Finding paths in graphs avoiding forbidden transitions

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Abstract

Let v be a vertex of a graph G; a *transition graph* $T(v)$ of v is a graph whose vertices are the edges incident with v. We consider graphs G with prescribed *transition systems* $T = \{\, T(v) \mid v \in V(G) \,\}$. A path P in G is called T*-compatible*, if each pair uv, vw of consecutive edges of P form an edge in $T(v)$. Let $\mathcal{A}$ be a given class of graphs (closed under isomorphism). We study the computational complexity of finding T-compatible paths between two given vertices of a graph for a specified transition system $T \subseteq \mathcal{A}$. Our main result is that a dichotomy holds (subject to the assumption P $\neq$ NP). That is, for a considered class $\mathcal{A}$, the problem is either (1) NP-complete, or (2) it can be solved in linear time. We give a criterion—based on vertex induced subgraphs—which decides whether (1) or (2) holds for any given class $\mathcal{A}$.

Key words: Transition, compatible path, complexity, NP-completeness, linear time algorithm, edge-colored graph, forbidden pairs.

1 Introduction

A *transition* in a graph is a pair of adjacent edges. We consider the problem of finding paths between two given vertices avoiding *forbidden transitions* (or equivalently, paths that use only *allowed transitions*); such paths are called *compatible*. For a vertex v, the set of allowed transitions define a graph $T(v)$ on the set of edges incident with v; the graph $T(v)$ is called the *transition graph* of v.

If, for example, all transition graphs are cliques, then the compatible-path-problem is trivial, since every path is compatible. It is natural to ask for the

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computational complexity of the compatible-path-problem with respect to a given class $\mathcal{A}$ of transition graphs. We give a complete answer to this question showing that—with respect to a given class $\mathcal{A}$—the problem is either

(1) NP-complete, or
(2) can be solved in linear time

(both cases exclude each other if P $\neq$ NP holds). We supply a 'forbidden induced subgraph'-criterion which decides whether (1) or (2) prevails. For the linear time cases we present algorithms which not only *decide* existence but actually *find* a compatible path (if one exists) in linear time.

In the final section we give an application of our linear time results to the problem of finding alternating paths in edge-colored graphs and extend a result by Bang-Jensen and Gutin [1].

Our work is closely related to [10], where the complexity of finding compatible 2-factors is studied. Compatible paths and cycles have been intensively studied in the context of eulerian graphs (see [5] for a survey on this topic).

2 Basic notions and notation

For graph theoretic terminology not defined in this paper, the reader is referred to [3]. All graphs considered are finite and simple. The set of vertices and the set of edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For a vertex $v \in V(G)$ we write $E_G(v)$ for the set of all edges of G which are incident with v. The degree of a vertex v is denoted by $d(v)$; for $d \geq 0$ we put $V_d(G) := \{ v \in V(G) \mid d(v) = d \}$. We write $H \leq G$ if H is a vertex-induced subgraph of G.

Let G be a graph. A *transition graph* $T(v)$ of a vertex $v \in V(G)$ is some graph whose vertices are the edges of G incident to v; i.e., $V(T(v)) = E_G(v)$. A *system of allowed transitions* (or *transition system*, for short) T is a set $\{ T(v) \mid v \in V(G) \}$ where $T(v)$ is a transition graph of v. A path $P = v_0, e_1, v_1, \ldots, e_k, v_k$ in G is T*-compatible* if $e_i e_{i+1} \in E(T(v_i))$ for every $1 \leq i \leq k-1$.

Instead of specifying allowed transitions we could also specify explictly the forbidden transitions by defining $F(v)$ to be the compliment graph of $T(v)$; this approach has been pursued, e.g., in [7,6,5]. However, we consider allowed transitions in order to be consistent with a closely related work by Kratochvíl and Poljak [10], who studied the complexity of finding compatible 2-factors.

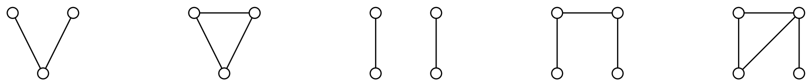

Fig. 1.

Let $\mathcal{A}$ be a class of graphs closed under isomorphism. We consider the following problem.

$\mathcal{A}$-COMPATIBLE PATH (or **$\mathcal{A}$-CP** for short)

Instance: a graph G with transition system $T \subseteq \mathcal{A}$, two distinct vertices $x, y \in V(G)$.

Question: is there a T-compatible path from x to y?

Obviously, this problem is in NP.

We refer by P_3, K_3, $K_2 + K_2$, P_4, and L_4 (the lollipop graph on four vertices) to the graphs depicted in Figure 1; the closure of a class $\mathcal{A}$ of graphs by taking vertex-induced subgraphs is denoted by $\mathcal{A}^{\text{ind}}$. Now we can formulate our main result.

Theorem 1 *The problem **$\mathcal{A}$-CP** is NP-complete if $\mathcal{A}^{\text{ind}}$ contains at least one of the sets*

$$\{P_3, K_2 + K_2\},\ \{K_3, K_2 + K_2\},\ \{P_4\},\ \{L_4\};$$

in all other cases the problem is solvable in linear time.

