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CRITICAL RANDOM WALKS ON TWO-DIMENSIONAL COMPLEXES WITH APPLICATIONS TO POLLING SYSTEMS

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We consider a time-homogeneous random walk $\Xi = \{\xi(t)\}$ on a two-dimensional complex. All of our results here are formulated in a constructive way. By this we mean that for any given random walk we can, with an expression using only the first and second moments of the jumps and the return probabilities for some transient one-dimensional random walks, conclude whether the process is ergodic, null-recurrent or transient. Further we can determine when pth moments of passage times τ_K to sets $S_K = \{x : \|x\| \le K\}$ are finite ($p > 0$, real). Our main interest is in a new critical case where we will show the long-term behavior of the random walk is very similar to that found for walks with zero mean drift inside the quadrants. Recently a partial case of a polling system model in the critical regime was investigated by Menshikov and Zuyev who give explicit results in terms of the parameters of the queueing model. This model and some others can be interpreted as random walks on two-dimensional complexes.

1. Introduction. This paper has two distinct parts. In the first and much the larger part we consider a time-homogeneous random walk $\Xi = \{\xi(t)\}$ on a two-dimensional complex. A two-dimensional complex is a union of a finite number of quarter plane lattices $\mathbf{Z}_+^2$ connected at boundaries where one component on each of a specified set of quadrants takes value 0. We are going to consider the specific case where each boundary belongs to only two quarter planes and the drifts combine to take the walk around a cycle. An easy case to picture is the union of the four quarter planes on $\mathbf{Z}^2$, see Figure 1, though for notational convenience we will actually embed a complex with n quarter planes in $\mathbf{R}^n$. Many features of the general case are discussed in [4] and [6] but our case is not treated there as additional techniques and ideas are needed. We will not discuss the general case here.

All of our results here are formulated in a constructive way. By this we mean that for any given random walk we can, with an expression using only the first and second moments of the jumps and the return probabilities for some transient one-dimensional random walks, conclude whether the process is ergodic, null-recurrent or transient. Further we can determine when pth moments of passage times τ_K to sets $S_K = \{x : \|x\| \le K\}$ are finite ($p > 0$, real). To do this we use the

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Lyapunov function method, that is, we demonstrate that a well chosen function of the underlying random walk is a sub(super)martingale and then use the results of [9] and [2] for one-dimensional discrete time stochastic processes. We have restricted ourselves to the two-dimensional case as it is only here that we can give precise necessary and sufficient conditions for recurrence or transience of the random walk and the finiteness or otherwise of the pth moments of the passage times.

Our main interest is in a new critical case where we will show the long-term behavior of the random walk is very similar to that found for walks with zero mean drift inside the quadrants in [5, 6, 3, 2] and for zero drift Brownian motion with reflections in [14] and [12]. In our problem the mean drifts are not zero so the local behavior of this process is quite different from that of the walk with zero mean drift inside the quadrants. Nevertheless, the global behavior of the system is similar to the zero mean drift case in that the tail of the stationary measure will decay polynomially, not exponentially, over the state space in the ergodic case. Further in this case the distribution of the process will converge over time polynomially, not exponentially, to the stationary measure. We do not show this explicitly here but it follows directly from our calculations of passage time moments via the results of [11]. It seems to us that this process is interesting as it exhibits this behavior without any regions of zero drift or heavy tailed jumps.

It is clear that the Lyapunov function method can be applied in more than two dimensions. For example, [10] contains an almost complete classification of random walks on $\mathbf{Z}_+^3$. The main result of that paper also appears as Theorem 4.4.5 of [6]. In the notation of that book the case which was not considered

$$\tilde{K} \equiv \left| \frac{v_2^{\{2,3\}}}{v_3^{\{2,3\}}} \frac{v_3^{\{1,3\}}}{v_1^{\{1,3\}}} \frac{v_1^{\{1,2\}}}{v_2^{\{1,2\}}} \right| = 1$$

is very similar to the main subject of this paper. The completion of the classification and even finding sufficient conditions for ergodicity of walks on $\mathbf{Z}_+^3$ when $\tilde{K} = 1$ is still an open problem.

The ideas in [2] for discrete processes were used in [12] to obtain criteria for finiteness of passage time moments for continuous nonnegative stochastic processes and Brownian motion with zero drift in a wedge. It seems to us that the ideas of our paper can be modified to treat reflected Brownian motion on two-dimensional complexes with internal skew reflections when the drift is nonzero.

In the second part of the paper we use the results of the first part to investigate the properties of polling and similar systems in the critical regime—in a polling system customers arrive at various nodes where they queue for service from a single server which visits the nodes, serving all tasks queueing at a node before moving on to another; see [7]. Recently a partial case of the polling system model was investigated in [13] which gives explicit results in terms of the parameters of the queueing model. This part of the present paper can be regarded as a generalization of the results of [13]. It is not our goal here to study a specific queueing model but just to show how such problems can be recast in the terminology of random walks on two-dimensional complexes. We only consider systems where there are two service nodes but we permit a variety of *regimes*, corresponding to the quarter planes of the complex, in which the behavior of the server and system are different. Each regime has its own set of system parameters, for example, arrival rates, service time distributions. These are restricted by the requirement that we can find an embedded point process which is a random walk on a two-dimensional complex. This allows for regimes where both nodes receive service simultaneously with service times being independent and exponentially distributed, as well as regimes where service times are nonexponential but one node receives all of the service. There may well be other possibilities.

2. Preliminaries and notation.

We start by specifying the way in which the planes are connected and need to assume some regularity properties of the one-step transition probabilities.

Faces and their connections.

1. For any set of integers $1 \le i_1 < \cdots < i_k \le n$, $k \ge 1$, a *face* of $\mathbf{R}^n_+$ is the set
$$B^\wedge \equiv \wedge(i_1, \dots, i_k) = \{x \in \mathbf{R}^n : x_i > 0, i \in \{i_1, \dots, i_k\};\ x_i = 0 \text{ otherwise}\}.$$
2. Ξ lives on the collection $\mathcal{L} = \mathcal{C} \cap \mathbf{Z}^n_+$ where
$$\mathcal{C} = \bigcup_\wedge B^\wedge \text{ with } \wedge \in \{\{i\}, i = 1, \dots, n; \{i, i+1\}, i = 1, \dots, n-1; \{n, 1\}\}.$$
We will use the notation $A_i \equiv B^\wedge \cap \mathbf{Z}^n_+$ where $\wedge = \{i\}$, $i = 1, \dots, n$, and $B_i \equiv B^\wedge \cap \mathbf{Z}^n_+$ where $\wedge = \{i, i+1\}$, $i = 1, \dots, n-1$, with $\wedge = \{n, 1\}$ for $i = n$. Note that we are interested in $\mathcal{L}$ as an essentially two-dimensional object and consider it as embedded in $\mathbf{R}^n$ for notational convenience only. When we refer to coefficients x_i of $x \in \mathcal{L}$ with $i = n + 1$ or $i = 0$ we will interpret this as referring to x_1 or x_n, respectively.
3. The one-step transition probabilities $p_{xy} \equiv P(\xi(t+1) = y | \xi(t) = x) = 0$ unless one of the following conditions holds:
   - $x \in B_i$, $y \in A_i \cup B_i \cup A_{i+1}$ (which means $A_n \cup B_n \cup A_1$ when $i = n$);
   - $x \in A_i$, $y \in B_{i-1} \cup A_i \cup B_i$ (meaning $B_n \cup A_n \cup B_1$ when $i = n$).

   When $n = 4$, $\mathcal{C}$ sits very nicely in the plane as is depicted in Figure 1.

Irreducibility and aperiodicity. We will assume throughout that Ξ is irreducible and aperiodic. This is not necessary for our arguments but no interesting phenomena are lost and it simplifies exposition.

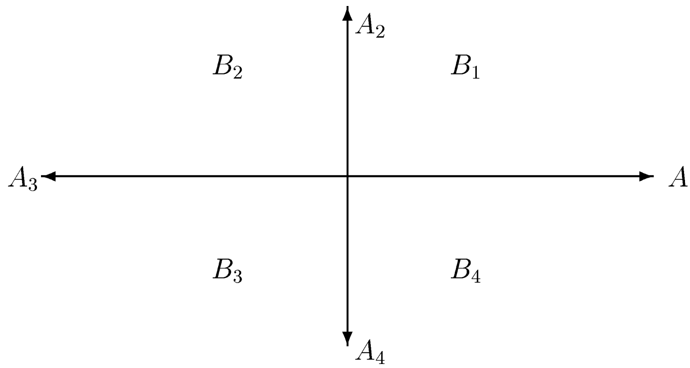

FIG. 1. *The space $\mathcal{C}$ when $n = 4$.*

Homogeneity of jumps. We assume homogeneity of jumps outside some finite set $\mathcal{B}_0$ containing the origin. If $x, \hat{x} \notin \mathcal{B}_0$ belong to the same face (A_i or B_i) and $\hat{y} - \hat{x} = y - x$ then we assume $p_{\hat{x}\hat{y}} = p_{xy}$.

Jumps are bounded below. We require $p_{xy} = 0$ when, for any i, $x \in B_i \setminus \mathcal{B}_0$ and $y - x$ has any component less than -1 or when $x \in A_i \setminus \mathcal{B}_0$ and $(y - x)_i < -\kappa$ for some finite $\kappa \in \mathbf{Z}_+$.

The assumption that jumps from B_i are bounded below by -1 is forced if we wish to maintain both homogeneity and restrict the chain to jumps between adjacent planes only by passing through their shared axis A_i.

