Noncommutative Planar Particle Dynamics with Gauge Interactions

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Abstract

We consider two ways of introducing minimal Abelian gauge interactions into the model presented in [1]. These two approaches are different only if the second central charge of the planar Galilei group is nonzero. One way leads to the standard gauge transformations and the other one to a generalized gauge theory with gauge transformations accompanied by time-dependent area-preserving coordinate transformations. Both approaches, however, are related to each other by a classical Seiberg-Witten map supplemented by a noncanonical transformation of the phase space variables for planar particles. We also

formulate the two-body problem in the model with our generalized gauge symmetry and consider the case with both CS and background electromagnetic fields, as it is used in the description of fractional quantum Hall effect.

1 Introduction

Recently there has been a lot of interest in considering quantum-mechanical and field-theoretic models with noncommutative space-time coordinates:

$$[\widehat{x}_\mu, \widehat{x}_\nu] = i\theta_{\mu\nu}(\widehat{x}) = i(\theta^{(0)}_{\mu\nu} + \theta^{(1)\rho}_{\mu\nu}\widehat{x}_\rho + \ldots) \tag{1.1}$$

If $\partial_\rho\theta_{\mu\nu}(\hat{x}) \neq 0$ the Poincaré symmetries with commutative translations do not preserve the relation (1.1) and so the only case invariant under classical translations $\hat{x}'_\mu = \hat{x}_\mu + a_\mu$ (a_μ - c-numbers) is provided by $\theta_{\mu\nu}(\hat{x}) = \theta^{(0)}_{\mu\nu}$. Such a deformation, first introduced on the grounds of quantum gravity by Doplicher, Fredenhagen and Roberts [2], was further justified in $D = 10$ string-theory moving in the background with a nonvanishing tensor field $B_{\mu\nu}$ [3,4]. However, it is easy to see that even for constant value of the commutator (1.1) the noncommutativity of space-time breaks Lorentz invariance, i.e. $\theta^{(0)}_{\mu\nu}$ is a constant tensor. If we assume that the relation (1.1) is valid in all classical Poincaré frames then this constant tensor should be described by a scalar parameter. The following two cases can be considered:

i) $D = 2$ relativistic theory, with classical Poincaré symmetries. In such a case

$$\theta^{(0)}_{\mu\nu} = \hbar\theta\, \varepsilon_{\mu\nu}\,, \tag{1.2}$$

where $\varepsilon_{\mu\nu}$ is a $D = 2$ covariant antisymmetric tensor.

ii) $D = 2+1$ nonrelativistic theory, with a classical time variable and relations (1.1) applied to the $D = 2$ space coordinates x_i ($i = 1, 2$). In this case one gets

$$\theta_{ij} = \hbar\theta\, \varepsilon_{ij}\,. \tag{1.3}$$

It is known that in a nonrelativistic Galilean-invariant theory the space-time coordinates can be related to the Galilean boosts by the following relation [5]

$$K_i = m\, X^L_i\,. \tag{1.4}$$

The formulae (1.3–1.4) in a $D = 2+1$ nonrelativistic theory imply that the Galilean symmetry is endowed with two central charges: one standard describing mass m, and the second "exotic", described by the parameter θ in (1.3). Moreover, if we consider the $(2+1)$–dimensional nonrelativistic $c \to \infty$ limit of a $(2+1)$–dimensional relativistic theory, the parameter θ determines the value of the nonrelativistic Abelian $D = 2$ spin [6].

The noncommutativity of position coordinates can be obtained as a consequence of canonical quantization of dynamical models. Such a result is valid for string–inspired noncommutativity and for the $(2 + 1)$–dimensional Galilean models with noncommutative spatial coordinates. In our previous paper [1] we have shown that a nonvanishing value of θ (see (1.3)) can be introduced by the following extension of the free classical $D = 2 + 1$ particle action ($\dot{a} \equiv \frac{d}{dt}a$):

$$L = \frac{m\dot{x}_i^2}{2} - k\varepsilon_{ij}\dot{x}_i\ddot{x}_j \,. \tag{1.5}$$

The action (1.5) contains higher derivatives and their presence leads, after canonical quantization, to the introduction of noncommutative position variables.

By comparison with formula (1.3), one can show that

$$k = -\frac{\theta m^2}{2}. \tag{1.5a}$$

The action (1.5), in the Hamiltonian approach, is characterized by a six-dimensional phase space with two canonical momenta

$$\begin{aligned} p_i &= \frac{\partial L}{\partial \dot{x}_i} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}_j} = m\dot{x}_i - 2k\varepsilon_{ij}\ddot{x}_j && \text{(1.6a)} \\ \widetilde{p}_i &= \frac{\partial L}{\partial \ddot{x}_i} = k\epsilon_{ij}\dot{x}_j && \text{(1.6b)} \end{aligned}$$

which leads to the Hamiltonian

$$H = -\frac{m}{2k^2}(\widetilde{p}_j)^2 + \frac{1}{k}\widetilde{p}_k\epsilon_{kl}p_l \,. \tag{1.7}$$

Introducing the variables

$$X_i^L = x_i - \frac{2}{m}\widetilde{p}_i, \qquad P_i = p_i, \qquad \widetilde{P}_i = \frac{k}{m}p_i + \epsilon_{ij}\widetilde{p}_j \tag{1.8}$$

we get

$$H = \frac{\overrightarrow{P}^2}{2m} - \frac{m\overrightarrow{\widetilde{P}}^2}{2k^2} \tag{1.9}$$

and, considering (1.6b) as a constraint, we see that we get the following symplectic structure [1]:

$$\{Y_A, Y_B\} = \Omega_{AB} \,, \tag{1.10}$$

where

$$\Omega = \begin{pmatrix} \frac{2k}{m^2}\varepsilon & 1_2 & 0 \\ -1_2 & 0 & 0 \\ 0 & 0 & \frac{k}{2}\varepsilon \end{pmatrix} \tag{1.11}$$

and where $Y_A = \{X_i^L, P_k, \tilde{P}_l\}$.

We see that

i) the parameter k introduces noncommutativity in the coordinate sector[1]

ii) the dynamics splits into the decoupled sum of the dynamics in the physical sector (X_i^L, P_i variables) and in the auxiliary sector ($\widetilde{P}_i$ variable).