3 NP-completeness results

The following proposition is the key for our NP-completeness results. In its proof we use an intermediary reduction of **3-SAT** (which is well-known to be NP-complete, see e.g., [9]) to the problem of finding paths in graphs avoiding forbidden pairs of vertices (i.e., paths that use at most one vertex of each prescribed forbidden pair). This reduction is based on a construction by Gabow et.al. [8]; however, to obtain our results we must proceed more painstakingly than in [8]. (We note in passing that in [8] the quoted NP-completeness result is formulated for directed graphs; though it is easy to observe that it applies as well to the same problem for undirected graphs. Garey and Johnson [9, Problem GT53, p. 203] quote only the directed version of the problem.)

If $X_1, \ldots, X_k$ are graphs, then we write $\langle X_1, \ldots, X_k \rangle$ for the smallest class of graphs closed under isomorphism containing $X_1, \ldots, X_k$.

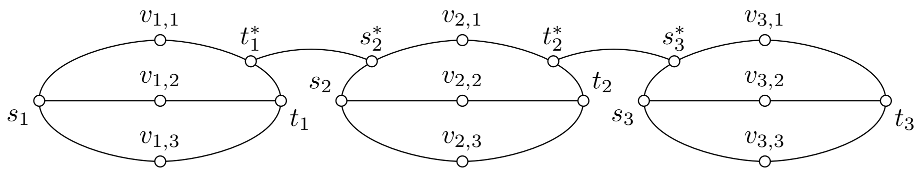

Fig. 2. The graph G obtained from a set of three clauses (cf. the proof of Proposition 2).

Proposition 2 *The problems $\langle K_3, K_2 + K_2\rangle$-**CP** and $\langle P_3, K_2 + K_2\rangle$-**CP** are NP-complete. In particular, these problems remain NP-complete if we consider only instances (G, T, x, y) such that*

- *$x, y \in V_3(G)$,*
- *there is matching M with $V(M) = V_3(G) \setminus \{x, y\}$, and*
- *for every $v \in V_3(G)$ with $E_G(v) = \{e_1, e_2, e_M\}$ and $e_M \in M$ we have $e_1e_M, e_2e_M \in E(T(v))$.*

PROOF. First we show that **3-SAT** reduces polynomially to $\langle K_3, K_2+K_2\rangle$-**CP**; in symbols,

$$\textbf{3-SAT} \propto \langle K_3, K_2 + K_2\rangle\textbf{-CP}. \tag{1}$$

Let A be a set of boolean variables and $\varphi = \{C_1, \ldots, C_n\}$ a collection of clauses (i.e., sets of literals) over A such that for $1 \le i \le n$ the clause C_i is of the form $\{x_{i,1}, x_{i,2}, x_{i,3}\}$ for three distinct literals $x_{i,j}$, $1 \le j \le 3$. We think of φ as a boolean formula in conjunctive normal form. We may assume that no clause of φ contains both x and $\overline{x}$ for any literal x, and that φ has no pure literals (literals that occur only positively or only negatively).

For each clause C_i we construct a graph G_i with $V(G_i) = \{s_i, t_i, v_{i,1}, v_{i,2}, v_{i,3}\}$ and $E(G_i) = \bigcup_{j=1}^{3}\{s_iv_{i,j}, v_{i,j}t_i\}$ (we assume that $V(G_i)$ and $V(G_{i'})$ are disjoint for $i \neq i'$). For $i = 1, \ldots, n-1$ we subdivide the edge $v_{i,1}t_i$ by introducing a new vertex t_i^* and similarly, for $i = 2, \ldots, n$ we subdivide the edge $s_iv_{i,1}$ by introducing a new vertex s_i^*. Let G_i^* denote the graph obtained by this construction. Now we join the graphs $G_1^*, \ldots, G_n^*$ by adding edges $t_i^*s_{i+1}^*$, $i = 1, \ldots, n-1$, and denote the obtained graph by G (for an example see Figure 2). We call a pair of vertices $\{v_{i,j}, v_{i',j'}\}$ *forbidden* if $x_{i,j} = \overline{x_{i',j'}}$ holds for the corresponding literals in φ; let F be the set of all forbidden pairs. An s_1–t_n path P in G is called *satisfying* if $V(P)$ does not contain both vertices of some forbidden pair. It is easy to check that to every truth assignment which satisfies φ there corresponds a satisfying path in G, and vice versa. Since φ contains no pure literals, each vertex $v_{i,j}$ ($1 \le i \le n$, $1 \le j \le 3$) is contained in at least one forbidden pair.

