$ be the number of minimal edge-cutsets of $C_{i,j}$ that contain fewer than l_i edges, let these sets be $E_1^{i,j}, E_2^{i,j}, \ldots, E_{r_{i,j}}^{i,j}$, and let $C_{m_1}^{i,j}$ and $C_{m_2}^{i,j}$ be the connected components of $C_{i,j} - E_m^{i,j}$. Let $\varepsilon_{i,j,m_p} = \sum_{v \in V(C_{m_p}^{i,j})} k_i - d_{G_i}(v)$.

Theorem 8 *Let n, t, K and L satisfy* (G1) *to* (G4) *and let $\alpha = n - m$. A t-edge-coloured K_m can be embedded in a (t, K, L)-factorization of K_n if and only if*

(I) *$d_{G_i}(v) \leq k_i$ for each $v \in V(K_m)$, for $1 \leq i \leq t$,*

(II) *$\varepsilon_{i,j} \geq l_i$ for $1 \leq i \leq t$, $1 \leq j \leq \omega_i$,*

(III) *$\alpha \geq \max\{\varepsilon_i / k_i : 1 \leq i \leq t\}$,*

(IV) *for $1 \leq i \leq t$, if $l_i = 1$, then $\alpha \geq \dfrac{2\omega_i - \varepsilon_i - 2}{k_i - 2}$,*

(V) *for $1 \leq i \leq t$, if $l_i = k_i$, l_i is odd and $\omega_i \geq 2$, then $\alpha \neq 2$, and*

(VI) *$\varepsilon_{i,j,m_p} \geq l_i - |E_m^{i,j}|$, for $1 \leq i \leq t$, $1 \leq j \leq \omega_i$, $1 \leq m \leq r_{i,j}$, $1 \leq p \leq 2$.*

Proof: Necessity: suppose that a t-edge-coloured K_m is embedded in an (t, K, L)-factorization of K_n (so each G_i is a subgraph of a k_i regular l_i-edge-connected graph F_i). We show that the conditions of the theorem hold.

For $1 \le i \le t$, as G_i is a subgraph of a k_i-regular graph, $d_{G_i}(v) \le k_i$ for each $v \in V(K_m)$. So (I) holds.

By definition, $\varepsilon_{i,j}$ is the number of edges incident with vertices of $C_{i,j}$ in $E(F_i) \setminus E(G_i)$. These edges all join $C_{i,j}$ to $V(K_n) \setminus V(K_m)$ and form an edge-cutset so there must be at least l_i of them. So (II) holds.

Similarly, ε_i is the number of edges incident with vertices of G_i in $E(F_i) \setminus E(G_i)$, and all these edges join G_i to one of the α vertices of $V(K_n) \setminus V(K_m)$ which each have degree k_i. Thus $\varepsilon_i \le k_i \alpha$. So (III) holds.

If $l_i = 1$, then from F_i form a graph J by merging vertices that belong to the same component in G_i and deleting any loops on these merged vertices. Thus J contains the α vertices of $V(K_n) \setminus V(K_m)$ plus ω_i vertices corresponding to the ω_i components of G_i. Its edge set contains ε_i edges corresponding to the ε_i edges in F_i that join vertices of $V(K_m)$ to vertices of $V(K_n) \setminus V(K_m)$. It also contains the edges of F_i joining pairs of vertices in $V(K_n) \setminus V(K_m)$; there are $(\alpha k_i - \varepsilon_i)/2$ such edges since there are α vertices with degree k_i and all but ε_i of the sum of their degrees is due to edges joining pairs of these vertices. As J is connected we must have that $|V(J)| \le |E(J)| + 1$. Thus $\alpha + \omega_i \le \varepsilon_i + (\alpha k_i - \varepsilon_i)/2 + 1$. Rearranging we see that (IV) holds.

Suppose that $\alpha = 2$ and $l_i = k_i$ is odd. If $\omega_i \ge 2$, then $C_{i,1}$ and $C_{i,2}$ are two components of G_i. There must be k_i distinct paths from $C_{i,1}$ to $C_{i,2}$ which each go through $V(K_n) \setminus V(K_m) = \{w_1, w_2\}$. We can assume that at least $\left\lceil \frac{k_i}{2} \right\rceil$ of these paths contain w_1. But then $d_{F_i}(w_1) \ge 2\left\lceil \frac{k_i}{2} \right\rceil = k_i + 1$, a contradiction. So (V) holds.