In this paper we consider the model (1.5) with electromagnetic interaction. Following the method of Faddeev and Jackiw [7, 8] we rewrite the Lagrangian (1.5) in the first-order form, and introduce noncommutative coordinates

$$X_i = X_i^L + \frac{2k}{m^2}\varepsilon_{ij}P_j \,, \tag{1.12}$$

which were recently introduced by Horvathy and Plyushchay [9]. The noncommutative coordinates (1.12) satisfy the relations (see (1.5a))

$$\{X_i, X_j\} = -\frac{2k}{m^2}\varepsilon_{ij} = \theta\varepsilon_{ij} \tag{1.13}$$

and transform with respect to the Galilean boosts as components of a Galilean two–vector.

The electromagnetic interaction with a magnetic potential can be introduced in two different ways:

i) By adding to the Lagrangian the term

$$L^{\text{int}} = eA_i(X_i, t)\dot{X}_i \,. \tag{1.14}$$

Such a way of introducing electromagnetic interaction can be interpreted as corresponding to the modification of the symplectic form of the system which determines the noncommutative phase-space geometry (1.10-1.11) [10].

[1] (1.10-11) describes classical Poisson brackets which are nonvanishing in the coordinate sector. For convenience, we will refect to this fact here and in the rest of this paper as 'noncommutativity' both in the quantum and in the classical case.

ii) One can introduce the minimal EM coupling by the replacement

$$H_0 = \frac{P^2}{2m} \to \frac{\vec{\mathcal{P}}^2}{2m} = \frac{1}{2m}\left(\overrightarrow{P} - e\overrightarrow{A}(X_i,t)\right)^2 \tag{1.15}$$

and preserve the symplectic structure (1.10-1.13). In such a way the interaction does not modify the noncommutative geometry, but changes Abelian gauge transformations.

The main aim of this paper is to consider the case ii), which is related to models describing the quantum Hall effect, with generalised gauge transformations accompanied by area - preserving transformations (see e.g. [11] – [13])[2] After considering in Sect. 2 the first order formalism for our model from [1] and the canonical structure of both models, i) and ii), we introduce the area reparametrization - invariant formalism. In Sect. 3 we show that both possibilities are related to each other by a classical Seiberg-Witten (SW) map [3] supplemented by a noncanonical transformation of phase space variables for planar particles. In such a way we recover the known definition of covariantized coordinates [16] describing the coordinate part of the noncanonical transformation in the phase space describing planar particles. In Sect 4 we consider the Chern-Simons (CS) gauge interactions of planar particles and formulate the dynamics of the corresponding two-body problem. This leads to the deformed anyonic dynamics which might then be applied to the description of the quantum Hall effect. In Sect 5 we consider our model with statistical CS fields in the electromagnetic background. We note that for the critical value of the magnetic background field strength we obtain the description of lowest Landau level for Quantum Hall Effect. In the last section we comment on the second quantization of our model [1] and outline the relativistic generalization to $D = 3+1$. Finally, in an appendix we introduce a gauge field-dependent dreibein formalism.

[2] Area-preserving transformations are the symmetry transformations for electrons in the lowest Landau level. They have been introduced in [14] and recently studied in [15].

2 Two Ways of Introducing Minimal Gauge Couplings

Following Faddeev-Jackiw's method of describing Lagrangians with higher order derivatives [8] we describe, equivalently, the action (1.5) as (see [1])[3]

$$
\begin{aligned}
L^{(0)} &= P_i(\dot{x}_i - y_i) + \frac{y_i^2}{2} + \frac{\theta}{2}\varepsilon_{ij} y_i \dot{y}_j = \\
&= P_i \dot{x}_i + \frac{\theta}{2}\varepsilon_{ij} y_i \dot{y}_j - H(y,P)\,, \qquad (2.1)
\end{aligned}
$$

where

$$
H(y,P) = -\frac{y^2}{2} + P_i y_i\,. \qquad (2.2)
$$

Using the variables [9]

$$
\begin{aligned}
Q_i &= \theta(y_i - p_i) \\
X_i &= x_i + \varepsilon_{ij} Q_j\,, \qquad (2.3)
\end{aligned}
$$

we see that our Lagrangian separates into two disconnected parts describing the "external" and "internal" degrees of freedom. Thus we have

$$
L^{(0)} = L^{(0)}_{\text{ext}} + L^{(o)}_{\text{int}}\,, \qquad (2.4)
$$

with

$$
\begin{aligned}
L^{(0)}_{\text{ext}} &= P_i \dot{X}_i + \frac{\theta}{2}\varepsilon_{ij} P_i \dot{P}_j - H^{(0)}_{\text{ext}} \qquad (2.5a) \\
L^{(0)}_{\text{int}} &= \frac{1}{2\theta}\varepsilon_{ij} Q_i \dot{Q}_j - H^{(0)}_{\text{int}} \qquad (2.5b)
\end{aligned}
$$

where

$$
H^{(0)}_{\text{ext}} = \frac{1}{2}\vec{P}^{\,2}\,, \qquad H^{(0)}_{\text{int}} = -\frac{1}{2\theta^2}\vec{Q}^{\,2}\,. \qquad (2.6)
$$

From (2.5a-2.6) we obtain the following Poisson brackets (PBs) of the independent sets of external and internal phase space variables:

$$
\{X_i, X_j\} = \theta\varepsilon_{ij}\,,
$$

[3] For simplicity we give for all the particles the same mass ($m = 1$ in appropriate units) and use θ defined by (1.5a) ($\theta = -2k$) instead of k as the second central charge.

$$
\begin{aligned}
\{X_i, P_j\} &= \delta_{ij}\,, \\
\{P_i, P_j\} &= 0\,,
\end{aligned}
\tag{2.7}
$$

and

$$
\{Q_i, Q_j\} = -\theta\,\varepsilon_{ij}\,, \tag{2.8}
$$

with all other PBs vanishing.