Notation for expected jump sizes. It is necessary to assume the existence of certain moments of the jump distributions within the faces and on the boundaries connecting them. Let

$$M(x) = \sum_{y \in \mathcal{L}} (y - x) p_{xy} \qquad \text{for every } x \in \mathcal{L}$$

so $M(x)$ is formally an n-vector and when $x \in B_i$, $M(x)$ is the drift vector of the process Ξ. $M(x)$ does not have any interpretation as a vector when $x \in A_i$ but its three potentially nonzero components $M(x)_{i-1}$, $M(x)_i$ and $M(x)_{i+1}$ are important in the critical case $K = 1$ to be defined below [our notational convention is that $M(x)_{i+1}$ refers to $M(x)_1$ when $i = n$]. By the homogeneity assumption, $M(x) = M(y)$ when x and y are in the same plane B_i or the same axis A_i and outside $\mathcal{B}_0$. For $p > 1$ let

$$M^{(p)}(x) = \sum_{y \in \mathcal{L}} \|y - x\|^p p_{xy} \qquad \text{for every } x \in \mathcal{L} \tag{1}$$

denote the pth moment of the jump length. Here $\|z\|$ denotes the standard Euclidean distance.

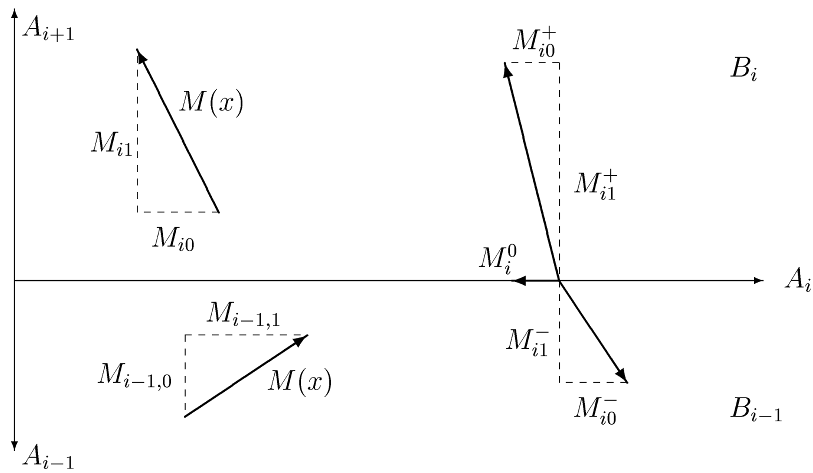

FIG. 2. *Drift vectors for the random walk.*

We need the following notation:

$$M_{i0} = M(x)_i \quad \text{and} \quad M_{i1} = M(x)_{i+1}, \qquad x \in B_i \setminus \mathcal{B}_0,$$

are the components of the drift vector $M(x)$ in the face B_i while for $x \in A_i \setminus \mathcal{B}_0$

$$M_{i0}^+ = \sum_{y \in B_i} (y-x)_i p_{xy}, \qquad M_{i1}^+ = \sum_{y \in B_i} (y-x)_{i+1} p_{xy} \qquad \text{(jumps into } B_i\text{)},$$

$$M_{i0}^- = \sum_{y \in B_{i-1}} (y-x)_i p_{xy}, \qquad M_{i1}^- = \sum_{y \in B_{i-1}} (y-x)_{i-1} p_{xy} \qquad \text{(jumps into } B_{i-1}\text{)}$$

and finally,

$$M_i^0 = \sum_{y \in A_i} (y-x)_i p_{xy} \qquad \text{(jumps along } A_i\text{)}$$

are the various expected jump sizes into the faces of $\mathcal{L}$ which are accessible from A_i. They are depicted in Figure 2.

Moment assumptions.

1. $M^{(\gamma)}(x)$ is finite for some $\gamma > 2$ for every $x \in \mathcal{L}$ [in particular $M(x)$ is finite for every x]. For $x \in B_i \setminus \mathcal{B}_0$ we will use the notation

$$\lambda_{i0} = \sum_{y \in \mathcal{L}} \big((y-x)_i\big)^2 p_{xy}, \qquad R_i = \sum_{y \in \mathcal{L}} \big((y-x)_i (y-x)_{i+1}\big) p_{xy},$$

$$\lambda_{i1} = \sum_{y \in \mathcal{L}} \big((y-x)_{i+1}\big)^2 p_{xy}$$

for second moments of the jumps within $A_i \cup B_i \cup A_{i+1}$.

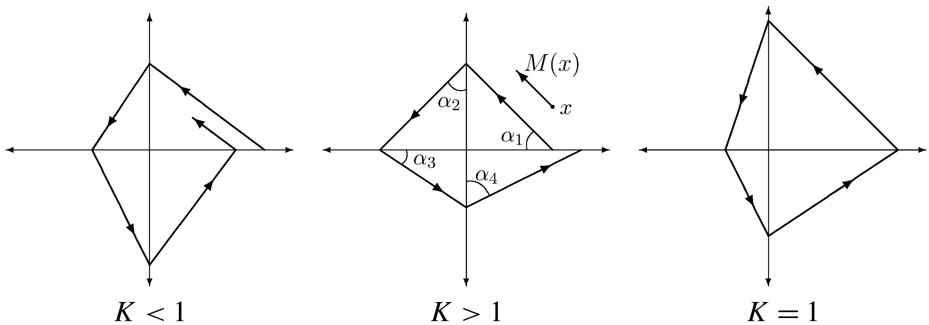

FIG. 3.

2. $M_{i0} < 0$, $M_{i1} > 0$ for every i. We define the angles $\alpha_i \equiv \arctan M_{i1}/|M_{i0}|$ and from these

$$K \equiv \prod_{i=1}^{n} \frac{M_{i1}}{|M_{i0}|} = \prod_{i=1}^{n} \tan \alpha_i. \tag{2}$$

We will give a complete classification of the long-term behavior of Ξ in the three cases $K < 1$ (ergodicity), $K > 1$ (transience) and $K = 1$. The main new results are for this critical case $K = 1$. Figure 3 shows geometrically that in this case, following a connected series of drift vectors through the faces B_i, ignoring what happens at the crossing of the axes A_i, we return to our starting point. It is this that makes Ξ critical.

3. The cases $K < 1$ and $K > 1$. We will make extensive use of a result of Lamperti [8] and its generalization by Aspandiiarov, Iasnogorodski and Menshikov in [2] to deal with the critical case. The noncritical cases with $K \neq 1$ can be resolved with some general Foster-type results from the book by Fayolle, Malyshev and Menshikov. To apply them we must first investigate the behavior of certain *induced* chains as defined in §4 of [6].

The Markov chain X_i induced on A_i is the one-dimensional Markov chain on $\mathbf{Z}$ with transition probabilities defined as follows. Let $L_{im}(j) = \{y \in B_i : y_{i+m} = j\}$ for $m = 0, 1$. Next for $x \in B_i \setminus \mathcal{B}_0$ and $k \geq -1$, define

$$q_i^{+}(k) = P\big(\xi_{i+1}(t+1) = x_{i+1} + k | \xi(t) = x\big) = \sum_{y \in L_{i1}(x_{i+1}+k)} p_{xy}. \tag{3}$$

For $x \in A_i \setminus \mathcal{B}_0$ define $q_i^0(0) = \sum_{y \in A_i} p_{xy}$ and

$$q_i^0(k) = \begin{cases} P\big(\xi_{i+1}(t+1) = k | \xi(t) = x\big) = \displaystyle\sum_{y \in L_{i1}(k)} p_{xy}, & k > 0, \\ P\big(\xi_{i-1}(t+1) = -k | \xi(t) = x\big) = \displaystyle\sum_{y \in L_{i-1,0}(-k)} p_{xy}, & k < 0, \end{cases}$$

while for $x \in B_{i-1} \setminus \mathscr{B}_0$ and $k \leq 1$, define

$$q_i^-(k) = P\big(\xi_{i-1}(t+1) = x_{i-1} - k | \xi(t) = x\big) = \sum_{y \in L_{i-1,0}(x_{i-1}-k)} p_{xy},$$

where homogeneity within the faces A_i and B_i ensures that these are all well defined. Now we define

$$P\big(X_i(t+1) = j + k | X_i(t) = j\big) = \begin{cases} q_i^+(k), & j > 0, k \geq -1, \\ q_i^0(k), & j = 0, \\ q_i^-(k), & j < 0, k \leq 1, \end{cases} \tag{4}$$

so that X_i captures the behavior of Ξ orthogonal to A_i on the collection of faces $B_{i-1} \cup A_i \cup B_i$.

Our assumptions on the moments of Ξ immediately imply that

$$\sum_k k q_i^+(k) = M_{i1} > 0 \quad \text{and} \quad \sum_k k q_i^-(k) = -M_{i-1,0} > 0$$

and as the state 0 cannot be absorbing it follows from homogeniety that 0 is a transient state for X_i. Hence the expected number of visits to 0 by X_i is finite and in the terminology of [6], each of the axes A_i are transient faces while the B_i are ergodic faces.

The following Theorem 1 states Theorems 2.1.2 and 2.1.10 of [6] which describe how the drifts of a process can be used to determine its transience or ergodicity.

Let $\{S_i, i \geq 0\}$ be a sequence of nonnegative random variables with S_0 constant and S_n measurable with respect to $\mathcal{F}_n = \sigma(\xi_0, \dots, \xi_n)$ for $n \geq 1$. Let $\{N_i, i \geq 0\}$ be an increasing sequence of stopping times adapted to $\{\mathcal{F}_n\}$ with $N_0 = 0$ and let $Y_0 = S_0$ and $Y_i = S_{N_i}$ for $i \geq 1$. Also, for constant $D > 0$, let $\tau = \min\{n \geq 1 : S_n \leq D\}$ and $\sigma = \min\{i \geq 1 : Y_i \leq D\}$. Finally, let $\{X_{i \wedge \sigma}\}$ denote a sequence $\{X_i\}$ stopped at σ and I_E be the indicator function of an event E.