For $1 \le i \le t$, $1 \le j \le \omega_i$, $1 \le m \le r_{i,j}$, there must be l_i distinct paths from $C^{i,j}_{m_1}$ to $C^{i,j}_{m_2}$. We know that $|E^{i,j}_m|$ of these paths are in $C_{i,j}$. The remainder must go through $V(K_n) \setminus V(K_m)$. Therefore there must be at least $l_i - |E^{i,j}_m|$ edges from each of $C^{i,j}_{m_1}$ and $C^{i,j}_{m_2}$ to $V(K_n) \setminus V(K_m)$. So (VI) holds as ε_{i,j,m_p} is the number of edges incident with vertices of $C^{i,j}_{m_p}$ in $E(F_i) \setminus E(G_i)$.

Sufficiency: to complete the proof we must show we can find an embedding if the six conditions hold. From K_m we form H and f, an outline

(t, K, L)-factorization of K_n. Let $V(H) = V(K_m) \cup \{v_0\}$. Let $f(v_0) = \alpha$, let $f(v) = 1$ for all $v \in V(K_m)$. The edge set of H contains the edges of K_m (which are already coloured) and

- for $1 \le i \le t$, for each $v \in V(K_m)$, there are $k_i - d_{G_i}(v)$ edges coloured i from v_0 to v, and

- for $1 \le i \le t$, there are $(\alpha k_i - \varepsilon_i)/2$ loops coloured i on v_0.

If we can prove that H and f are an outline (t, K, L)-factorization of K_n, then the proof is completed by applying Theorem 2 since any (t, K, L)-factorization $F_1, \dots, F_t$ of K_n of which H and f is an amalgamation is such that G_i is a subgraph of F_i.

First we check that the number of loops added of each colour is an integer. As $\alpha = n - m$,

$$\begin{aligned}\frac{\alpha k_i - \varepsilon_i}{2} &= \frac{(n-m)k_i - \varepsilon_i}{2} \\ &= \frac{k_i n}{2} - \varepsilon_i - \frac{k_i m - \varepsilon_i}{2}\end{aligned}$$

which is an integer since $k_i n$ is even (by (G2)) and $(k_i m - \varepsilon_i)/2 = |E(G_i)|$.

We must show that H and f satisfy (B1) to (B5).

For $v, w \in V(K_m)$, there is $1 = f(v)f(w)$ edge joining v to w. For $v \in V(K_m)$, the number of edges from v to v_0 is

$$\begin{aligned}\sum_{i=1}^{t}(k_i - d_{G_i}(v)) &= \sum_{i=1}^{t} k_i - \sum_{i=1}^{t} d_{G_i}(v) \\ &= (n-1) - (m-1) \\ &= \alpha \\ &= f(v)f(v_0).\end{aligned}$$

So (B1) is satisfied.

For $v \in V(K_m)$ there are $0 = \binom{f(v)}{2}$ loops on v. The number of loops on v_0 is

$$\sum_{i=1}^{t} \frac{\alpha k_i - \varepsilon_i}{2} = \sum_{i=1}^{t} \frac{\alpha k_i}{2} - \sum_{i=1}^{t} \sum_{v \in V(K_m)} \frac{k_i - d_{G_i}(v)}{2}$$

by the definition of ε_i. The order in which we evaluate the double sum is not important, and

$$\begin{aligned}
\sum_{i=1}^{t}\frac{\alpha k_i}{2} - \sum_{v\in V(K_m)}\sum_{i=1}^{t}\frac{k_i - d_{G_i}(v)}{2} &= \frac{\alpha(n-1)}{2} - \sum_{v\in V(K_m)}\frac{(n-1)-(m-1)}{2}\\
&= \frac{\alpha(n-1)}{2} - \sum_{v\in V(K_m)}\frac{\alpha}{2}\\
&= \frac{\alpha(n-1)}{2} - \frac{\alpha m}{2}\\
&= \frac{\alpha(n-1-m)}{2}\\
&= \frac{\alpha(\alpha-1)}{2}\\
&= \binom{\alpha}{2}\\
&= \binom{f(v_0)}{2}.
\end{aligned}$$

So (B2) is satisfied.