Next we modify G and F such that each vertex is contained in at most one forbidden pair. Assume that a vertex v is contained in $q > 1$ forbidden pairs, say $\{v, v_i\}$, for $i = 1, \ldots, q$. Let u, u' be the two vertices adjacent with v. We remove v from G and add instead a u–u' path of length $q+1$ introducing new vertices $w_1, \dots, w_q$. For $i = 1, \dots, q$ we replace the forbidden pair $\{v, v_i\}$ by $\{w_i, v_i\}$. We repeat this construction for all vertices contained in more than one forbidden pair and denote the resulting graph and the new set of forbidden pairs by G' and F', respectively. It still holds that G' contains a satisfying path if and only if G contains a satisfying path. Note that all vertices of G' have degree 2 or 3; if $d(v) = 2$ then v is contained in exactly one pair of F'; if $d(v) = 3$, then $v \in \{s_i, t_i, s_i^*, t_i^*\}$ for some $1 \le i \le n$.

Let G^* be the graph obtained by identifying vertices which form a forbidden pair; i.e., if $\{v_1, v_2\} \in F'$, $E_{G^*}(v_1) = \{e_1, f_1\}$, and $E_{G^*}(v_2) = \{e_2, f_2\}$, then we remove v_1, v_2, and make e_1, e_2, f_1, f_2 incident with a new vertex $v_{1,2}$. Note that all vertices in the new graph G^* have degree 3 or 4. For each new vertex $v_{1,2}$ we define a transition graph $T^*(v_{1,2}) \in \langle K_2 + K_2 \rangle$ by setting $E(T^*(v_{1,2})) := \{e_1 f_1, e_2 f_2\}$; for the remaining vertices of G^* we choose $T^*(v) \in \langle K_3 \rangle$. We have defined a transition system $T^* = \{\, T^*(v) \mid v \in V(G^*) \,\} \subseteq \langle K_3, K_2 + K_2 \rangle$.

It is easy to verify that a satisfying path in G' corresponds to a T^*-compatible s_1–t_n path in G^*, and vice versa. Since our constructions can be carried out in polynomial time, (1) follows.

Next we show that we can replace T^* by a transition system $T^{**} \subseteq \langle P_3, K_2 + K_2 \rangle$. For all $v \in V_4(G^*)$ we put $T^{**}(v) := T^*(v)$. Since the transition graphs of s_1 and t_n are irrelevant with respect to compatible s_1–t_n paths, we can choose $T^{**}(s_1), T^{**}(t_n) \in \langle P_3 \rangle$ arbitrarily. For each vertex $v \in V_3(G^*) \setminus \{s_1, t_n\}$ we select one of its incident edges h_v:

$$\begin{array}{llll} v = s_i & \Rightarrow & h_v := s_i s_i^* & (2 \le i \le n) \\ v = s_i^* & \Rightarrow & h_v := s_i^* t_{i-1}^* & (2 \le i \le n) \\ v = t_i & \Rightarrow & h_v := t_i t_i^* & (1 \le i \le n-1) \\ v = t_i^* & \Rightarrow & h_v := t_i^* s_{i+1}^* & (1 \le i \le n-1). \end{array}$$

We observe that for every T^{**}-compatible s_1–t_n path P in G^* and $v \in V_3(G^*) \setminus \{s_1, t_n\}$ it follows that

$$v \in V(P) \quad \text{implies} \quad h_v \in E(P). \tag{2}$$

For $v \in V_3 \setminus \{s_1, t_n\}$ with $E_{G^*}(v) = \{e, f, h_v\}$ we put

$$E(T^{**}(v)) := \{e h_v, f h_v\}.$$

By (2) it follows that every T^*-compatible s_1–t_n path is also T^{**}-compatible; the converse holds trivially. Since $T^{**} \subseteq \langle P_3, K_2 + K_2 \rangle$ by definition, we have shown that

$$\langle K_3, K_2 + K_2 \rangle\textbf{-CP} \quad \propto \quad \langle P_3, K_2 + K_2 \rangle\textbf{-CP}.$$

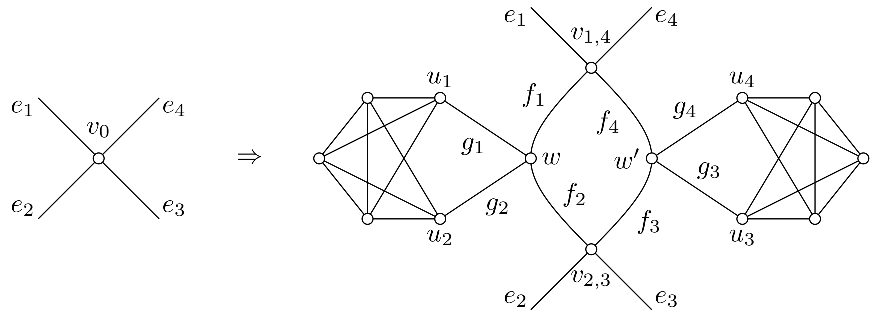

Fig. 3. Illustration of Construction 1

Thus, the first statement of the proposition holds. Note that

$$M = \{ \, t_i t_i^* \mid 1 \leq i \leq n-1 \, \} \cup \{ \, s_i s_i^* \mid 2 \leq i \leq n \, \}$$

is a matching in G^*. Hence it is easy to check that the instances obtained by the above constructions satisfy the second statement of the proposition. $\square$

In view of the proof of the preceding proposition, K_3 and P_3 can be considered as 'branching' transition graphs, whereas $K_2 + K_2$ as an 'exclusive' transition graph. It turns out, that in all NP-complete cases of $\mathcal{A}$**-CP**, $\mathcal{A}$ must contain both, branching transition graphs and exclusive transition graphs. Lemma 3 below shows that any graph on four vertices containing K_3 or P_3 as vertex-induced subgraph can be used as branching transition graph. On the other hand, Lemma 4 shows that P_4 and L_4 are exclusive transition graphs. Finally, Lemma 6 shows that any graph containing K_3 or P_3 (respectively, $K_2 + K_2$, P_4, or L_4) as vertex-induced subgraph can be used as branching (respectively, exclusive) transition graph.