THEOREM 1. (i) *If $S_0 > D$ and for some $\varepsilon > 0$ and all $n \geq 0$,*

$$\mathbf{E}\big(Y_{(n+1)\wedge\sigma} | \mathcal{F}_{N_{n\wedge\sigma}}\big) \leq Y_{n\wedge\sigma} - \varepsilon \mathbf{E}\big(N_{(n+1)\wedge\sigma} - N_{n\wedge\sigma} | \mathcal{F}_{N_{n\wedge\sigma}}\big) \qquad a.s.,$$

then $\mathbf{E}(\tau) \leq S_0/\varepsilon < \infty$.

(ii) *If $S_0 > D$, the jumps $S_{n+1} - S_n$ are uniformly bounded below and there exists $\varepsilon > 0$ and a positive constant b such that for every $n \geq 0$,*

$$\mathbf{E}\big((S_{n+1} - S_n) I_{\{S_{n+1} - S_n < b\}} | \mathcal{F}_n\big) \geq \varepsilon \qquad a.s.,$$

then $\mathbf{P}(\tau = \infty) > 0$.

Now we will use Theorem 1 to prove the main result of this section.

THEOREM 2. *If $K < 1$, then Ξ is ergodic while if $K > 1$ then Ξ is transient.*

PROOF. We will use Theorem 1 with $S_n = f(\xi_n)$ for a suitable Lyapunov function f. Let $\pi_i = \prod_{j=1}^{i-1} \tan\alpha_j$ and consider the function f, where for $x \in A_i \cup B_i$, $1 \le i \le n-1$,

$$f(x) = \frac{x_i}{\pi_i u^{i-1}} + \frac{x_{i+1}}{\pi_{i+1} u^i} \tag{5}$$

with $f(x) = ux_n + x_1$ for $x \in A_n \cup B_n$ where $u = K^{-1/n}$. This function is nonnegative, piecewise linear and for $x \in B_i \setminus \mathscr{B}_0$ and $v = M(x)$ we calculate for each i

$$\begin{aligned}\delta_i &\equiv \mathbf{E}\big(f(\xi(t+1)) - f(\xi(t))|\xi(t) = x\big)\\ &= f(x+v) - f(x) = \frac{1}{\pi_{i+1}u^i}(uM_{i0}\tan\alpha_i + M_{i1}).\end{aligned}$$

Hence for all i, if $K < 1$ then $u > 1$ and so $\delta_i < 0$ while for $K > 1$ we have $u < 1$ and so $\delta_i > 0$. Figure 4 shows how the contours of f relate to the drift vectors $M(x)$, $x \in B_i \setminus \mathscr{B}_0$ for each i in the case $K < 1$.

Let $\Delta_t f = f(\xi(t+1)) - f(\xi(t))$ and $a_i = \mathbf{E}(\Delta_t f|\xi(t) = x)$ for $x \in A_i \setminus \mathscr{B}_0$. In the case $K < 1$, for any integer $k > 0$ and $x \in A_i$ with $\|x\|$ large enough that $\xi(t+j) \notin \mathscr{B}_0$ a.s. for $0 \le j \le k-1$,

$$\begin{aligned}&\mathbf{E}\big(\Delta_{t+j} f|\xi(t) = x\big)\\ &\quad= \mathbf{E}\big(\Delta_{t+j} f\big(I_{\{\xi(t+j)\in A_i\}} + I_{\{\xi(t+j)\in B_{i-1}\cup B_i\}}\big)|\xi(t) = x\big)\\ &\quad\le a_i\mathbf{E}\big(I_{\{\xi(t+j)\in A_i\}}|\xi(t) = x\big) + \delta\big(1 - \mathbf{E}\big(I_{\{\xi(t+j)\in A_i\}}|\xi(t) = x\big)\big)\end{aligned}$$

by the tower property of conditional expectation, where $\delta = \max_{1\le i\le n}\delta_i < 0$. Telescoping these together we have

$$\begin{aligned}&\mathbf{E}\big(f(\xi(t+k)) - f(\xi(t))|\xi(t) = x\big)\\ &\quad= \sum_{j=0}^{k-1}\mathbf{E}\big(\Delta_{t+j} f|\xi(t) = x\big)\\ &\quad\le k\delta + (a_i - \delta)\mathbf{E}\big(V_i(k)|\xi(t) = x\big),\end{aligned}$$

where $V_i(k)$ denotes the number of times Ξ visits A_i between times t and $t+k-1$. As A_i is transient, $\mathbf{E}(V_i(k)|\xi(t) = x)$ is uniformly bounded for every k and so for some large enough k_i and $\varepsilon = -\delta/2$, we have

$$\mathbf{E}\big(f(\xi(t+k_i)) - f(\xi(t))|\xi(t) = x\big) \le -\varepsilon k_i$$

for $x \in A_i \setminus \mathscr{B}_0$ with $\|x\|$ large enough. Hence, by choosing

$$k(x) = \begin{cases} 1, & x \in B_i, i = 1, \dots, n,\\ k_i, & x \in A_i, i = 1, \dots, n,\end{cases}$$

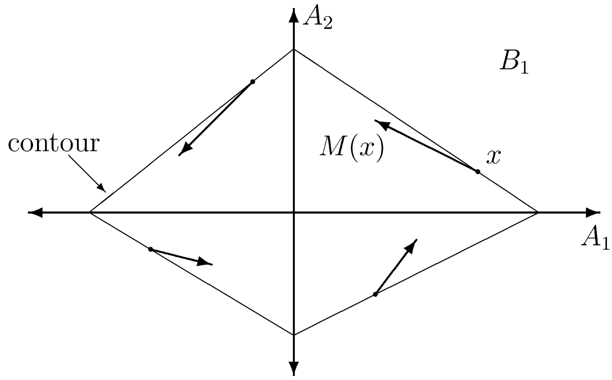

FIG. 4. *Drift vectors relative to contour of f in the case $K < 1$.*

and setting $S_t = f(\xi(t))$ and $N_{i+1} = N_i + k(\xi(N_i))$, $i \geq 1$, the conditions of the first part of Theorem 1 are satisfied. As f satisfies $c_1\|x\| \leq |f(x)| \leq c_2\|x\|$ for some constants $c_2 > c_1 > 0$, it follows that the expected hitting time of Ξ to $\{x \in \mathcal{L} : \|x\| \leq D\}$ (with D suitably large) is finite and so by irreducibility and aperiodicity Ξ is ergodic.

In the case $K > 1$ we can, by integrability and homogeneity, choose a truncation level b so that for $x \in B_i \setminus \mathcal{B}_0$,

$$\tilde{\delta}_i = \mathbf{E}\big(\Delta_t f I_{\{\Delta_t f < b\}} \big| \xi(t) = x\big) > \delta_i/2 > 0$$

for $i = 1, \dots, n$. Now take $\varepsilon = \min_{1 \leq i \leq n} \tilde{\delta}_i > 0$ and as above, choose $k(x) = 1$ for $x \in B_i$ while for $x \in A_i$ for each i, choose $k(x) = k_i$ large enough that

$$\mathbf{E}\big(f(\xi(t + k_i)) - f(\xi(t)) | \xi(t) = x\big) \geq k_i\tilde{\delta} + (a_i - \tilde{\delta})\mathbf{E}\big(V_i(k_i) | \xi(t) = x\big) \geq \varepsilon.$$

Defining the stopping times N_i as in the ergodic case but this time setting $S_i = f(\xi(N_i))$, we see by the second part of Theorem 1 that the hitting time to $\{x \in \mathcal{L} : \|x\| \leq D\}$ from x_0 with $\|x_0\| > D$ for the imbedded process $\xi(N_i)$ is infinite with positive probability. By lower boundedness of the jumps and uniform boundedness of the $N_{i+1} - N_i$ it follows from standard results that Ξ is transient. $\square$

REMARK. Only the existence of a first moment, that is, $M(x)$ is required for this result. The theorem shows that (under the moment condition) the jump behavior from the faces A_i does not affect the transience/ergodicity of the process. Our $\gamma > 2$ moment condition is required to deal with the critical case $K = 1$ of the next section. In this critical case the jump behavior on the A_i is crucially important as no f allowing us to use Theorem 1 exists.

4. Transience and recurrence in the critical case. When $K = 1$ there are various possibilities for the behavior of the process Ξ. We will find conditions which ensure the transience, null-recurrence or ergodicity respectively of Ξ.

Our method is to produce a Lyapunov function f of the states x such that the process $f(\Xi)$ satisfies exactly the right Lamperti conditions on its mean drifts as specified in the following result of Menshikov and Zuyev (see [13]) which is in turn a variant of Theorem 3.2 in [8]. No function satisfying the conditions of Theorem 1 exists in this critical case.

THEOREM 3. *Let X be a time-homogeneous Markov chain on a countable state space $\mathcal{A} = \{\alpha_n\}$. Assume that there exists a positive function $f : \mathcal{A} \to \mathbf{R}_+$ with the following properties*:

$$\lim_{n\to\infty} f(\alpha_n) = +\infty \tag{6}$$

and there exist constants $C_0, \delta > 0$ such that

$$\mathbf{E}\big(|f(X(t+1)) - f(X(t))|^{2+\delta} | X(t) = \alpha_n\big) \le C_0 \qquad \textit{for all } n. \tag{7}$$

Let

$$a_n = f(\alpha_n) \cdot \mathbf{E}\big(f(X(t+1)) - f(X(t)) | X(t) = \alpha_n\big),$$
$$b_n = \mathbf{E}\big(\big(f(X(t+1)) - f(X(t))\big)^2 \big| X(t) = \alpha_n\big).$$

Then X is recurrent if $\limsup(2a_n - b_n) < 0$, *and X is transient if* $\liminf(2a_n - b_n) > 0$.