Having separated off the "internal" degrees of freedom (i.e. L_{int}) we now proceed to couple in the electromagnetic field. We couple it to the "external" sector only. Hence in the remainder of this paper we shall not be concerned with the "internal" sector of the theory (described by Q_i and $L^{(0)}_{int}$). We note first that the action (2.5a) describes the model by Duval and Horvathy [10], with the symplectic structure given by the following Liouville form

$$
\Omega = P_i dX_i + \frac{\theta}{2}\varepsilon_{ij} P_i dP_j - H^{(0)}_{\mathrm{ext}} dt\,. \tag{2.9}
$$

The minimal coupling to the gauge field $A_\mu(\vec{x},t) = (A_i(\vec{x},t), A_0(\vec{x},t))$ can be introduced in the following two ways:

2.1 Duval-Horvathy model

One replaces the one-form (2.9) by:

$$
\Omega \rightarrow \Omega_e = \Omega + e(A_i dX_i + A_0 dt)\,, \tag{2.10}
$$

which corresponds to the addition of (1.14). Introducing $dX_\mu = (dX_i, dt)$ the modification (2.10) leads to the symplectic form with a standard addition corresponding to the minimal EM coupling

$$
\begin{aligned}
\omega &= d\Omega = dP_i \wedge dX_i + \frac{\theta}{2}\varepsilon_{ij} dP_i \wedge dP_j - dH^{(0)}_{\mathrm{ext}} dt \\
&\quad + e(\frac{1}{2} F_{ij} dX_i \wedge dX_j - E_i dX_i \wedge dt)\,,
\end{aligned}
\tag{2.11}
$$

where

$$
F_{ij} = \partial_i A_j - \partial_j A_i = \varepsilon_{ij} B\,, \qquad E_i = \partial_i A_0 - \partial_t A_i\,. \tag{2.12}
$$

It is easy to see that the symplectic form (2.11) is invariant under standard gauge transformations

$$
A_i \rightarrow A'_i = A_i + \partial_i \Lambda\,, \qquad A_0 \rightarrow A'_0 = A_0 + \partial_t \Lambda\,. \tag{2.13}
$$

The Lagrangian corresponding to (2.10) now becomes

$$L_{\rm ext} = L_{\rm DH} = (P_i + eA_i)\dot{X}_i + \frac{\theta}{2}\varepsilon_{ij}P_i\dot{P}_j - \frac{1}{2}\overrightarrow{P}^2 + eA_0 \ , \tag{2.14}$$

which may be brought by the point transformation

$$P_i \to P_i' = P_i + eA_i \, , \tag{2.15}$$

to the equivalent form:

$$L_{\rm DH} = P_i'\dot{X}_i + \frac{\theta}{2}\varepsilon_{ij}(P_i' - eA_i)(\dot{P}_j' - e\frac{d}{dt}A_j) + eA_0 - \frac{1}{2}(P_i' - eA_i)^2. \tag{2.16}$$

The Lagrangian (2.14) is quasi-invariant under standard local gauge transformations (2.13):

$$L_{\rm ext} \to L_{\rm ext}' = L_{\rm ext} + \partial_i\Lambda\dot{X}_i + \partial_t\Lambda = L_{\rm ext} + \frac{d}{dt}\Lambda \, . \tag{2.17}$$

The modification (2.10), (2.11) has been considered in [10] and it leads to the modification of the PB structure (2.7) [10]:

$$\begin{aligned} \{X_i, X_j\} &= \frac{\theta\varepsilon_{ij}}{1 - e\theta B} \, , \\ \{X_i, P_j\} &= \frac{\delta_{ij}}{1 - e\theta B} \, , \\ \{P_i, P_j\} &= \frac{eB\varepsilon_{ij}}{1 - e\theta B} \, . \end{aligned} \tag{2.18}$$

2.2 Model with generalized gauge transformations

The other possibility of a minimal coupling follows from the assumption that the symplectic structure (2.7) remains unchanged. This is the case if we insert the minimal substitution[4]

$$P_i \to \mathcal{P}_i \ = \ P_i - e\hat{A}_i \tag{2.19}$$

$$H_{\rm ext}^{(0)} \to H_{\rm ext}^{(0)} - e\hat{A}_0$$

[4] The gauge fields in this model we shall denote by hat $(\hat{A}_\mu, \hat{F}_{n\nu})$ in order to distinguish them from the corresponding quantities in the model of Duval and Horvathy [10].

into the free Hamiltonian $H^{(0)}_{\rm ext}$ only.

In this way we get, in place of (2.16), the following Lagrangian

$$\widetilde{L}_{\rm ext} = P_i\dot{X}_i + \frac{\theta}{2}\varepsilon_{ij}P_i\dot{P}_j - \frac{1}{2}(P_i - e\hat{A}_i)^2 + e\hat{A}_0\,. \tag{2.20}$$

The difference between both Lagrangians is in the 2nd term of (2.20). $L_{\rm DH}$ (2.16) arises from $L^{(0)}_{ext}$ by performing the minimal substitution (2.19) not only in $H^{(0)}_{\rm ext}$ but also in the second term of $L^{(0)}_{ext}$.

We note that the symplectic structure described by (2.7) is invariant under the following infinitesimal time-dependent area - preserving - local coordinate transformations

$$\delta X_i = -e\theta\varepsilon_{ij}\partial_j\Lambda(\overrightarrow{X},t) \qquad \delta P_i = e\partial_i\Lambda(\overrightarrow{X},t), \tag{2.21}$$

where Λ is infinitesimal.

If we supplement (2.21) by the transformation of the gauge fields

$$\begin{aligned}\delta\hat{A}_\mu(\overrightarrow{X},t) : \;&=\; \hat{A}'_\mu(\overrightarrow{X}+\delta\overrightarrow{X},t) - \hat{A}_\mu(\overrightarrow{X},t)\\ &=\; \partial_\mu\Lambda(\overrightarrow{X},t)\,,\end{aligned} \tag{2.22}$$

it is easy to check that the Lagrangian (2.20) is quasi-invariant

$$\delta\widetilde{L}_{\rm ext} = e\frac{d}{dt}(\Lambda + \frac{\theta}{2}\varepsilon_{ij}\partial_i\Lambda P_j)\,. \tag{2.23}$$

We note that (2.22) differs from the standard gauge transformation (2.13) by the simultaneous coordinate transformation[5] (2.21). For the corresponding change $\delta_0\hat{A}_\mu$ of the gauge field at fixed $\vec{X}$ we obtain from (2.21-22)

$$\begin{aligned}\delta_0\hat{A}_\mu(\vec{X},t) &:= \hat{A}'_\mu(\vec{X},t) - A_\mu(\vec{X},t)\\ &= \partial_\mu\Lambda(\vec{X},t) + e\{A_\mu(\vec{X},t),\Lambda(\vec{X},t)\}\end{aligned} \tag{2.24}$$

in place of (2.13). Therefore we call the transformation (2.24) a generalised gauge transformation. In deriving (2.24) from (2.22) we have used the PBs (cp. (2.7))

$$\{g,f\} := \theta\,\epsilon_{ij}\,\partial_i g\,\partial_j f \tag{2.25}$$

for two generic functions f and g.

