Test of gauged $\mathcal{N} = 8$ SUGRA / $\mathcal{N} = 1$ SYM duality at sub-leading order.

Paul Mansfield[a] and David Nolland[b]

[a] *Department of Mathematical Sciences*
University of Durham
South Road
Durham, DH1 3LE, England
e-mail: p.r.w.mansfield@durham.ac.uk

[b] *Department of Computing and Mathematics*
University of Northumbria
Ellison Building, Ellison Place
Newcastle Upon Tyne, NE1 8ST, England
e-mail: david.nolland@unn.ac.uk

Abstract

An infra-red fixed point of $\mathcal{N} = 1$ super-Yang-Mills theory is believed to be dual to a solution of five-dimensional gauged $\mathcal{N} = 8$ supergravity. We test this conjecture at next to leading order in the large N expansion by computing bulk one-loop corrections to the anomaly coefficient $a - c$. The one-loop corrections are non-zero for all values of the bulk mass, and not just special ones as claimed in previous work.

There have been many successful tests of Maldacena's conjecture [1] that IIB string theory compactified on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ super-Yang-Mills theory with gauge group $SU(N)$ on the boundary of AdS_5. This has led to extensive studies of other conjectured holographic dualities.

A particularly interesting possibility is that an infra-red fixed point of massive $\mathcal{N} = 1$ super-Yang-Mills theory can be described by a solution of five-dimensional gauged $\mathcal{N} = 8$ supergravity, [2], [3], [4]. The purpose of the present letter is to test this correspondence at next to leading order in the large-N expansion.

The $AdS_5 \times S^5$ compactification [5] pertinent to Maldacena's original conjecture is constructed by assuming that in the field theory limit only the metric and five-form field strength have non-zero vacuum expectation values. Kaluza-Klein decomposition of the fields on S^5 gives an infinite number of supermultiplets of $U(2, 2/4)$ propagating in AdS_5. Assuming that this theory can be consistently truncated to its 'massless' multiplet leads to a description in terms of five-dimensional gauged $\mathcal{N} = 8$ supergravity [6],[7]. Allowing certain scalars also to be non-zero introduces mass terms into the boundary theory and breaks $\mathcal{N} = 4$ down to $\mathcal{N} = 1$. This deforms the $AdS_5 \times S^5$ structure of the bulk theory, but when the scalars are at the critical point of their potential the bulk spacetime is again AdS_5 and so the boundary theory must be conformally invariant. It is thought to be the infra-red fixed point of the renormalisation group flow driven by the mass terms. Evidence for this comes from a comparison of symmetries in the bulk and boundary theories, and a comparison of the results of tree-level computations in the bulk theory, (valid to leading order in large N) with exactly computable quantities in the boundary theory. For example the bulk tree-level mass spectrum has been compared with scaling dimensions in the boundary theory [2], and the trace anomaly coefficients of the boundary theory (conventionally called a and c) have been correctly reproduced to leading order in N using the saddle-point method of [8] in the bulk theory [2],[4],[13]. In this letter we show that the latter test holds also at the next order in the large-N expansion. We consider only the combination $a - c$, as was done for the Maldacena conjecture itself in [9].

When a four dimensional gauge theory is coupled to a non-dynamical, external metric, g_{ij}, the Weyl anomaly, $\mathcal{A}$, is the response of the logarithm of the partition function, $F[g_{ij}]$, to a scale transformation of that metric:

$$\delta F = \int d^4x\, \sqrt{g}\, \delta\rho\, \langle T^i_i \rangle = \int d^4x\, \sqrt{g}\, \delta\rho\, \mathcal{A}, \qquad \delta g_{ij} = \delta\rho\, g_{ij}, \tag{1}$$