PROOF. See Theorem 1 in [13]. The proof uses the well-known Foster's criteria which can be found in [1] or [6]. □

To exhibit our Lyapunov function we need to calculate a couple of properties of the one-dimensional walk induced from Ξ onto A_i as introduced in the previous section.

LEMMA 1. *Let X_i denote the one-dimensional random walk on $\mathbf{Z}$ induced from Ξ onto A_i with one-step transition probabilities as defined in* (4). *Then X_i is transient and there is a unique constant $s_i \in [0, 1)$ such that*

$$P\big(X_i \textit{ ever hits } 0 | X_i(0) = j\big) = s_i^j, \qquad j > 0.$$

Further, the process $s_i^{X_i(t)}$ satisfies $\mathbf{E}(s_i^{X_i(t+1)} | X_i(t)) = s_i^{X_i(t)}$ on $\{X_i(t) > 0\}$ and

$$u_i \equiv P(X_i \textit{ exits } 0 \textit{ and never returns}) = \sum_{j=1}^{\infty} q_i^0(j)(1 - s_i^j) > 0. \tag{8}$$

PROOF. These are routine calculations. When Ξ cannot make jumps toward A_i from within B_i, that is, $q_i^+(-1) = 0$ we will set $s_i = 0$. Otherwise s_i is calculated as the unique root in $(0, 1)$ of the equation $G_i(z) = 1$ where $G_i(z) = \sum_{x=-1}^{\infty} q_i^+(x)z^x$ is the one-step transition probability generating function of X_i from states $j > 0$. Our moment assumptions ensure that G_i is a smooth function of z with $G_i'(s_i) < 0$ when $s_i > 0$ so for $z \in (s_i, 1)$ we have the inequality $\mathbf{E}(z^{X_i(t+1)}|X_i(t)) < z^{X_i(t)}$ on $\{X_i(t) > 0\}$, while for $z \in (0, s_i)$ we have $\mathbf{E}(z^{X_i(t+1)}|X_i(t)) > z^{X_i(t)}$.

Note that X_i, when started from state $j < 0$, is certain to visit state 0 eventually so transitions to such states make no contribution to u_i. As the number of upward excursions from 0, including the eventual infinite excursion, is geometrically distributed we can interpret $1/u_i$ as the expected number of such excursions from 0 for the process X_i. The analogue of these for Ξ are excursions from A_i into B_i. $\square$

In order to state the main theorem of this section we need to define two sets of constants that express the crucial properties of the underlying random walk Ξ. Let $\pi_i = \prod_{j=1}^{i-1} \tan\alpha_j$ with $\pi_0 = 1/\tan\alpha_n$, $\pi_{n+1} = \pi_1 = 1$ and define C_i, $i = 1, \dots, n$, to be the unique constants satisfying

$$\left[\frac{M_{i1}^-}{\pi_{i-1}} + \frac{M_{i0}^- + M_i^0 + M_{i0}^+}{\pi_i} + \frac{M_{i1}^+}{\pi_{i+1}}\right] + C_i u_i = 0. \tag{9}$$

Further for any constant $p \in [0, 1]$ define for each $i = 1, \dots, n$,

$$h_i(p) = \left(p - \frac{1}{2}\right)\left(\frac{\lambda_{i0}\tan\alpha_i + 2R_i + \lambda_{i1}/\tan\alpha_i}{\pi_i\rho_i \sin\alpha_i}\right), \tag{10}$$

where $\rho_i = \|M(x)\|$ for $x \in B_i \setminus \mathscr{B}_0$.

THEOREM 4. *If* $\sum_{i=1}^n (C_i - h_i(0)) > 0$ *then the process* Ξ *is recurrent while if* $\sum_{i=1}^n (C_i - h_i(0)) < 0$ *then* Ξ *is transient.*

REMARKS. The presence of the π_i in the formulas for C_i and $h_i(p)$ gives the impression that this result somehow depends upon where we choose to start numbering the faces. In fact the choice of starting axis only affects $\sum C_i - h_i$ by a positive multiplicative scaling which has no influence on our result. A detailed investigation of the recurrent case is the subject of Section 5. From this point on we will neglect to mention $\mathscr{B}_0$ as all our calculations depend on $\|x\|$ being sufficiently large.

PROOF OF THEOREM 4. We will treat the transient case in detail and only indicate the minor changes needed for the recurrent case.

The piecewise linear Lyapunov function used in the cases $K \neq 1$ is not adequate for this task. There are two important modifications required of which the most involved is the replacement of the linear martingale term with a nearly linear transformation which we describe now. For any constants h (positive or negative) define, for $i = 1, \dots, n$,

$$
\begin{aligned}
F_i(x, h) &= \frac{x_i}{\pi_i} + \frac{x_{i+1}}{\pi_{i+1}} - h\frac{x_{i+1}/\pi_{i+1}}{x_i/\pi_i + x_{i+1}/\pi_{i+1}} \\
&= \frac{x_i}{\pi_i} + \frac{x_{i+1}}{\pi_{i+1}} - h\frac{x_{i+1}}{x_i \tan\alpha_i + x_{i+1}}
\end{aligned}
\tag{11}
$$

for $x \in A_i \cup B_i \cup A_{i+1}$, $i \le n-1$, with the obvious replacement of x_{n+1} with x_1 for $x \in A_n \cup B_n \cup A_1$. For any positive constant v we see that $-h \le F_i(x, h) - v \le 0$ along the line $x_i \tan\alpha_i + x_{i+1} = v\pi_{i+1}$ and similarly on the level curve $\{x : F_i(x, h) = v\}$, $0 \le \frac{x_i}{\pi_i} + \frac{x_{i+1}}{\pi_{i+1}} - v \le h$ so the F_i are "almost linear" functions. They do *not* fit together smoothly on the A_i where we see that $F_i(x, h_i) = F_{i-1}(x, h_{i-1}) + h_{i-1}$ and we need to remember this at a later point in our calculation.

Expanding F_i using Taylor's formula to second order we find that for x, $x + y \in A_i \cup B_i \cup A_{i+1}$

$$
\begin{aligned}
&F_i(x + y, h) - F_i(x, h) \\
&\quad = \left(\frac{y_i}{\pi_i} + \frac{y_{i+1}}{\pi_{i+1}}\right) - h\tan\alpha_i \frac{x_i y_{i+1} - x_{i+1} y_i}{(x_i \tan\alpha_i + x_{i+1})^2} \\
&\qquad + h\tan\alpha_i \frac{(x_i y_{i+1} - x_{i+1} y_i)(y_i \tan\alpha_i + y_{i+1})}{(x_i \tan\alpha_i + x_{i+1} + \theta(y_i \tan\alpha_i + y_{i+1}))^3},
\end{aligned}
\tag{12}
$$

where $\theta \in [0, 1]$ depends upon x and y. As y will represent possible jumps from $x \in B_i$, we can assume its components are bounded below by -1 and so the residual term's denominator $(x_i \tan\alpha_i + x_{i+1} + \theta(y_i \tan\alpha_i + y_{i+1}))^3$ is bounded below by

$$
\bigl((x_i - 1)\tan\alpha_i + x_{i+1} - 1\bigr)^3 = \pi_{i+1}^3 F_i(x, h)^3 + O\bigl(F_i(x, h)^2\bigr).
$$

Taking expectations and noting the relations $M_{i0} = -\rho_i \cos\alpha_i$ and $M_{i1} = \rho_i \sin\alpha_i$ we see that when $x \in B_i$ (so jumps end inside $A_i \cup B_i \cup A_{i+1}$) and $F_i(x, h)$ is large

$$
\begin{aligned}
&E\bigl(F_i\bigl(\xi(t+1), h\bigr) - F_i\bigl(\xi(t), h\bigr) | \xi(t) = x\bigr) \\
&\quad = 0 - h\tan\alpha_i \frac{x_i \rho_i \sin\alpha_i + x_{i+1}\rho_i \cos\alpha_i}{(x_i \tan\alpha_i + x_{i+1})^2} + O\left(\frac{\|x\| \cdot M^{(2)}(x)}{F_i(x, h)^3}\right) \\
&\quad = -\frac{h\rho_i \sin\alpha_i}{x_i \tan\alpha_i + x_{i+1}} + O\bigl(F_i(x, h)^{-2}\bigr) \\
&\quad = -\frac{h\rho_i \cos\alpha_i}{\pi_i F_i(x, h)} + O\bigl(F_i(x, h)^{-2}\bigr),
\end{aligned}
\tag{13}
$$

where in obtaining the final expression we have used the binomial theorem to write

$$\frac{1}{x_i \tan\alpha_i + x_{i+1}} = \frac{1}{\pi_i \tan\alpha_i F_i(x,h)} + O\big(F_i(x,h)^{-2}\big).$$

Using a very similar argument but with Taylor's formula only to first order we calculate

$$\begin{aligned} &E\big(\big[F_i\big(\xi(t+1),h\big) - F_i\big(\xi(t),h\big)\big]^2 \big| \xi(t)=x\big) \\ &\quad = \frac{1}{\pi_i^2}\left(\lambda_{i0} + \frac{2R_i}{\tan\alpha_i} + \frac{\lambda_{i1}}{\tan^2\alpha_i}\right) + O\big(F_i(x,h)^{-1}\big). \end{aligned} \tag{14}$$