Next we define a construction which we shall use several times in subsequent proofs.

Construction 1 Let G be a graph and v_0 a vertex of G with $E_G(v_0) = \{e_1, \ldots, e_4\}$. We obtain a graph G' as follows (see Figure 3 for an illustration). We split v_0 into vertices $v_{1,4}$ and $v_{2,3}$ such that e_1, e_4 are incident with $v_{1,4}$ and e_2, e_3 are incident with $v_{2,3}$. We add new vertices w, w' and the edges $f_1 = v_{1,4}w$, $f_2 = wv_{2,3}$, $f_3 = v_{2,3}w'$, and $f_4 = w'v_{1,4}$. Now we take two copies of K_5 (say K_5 and K_5') and remove one edge of each copy. Now each copy has two vertices of degree 3, say u_1, u_2 and u_3, u_4, respectively. We add edges $g_1 = u_1w$, $g_2 = u_2w$, $g_3 = u_3w'$, $g_4 = u_4w'$, and denote the new graph by G'. Note that all vertices in $V(G') \setminus V(G)$ have degree 4.

Lemma 3 *Let Z be a graph with at most four vertices such that $K_3 \leq Z$ or $P_3 \leq Z$. Then $\langle Z, K_2 + K_2 \rangle$-**CP** is NP-complete.*

PROOF. Let $X \in \{K_3, P_3\}$ such that $X \leq Z$. If $|V(Z)| = 3$, then the lemma follows directly from Proposition 2; hence assume $|V(X)| = 4$. Writing $\langle X, K_2 + K_2 \rangle$-**CP*** for the problem $\langle X, K_2 + K_2 \rangle$-**CP** restricted to instances which satisfy the additional properties stated in the second claim of Proposition 2, we show that

$$\langle X, K_2 + K_2 \rangle\textbf{-CP*} \quad \propto \quad \langle Z, K_2 + K_2 \rangle\textbf{-CP}. \tag{3}$$

Let (G, T, x, y) be an instance of $\langle X, K_2 + K_2 \rangle$-**CP*** and M a matching of G according to Proposition 2. We apply local replacements to eliminate successively vertices whose transition graph is in $\langle Z \rangle$.

Let $e_M = vv' \in M$ with $E_G(v) = \{e_1, e_4, e_M\}$ and $E_G(v') = \{e_2, e_3, e_M\}$. Note that

$$e_1 e_M, e_4 e_M \in E(T(v)) \quad \text{and} \quad e_2 e_M, e_3 e_M \in E(T(v')). \tag{4}$$

We remove e_M from G and identify v and v', denoting the vertex obtained by identifying v and v' by v_0. Now we apply Construction 1 w.r.t. v_0 and obtain a graph G'. A transition system T' of G' can be defined as follows. For the new vertices $v \in V(G') \setminus V(G)$ we choose transition graphs from $\langle Z \rangle$ such that

$$\begin{aligned} e_1 f_1, e_4 f_1 &\in E(T'(v_{1,4})), \\ f_2 e_2, f_2 e_3 &\in E(T'(v_{2,3})), \\ f_1 f_2 &\in E(T'(w)). \end{aligned}$$

For the remaining vertices v we put $T'(v) = T(v)$. In view of (4), we observe that G contains a T-compatible x–y path if and only if G' contains a T'-compatible x–y path. We apply this construction for each $e_M \in M$ and end up with a graph G'' and a transition system T'' such that $T''(v) \in \langle Z, K_2 + K_2 \rangle$ for all $v \in V(G'') \setminus \{x, y\}$, and $T''(z) \in \langle X \rangle$ for $z \in \{x, y\}$. Hence it remains to alter the transition graphs of x and y.