[5]For the mixing of gauge and coordinate transformation see also Jackiw et al. [17]

The equations of motion (EOM) derived from (2.20) are given by

$$
\begin{aligned}
\dot{X}_i &= -\theta\varepsilon_{ij}[e(P_k - e\hat{A}_k)\partial_j\hat{A}_k + e\partial_j\hat{A}_0] + P_i - e\hat{A}_i\,, \\
\dot{P}_i &= e(P_k - e\hat{A}_k)\partial_i\hat{A}_k + e\partial_i\hat{A}_0\,,
\end{aligned}
\tag{2.26}
$$

which, having made use of (2.7), can be put into the Hamiltonian form

$$
\dot{X}_i = \{X_i, H\}\,, \qquad \dot{P}_i = \{P_i, H\}\,, \tag{2.27}
$$

where

$$
H = \frac{1}{2}(P_i - e\hat{A}_i)^2 - e\hat{A}_0\,. \tag{2.28}
$$

Let us rewrite the EOM (2.26) in terms of our new variable $\mathcal{P}_i$ (2.19). We obtain

$$
\dot{\mathcal{P}}_i = e(\widehat{F}_{ik}\mathcal{P}_k + \widehat{F}_{i0})\,, \tag{2.29}
$$

with the invariant field strength[6]

$$
\widehat{F}_{\mu\nu} := \partial_\mu\widehat{A}_\nu - \partial_\nu\widehat{A}_\mu + e\{\hat{A}_\mu, \hat{A}_\nu\} \tag{2.30}
$$

and

$$
\dot{X}_i + e\theta\varepsilon_{ij}\partial_j\widehat{A}_0 = \mathcal{P}_k(\delta_{ik} - e\theta\varepsilon_{ij}\partial_j A_k)\,. \tag{2.31}
$$

3 Seiberg-Witten (SW) Map and the Equivalence of the Two Planar Particle Models with Noncommutative Structure

In this section we show that our model, (2.20), and the one of Duval et al., (2.14), are related to each other by a noncanonical transformation of the phase space variables $(X_i, P_i) \to (\eta_i, \mathcal{P}_i)$ supplemented by a classical SW map between the corresponding gauge potentials.

Let us introduce, besides the invariant $\mathcal{P}_i$ given by formula (2.19), the invariant particle coordinates as follows[7] (cp [13]):

$$
\eta_i(\overrightarrow{X}, t) := X_i + e\theta\varepsilon_{ij}\widehat{A}_j(\overrightarrow{X}, t)\,. \tag{3.1}
$$

[6] We draw attention to the difference from the model of Duval et al. [10] which has a standard Abelian field strength.

[7] cp. [16,19] for the case of noncommutative gauge theories

Clearly from (2.21-22) we obtain

$$\delta\eta_i = 0 \tag{3.2}$$

but at fixed $\vec{X}$ the fields η_i transform as

$$\delta\eta_i \;=\; e\{\eta_i, \Lambda\}. \tag{3.3}$$

It is easy to check that the new phase-space variables $(\eta_i, \mathcal{P}_i)$ satisfy the noncanonical Poisson brackets (2.18)

$$\{\eta_i, \eta_j\} = \frac{\theta\epsilon_{ij}}{1 - e\theta B(\vec{\eta}, t)}\ , \tag{3.4}$$

$$\{\eta_i, \mathcal{P}_j\} = \frac{\delta_{ij}}{1 - e\theta B(\vec{\eta}, t)}\ , \quad \{\mathcal{P}_i, \mathcal{P}_j\} = \frac{e\epsilon_{ij} B(\vec{\eta}, t)}{1 - e\theta B(\vec{\eta}, t)}$$

with the field B defined by (cp. [20])

$$B(\vec{\eta}, t) = \frac{\hat{B}(\vec{X}, t)}{1 + e\theta\hat{B}(\vec{X}, t)} \tag{3.5}$$

where X_i is a function of η_i as follows from (3.1).

The relations (3.4) as well as (2.7) describe, after quantization, two different quantum phase spaces with noncommutative position sectors.

With $(\eta_i, \mathcal{P}_i)$ as the new noncanonical phase-space variables our L (2.20) becomes

$$L = \widehat{L}_{\mathrm{part}} + \frac{\theta}{2}\varepsilon_{ij}\mathcal{P}_i\dot{\mathcal{P}}_j\ , \tag{3.6}$$

where $\widehat{L}_{\mathrm{part}}$ is given by the θ-deformed particle Lagrangian in the presence of gauge fields defined in [13], i.e.

$$\widehat{L}_{\mathrm{part}} = \mathcal{P}_i\dot{\eta}_i - \frac{1}{2}\mathcal{P}_i^2 + e(\widehat{A}_i\dot{X}_i + \widehat{A}_0 + \frac{e\theta}{2}\varepsilon_{ij}\widehat{A}_i\frac{d}{dt}\widehat{A}_j) - \frac{1}{2}\frac{d}{dt}(e\theta\varepsilon_{ij}\mathcal{P}_i\widehat{A}_j)\,. \tag{3.7}$$

Moreover, we neglect the total time-derivative term which is irrelevant for EOM.

In order to express L in terms of $(\eta_i, \mathcal{P}_i)$ we have to introduce a map

$$\widehat{A}_\mu(\overrightarrow{x}, t) \to A_\mu(\overrightarrow{\eta}, t)\,. \tag{3.8}$$

In accordance with [13] we define (3.8) by the requirement

$$\widehat{A}_i\dot{X}_i + \widehat{A}_0 + \frac{\theta}{2}\varepsilon_{ij}\widehat{A}_i\frac{d}{dt}\widehat{A}_j = A_i(\overrightarrow{\eta},t)\dot{\eta}_i + A_0(\overrightarrow{\eta},t)\,. \tag{3.9}$$

Eliminating at the l.h.s. of (3.9) $\dot{X}_i$ in favour of $\dot{\eta}_i$ we obtain, by comparing the coefficients of $\dot{\eta}_i$ as well as of unity at both sides of (3.9), the relations

$$A_k\left(\overrightarrow{\eta}(\overrightarrow{X},t),t\right) = \frac{1}{2}\hat{A}_l(\overrightarrow{X},t)\left(\delta_{kl} + \frac{e_{kl}(\overrightarrow{X},t)}{1+e\theta\widehat{B}}\right) \tag{3.10}$$

$$A_0\left(\overrightarrow{\eta}(\overrightarrow{X},t),t\right) = \hat{A}_0(\overrightarrow{X},t) - \frac{e\theta}{2(1+e\theta\widehat{B})}\widehat{A}_l(\overrightarrow{X},t)\varepsilon_{kj}\partial_t\widehat{A}_j(\overrightarrow{X},t)e_{kl}(\overrightarrow{X},t)\,, \tag{3.11}$$

expressed in terms of the inverse dreibein, which we discuss in more detail in the Appendix (see (A.6) and also [13], Eq. (24)).