Further modifications to the Lyapunov function of Theorem 2 are needed to deal with the behavior of Ξ at the axes A_i. Let $q_i^- = \sum_{y\in B_{i-1}} p_{xy}$, where $x \in A_i$, $i = 1,\dots,n$, and let

$$\delta \equiv \frac{\sum_{i=1}^n (C_i - h_i(0))}{2n + 2\sum_{i=1}^n q_i^-/u_i},$$

where u_i was defined in (8). Note that we have $\delta < 0$ in the transient case but $\delta > 0$ in the recurrent case. Next define for each i

$$C_i' = C_i - 2\delta\frac{q_i^-}{u_i}, \qquad \tilde{C}_i = C_i' - \delta \quad \text{and} \quad \tilde{h}_i = h_i(0) + \delta \tag{15}$$

with $h_i(0)$ as in (10) so that $\sum_{i=1}^n(\tilde{C}_i - \tilde{h}_i) = 0$. For each i we now use the functions $T_i(x) = \beta_i^{x_{i+1}}$ for $x \in A_i \cup B_i$, where the β_i are constants to be chosen, to construct the process $T(\Xi)$ satisfying

$$T(\xi(t)) = T_i(x) \qquad \text{when } \xi(t) = x \in A_i \cup B_i. \tag{16}$$

To establish transience of Ξ it turns out that we will require

$$C_i'\mathbf{E}\big(1 - T(\xi(t+1))|\xi(t)=x\big) > C_i'\big(1 - T_i(x)\big) \qquad \text{when } x \in B_i.$$

The results of Lemma 1 imply that when $C_i' > 0$ we must choose $\beta_i \in (0, s_i)$ and when $C_i' < 0$ we require $\beta_i \in (s_i, 1)$. For any such choice of the β_i we define now for later use

$$\Gamma_i \equiv \Gamma_i(\beta_i) = -C_i'\mathbf{E}\big(T(\xi(t+1)) - T(\xi(t))|\xi(t)=x\big)/T_i(x)$$

for any $x \in B_i$ with $x_i > 1$ and note that $\Gamma_i > 0$. Further for $x \in A_i$ with $x_i > 1$ and u_i as defined in (8), we define

$$\Delta_i \equiv \Delta(\beta_i) = C_i'\mathbf{E}\big(I_{B_i}(t+1)\big[1 - T(\xi(t+1))\big]\big|\xi(t)=x\big) - C_i'u_i,$$

where the notation $I_E(t)$ denotes the indicator function of the event $\{\xi(t) \in E\}$ for any time t. The results of Lemma 1 imply that $\Delta_i < 0$. When considering the recurrence problem we choose β_i to ensure that $\Gamma_i < 0$ which then makes $\Delta_i > 0$.

With these preliminary calculations in place we now proceed to exhibit a function f on $\mathcal{L}$ such that the jump moments

$$\mu_1(f|x) = \mathbf{E}\big(f(\xi(t+1)) - f(\xi(t))|\xi(t) = x\big)$$

and

$$\mu_2(f|x) = \mathbf{E}\big(\big[f(\xi(t+1)) - f(\xi(t))\big]^2|\xi(t) = x\big)$$

of $f(\xi(t))$ satisfy the transience conditions of Theorem 3. Specifically we will show that when $\sum_i C_i - h_i(0) < 0$ there is a function f and an $\varepsilon > 0$ such that $2f(x)\mu_1(f|x) - \mu_2(f|x) \geq \varepsilon$ for all x outside some finite set near the origin. The conditions for recurrence when $\sum_i C_i - h_i(0) > 0$ are established in a nearly identical fashion.

Our Lyapunov function f is defined by

$$f(x) = F_i(x, \tilde{h}_i) + \sum_{j=1}^{i-1}(\tilde{C}_j - \tilde{h}_j) + C_i'\big(1 - T_i(x)\big), \qquad x \in A_i \cup B_i, \tag{17}$$

for $i = 1, \dots, n$ (with the usual replacement of x_{n+1} by x_1 when this is necessary) and F_i is defined in (11). The β_i are as yet not specified but we will show later how they should be chosen.

We start with $\mu_2(f|x)$. Our assumptions of homogeneity within faces and $M^{(\gamma)}(x) < \infty$ for some $\gamma > 2$ and all $x \in \mathcal{L}$ imply that μ_2 is uniformly bounded on $\mathcal{L}$. We now show that within B_i, for each i, μ_2 converges to a constant we can calculate as x_{i+1} becomes large. For all $x \in B_i$ we have

$$\mu_2(f|x) = \frac{1}{\pi_i^2}\left(\lambda_{i0} + \frac{2R_i}{\tan\alpha_i} + \frac{\lambda_{i1}}{\tan^2\alpha_i}\right) + O\big(F_i(x, \tilde{h}_i)^{-1}\big) + W_i^a(x) + W_i^b(x),$$

where

$$W_i^a(x) = (C_i')^2\mathbf{E}\big(\big[T_i(x) - T_i(\xi(t+1))\big]^2|\xi(t) = x\big)$$

and

$$W_i^b(x) = 2C_i'\mathbf{E}\big(\big[F_i\big(\xi(t+1), \tilde{h}_i\big) - F_i\big(\xi(t), \tilde{h}_i\big)\big] \cdot \big[T_i(x) - T_i(\xi(t+1))\big]\big|\xi(t) = x\big).$$

As $T_i(x) = \beta_i^{x_{i+1}}$ and jumps are bounded below we have the relation $T_i(x) - T_i(y) = O(T_i(x))$ for $x \in B_i$ and y such that $p_{xy} > 0$. Further $T_i(x) \to 0$ as $x_{i+1} \to \infty$ so we see that for $x \in B_i$ with x_{i+1} not too small, we have

$$W_i^a(x) = O\big(T_i(x)^2\big) \quad \text{and} \quad W_i^b(x) = O(T_i(x)).$$

At this point note that $f(x) = F_i(x, h_i) + O(1)$ so that when considering asymptotics $F_i(x, h_i)^{-1} = f(x)^{-1} + O(f(x)^{-2})$. Combining these estimates we have

$$\begin{aligned}\mu_2(f|x) &= \frac{1}{\pi_i^2}\left(\lambda_{i0} + \frac{2R_i}{\tan\alpha_i} + \frac{\lambda_{i1}}{\tan^2\alpha_i}\right) + O\big(f(x)^{-1}\big) + O(T_i(x)) \\ &= -2\frac{\rho_i\cos\alpha_i}{\pi_i} h_i(0) + O\big(f(x)^{-1}\big) + O(T_i(x)).\end{aligned} \tag{18}$$

It remains to compute μ_1 and show that under the condition $\sum_i (C_i - h_i(0)) < 0$ we have $\liminf 2f(x)\mu_1(f|x) - \mu_2(f|x) > 0$. To establish the required inequalities we consider each $A_i \cup B_i$ in four pieces: A_i; a strip of B_i adjacent to A_i; $\{x \in B_i : x_i = 1\}$; the rest of B_i. We now consider these in the reverse of the order listed.

Using the expansion (12) and the consequent estimate (13) above we have for $x \in B_i$ such that $x_i > 1$ (so all jumps end in $A_i \cup B_i$),

$$\begin{aligned}\mu_1(f|x) &= -\frac{\tilde{h}_i\rho_i\cos\alpha_i}{\pi_i F_i(x, \tilde{h}_i)} + O\big(F_i^{-2}(x, \tilde{h}_i)\big) \\ &\quad + C_i'\big[\mathbf{E}\big(1 - T_i(\xi(t+1))|\xi(t) = x\big) - \big(1 - T_i(x)\big)\big] \\ &= -\big(h_i(0) + \delta\big)\frac{\rho_i\cos\alpha_i}{\pi_i f(x)} + O\big(f(x)^{-2}\big) + \Gamma_i T_i(x).\end{aligned} \tag{19}$$

From (19) and (18) it follows that for $x \in B_i$ such that $x_i > 1$,

$$\begin{aligned}&2f(x)\cdot\mu_1(f|x) - \mu_2(f|x) \\ &\quad = -2\big(h_i(0) + \delta\big)\frac{\rho_i\cos\alpha_i}{\pi_i} \\ &\qquad + 2\Gamma_i T_i(x) f(x) + 2h_i(0)\frac{\rho_i\cos\alpha_i}{\pi_i} + O\big(f(x)^{-1}\big) \\ &\quad = -2\delta\frac{\rho_i\cos\alpha_i}{\pi_i} + 2\Gamma_i T_i(x) f(x) - O(T_i(x)) + O\big(f(x)^{-1}\big)\end{aligned}$$

so that, since $\delta < 0$ and $\Gamma_i > 0$, there exists j_i such that for some $\varepsilon > 0$, $2f\mu_1 - \mu_2 \geq \varepsilon$ for x with $x_{i+1} \geq j_i$.