Let G''_1 be a disjoint copy G'', where $x_1, y_1 \in V(G''_1)$ are the vertices which correspond to $x, y \in V(G'')$, respectively. Further, let T''_1 be the transition system of G''_1 that works on G''_1 exactly as T'' works on G''. We obtain a graph H from the union of G'' and G''_1 by adding edges xx_1 and yy_1. For $z \in \{x, y\}$ (respectively, $z \in \{x_1, y_1\}$) we choose $T_H(z) \in \langle Z \rangle$ such that $T''(z) \leq T_H(z)$ (respectively, $T''_1(z) \leq T_H(z)$). For all other vertices v of H we keep the inherited transition graphs from T'' or T''_1, i.e., $T_H(v) = T''(v)$ or $T_H(v) = T''_1(v)$. By construction, $T_H \subseteq \langle Z, K_2 + K_2 \rangle$. It is easy to check that H contains a T_H-compatible x–y path if and only if G'' contains a T''-

compatible x–y path. Hence (3) holds. Since $\langle X, K_2{+}K_2\rangle$**-CP*** is NP-complete by Proposition 2, the lemma now follows. □

Lemma 4 *Let Z be a graph with at most four vertices such that $K_3 \leq Z$ or $P_3 \leq Z$. The problems $\langle Z, P_4\rangle$***-CP** *and $\langle Z, L_4\rangle$***-CP** *are NP-complete.*

PROOF. We show that

$$\begin{aligned} \langle Z, K_2+K_2\rangle\text{-}\mathbf{CP} \propto \langle Z, P_4\rangle\text{-}\mathbf{CP}, \text{ and} \qquad (5)\\ \langle Z, P_4\rangle\text{-}\mathbf{CP} \propto \langle Z, L_4\rangle\text{-}\mathbf{CP}. \qquad (6)\end{aligned}$$

Let (G, T, x, y) be an instance of $\langle Z, K_2+K_2\rangle$-**CP**. If the transition graph of x or y is isomorphic to K_2+K_2, then we can replace it by some transition graph isomorphic to P_4 without effect to the existence of compatible x–y paths. Hence we have only to consider $v_0 \in V(G)\setminus\{x,y\}$ with $T(v_0) \in \langle K_2+K_2\rangle$. Let $E(T(v_0)) = \{e_1e_2, e_3e_4\}$. We apply Construction 1 w.r.t. v_0 and obtain a graph G'. The transition graphs of the new vertices can be chosen from $\langle P_4\rangle$ such that $f_1f_2 \in E(T'(w))$, $f_3f_4 \in E(T'(w'))$, and

$$\begin{aligned} E(T'(v_{1,4})) &= \{e_1f_1, f_1f_4, f_4e_4\},\\ E(T'(v_{2,3})) &= \{e_2f_2, f_2f_3, f_3e_3\}.\end{aligned}$$

For the remaining vertices $v \in V(G)\setminus V(G')$ we put $T'(v) = T(v)$. We observe that there is a T-compatible x–y path in G if and only if there is a T'-compatible x–y path in G'. By multiple applications of this construction we can eliminate successively all vertices v with $T(v) \in \langle K_2+K_2\rangle$ such that we end up with an instance (G'', T'', x, y) of $\langle Z, P_4\rangle$-**CP**. Obviously, this construction can be carried out in polynomial time, whence (5) follows.

For (6) we proceed likewise. Let (G, T, x, y) be an instance of $\langle Z, P_4\rangle$-**CP**. Again, we replace the transition graphs of x or y by graphs from $\langle L_4\rangle$, if necessary. Now consider some $v_0 \in V(G)\setminus\{x,y\}$ with $T(v_0) \in \langle P_4\rangle$, say $E(T(v_0)) = \{e_1e_2,\ e_1e_3,\ e_3e_4\}$. As above, we construct a graph G' applying Construction 1 and we define a transition system T' for G' as follows. For the new vertices $v \in V(G')\setminus V(G)$ we choose transition graphs from $\langle L_4\rangle$ such that $f_1f_2 \in E(T'(w))$, $f_3f_4 \in E(T'(w'))$, and

$$\begin{aligned} E(T'(v_{1,4})) &= \{e_1f_1, f_1f_4, f_4e_4, e_1f_4\},\\ E(T'(v_{2,3})) &= \{e_2f_2, f_2f_3, f_3e_3, e_3f_2\}.\end{aligned}$$

For the remaining vertices $v \in V(G)\setminus V(G')$ we put $T'(v) = T(v)$. Again, it can be verified that there is a T-compatible x–y path in G if and only if there is a T'-compatible x–y path in G'. Charring out the above construction for every $v \in V(G) \setminus \{x, y\}$ with $T(v) \in \langle P_4 \rangle$ we obtain an instance (G'', T'', x, y) of $\langle Z, L_4 \rangle$-**CP**; whence (6) holds true. In view of Lemma 3 and (5), the result now follows. □

In order to show Lemma 6 below, we need the following result, which is based on an observation by Fleischner [4].

Lemma 5 *Let n, k be positive integers with $n > k$ and $n \equiv k \pmod 2$. Then there is a graph $G_{n,k}$ and $v^* \in V(G_{n,k})$ such that*

(1) $d(v) = n$ for all $v \in V(G_{n,k}) \setminus \{v^*\}$,
(2) $d(v^*) = k$, and
(3) $|V(G_{n,k})| \leq 2n + 2$.

PROOF. *Case 1:* k is even (n may be even or odd). Take K_{n+1} and a matching $M \subseteq E(K_{n+1})$ of size $k/2$. Subdivide the edges in M and identify the new vertices. The graph obtained by this construction has obviously properties 1–3.