From (3.10-11) we derive a simple relation between the corresponding field strengths (cp. [20])

$$F_{\mu\nu}(\vec{\eta},t) = \frac{\hat{F}_{\mu\nu}(\vec{X},t)}{1+e\theta\hat{B}(\vec{X},t)}\ . \tag{3.12}$$

The relations (3.10) and (3.11) are just the classical limits of an inverse SW-map defined by replacing in the SW differential equation ([3], eq. (3.8)) star products by ordinary products (cp. ([21], sect. 2) and ([20], sect. 4.1)). They give us the required relation between our Lagrangian given by (2.20) and the one of Duval and Horvathy denoted by L_{DH} and given by (2.14)

$$L\left(\widehat{A}_\mu(\overrightarrow{X},t),\dot{\overrightarrow{X}},\overrightarrow{X},\overrightarrow{P},\dot{\overrightarrow{P}})\right) = L_{DH}(A_\mu(\overrightarrow{\eta},t),\overrightarrow{\eta},\dot{\overrightarrow{\eta}},\overrightarrow{\mathcal{P}},\dot{\overrightarrow{\mathcal{P}}})). \tag{3.13}$$

Thus we see that the relations (3.10) and (3.11), supplemented by the transformation (3.1) and (2.19), describe within a classical framework the SW map relating the planar particle dynamics in the presence of Abelian gauge fields in two different noncanonical phase spaces with two different symplectic structures. These symplectic structures are either gauge field independent (cp. (2.7)) or gauge field dependent (cp. (3.4)), (cp. [20,21]). A characterization of the SW map as relating two different symplectic structures has been considered also earlier (see e.g. [21,22]) and provides an extension of the original formulation in terms of infinitesimal gauge transformations [3] in the presence of particle coordinates.

The relation (3.13) is the central result of our paper. We see that the two models describing different possibilities of introducing minimal electromagnetic interaction, one with the standard gauge transformations (see (2.13)) and the other one with the generalized gauge transformations (see (2.24)), may be transformed into each other by a local Seiberg-Witten transformation accompanied by a change of phase space variables in the particle sector. It should be stressed that if $\theta \neq 0$, in both phase spaces, the Poisson brackets in the coordinate sector imply noncommutative space coordinates. In this way we have achieved an extension to $\theta \neq 0$ of a classical SW map for standard point particles with commuting space coordinates considered in [13].

The total action is obtained if we further add a pure gauge part of the action (Maxwell, Chern-Simons etc.), with corresponding symplectic structures (and, ultimately, one can add also our "internal" Lagrangian $L^{(0)}_{int)}$). In particular if the gauge field actions transform into each other by the SW-map (3.10-3.11), the particle trajectories with gauge interaction in the respective phase-spaces are classically equivalent i.e. may be expressed equivalently in two noncanonical phase space frameworks. It should be added that such a classical equivalence might become invalid after quantization due to the operator ordering problems providing θ-dependent quantum corrections to the particle interactions.

It is worth noting that, using arguments similar to ours, Jackiw et al. have presented in a very recent paper [19] the Seiberg-Witten map relating the Lagrange and Euler pictures in the presence of gauge fields for another dynamical model: the field-theoretical formulation of fluid mechanics.

4 Chern-Simons Gauge Interaction and the Two-Body Problem

In this Section we derive the dynamics for two identical particles described by our model (2.20) interacting via Chern-Simons (CS) gauge interactions.

Let us start with the CS-action of a $\widehat{A}_\mu$ field invariant with respect to the generalized gauge transformation (2.24).

We have (cp. [11], [13])

$$L_{\rm CS} = \frac{\kappa}{2}\int d^2x\,\varepsilon^{\mu\nu\rho}\widehat{A}_\mu\left(\partial_\nu\widehat{A}_\rho + \frac{e}{3}\{\hat{A}_\nu,\hat{A}_\rho\}\right)\,. \tag{4.1}$$

The extra (unusual) term in this expression is required by our generalised gauge invariance as discussed in [11] and [13]. Its origin can be traced to the appearance of an extra term in (2.30).

Next we consider the following total Lagrangian

$$L_{\rm tot} = \sum_{\alpha=1}^{2}\widetilde{L}_{{\rm ext},\alpha} + L_{\rm CS} \tag{4.2}$$

with each of $\widetilde{L}_{\rm ext}$ given by (2.20).

The variation of $L_{\rm CS}$ with respect to the Lagrange multiplier field $\widehat{A}_0$ leads to the well known Gauss constraint

$$\epsilon_{ij}\widehat{B}(\overrightarrow{x},t) = \hat{F}_{ij} = -\epsilon_{ij}\frac{e}{\kappa}\sum_{\alpha=1}^{2}\delta(\overrightarrow{x}-\overrightarrow{X}_\alpha)\,. \tag{4.3}$$

Modulo asymptotic parts, which do not contribute to the Hamiltonian describing relative particle motion, we obtain a solution of (4.3) for $\widehat{A}_k$ at the particle position $\overrightarrow{x} = \overrightarrow{X}_{\frac{1}{2}}$ in the form [13]

$$\widehat{A}_k(X_{\frac{1}{2}}) = \pm\varepsilon_{kj}(X_1 - X_2)_j\,\chi(|\overrightarrow{X}_1 - \overrightarrow{X}_2|)\,, \tag{4.4}$$

with

$$\chi(R) = \frac{1}{e\theta}\left(1-\left(1-\frac{\widetilde{\theta}}{R^2}\right)^{1/2}\right) = \frac{1}{2}\frac{e}{\pi\kappa}\frac{1}{R^2}\left(1+\frac{1}{4}\tilde{\theta}\frac{1}{R^2}+0(\theta^2)\right)\,, \tag{4.5}$$