with T_{ij} the stress-tensor. On general grounds $\mathcal{A}$ must be a linear combination of the Euler density, E, and the square of the Weyl tensor, I, so $\mathcal{A} = a\,E + c\,I$. The coefficients a, c are known exactly both for the $\mathcal{N} = 4$ gauge theory that is the subject of Maldacena's conjecture and for the infra-red fixed point of the $\mathcal{N} = 1$ theory. In both cases $a = c$. The numerical values of a and c are reproduced to leading order in large N by tree-level calculations in the appropriate bulk supergravity theories using just the Einstein-Hilbert part of the action. In [9] it was shown that for the Maldacena conjecture $a = c$ continued to hold at next to leading order in N when bulk one-loop effects contribute for each of the fields in the supergravity theory. The results of [9] can be summarised in the statement that to this order $a - c$ is proportional to $\Phi \equiv \sum(\Delta - 2)\,\alpha$, where Δ depends on the mass of the bulk field and coincides with the scaling dimension of the boundary Green's function and α is a numerical coefficient occuring in the short-distance expansion of an appropriate heat-kernel. The sum runs over all the fields in the bulk theory. The values of α are: 1 for a real scalar, ϕ; 7/2 for a complex spinor, χ; -11 for a vector, A_μ; 33 for a two-form, $B_{\mu\nu}$; $-219/2$ for a complex Rarita-Schwinger field, ψ_μ; and 189 for a symmetric second rank tensor, $h_{\mu\nu}$. Faddeev-Popov ghosts must be included when there is a gauge symmetry. This occurs for vector fields when $\Delta - 2 = 1$, when the net effect of the ghosts is an additional contribution to Φ of -2. Similarly, the Rarita-Schwinger field has a gauge symmetry when $\Delta - 2 = 3/2$ and the ghosts contribute $-35/4$, and the graviton has a gauge symmetry for $\Delta - 2 = 2$ in which case the ghosts contribute 33. When Φ was computed for the Maldacena conjecture [9] it was found that the sum over each of the infinite number of $U(2, 2/4)$ supermultiplets vanished so that $a = c$ for the full theory agreeing with the boundary theory result.

The supergravity theory conjectured to be dual to the infra-red fixed point of the $\mathcal{N} = 1$ gauge theory is a truncation of the ten-dimensional theory and its fields are organised into a finite number of supermultiplets of $SU(2, 2|1)$. Φ is readily computed from the mass spectrum. The ingredients of the calculation are given in tables 1 and 2 corresponding to tables 6.1 and 6.2 of [2]. Each field is in a representation of $SU(2)_I$. The dimensions of the representations are given in the tables and contribute to Φ. When the representation is complex the contribution to Φ is doubled for the bosonic fields. The fermionic fields are assumed to be complex already so a factor of 1/2 is included for real representations. We should note that the sign of the contribution of the first scalar in Table 1 is taken in accordance with the comments of [2] so as to fit the standard relation for the scaling dimensions of chiral primaries. Summing Φ over all the supermultiplets in the tables gives

$$\Phi = 0 - 135/4 + 225/2 + 45/4 + 225 + 0 - 675/2 + 45/2 = 0\,,$$

so that $a = c$ to next to leading order in N in the bulk theory, in agreement with the exact result in the boundary theory to which it is conjectured to be dual. This result also provides a check on the spectrum of [2].

Finally we outline the derivation of Φ. It is easy to solve Einstein's equations with cosmological constant $\Lambda = -6/l^2$ in the bulk when the boundary metric is Ricci flat to obtain

$$ds^2 = \frac{1}{t^2}\left(l^2\,dt^2 + \tau'^2 \sum_{i,j} g_{ij}\,dx^i\,dx^j\right), \qquad t \geq \tau' \tag{2}$$

where τ' is a regulator that ultimately should be taken to zero. For a Ricci-flat boundary $E = -I$ so that $\mathcal{A} = (a - c)\,E$, and working with this metric will only reveal the combination $a - c$. The central object of interest in the AdS/CFT correspondence is the 'partition function' given as a functional integral for the bulk theory in which the fields have prescribed values, φ, on the boundary at $t = \tau'$ [10, 11]. The regulator is necessary even in tree-level calculations but at one-loop we also need a large t cut-off τ; this introduces another boundary, and the functional integral should be performed with the fields taking prescribed values, $\tilde{\varphi}$, there as well. Consequently the partition function is the limit as the cut-offs are removed of a functional $\Psi_{\tau,\tau'}[\tilde{\varphi}, \varphi]$. The exponential of $F[g_{ij}]$