Case 2: both n and k are odd. If $k = 1$ we obtain $G_{n,n-1}$ by the above construction; let $u^* \in V(G_{n,n-1})$ with $d(u^*) = n - 1$. Add a new vertex v^* and the edge v^*u^*. Again we have constructed a graph $G_{n,k}$ with the claimed properties. Now assume $k > 1$. By the construction applied in Case 1 we obtain (disjoint) graphs $G_{n,k-1}$ and $G_{n,n-1}$. Let $v^* \in V(G_{n,k-1})$ and $u^* \in V(G_{n,n-1})$ with $d(v^*) = k-1$ and $d(u^*) = n-1$, respectively. We connect the two graphs by adding the edge v^*u^*. The obtained graph satisfies properties 1–3 as well, whence the lemma is shown true. □

Note that if k is odd and $n > k$ is even, then a graph $G_{n,k}$ cannot exist (by the Handshaking Lemma, the number of vertices of odd degree is always even). Therefore, in the proof of the next lemma, we have to apply Lemma 3 as an intermediate step before we can apply the above construction.

Lemma 6 *Let $X \in \{K_3, P_3\}$, $Y \in \{K_2 + K_2, P_4, L_4\}$ and $\mathcal{A}$ a class of graphs closed under isomorphism such that $X, Y \in \mathcal{A}^{\text{ind}}$. Then $\mathcal{A}^{\text{ind}}$-**CP** is NP-complete.*

PROOF. Let $X', Y' \in \mathcal{A}$ such that $X \leq X'$ and $Y \leq Y'$. If $X = X'$, then let $Z := X$; otherwise, let Z be a graph with $|V(Z)| = 4$ and $X \leq Z \leq X'$. We show that

$$\langle Z, Y \rangle\textbf{-CP} \propto \langle X', Y' \rangle\textbf{-CP}. \tag{7}$$

Let (G, T, x, y) be an instance of $\langle Z, Y\rangle$**-CP**. Consider $v_0 \in V(G)$ with $T(v_0) \notin \langle X', Y'\rangle$ (consequently $d(v_0) = 4$). In order to assign to v_0 a transition graph which belongs to $\langle X', Y'\rangle$, we have to increase its degree first. If $T(v_0) \in \langle Z\rangle$, then we put $n := |V(X')|$, otherwise we put $n := |V(Y')|$. Next we take $G_{n,n-4}$ (see Lemma 5) with $v^* \in V(G_{n,n-4})$, $d(v^*) = n - 4$, and we obtain a graph G' from G by adding $G_{n,n-4}$ and identifying v^* and v_0. Let v_0^* denote the vertex obtained by identification of v^* and v_0 (thus $d(v_0^*) = n$ follows). We define a transition system T' of G': if $T(v_0) \in \langle Z\rangle$, then we choose $T'(v_0^*) \in \langle X'\rangle$, otherwise we choose $T'(v_0^*) \in \langle Y'\rangle$; in any case we assure $T(v_0) \leq T'(v_0^*)$; for $v \in V(G) \cap V(G')$ we put $T'(v) := T(v)$, and for $v \in V(G_{n,n-4}) \setminus \{v^*\}$ we choose $T'(v) \in \langle T(v_0^*)\rangle$ arbitrarily.

Applying this construction to all vertices of G whose transition graph is not contained in $\langle X', Y'\rangle$, we end up with an instance (G'', T'', x'', y'') of $\langle X', Y'\rangle$-**CP** such that G contains a T-compatible x–y path if and only if G'' contains a T''-compatible x''–y'' path. Evidently, this construction can be carried out in polynomial time; thus (7) holds true. In view of Lemmas 3 and 4 the result now follows. □

Since $P_3 \leq P_4$ and $P_3 \leq L_4$, Lemma 6 yields the NP-completeness part of Theorem 1.

4 Linear time results

For a graph G we write G° for the graph obtained by removing isolated vertices from it.

Lemma 7 *Let $\mathcal{A}$ be a class of graphs closed under isomorphism such that none of the sets*

$$\{P_3, K_2 + K_2\},\ \{K_3, K_2 + K_2\},\ \{P_4\},\ \{L_4\}$$

is contained in $\mathcal{A}^{\text{ind}}$. Then at least one of the following holds.

(1) X° is a matching for every $X \in \mathcal{A}^{\text{ind}}$.
(2) X° is a complete multipartite graph for every $X \in \mathcal{A}^{\text{ind}}$.

PROOF. The assumption implies that at least one of the following cases prevails.

$$\{P_3, K_3\} \cap \mathcal{A}^{\mathrm{ind}} = \emptyset; \tag{8}$$
$$\{K_2 + K_2, P_4, L_4\} \cap \mathcal{A}^{\mathrm{ind}} = \emptyset. \tag{9}$$

If (8) holds, then we observe that each $X \in \mathcal{A}$ has only vertices of degree 0 or 1; i.e., X° is a matching.