where $R = |\overrightarrow{X}|$ and

$$\widetilde{\theta} := \frac{e^2\theta}{\pi\kappa}.$$

With (4.4-4.5) and the position and momentum variables for the relative motion

$$\overrightarrow{X} := \overrightarrow{X}_1 - \overrightarrow{X}_2\,, \qquad \overrightarrow{P} := \frac{1}{2}\left(\overrightarrow{P}_1 - \overrightarrow{P}_2\right)\,, \tag{4.6}$$

and by applying the Legendre transformation to (4.2) and using the Gauss - constraint (4.3) we obtain the following Hamiltonian for the relative motion

$$
\begin{aligned}
H &= \overrightarrow{P}^2 + 2e\left(\varepsilon_{ij}X_iP_j + \frac{R^2}{\theta}\right)\chi(X_k) - \frac{e^2}{\pi\kappa\theta} \\
&= \overrightarrow{P}^2 + \frac{e^2}{\pi\kappa}\varepsilon_{ij}X_iP_j\frac{1}{R^2} + \frac{e^4}{4\pi^2\kappa^2R^2} + O(\theta)\,,
\end{aligned}
\tag{4.7}
$$

i.e. in the leading order of the θ-expansion we reproduce the known anyonic Hamiltonian.

The phase-space variables for the relative motion (4.6) obey, according to (2.7), the Poisson bracket relations

$$
\begin{aligned}
\{X_i, X_j\} &= 2\theta\varepsilon_{ij}\,, \\
\{X_i, P_j\} &= \delta_{ij}\,, \\
\{P_i, P_j\} &= 0\,.
\end{aligned}
\tag{4.8}
$$

In order to quantize the Hamiltonian system (4.7-8) we proceed in three steps:

i) We replace the classical structure (4.8) by commutators of the corresponding operators

$$
\{A,\, B\} \quad \to \quad \frac{1}{i\hbar}[\hat{A},\, \hat{B}],
\tag{4.9}
$$

where $\hat{A}$, $\hat{B}$ denote the quantized variables.[8]

ii) We solve the ordering problem arising from the noncommuting position and momentum variables by symmetrization

$$
P_i\chi(X_k) \;\to\; \frac{1}{2}\left(\hat{P}_i\hat{\chi}(\hat{X}_k) \;+\; \hat{\chi}(\hat{X}_k)\hat{P}_i\right).
\tag{4.10}
$$

iii) We replace the operator-valued functions $\hat{f}(\hat{X}_k)$, $\hat{g}(\hat{X}_k)$ of noncommuting position variables $\hat{X}_k$ with local multiplication by functions $f(y_k)$, $g(y_k)$ depending on commuting position variables y_k and the nonlocal Moyal-star product

$$
\hat{f}(\hat{X}_k)\hat{g}(\hat{X}_k) \;\longleftrightarrow\; f(y_k) * g(y_k) :=
$$

[8] We hope that there is no confusion here with the hat introduced before - for the field quantities of our model

$$f(y_k)\exp\left(i\hbar\theta\epsilon_{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)g(y_k) =: f\left(y_k-\theta\varepsilon_{kl}\hat{P}_l\right)g(y_k), \tag{4.11}$$

where

$$\hat{P}_i := \frac{\hbar}{i}\partial_i, \tag{4.12}$$

with $\partial_i := \frac{\partial}{\partial y_i}$.

Such a quantization procedure leads to the Schrödinger equation

$$(-\hbar^2\Delta - \frac{e^2}{\pi k\theta} - E)\psi + 2e\epsilon_{ij}(y_i\chi(y)) * \hat{P}_j\psi + \frac{2e}{\theta}(y^2\chi(y)) * \psi = 0. \tag{4.13}$$

In deriving (4.13) we have used the property that χ is a function of only $y := |\vec{y}|$ (see (4.5)) and thus

$$\epsilon_{ij}\, y_i\, (\hat{P}_j\chi) = 0. \tag{4.14}$$

In this Section we have been considering the gauge interaction between two identical particles, with the same charge e. An interesting question now arises, as to whether the Poisson bracket (4.8) for relative coordinates should depend on the choice e_1, e_2 of charges at the points $\vec{X}_1, \vec{X}_2$. If we observe that θ is geometrically similar to the mass parameter, which is also a Galilean central charge, one can assume, by analogy, that θ differs for particles with different electric charges. In order to obtain for N planar particles the invariant action (4.2) we are led to the consistant replacement $\theta \to \frac{\theta}{e}$ in the formulae of Sect. 2-4. In such a case one gets for relative coordinates (4.6) in the $N = 2$ case the following modification of the first formula (4.8)

$$\{X_1, X_i\} = \theta\left(\frac{1}{e_1} + \frac{1}{e_2}\right) , \tag{4.15}$$

i.e. if $e_1 = -e_2$ we obtain $\{X_1, X_2\} = 0$, in agreement with the conclusions of [23].

5 Application: Statistical planar CS gauge action and external electromagnetic background fields

5.1 Physical background

It is known that CS gauge transformations as well as CS gauge fields in the $D = 2 + 1$ Hamiltonian framework are used for the description of the

Fractional Quantum Hall Effect (FQHE) (see e.g. [24,25]) and represent flux tubes attached to electrons forming basic fermionic quasiparticles - composite fermions (CF). However, formally such CS gauge fields are gradients, i.e. pure gauge, the gauge functions are multivalued and from the Stokes theorem it follows that the CS gauge field strength is nonzero. In what follows these gauge fields A_μ^{CS}, which dress the electrons in the Hamiltonian formulation of FQHE, will be called statistical CS fields.

In a general case one can embedd the system of CFs in an external electromagnetic background field $A_\mu^{ext}(X)$, i.e. add to the CS actions considered in sect. 2 additional gauge field couplings. One can proceed in two ways:

i) By modifying the minimal substitution (2.19) in the Hamiltonian

$$P_i \to P_i - e\hat{A}_i \;\longrightarrow\; P_i \to P_i - e\hat{A}_i^{tot} \;, \tag{5.1}$$

where $\hat{A}_i^{tot}$ turns out to be a nonlinear function of $\hat{A}_i^{CS}$ and $\hat{A}_i^{ext}$ as given below.

ii) By adding to the Lagrangian (2.20) the background field term in the form of (1.14).

We shall consider below these two couplings in our model, (2.20), which is invariant with respect to the area-preserving coordinate transformations (2.21-22).

5.2 Minimal coupling (5.1)

Our main point here is, that for such a coupling, the gauge fields in our model (for $\theta \neq 0$) are nonadditive.