Next when $x \in B_i$ with $x_i = 1$ (so it is possible to reach A_{i+1} in a single jump) the same expansion holds except that $F_i(x, \tilde{h}_i)$ is replaced by $F_{i+1}(x, \tilde{h}_{i+1})$ on A_{i+1}. We simply substitute $F_{i+1}(x, \tilde{h}_{i+1}) = F_i(x, \tilde{h}_i) + \tilde{h}_i I_{A_{i+1}}(t+1)$ and proceed as before [as before $I_E(t)$ denotes the indicator function of the event

$\{\xi(t) \in E\}]$. This time

$$
\begin{aligned}
\mu_1(f|x) &= -\frac{\tilde{h}_i \rho_i \cos\alpha_i}{\pi_i F_i(x, \tilde{h}_i)} + O\big(F_i^{-2}(x, \tilde{h}_i)\big) \\
&\quad + C_i' \mathbf{E}\big(I_{B_i}(t+1)\big[1 - T_i(\xi(t+1))\big] | \xi(t) = x\big) \\
&\quad + (\tilde{h}_i + \tilde{C}_i - \tilde{h}_i)\mathbf{E}\big(I_{A_{i+1}}(t+1) | \xi(t) = x\big) - C_i'[1 - T_i(x)] \\
&= -\frac{\tilde{h}_i \rho_i \cos\alpha_i}{\pi_i f(x)} + O\big(f(x)^{-2}\big) - \delta \mathbf{E}\big(I_{A_{i+1}}(t+1) | \xi(t) = x\big) \\
&\quad + C_i'\big[T_i(x) - \mathbf{E}\big(I_{B_i}(t+1) T_i(\xi(t+1)) | \xi(t) = x\big)\big] \\
&= -\delta \mathbf{E}\big(I_{A_{i+1}}(t+1) | \xi(t) = x\big) + O\big(f(x)^{-1}\big) + O(T_i(x))
\end{aligned} \tag{20}
$$

and now (20) and the uniform boundedness of μ_2 imply

$$
\begin{aligned}
&2f(x) \cdot \mu_1(f|x) - \mu_2(f|x) \\
&\quad = -2f(x) \cdot \delta \mathbf{E}\big(I_{A_{i+1}}(t+1) | \xi(t) = x\big) + O\big(f(x) T_i(x)\big) + O(1)
\end{aligned}
$$

so that again, since $\delta < 0$, there exists $\varepsilon > 0$ such that $2f(x)\mu_1(f|x) - \mu_2(x) \geq \varepsilon$ when x_{i+1} and hence $f(x)$ are large.

When $x \in B_i$ with $x_{i+1} = j$ for values $j = 1, \dots, j_i$ we know that μ_2 is uniformly bounded while (19) for each such j says

$$
\mu_1(f|x) = -\frac{\tilde{h}_i \rho_i \cos\alpha_i}{\pi_i f(x)} + \Gamma_i \beta_i^j + O\big(f(x)^{-2}\big)
$$

so that as $\Gamma_i > 0$, $\beta_i > 0$ and $f(x) \to \infty$ as $x_i \to \infty$ there exist constants N_j and $\varepsilon > 0$ such that $2f\mu_1 - \mu_2 \geq \varepsilon$ when $x_i \geq N_j$. Hence for x with $x_i \geq \max_{j \leq j_i} N_j$ we can be sure that $2f(x)\mu_1(f|x) - \mu_2(f|x) \geq \varepsilon$.

This only leaves the axes A_i. We again employ the argument that leads to the Taylor expansions in (13) and (14), but as Ξ can jump into either B_{i-1} or B_i from A_i we must consider both F_i and F_{i-1}, combining them with appropriate indicator functions. The sum to be considered is quite long so temporarily we will abreviate our notation writing $\mathbf{E}_x$ for expectation conditioned on $\{\xi(t) = x\}$ and sometimes ξ_i for $\xi_i(t+1)$, $I_{B_{i-1}}$ for $I_{B_{i-1}}(t+1)$ and so on. Considering just the linear components of f we see that for $x \in A_i$ we have

$$
\mathbf{E}_x\left(I_{B_{i-1}} \frac{\xi_{i-1}}{\pi_{i-1}} + \frac{\xi_i}{\pi_i} + I_{B_i} \frac{\xi_{i+1}}{\pi_{i+1}}\right) - \frac{x_i}{\pi_i} = \frac{M_{i1}^-}{\pi_{i-1}} + \frac{M_{i0}^- + M_i^0 + M_{i0}^+}{\pi_i} + \frac{M_{i1}^+}{\pi_{i+1}},
$$

which we saw before in (9). For the nonlinear terms in F_i, F_{i-1} note that, for example,

$$
\mathbf{E}_x\left(\frac{\xi_{i+1}(t+1)}{\xi_i(t+1)\tan\alpha_i + \xi_{i+1}(t+1)}\right) = O\big(f(x)^{-1}\big)
$$

by the Taylor argument leading to (13) as $\xi_{i+1}(t+1) \geq 0$ and $\xi_i(t+1) \geq x_i - \kappa > 0$ (recall the assumption that jumps are bounded below). Further for $x \in A_i$, $1 - T_i(x) = 0$ and we defined Δ_i such that

$$C_i' \mathbf{E}_x\big(I_{B_i}[1 - T_i(\xi)]\big) = C_i' u_i + \Delta_i$$

with u_i as in (8). Taking care to include the constant terms as well, we have, for $i = 2, \dots, n$,

$$\begin{aligned}
\mu_1(f|x) &= \mathbf{E}_x\big(f(\xi(t+1))\big) - f(x) \\
&= \mathbf{E}_x\big(I_{A_i \cup B_i} F_i + I_{B_{i-1}} F_{i-1}\big) - F_i(x, \tilde{h}_i) - (\tilde{C}_{i-1} - \tilde{h}_{i-1})\mathbf{E}_x(I_{B_{i-1}}) \\
&\quad + C_i' \mathbf{E}_x\big(I_{B_i}[1 - T_i(\xi)]\big) + C_{i-1}' \mathbf{E}_x\big(I_{B_{i-1}}[1 - T_{i-1}(\xi)]\big) \\
&= \left[\frac{M_{i1}^-}{\pi_{i-1}} + \frac{M_{i0}^- + M_i^0 + M_{i0}^+}{\pi_i} + \frac{M_{i1}^+}{\pi_{i+1}}\right] + C_i' \mathbf{E}_x\big(I_{B_i}(1 - T_i)\big) \\
&\quad - \tilde{h}_i \mathbf{E}_x\left(I_{B_i} \frac{\xi_{i+1}}{\xi_i \tan\alpha_i + \xi_{i+1}}\right) \\
&\quad + C_{i-1}' \mathbf{E}_x\big(I_{B_{i-1}}(1 - T_{i-1})\big) - (\tilde{C}_{i-1} - \tilde{h}_{i-1})\mathbf{E}_x(I_{B_{i-1}}) \qquad (21) \\
&\quad - \tilde{h}_{i-1} \mathbf{E}_x\left(I_{B_{i-1}} \frac{\xi_i}{\xi_{i-1} \tan\alpha_{i-1} + \xi_i}\right) \\
&= (C_i' - C_i)u_i + \Delta_i + O\big(f(x)^{-1}\big) + (C_{i-1}' - \tilde{C}_{i-1})\mathbf{E}_x(I_{B_{i-1}}) \\
&\quad - C_{i-1}' \mathbf{E}_x\big(I_{B_{i-1}} T_{i-1}\big) + \tilde{h}_{i-1} \mathbf{E}_x\left(I_{B_{i-1}} \frac{\xi_{i-1} \tan\alpha_{i-1}}{\xi_{i-1} \tan\alpha_{i-1} + \xi_i}\right) \\
&= -2\delta q_i^- + \Delta_i + \delta q_i^- + O(T_{i-1}(x)) + O\big(f(x)^{-1}\big) \\
&= -\delta q_i^- + \Delta_i + O(T_{i-1}(x)) + O\big(f(x)^{-1}\big),
\end{aligned}$$

where $(C_i' - C_i)u_i = -2\delta q_i^-$ from (15). We know that $-\delta > 0$ but $\Delta_i = \Delta(\beta_i) < 0$. By choosing β_i close enough to s_i so that

$$-\delta q_i^- + \Delta_i > 0$$

we can guarantee that $\mu_1(f|x)$ is bounded away from 0 for large enough $x \in A_i$. Examining the results of Lemma 1 we see that this choice of β_i is possible under the assumption that $M_{i1}^+ < \infty$, that is, the jumps in direction x_{i+1} from A_i have finite mean. On the axis A_1 exactly the same argument applies with $i - 1$ replaced by n throughout as crucially $\sum_{i=1}^n (\tilde{C}_i - \tilde{h}_i) = 0$. Finally, $\mu_2(f|x)$ is uniformly bounded everywhere so that since $f(x) \to \infty$ as x increases we have shown that $2f\mu_1 - \mu_2 \geq \varepsilon > 0$ for large enough $x \in A_i$.

As our process Ξ lives on a countable state space any sensible labelling $\{\alpha_n\}$ allows us to apply Theorem 3 directly as we have shown that the function f defined in (17) satisfies the condition $2f \cdot \mu_1 - \mu_2 \geq \varepsilon$ everywhere outside some finite region around the origin and hence $\liminf_n 2f \cdot \mu_1 - \mu_2 > 0$.

To modify the proof to establish recurrence when $\sum_i C_i - h_i > 0$ we observe that now $\delta > 0$ and we simply select the β_i so that $\Gamma_i < 0$, $\Delta_i > 0$ for each i and proceed exactly as above to show that $2f\mu_1 - \mu_2$ is bounded below 0 for all suitably large x. $\square$

5. Existence of return time moments. It is possible to give a more refined description of the possibilities in the critical case by considering the finiteness of moments of the first-hitting time of finite subsets for our random walk Ξ. Given a finite set $S_K = \{x \in \mathcal{L} : \|x\| \leq K\}$ for some constant K, the first-hitting time of S_K from $x_0 \notin S_K$ is

$$\tau_K(x_0) = \inf\{t : \xi(0) = x_0,\ \xi(t) \in S_K\}.$$

General results showing when moments of hitting times are finite and when infinite were established in [2] for the critical process with asymptotically zero drifts. We start with a technical lemma based on the results there, which is important in this section as it will enable us to apply our earlier calculations concerning the sign of

$$2f(x)\mu_1(f|x) - \mu_2(f|x)$$

to processes of the form $f(\Xi)^{2p}$, which is the key to learning about the finiteness or otherwise of $\mathbf{E}(\tau_K(x_0)^p)$.