Now we assume that (9) holds. We will show that X° is a complete multipartite graph for every $X \in \mathcal{A}^{\mathrm{ind}}$. Suppose to the contrary that there is some $X \in \mathcal{A}^{\mathrm{ind}}$ such that X° is not a complete multipartite graph. It follows that there are three distinct vertices $u, u', v \in V(X^\circ)$ with $uu' \in E(X^\circ)$ but neither $uv \in E(X^\circ)$ nor $u'v \in E(X^\circ)$. Since X° contains no isolates, v must be adjacent with a fourth vertex v'. Consider the subgraph Y of X° induced by $\{u, u', v, v'\}$. Clearly $Y \in \mathcal{A}^{\mathrm{ind}}$. We have to consider four possibilities for $E_Y(v')$, namely $\{v'v\}$, $\{v'v, v'u\}$, $\{v'v, v'u'\}$, and $\{v'v, v'u, v'u'\}$. We observe that in the first case Y is isomorphic to $K_2 + K_2$, in the second and third case to P_4, and in the last case to L_4. In any case we have a contradiction to (9). Hence X° is complete multipartite for every $X \in \mathcal{A}^{\mathrm{ind}}$. $\square$

Proposition 8 *Let $\mathcal{A}$ be a class of graphs closed under isomorphism such that X° is a matching for every $X \in \mathcal{A}^{\mathrm{ind}}$. Then $\mathcal{A}$-***CP** *can be solved in linear time. In particular, for an instance (G, T, x, y) of $\mathcal{A}$-***CP** *we need at most $\mathcal{O}(|E(G)|)$ time to find a T-compatible x–y path if it exists, or to determine that it does not exist.*

PROOF. Let (G, T, x, y) be an instance of $\mathcal{A}$-**CP**. We set $v_0 = x$, $E_G(v_0) = \{f_1, \ldots, f_r\}$, and trace out a T-compatible path $P = v_0, e_1, v_1, e_2, v_2 \ldots$ as follows. Starting in v_0 we have $r = |E_G(v_0)|$ possibilities for choosing the first edge v_0v_1. However, for every v_i with $i > 0$ there is at most one choice for the next edge; namely, since $T(v_i)^\circ$ is a matching, either $e_i = v_{i-1}v_i$ is an isolated vertex of $T(v_i)$ (and we cannot extend P), or there is a unique edge $v_iw \in E_G(v_i)$ which forms an allowed transition with e_i. In the latter case, if $w \notin \{v_0, \ldots, v_{i-1}\}$, then we extend the path putting $v_{i+1} := w$ (otherwise, if $w \in \{v_0, \ldots, v_{i-1}\}$, then we cannot extend the path). We continue this process until the path cannot be extended. The selection of the first edge is the only nondeterministic step in the above construction; hence we conclude that there are exactly r non-extendable T-compatible paths (say, $P_1, \ldots, P_r$) which start in x. If there is any T-compatible x–y path, then it is evidently a subpath of some P_i, $1 \le i \le r$.

We claim that every edge of G lies on at most two paths P_j, P_k, $(1 \le j < k \le r)$. We suppose the contrary and conclude that some edge of G must be traversed—starting in x—by at least two paths P_j, P_k, in the same direction $(1 \le j < k \le r)$. Let $P_j = v_0, e_1, v_1, e_2, v_2 \ldots e_s, v_s$ and choose the minimal index $i \in \{1, \ldots, s\}$ such that e_i is traversed by P_j and P_k in the same direction.

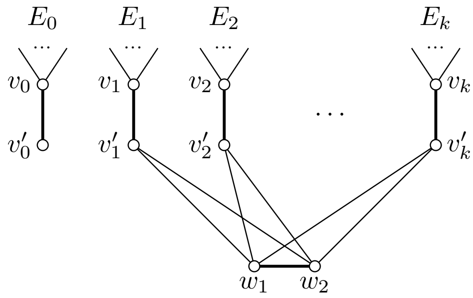

Fig. 4. Illustration for the proof of Proposition 9; the matching $M(v)$ is indicated by bold lines.

By construction of P_j and P_k, $i \neq 1$ follows. Since P_k is T-compatible, some edge f of P_k forms an allowed transition with e_i; thus $fe_i \in E(T(v_{i-1}))$. However, we also have $e_{i-1}e_i \in E(T(v_{i-1}))$. By assumption, $T(v_{i-1})^\circ$ is a matching, hence $e_{i-1} = f$ and so $e_{i-1} \in E(P_k)$ follows. Hence e_{i-1} is traversed by P_j and P_k in the same direction; a contradiction to the particular choice of i. Whence the claim is shown true, and we have

$$\sum_{i=1}^{k} |E(P_i)| \quad = \quad \mathcal{O}(E(G)). \tag{10}$$

Evidently, the construction of each individual path P_i, $1 \leq i \leq r$, can be carried out in $\mathcal{O}(|E(P_i)|)$ time. Hence the proposition follows from (10). $\square$

Proposition 9 *Let $\mathcal{A}$ be a class of graphs closed under isomorphism such that X° is a complete multipartite graph for every $X \in \mathcal{A}^{\text{ind}}$. Then* $\mathcal{A}$-**CP** *can be solved in linear time. In particular, for an instance (G, T, x, y) of* $\mathcal{A}$-**CP** *we need at most $\mathcal{O}(|V(G)| + |E(G)|)$ time to find a T-compatible x–y path if it exists, or to determine that it does not exist.*