For $v \in V(K_m)$ there are $d_{G_i}(v) + (k_i - d_{G_i}(v)) = k_i = k_i f(v)$ edges of each colour incident with v. The number of edges of each colour incident with v_0 is

$$\begin{aligned}
\left(\sum_{v\in V(K_m)}(k_i - d_{G_i}(v))\right) + \alpha k_i - \varepsilon_i &= \varepsilon_i + \alpha k_i - \varepsilon_i\\
&= \alpha k_i\\
&= k_i f(v_0).
\end{aligned}$$

So (B3) is satisfied.

As $\sum_{v\in V(H)} f(v) = m + \alpha = m + n - m = n$, (B4) is satisfied.

Finally to show that (B5) is satisfied we must show that each H_i has an l_i-edge-connected k_i-regular detachment. Thus we must show that each H_i satisfies (X1) to (X4).

First we show that each H_i is l_i-edge-connected. Suppose that H_i is not l_i-edge-connected. Then there is a minimal edge-cutset E such that $|E| < l_i$. As E is minimal it will contain only edges from one component of G_i, say $C_{i,1}$, and perhaps also edges from v_0 to $C_{i,1}$. It cannot contain only edges from v_0 to $C_{i,1}$ since it would need to contain all of them and there are $\sum_{v\in V(C_{i,j})} (k_i - d_{G_i}(v)) = \varepsilon_{i,j}$ such edges and, by (II), $\varepsilon_{i,j} \geq l_i$. The edges of E contained in $C_{i,1}$ form one of its minimal separating sets, say $E_1^{i,1}$, and we can assume that the two components of $H_i - E$ are $C_{1_1}^{i,1}$ and $H_i - C_{1_1}^{i,1}$. Therefore E must also contain all the edges from $C_{1_1}^{i,1}$ to v_0. There are $\sum_{v\in V(C_{i,1}^{1_1})} (k_i - d_{G_1}(v)) = \varepsilon_{i,1,1_1}$ such edges. So

$$\begin{aligned} |E| &= |E_1^{i,1}| + \varepsilon_{i,1,1_1} \\ &\geq l_i, \end{aligned}$$

by (VI), a contradiction. So each H_i satisfies (X1).

We show that (X2) is satisfied. First consider $V \subseteq V(H_i)$ such that $v_0 \notin V$. Thus $f(V) = |V|$. From H_i form a graph J by merging vertices that belong to the same component of $H_i - V$ and deleting any loops on these merged vertices. Thus J has $f(V) + c(V)$ vertices and as it is connected,

$$\begin{aligned} f(V) + c(V) = |V(J)| &\leq |E(J)| + 1 \\ &\leq e(V) + 1. \end{aligned}$$

So (X2) is satisfied in this case. Now let $V = \{v_0\}$. So $f(v) = \alpha$, $c(V) = \omega_i$ and $e(V) = \varepsilon_i + (k_i\alpha - \varepsilon_i)/2$. Then (X2) can be shown to hold by rearranging the inequality given in (IV). If $\{v_0\} \subset V$, then label the other vertices of V so that $V = \{v_0, v_1, \ldots, v_s\}$. We have just seen that $\{v_0\}$ satisfies (X2) so we can show that V satisfies (X2) by proving that if $V' = \{v_0, v_1, \ldots, v_\sigma\}$, $\sigma < s$, satisfies (X2), then so does $V'' = \{v_0, v_1, \ldots, v_{\sigma+1}\}$. This is done by examining how f, c and e change when the argument V' is replaced by V''. The change in f is clearly $+1$. Let C be the component of $J - V'$ containing

$v_{\sigma+1}$, and let x be the number of components of $C - v_{\sigma+1}$. So the change in c is $+(x-1)$. As $v_{\sigma+1}$ is joined by at least one edge to each of the x components of $C - v_{\sigma+1}$, there at least x edges incident with V'' but not with V'. So the change in e is at least $+x$. So e increases by at least as much as $f + c$. Thus (X2) remains satisfied.

The only vertex that can have degree $2k_i$ in H_i is v_0. It is a cutvertex if G_i has more than one component. By (V), if $l_i = k_i$ and l_i is odd, then $\alpha \neq 2$ and so $d_{H_i}(v_0) \neq 2l_i$. So (X3) is satisfied.

Finally, (X4) is satisfied since each H_i contains only one vertex with degree greater than k_i. □

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