LEMMA 2. *Suppose that $\{X_n\}$ is a time-homogeneous $\mathcal{F}_n$-adapted nonnegative stochastic process such that for some $\gamma > 2$ and some finite B,*

$$\mathbf{E}(|X_{n+1} - X_n|^\gamma | \mathcal{F}_n) < B$$

uniformly in n. For $k = 1, 2$ let $\mu_k(\mathcal{F}_n) = \mathbf{E}([X_{n+1} - X_n]^k | \mathcal{F}_n)$. Then for $p > 0$ such that $2p < \gamma$ and any $\varepsilon_0 > 0$ there exists a constant K such that on $\{X_n > K\}$,

$$\big|\mathbf{E}\big(X_{n+1}^{2p} - X_n^{2p} | \mathcal{F}_n\big) - pX_n^{2p-2}\big(2X_n\mu_1(\mathcal{F}_n) + (2p-1)\mu_2(\mathcal{F}_n)\big)\big| < \varepsilon_0 X_n^{2p-2}.$$

PROOF. Here $\{\mathcal{F}_n\}$ is any suitable filtration—we will only need to use the natural filtration $\mathcal{F}_n = \sigma(\xi_0, \ldots, \xi_n)$, $n \geq 0$, in what follows. The term

$$2X_n\mu_1(\mathcal{F}_n) + (2p-1)\mu_2(\mathcal{F}_n)$$

will be central to our subsequent arguments as its sign essentially determines whether X_n^{2p} is a submartingale or supermartingale. A result analogous to this lemma was proved in Lemma 10 of the Appendix to [2] under weaker moment conditions but for a Markov chain. In fact the proof there translates directly to our case so we refer the reader to that lemma—it is written in terms of Markov chains there because the Appendix to [2] is concerned with showing exactly how the results of that paper extend those of in [9].

So far we have assumed [see (1) and following] that the jumps of our random walk Ξ satisfy the moment condition $M^{(\gamma)}(x) < \infty$ for all x for some $\gamma > 2$. We are now concerned with exactly what moment conditions obtain so we introduce $\hat{\gamma} = \sup\{\gamma : M^{(\gamma)}(x) < \infty \text{ for all } x \in \mathcal{L}\}$ ($\hat{\gamma} = \infty$ is possible but we have no need to consider it separately). For our intended applications of Lemma 2 above note that any function g which differs from a linear function only by uniformly bounded terms satisfies $g(x) = O(\|x\|)$ and so the stochastic process $g(\Xi)$ will satisfy Lemma 2's moment conditions for $p < \frac{1}{2}\hat{\gamma}$. $\square$

Establishing the existence of moments of return times proceeds in basically the same way as establishing recurrence and we show this first.

THEOREM 5. *If* $\sum_i C_i - h_i(0) > 0$ *then there is a unique* $p_0 > 0$ *such that* $\sum_i C_i - h_i(p_0) = 0$. *If* $2p_0 < \hat{\gamma}$ *then for* $0 < p < p_0$ *we have, for sufficiently large sets* S_K *and any* $x \notin S_K$

$$\mathbf{E}\big(\tau_K(x)^p\big) = O(\|x\|^{2p}).$$

This is also true for $p = p_0$ *when* $p_0 > 1$.

PROOF. The existence of p_0 is evident from definitions (9) and (10) which imply the expression $\sum_i C_i - h_i(p)$ is linear in p and decreasing.

It's necessary to modify the Lyapunov function constructed in the previous section. For $0 < p < p_0$ let

$$\delta(p) \equiv \frac{\sum_{i=1}^n (C_i - h_i(p))}{2n + 2\sum_{i=1}^n q_i^- / u_i} \qquad \text{so } \delta(p) > 0$$

and define for each i

$$C_i'(p) = C_i - 2\delta(p)\frac{q_i^-}{u_i},$$

$$\tilde{C}_i(p) = C_i'(p) - \delta(p) \quad \text{and} \quad \tilde{h}_i(p) = h_i(p) + \delta(p).$$

Our modified Lyapunov function is

$$\begin{aligned} f_p(x) = {} & F_i\big(x, \tilde{h}_i(p)\big) \\ & + \sum_{j=1}^{i-1} \big(\tilde{C}_j(p) - \tilde{h}_j(p)\big) + C_i'(p)\big(1 - T_i(x)\big), \qquad x \in A_i \cup B_i, \end{aligned}$$

with $T_i(x)$ as defined for (16). The leading terms of the expansions of $\mu_2(f|x)$ and $\mu_2(f_p|x)$ are identical so (18) also holds for f_p. To achieve cancellation of leading terms it is necessary to consider $2f_p\mu_1(f_p|x) + (2p-1)\mu_2(f_p|x)$ now.

The reader can readily check that the proof of Theorem 4 when applied to $f_p(\Xi)$ demonstrates that

$$2f_p(x) \cdot \mu_1(f_p|x) + (2p-1)\mu_2(f_p|x) \le -\varepsilon < 0 \tag{22}$$

for some $\varepsilon > 0$ and all suitably large $x \in \mathcal{L}$. It is clear that f_p can be defined for $p > p_0$ but as $\delta(p) < 0$ for such p, (22) will not hold.

For $p < p_0$ the process $f_p(\Xi)$ satisfies the conditions of Lemma 2 and we can hence deduce that for some $d > 0$,

$$\mathbf{E}\big(f_p(\xi(t+1))^{2p} - f_p(\xi(t))^{2p} | \xi(t) = x\big) \le -d f_p(x)^{2p-2}$$

for sufficiently large x. The result now follows from Theorem 1 of [2] which gives the estimate $\mathbf{E}(\tau_K^p(x)) = O(f_p(x)^{2p})$ which we combine with our earlier observation about essentially linear functions like f_p. $\square$

Showing which return time moments are infinite requires the introduction of a process very similar to $f(\Xi)$ but different in a couple of important respects. This companion process need not be a submartingale so we can be rather less careful about the exact sign of various terms but we must be able to get an estimate of how long it takes to reach a set S from any suitably distant starting state in order to employ some general results from [2].

THEOREM 6. *For $p > \min\{p_0, \frac{1}{2}\hat{\gamma}\}$, with p_0 as defined in Theorem 5, we have*

$$\mathbf{E}\big(\tau_K(x)^p\big) = \infty$$

for sufficiently large sets S_K and x where $\|x\| \ge K(1+\delta)$ for some $\delta > 0$.

PROOF. Suppose first that $2p_0 < \hat{\gamma}$. Let $\hat{T}_i(x) = s_i^{x_{i+1}}$ for $i = 1, \dots, n$ with the s_i as defined in Lemma 1. Next, using the constants π_i and C_i as defined in (9) and its preceding paragraph, define the function $\hat{f} : \mathcal{L} \to \mathbf{R}$ by

$$\hat{f}(x) = \frac{x_i}{\pi_i} + \frac{x_{i+1}}{\pi_{i+1}} + C_i\left(1 - \hat{T}_i(x) - \frac{x_{i+1}/\pi_{i+1}}{x_i/\pi_i + x_{i+1}/\pi_{i+1}}\right), \qquad x \in A_i \cup B_i,$$

for $i = 1, \dots, n$ (reading $i+1$ as 1 when $i = n$). With this set up we consider the process $\hat{f}(\Xi)$.

We want to employ Theorem 2 of [2] but this requires we first show that the conditions of their Lemma 2 (which shows how long it takes to reach a set like S_K from starting points sufficiently far away) are satisfied. To do this we show that $\mathbf{E}(\hat{f}(\xi(t+1))^2 - \hat{f}(\xi(t))^2 | \xi(t) = x)$ is bounded below uniformly in x and that $\mathbf{E}(\hat{f}(\xi(t+1))^{2r} - \hat{f}(\xi(t))^{2r} | \xi(t) = x) \le d\hat{f}(x)^{2r-2}$ for some $r > 1$, constant $d > 0$ and suitable x.

Replacing $\Gamma_i, \Delta_i, \delta$ with 0 and $\tilde{h}_i, C_i', \tilde{C}_i$ with C_i and adapting the calculations leading to (19) and (21) we see that

$$\mu_1(\hat{f}|x) = \begin{cases} O\big(\hat{f}(x)^{-1}\big), & x \in B_i, \\ O\big(\hat{T}_{i-1}(x)\big) + O\big(\hat{f}(x)^{-1}\big), & x \in A_i, \end{cases}$$

so that, since $s = \max_i\{s_i\} < 1$, $2\hat{f}\mu_1(\hat{f}|x) + (2p-1)\mu_2(\hat{f}|x) = O(1)$ by our moment assumptions on Ξ. By using Lemma 2 with $X_t = \hat{f}(\xi(t))$ and $p = 1$ the boundedness of $\mathbf{E}(\hat{f}(\xi(t+1))^2 - \hat{f}(\xi(t))^2|\xi(t) = x)$ is established. Again applying Lemma 2 to the process $\hat{f}(\Xi)$ we see that for $p < \frac{1}{2}\hat{\gamma}$ there exists a positive constant d such that for sufficiently large x

$$\mathbf{E}\big(\hat{f}(\xi(t+1))^{2p} - \hat{f}(\xi(t))^{2p}|\xi(t) = x\big) \le d\hat{f}(x)^{2p-2}.$$

This shows that $\hat{f}(\Xi)$ satisfies the conditions of [2] Lemma 2.