PROOF. We are going to reduce $\mathcal{A}$-**CP** to the problem of finding an augmenting path in a graph w.r.t. some matching. Let (G, T, x, y) be an instance of $\mathcal{A}$-**CP**. Isolated vertices of G can be deleted in

$$\mathcal{O}(|V(G)|) \tag{11}$$

time; if x or y is deleted, then G has no T-compatible x–y path by trivial reasons. Hence, w.l.o.g., we assume that G has no isolated vertices. For each $v \in V(G) \setminus \{x, y\}$ we apply the following construction (see Figure 4) which is an extension of a construction used by Tutte [13]. Let $E_0 \subseteq V(T(v))$ be the set of isolated vertices of $T(v)$. Since $T(v)^\circ = T(v) - E_0$ is a complete multipartite graph, there is a partition of $E_G(v) \setminus E_0$ into mutually disjoint sets $E_1, \ldots, E_k$ such that $ef \in E(T(v)^\circ)$ if an only if $e \in E_i$ and $f \in E_j$ for some $i \neq j$. We split v into vertices $v_0, \ldots, v_k$ such that the edges in E_i are incident with v_i, $0 \leq i \leq k$. For $i = 0, \ldots, k$ we add a new vertex v_i' and the edge v_iv_i'. Finally, we add two new vertices w_1 and w_2, the edge w_1w_2, and for each $1 \leq i \leq n$, $1 \leq j \leq 2$, the edge $v'_i w_j$ (note that v'_0 is not joined to w_1 or w_2 and is therefore an end vertex). Evidently, the set $M(v) = \{v_0v'_0, \ldots, v_kv'_k, w_1w_2\}$ is a matching.

We form a graph G' by applying this construction to every $v \in V(G) \setminus \{x, y\}$. Since for every $v \in V(G)$ at most $4d(v)$ new edges have been added, we have

$$|E(G')| = \mathcal{O}(|E(G)|). \tag{12}$$

Assuming the use of suitable data structures, G' can be constructed from G in $\mathcal{O}(|E(G)|)$ time. We put $M = \bigcup_{v \in V(G)\setminus\{x,y\}} M(v)$ and observe that M is a matching of G' which covers all vertices of G' except x and y. It is easy to verify that G contains a T-compatible x–y path if and only if G' contains an M-augmenting path (see Figure 4); moreover, if some some M-augmenting path P' of G' is found, then it can be transformed efficiently into a T-compatible x–y path P in G.

One needs at most $\mathcal{O}(|E(G')|)$ time to detect an M-augmenting path P' of G', and to construct P' if it exists (see [12, pp. 121]). Hence, by (11), (12) and the preceding considerations, the proposition follows. $\square$

In view of Lemma 7, the remaining part of Theorem 1 now follows from Propositions 8 and 9. Whence Theorem 1 is shown true.

5 An application to edge-colored graphs

Let $c \geq 2$ be an integer. A graph G is called *c-edge-colored* if every edge of G has one of c colors; formally, a map $\chi_G : E(G) \to \{1, \ldots, c\}$ is specified (note that the coloring need not be a 'proper' edge-coloring). A path P (or cycle C) in a c-edge-colored graph G is called *properly colored* if incident edges of P (or C, respectively) differ in color. Numerous results on c-edge-colored graphs (and applications of properly colored paths and cycles to genetics) can be found in Chapter 11 of Bang-Jensen and Gutin's book [2].

We consider the following problem.

PROPERLY COLORED PATH

Instance: a c-edge-colored graph G, two distinct vertices $x, y \in V(G)$.
Question: does G contain a properly colored path from x to y?

This problem is known to be solvable in linear time for $c = 2$ ([1], see also [2]).

Applying Proposition 9, we extend this result to arbitrary $c \geq 2$. (It seems to be feasible to apply the construction defined in the proof of Proposition 9 to generalize other results on c-edge-colored graphs from $c = 2$ to the general case; one such application can be found in [11].)

Let G be a c-edge-colored graph. For each $v \in V(G)$ the c-edge-coloring induces a partition of $E_G(v)$ into (nonempty and mutually disjoint) classes $E_1, \ldots, E_k$, $k \leq c$, such that for $e, f \in E_G(v)$, $\chi_G(e) = \chi_G(f)$ if and only if e and f belong to the same class E_i, $1 \leq i \leq k$. This partition defines a complete multipartite graph $T_\chi(v)$ with $V(T_\chi(v)) = E_G(v)$ and $E(T_\chi(v)) = \{ ef \mid \chi_G(e) \neq \chi_G(f) \}$. Let $T_\chi = \{ T_\chi(v) \mid v \in V(G) \}$ be the corresponding transition system. It can be verified easily that properly colored paths and T_χ-compatible paths coincide in G. Whence Proposition 9 yields the following corollary.

Corollary 10 *The problem* **PROPERLY COLORED PATH** *can be solved in linear time.*

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