Table 1: Φ for the five short $SU(2,2|1)$ representations

Representation	$\Delta-2$	ϕ	χ	A_μ	$B_{\mu\nu}$	ψ_μ	$h_{\mu\nu}$	Φ
$\mathcal{D}(3/2,0,0;1)$	$-1/2$	**3**						-3
	0		**3**					0
complex	$1/2$	**3**						3
Total								**0**
$\mathcal{D}(2,0,0;0)$	0	**3**						0
	$1/2$		$\mathbf{3}\oplus\mathbf{3}$					$21/4$
real	1			**3**				-39
Total								$\mathbf{-135/4}$
$\mathcal{D}(9/4,1/2,0;3/2)$	$1/4$		**2**					$7/4$
	$3/4$	**2**			**2**			102
complex	$5/4$		**2**					$35/4$
Total								$\mathbf{225/2}$
$\mathcal{D}(3,0,0;2)$	1	**1**						2
	$3/2$		**1**					$21/4$
complex	2	**1**						4
Total								$\mathbf{45/4}$
$\mathcal{D}(3,1/2,1/2;0)$	1			**1**				-13
	$3/2$					$\mathbf{1}\oplus\mathbf{1}$		-173
real	2						**1**	411
Total								**225**

is the field independent part of this partition function. With the regulators in place the free energy becomes a function of τ, τ', and the Weyl anomaly can be found by exploiting the invariance of the five-dimensional metric (2) under $t \to (1+\delta\rho/2)t$, $g_{ij} \to (1+\delta\rho)g_{ij}$, with $\delta\rho$ constant. So, for a constant Weyl scaling

$$\delta F = \int d^4x\,\sqrt{g}\,\delta\rho\,\mathcal{A} = -\frac{\delta\rho}{2}\left(\tau\frac{\partial F}{\partial\tau} + \tau'\frac{\partial F}{\partial\tau'}\right) \tag{3}$$

At one-loop we only need the quadratic fluctuations in the action, so the fields are essentially free. In [12] we computed the Weyl anomaly for free scalar and spin-half particles for the metric (2), not by performing a functional integration but by interpreting $\Psi_{\tau,\tau'}[\tilde\varphi,\varphi]$ (after Wick rotation of g_{ij}) as the Schrödinger functional, i.e. the matrix element of the time evolution operator between eigenstates of the field, $\langle\,\tilde\varphi\,|\,T\,\exp(-\int_{\tau'}^{\tau} dt\,H(t))\,|\,\varphi\,\rangle = \Psi_{\tau,\tau'}[\tilde\varphi,\varphi]$.

To illustrate this consider a massless scalar field propagating in the metric (2). Ψ satisfies the functional Schrödinger equation

$$\frac{\partial}{\partial\tau}\Psi_{\tau,\tau'}[\tilde\varphi,\varphi] = \frac{1}{2}\int d\mathbf{x}\,\left(\tau^3\,\frac{\delta^2}{\delta\tilde\varphi^2} + \tau^{-3}\,\tilde\varphi\nabla\cdot\nabla\tilde\varphi + 2\,\delta^4(0)/\tau\right)\Psi_{\tau,\tau'}[\tilde\varphi,\varphi], \tag{4}$$

Table 2: Φ for the remaining $SU(2,2|1)$ representations

Representation	$\Delta - 2$	ϕ	χ	A_μ	$B_{\mu\nu}$	ψ_μ	Φ
$\mathcal{D}(\sqrt{7}+1,0,0;0)$	$\sqrt{7}-1$	$\mathbf{1}$					$\sqrt{7}-1$
	$\sqrt{7}-1/2$		$\mathbf{1}\oplus\mathbf{1}$				$7(\sqrt{7}-1/2)/2$
real	$\sqrt{7}$	$\mathbf{1}\oplus\mathbf{1}$		$\mathbf{1}$			$-9\sqrt{7}$
	$\sqrt{7}+1/2$		$\mathbf{1}\oplus\mathbf{1}$				$7(\sqrt{7}+1/2)/2$
	$\sqrt{7}+1$	$\mathbf{1}$					$\sqrt{7}+1$
Total							$\mathbf{0}$
$\mathcal{D}(11/4,1/2,0;1/2)$	$3/4$		$\mathbf{2}$				$21/4$
	$5/4$	$\mathbf{2}$		$\mathbf{2}$	$\mathbf{2}$		115
complex	$7/4$		$\mathbf{2}\oplus\mathbf{2}$			$\mathbf{2}$	$-1435/4$
	$9/4$			$\mathbf{2}$			-99
Total							$\mathbf{-675/2}$
$\mathcal{D}(3,0,1/2;1/2)$	1		$\mathbf{1}$				$7/2$
	$3/2$			$\mathbf{1}$	$\mathbf{1}$		66
complex	2		$\mathbf{1}$			$\mathbf{1}$	-212
	$5/2$				$\mathbf{1}$		165
Total							$\mathbf{45/2}$