Firstly, let us observe that in the DH Lagrangian (2.14) the gauge fields are coupled linearly, i.e. one gets Abelian addition formula

$$A_\mu^{tot} = A_\mu^{CS} + A_\mu^{ext} \tag{5.2}$$

but the gauge fields $\hat{A}_\mu^{tot}$ in our model will be the solution of the relations (3.10-11) and so (see (3.13))

$$L(\hat{A}_\mu^{tot}(\vec{X},t), \vec{X}, \dot{\vec{X}}, \vec{P}, \dot{\vec{P}}) = L_{DH}(A_\mu^{CS}(\vec{\eta},t) + A_\mu^{ext}(\vec{\eta},t)\;, \vec{\eta}, \dot{\vec{\eta}}, \vec{\mathcal{P}}, \dot{\vec{\mathcal{P}}}) \tag{5.3}$$

In order to have insight into the nonlinear structure of our decomposition of $\hat{A}_\mu^{tot}$ we determine the SW map (3.10-11) for $\hat{A}_\mu^{tot}$ in the lowest order of the θ expansion using (3.1) (cp. [3]):

$$\hat{A}_\mu^{tot}(\vec{x},t) \;=\; A_\mu^{tot}(\vec{x},t) \;-\; \frac{e\theta}{2}\epsilon_{ik}\, A_i^{tot}(\partial_k A_\mu^{tot} \;+\; F_{k\mu}^{tot}) \;+\; O(\theta^2), \tag{5.4}$$

where A_μ^{tot} is given by (5.2) and the field strength $F_{k\mu}^{tot}$ is related to A_μ^{tot} by (2.12).

The analogue of (5.4) for the field strength has been given in a closed form in (3.12), i.e. we have

$$\begin{aligned} \hat{B}^{tot}(\vec{X},t) &= \tfrac{B^{tot}(\vec{\eta},t)}{1-e\theta B^{tot}(\vec{\eta},t)}, \\ \hat{E}_i^{tot}(\vec{X},t) &= \tfrac{E_i^{tot}(\vec{\eta},t)}{1-e\theta B^{tot}(\vec{\eta},t)}, \end{aligned} \tag{5.5}$$

with $\vec{\eta}$ defined by (3.1) and $F_{\mu\nu}^{tot}$ decomposing additively

$$F_{\mu\nu}^{tot} = F_{\mu\nu}^{CS} + F_{\mu\nu}^{ext}. \tag{5.6}$$

As an obvious consequence of this procedure we see that the minimal substitution (5.1) for the total gauge field defined by (5.3) leaves the symplectic structure (2.7) unchanged.

5.3 Hybrid coupling

In this case we couple the CS and external fields differently, introducing A_μ^{ext} into the symplectic form as in (2.10). We assume

$$L = \tilde{L}_{ext}^{CS} + e(\hat{A}_i^{ext}\dot{X}_i + \hat{A}_0^{ext}), \tag{5.7}$$

where $\tilde{L}_{ext}^{CS}$ is given by (2.20) with $\hat{A}_\mu$ replaced by $\hat{A}_\mu^{CS}$. In this coupling scheme, which we call hybrid, the CS field is coupled via the minimal substitution rule (2.19) while the electromagnetic background field is coupled like in the Duval-Horvathy model.

If we consider the case of constant external fields $\hat{B}^{ext}$ and $\hat{E}^{ext}$ we find that the second term in (5.7) becomes, modulo a gauge dependent total time-derivative term,

$$e\left(\frac{1}{2}\hat{B}^{ext}\,\epsilon_{ij}\,X_i\,\dot{X}_j + \hat{E}_i^{ext}\cdot X_i\right). \tag{5.8}$$

Note that from the two terms in (5.8) the first one is known to be invariant with respect to the time-independent area preserving coordinate transformations ([12], [26]), but the second is not invariant. However, we can further modify the action by adding the following term proportional to θ:

$$-\frac{e^2\theta}{2}\,\epsilon^{\mu\nu\rho}\,\hat{F}_{\nu\rho}^{ext}\,\hat{A}_\mu^{CS}. \tag{5.9}$$

With such a term we obtain instead of (5.8)

$$\frac{e\hat{B}^{ext}}{2}(\epsilon_{ij}X_i\dot{X}_j \,-\, 2e\theta\hat{A}_0^{CS}) + e\hat{E}_i^{ext}(X_i \,+\, e\theta\,\epsilon_{ij}\hat{A}_j^{CS}), \tag{5.10}$$

and we see that in the second term of (5.8) X_i has become replaced by the invariant coordinate $X_i \,+\, e\theta\,\epsilon_{ij}\hat{A}_j^{CS} = \eta_i(\vec{X},t)$.

Note that (5.10) is quasi-invariant with respect to time-dependent area-preserving transformations (2.21-22)

$$\delta(\epsilon_{ij}X_i\dot{X}_j \,-\, 2e\theta\hat{A}_0^{CS}) \;=\; e\theta\,\frac{d}{dt}\,(X_i\partial_i\Lambda \,-\, 2\Lambda). \tag{5.11}$$

So we have

$$L_{hyb} \;=\; \tilde{L}_{ext}^{CS} \,+\, \frac{e\hat{B}^{ext}}{2}(\epsilon_{ij}X_i\dot{X}_j \,-\, 2e\theta\hat{A}_0^{CS}) \,+\, e\hat{E}_i^{ext}\cdot\eta_i. \tag{5.12}$$

We would like to make the following comments:

(i) The additional terms (5.10) lead to the change of the symplectic structure from (2.7) to (2.18) with $B = \hat{B}^{ext}$.

(ii) Expression (5.9) looks like the interaction of an induced current

$$J_\theta^\mu := -\frac{e\theta}{2}\,\epsilon^{\mu\nu\rho}\,\hat{F}_{\nu\rho}^{ext} \tag{5.13}$$

with the CS-gauge potential $\hat{A}_\mu^{CS}$. Obviously, the current J_θ^μ is conserved.

(iii) Arbitrary time-dependence of $\hat{E}_i^{ext}$ preserves the quasi-invariance of L_{hyb} with respect to the transformations (2.21-22). However, any space-dependence of $\hat{F}_{\mu\nu}^{ext}$ or time-dependence of $\hat{B}^{ext}$ spoils it.

One can consider L_{hyb} given by (5.12) for the critical value of the B field i.e. at

$$\hat{B}_{crit}^{ext} = (e\theta)^{-1}. \tag{5.14}$$

Then,

- the two terms being proportional to $\hat{A}_0^{CS}$ in (5.12) add up to zero and so, due to the Gauss constraint, the $\hat{A}_i^{CS}$ becomes trivial, i.e. the CS field decouples from our particles.