By repeating the argument leading to (22), but for some $p \in (p_0, \frac{1}{2}\hat{\gamma})$, we can show that

$$2f_p(x)\cdot\mu_1(f_p|x) + (2p-1)\mu_2(f_p|x) \ge \varepsilon$$

for some $\varepsilon > 0$ and all suitably large $x \in \mathcal{L}$ and hence, via Lemma 2, that for some $d > 0$,

$$\mathbf{E}\big(f_p(\xi(t+1))^{2p} - f_p(\xi(t))^{2p}\big|\xi(t) = x\big) \ge df_p(x)^{2p-2} > 0 \tag{23}$$

for sufficiently large x.

We now recall the hitting time $\tau_K(x)$ and introduce for $D > 0$ the hitting times

$$\sigma_D(x) = \min\{t : \hat{f}(\xi(t)) \le D, \xi(0) = x\}.$$

We have established that the process $f_p(\Xi)^{2p}$ when stopped at a suitably large set S is a submartingale. As the processes $f_p(\Xi)$ and $\hat{f}(\Xi)$ also satisfy the other conditions of [2] Theorem 2 its conclusion holds, namely that for large enough D and starting points x sufficiently far from $\hat{S}_D \equiv \{x : \hat{f}(x) \le D\}$, we have $\mathbf{E}(\sigma_D(x)^p) = \infty$. As $\hat{f}$ is essentially a linear function we can choose D for any sufficiently large K so that $S_K \subseteq \hat{S}_D$ which implies $\tau_K(x) \ge \sigma_D(x)$ and hence,

$$\mathbf{E}\big(\tau_K(x)^p\big) = \infty.$$

The nonfiniteness of higher moments in this case follows by the usual Hölder type argument.

It remains to consider $p > \frac{1}{2}\hat{\gamma}$ when $\hat{\gamma}$ is finite. By assumption $\hat{\gamma} > 2$ so the argument leading to (23) is valid for some $r \in (1, \frac{1}{2}\hat{\gamma})$ and [2], Lemma 2 implies that for any $q \in (0, 1)$ there exist positive ε and δ such that

$$\mathbf{P}\big(\sigma_D(x) > \varepsilon\hat{f}(x)^2\big|\xi(t) = x\big) \ge q \qquad \text{whenever } \|x\| \ge D(1+\delta).$$

For $p > \frac{1}{2}\hat{\gamma}$ let F denote the union of the faces A_i, B_i on which $M^{(2p)}(x) = \infty$. Consider starting our random walk from an $x_0 \in F$ such that $\|x_0\| \ge D(1+\delta)$. Clearly $\sigma_D(x_0)^p = \sum_x (1 + \sigma_D(x))^p I_{\{x\}}(1)$ and hence

$$\mathbf{E}\big(\sigma_D(x_0)^p\big) \ge q\varepsilon^p \sum_x \hat{f}(x)^{2p}\mathbf{P}\big(\xi(1) = x|\xi(0) = x_0\big) = \infty$$

because by assumption $M^{(2p)}(x_0) = \infty$ for $x_0 \in F$ and $|\hat{f}(x)| \geq C\|x\|$ for some positive constant C when $\|x\|$ is large. For starting points $x_0 \notin F$ it is clear from our homogeneity and drift assumptions that there are no x with $\|x\| \geq D(1+\delta)$ from which the random walk will with probability 1 hit $\hat{S}_D$ before reaching F so that $\mathbf{E}(\sigma_D(x)^p) = \infty$ for all x with $\|x\| \geq D(1+\delta)$. $\square$

COROLLARY 1. *When $\sum_{i=1}^n C_i - h_i(0) > 0$, Ξ is null-recurrent if $p_0 < 1$ and ergodic if $p_0 > 1$, where $p_0 > 0$ is defined in Theorem* 5.

PROOF. As $p_0 > 0$, Ξ is recurrent by Theorem 4. If $p_0 < 1$ then, by Theorem 6, $\mathbf{E}(\tau_K(x)) = \infty$ for all $x \notin S_K$ with some large enough K and so Ξ is null-recurrent. If $p_0 > 1$, Theorem 5 implies that for $x \notin S_K$ we have $\mathbf{E}(\tau_K(x)^p) < \infty$ for $1 \leq p < \min(p_0, \frac{1}{2}\hat{\gamma})$ and hence Ξ is ergodic. $\square$

6. Applications. The paper [13] considers polling systems with Poisson arrival streams and the exhaustive service discipline in the critical case where the mean load on the system equals the mean arrival rate. We quickly describe their model and results. Independent homogeneous Poisson streams of customers arrive at a pair of nodes, each with an infinite buffer, at rates λ_1 and λ_2, respectively. A single server visits (polls) the nodes performing the service of the customers in their buffers. The service times of the customers at node i, $i = 1, 2$, form a sequence of i.i.d. random variables each distributed as a variable σ_i with the mean m_i and independent of the system's state. The served customers leave the system. After the server has started service at node i it continues there until the buffer is empty (this is the so-called *exhaustive service discipline*). Then the server stops serving the i_1th node and moves to the other node i_2. The time taken for this switch between nodes i_1 and i_2 is a random variable $r_{i_1 i_2}$ assumed independent of the number of customers present in the whole system at the moment the i_1th node is exhausted.

Using essentially the same techniques as in this paper they give explicit conditions for null-recurrence or transience of the system in terms of mean switching times, the arrival rates and the first two moments of the service times in the case where $\rho = \lambda_1 m_1 + \lambda_2 m_2 = 1$. They go on to give partial results for systems with three or more nodes.

The results of this paper apply to various generalizations of the standard polling system model as discussed in [13]. We only consider cases where there are two service nodes but we permit a variety of *regimes* in which the behavior of the server and system are different. Each regime has its own set of system parameters, for example, arrival rates, service time distributions. These are restricted by the requirement that we can find an embedded point process which is a random walk on a two-dimensional complex. This allows for regimes where both nodes receive service simultaneously with service times being independent and exponentially distributed, as well as regimes where service times are nonexponential but one node receives all of the service. There may well be other possibilities.

Changes of regime can occur at any time when there are no customers at one or other node according to the following rule. There are $2n$ regimes and in regime r the customers are put into class types c_{ri} according to the node they are waiting at. If r is odd, customers at node 1 have class $c_{r1} = 1$ and those at node 2 class $c_{r2} = 2$, while if r is even, $c_{r1} = 2$ and $c_{r2} = 1$. Following regime r, the next regime is randomly chosen from regimes r, $r - 1$ if there are no customers of class c_{r2}, while if there are no customers of type c_{r1} it is randomly chosen from regimes r, $r + 1$. During changes of regime we assume only that the changes to queue lengths have finite mean so customers may accumulate or indeed be lost or served by some other mechanism.

The earlier sections of this paper are relevant to this generalized polling system model only in the case where, for every r, regime r's service rate for customers of class c_{r1} is greater than their arrival rate, while for class c_{r2} the arrival rate is greater than the service rate. Methods for deciding the transience or recurrence/ergodicity of the general case appeared in [10] and are treated at some length in Chapters 4 and 5 of [6].

A further type of generalization is possible in which completed tasks are routed back into the input buffers independently but with probabilities that depend on regime and node. This would be a multi-regime open two node Jackson network.

REFERENCES

[1] ASMUSSEN, S. (1987). *Applied Probability and Queues*. Wiley, Chichester.

[2] ASPANDIIAROV, S., IASNOGORODSKI, R. and MENSHIKOV, M. (1996). Passage-time moments for nonnegative stochastic processes and an application to reflected random walks in a quadrant. *Ann. Probab.* **24** 932–960.

[3] ASYMONT, I., FAYOLLE, G. and MENSHIKOV, M. (1995). Random walks in a quarter plane with zero drifts: Transience and recurrence. *J. Appl. Probab.* **32** 941–955.

[4] FAYOLLE, G., IGNATYUK, I. A., MALYSHEV, V. A. and MENSHIKOV, M. (1991). Random walks in two-dimensional complexes. *Queueing Syst. Theory Appl.* **9** 269–300.

[5] FAYOLLE, G., MALYSHEV, V. and MENSHIKOV, M. (1992). Random walks in a quarter plane with zero drift. *Ann. Inst. H. Poincaré* **28** 179–195.

[6] FAYOLLE, G., MALYSHEV, V. and MENSHIKOV, M. (1995). *Topics in the Constructive Theory of Countable Markov Chains*. Cambridge Univ. Press.

[7] FOSS, S. and LAST, G. (1996). Stability of polling systems with exhaustive service policies and state-dependent routing. *Ann. Appl. Probab.* **6** 116–137.

[8] LAMPERTI, J. (1960). Criteria for the recurrence and transience of stochastic processes I. *J. Math. Anal. Appl.* **1** 314–330.

[9] LAMPERTI, J. (1963). Criteria for stochastic processes II: Passage time moments. *J. Math. Anal. Appl.* **7** 127–145.

[10] MALYSHEV, V. and MENSHIKOV, M. (1981). Ergodicity, continuity and analyticity of countable Markov chains. *Trans. Moscow Math. Soc.* **39** 1–48.

[11] MENSHIKOV, M. and POPOV, S. YU. (1995). Exact power estimates for countable Markov chains. *Markov Process. Related Fields* **1** 57–78.

[12] MENSHIKOV, M. and WILLIAMS, R. J. (1996). Passage-time moments for continuous nonnegative stochastic processes and applications. *Adv. in Appl. Prob.* **28** 747–762.
[13] MENSHIKOV, M. and ZUYEV, S. (2001). Polling systems in the critical regime. *Stoch. Process. Appl.* **92** 201–218.
[14] VARADHAN, S. R. S. and WILLIAMS, R. J. (1985). Brownian motion in a wedge with oblique reflection. *Comm. Pure Appl. Math.* **38** 405–443.

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