with a similar equation for the τ' dependence, and an appropriate initial condition as τ' approaches τ.[1] $\log\Psi$ has the form $F+\int d^d\mathbf{x}\left(\frac{1}{2}\tilde{\varphi}\,\Gamma_{\tau,\tau'}\,\tilde{\varphi}+\tilde{\varphi}\,\Xi_{\tau,\tau'}\,\varphi+\frac{1}{2}\varphi\,\Upsilon_{\tau,\tau'}\,\varphi\right)$. The kernels can be expressed in terms of simpler operators $\Gamma_{\tau,0}\equiv\Gamma(\Omega)/\tau^3$, $\Xi_{\tau,0}\equiv\Xi(\Omega)/\tau^3$ and $\Upsilon_{\tau,0}\equiv\Upsilon(\Omega)/\tau^3$, where $\Omega\equiv-\tau^2\nabla^2$ by using the self-reproducing property $\int\Psi_{\tau,\tau'}[\tilde{\varphi},\phi]\,\mathcal{D}\phi\,\Psi_{\tau',\tau''}[\phi,\varphi]=\Psi_{\tau,\tau''}[\tilde{\varphi},\varphi]$:

$$\Gamma_{\tau,\tau'}=\frac{1}{\tau^4}\left(\Gamma(\Omega)+\left(\left(\frac{\tau}{\tau'}\right)^4\Upsilon(\Omega')-\Upsilon(\Omega)\right)^{-1}\Xi^2(\Omega)\right),$$

$$\Upsilon_{\tau,\tau'}=\frac{1}{(\tau'+\epsilon)^4}\left(-\Gamma(\Omega')+\left(\Upsilon(\Omega')-\left(\frac{\tau'+\epsilon}{\tau}\right)^4\Upsilon(\Omega)\right)^{-1}\Xi^2(\Omega')\right),$$

where $\Omega'\equiv-\tau'^2\nabla^2$. The ϵ prescription is needed to ensure that this last expression reduces to $\Upsilon(\Omega)/\tau^3$ as $\tau'\to 0$. The Schrödinger equation relates $\tau'\partial F/\partial\tau'$ to the functional trace of $(\tau'+\epsilon)^4\Upsilon_{\tau,\tau'}$. When this is regulated with a cut-off on Ω' then it simplifies as $\tau'\to 0$ and $\tau\to\infty$

$$-\Gamma(\Omega')+\left(\Upsilon(\Omega')-\left(\frac{\tau'+\epsilon}{\tau}\right)^4\Upsilon((\tau^2/\tau'^2)\Omega')\right)^{-1}\Xi^2(\Omega')\to-\Gamma(\Omega')$$

[1] This initial condition is quite subtle. In flat space we would get a delta-functional in the $\tau\to\tau'$ limit. The solution we found in [9] has the delta-functional property in this limit provided we use the ϵ prescription discussed here.

because for large argument $\Upsilon((\tau^2/\tau'^2)\Omega) \sim ((\tau^2/\tau'^2)\Omega)^2$. Similarly $\tau\partial F/\partial\tau$ is related to the functional trace of $\tau^4\Gamma_{\tau,\tau'}$ which must be regulated with a cut-off on Ω and simplifies as $\tau' \to 0$ and $\tau \to \infty$

$$\Gamma(\Omega) + \left(\left(\frac{\tau}{\tau'}\right)^4 \Upsilon((\tau'^2/\tau^2)\Omega') - \Upsilon(\Omega)\right)^{-1} \Xi^2(\Omega) \to \Gamma(\Omega)$$

Γ is obtained as a power series in Ω from the Schrödinger equation, and the regulated traces are calculated using the heat kernel for Ω. The short distance expansion of the heat kernel finally gives the trace as being proportional to $-\Gamma(0) = 2(\Delta - 2)$. This is readily extended to massive scalars, and, with some work, to all the other fields in the theory.

The simplification of the traces of $\Gamma_{\tau,\tau'}$ and $\Upsilon_{\tau,\tau'}$ to that of Γ occurs for all values of the mass, not just special values as claimed in [9, 12]. This is a result of the ϵ prescription introduced above. The prescription may be understood by writing the Schrödinger functional in terms of field variables which give the expected flat-space behaviour in the $\tau \to \tau'$ limit. A similar prescription holds for fermions.

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