- By the point transformation [10]

$$X_i \;\rightarrow\; q_i := \; X_i \;+\; \theta\,\epsilon_{ik}\,P_k \tag{5.15}$$

one finds as derived by Duval et al [10] that

$$L_{hyb} \;=\; \frac{1}{2\theta}\,\epsilon_{ij}\,q_i\dot{q}_j \tag{5.16}$$

i.e. the particle phase-space reduces to two degrees of freedom. Furthermore, the particle EOM reduce to the Hall constraint [10]

$$P_i = e\theta\epsilon_{ij}E_j. \tag{5.17}$$

We see, therefore, that in the critical case (5.14), even in the presence of a CS-coupling, the Hilbert space reduces to the well known subspace of the lowest Landau level describing the Quantum Hall Effect.

6 Outlook

The aim of this paper has been to discuss the couplings with a gauge field of our planar particle model [1,9] which provides, via canonical quantization, noncommutative position coordinates (see (2.7)). The relations (2.7) are invariant under time-dependent area-preserving transformations (2.21).

In our paper we have presented a coupling of Abelian gauge fields which transform under generalized gauge transformations (see (2.24)) in a way which implies the invariance of the action under the joint transformations (2.21) and (2.22). We have shown that after changing the phase space variables for point planar particles and introducing classical SW transformation for gauge fields one can identify our model with the one containing gauge coupling as presented by Duval and Horvathy [9,10]. We would like to stress here that our classical SW transformation (see (3.10-12)) relates the gauge fields formulated on two noncommutative coordinate spaces (see (2.18) and (2.7)) which, only to the first order in θ, coincides with the standard SW transformations.

Our results on the two-body problem, with the inclusion of an external magnetic field, should be further extended. Detailed quantum mechanical calculations along the lines given in a recent paper by Correa et al [27] are called for.

The considerations presented in this paper describe nonrelativistic dynamics in $2+1$ dimensions. In such a case the action (1.5) is Galilean-invariant. The analogous relativistic model can be constructed in $D=1+1$. In a general D-dimensional relativistic case we could introduce the following extension of the action for a relativistic massless particle

$$\mathcal{L} = \frac{1}{e}\dot{X}_\mu^2 - \frac{k}{e^2}\dot{X}_\mu \ddot{X}_\nu \theta^{\mu\nu}, \tag{6.1}$$

where $\dot{X}_\mu \equiv \frac{dX_\mu}{ds}$ and s describes a parametrization of the particle trajectory and e is an einbein variable transforming under reparametrization $s' = s'(s)$ by the formula $e'(s') = \left(\frac{ds'}{ds}\right)^{-1} e(s)$. Unfortunately, if $\theta^{\mu\nu}$ is a constant, the action (6.1) breaks the D-dimensonal Lorentz invariance[9].

One of the questions which should be also addressed is the second quantization of the model (1.5), i.e. the passage from the classical and quantum mechanics to the corresponding field-theoretic model.

The required $D = 2+1$-dimensional field-theoretic model should have the following properties[10]:

i) In the limit $\theta \to 0$ it should become the Schrödinger theory for free nonrelativistic $D=2$ particles.

ii) For $\theta \neq 0$ it should be invariant under the Galilei group with two central charges, m and θ, and should lead to the nonvanishing value of θ from the commutator of generators of Galilei boosts.

Finally we would like to observe that in this paper we have dealt only with the couplings of Abelian gauge fields. In order to consider coupled non-Abelian gauge fields we would have to extend our model from [1] by supplementing the space-time geometry by new degrees of freedom describing non Abelian charge space coordinates (see [31-34]).

---

[9] We would like to mention that the relativistic invariance can be restored if we promote the constant $\theta^{\mu\nu}$ to a one-dimensional field $\theta^{\mu\nu}(s)$ (see e.g. [28]).

[10] Such a model would help to solve the problem of the relation between the second Galilean central charge and spin, recently discussed by Hagen [29]. A first attempt to construct such a model has been done very recently by Horvathy et al. [30].

Appendix - Gauge Field Dependent Dreibein Formalism

In this appendix we would like to derive a gauge field-dependent dreibein formalism.

We solve (2.31) for $\mathcal{P}_k$ and so get

$$\mathcal{P}_k = \dot{X}_i E_{ik} + E_{0k} \,, \tag{A.1}$$

where

$$\begin{aligned} E_{ik} &= (1 + e\theta\widehat{B})^{-1}(\delta_{ik} + e\theta\varepsilon_{kj}\partial_i\widehat{A}_j), & \text{(A.2)} \\ E_{0k} &= e\theta\varepsilon_{ij}\partial_j\widehat{A}_0 E_{ik} & \text{(A.3)} \end{aligned}$$

describes a dreibein differing from the one proposed in [13], in the case of components (A.2), only by an invariant factor. The dreibein components (A.2-A.3) transform with respect to the transformations (2.21-22) as follows:

$$\delta E_{\mu k} = e\theta\varepsilon_{ij}(\partial_\mu\partial_j\Lambda)E_{ik} \,, \tag{A.4}$$

which is a special case of the general transformation formula for a generic field $f(\overrightarrow{X}, t)$ [18]

$$\delta(\partial_\mu f) = \partial_\mu \delta f + e\theta\varepsilon_{kj}(\partial_\mu\partial_j\Lambda)\partial_k f \,. \tag{A.5}$$

The formulae (A.2-A.3) can be treated as the modification, with non-vanishing torsion, of the torsion-less θ-dependent dreibein presented in [13] (see [13], formula (20)), with the components $E_{00} = 1$ and $E_{k0} = 0$ kept unchanged.

The inverse dreibeins $e_\mu^\rho E_\rho{}^\nu = \delta_\mu{}^\nu$ have a simple form ($e_\mu{}^k \equiv e_{\mu k}$)

$$\begin{gathered} e_0{}^0 = 1 \,, \qquad e_i{}^0 = 0 \\ e_{\mu k} = \delta_{\mu k} + e\theta\varepsilon_{ik}\partial_i\widehat{A}_\mu \end{gathered} \tag{A.6}$$

and provide the formula for the derivative

$$D_\mu = e_\mu{}^\nu \partial_\nu \,, \tag{A.7}$$

which is invariant under the local transformations (2.21-22).

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