VERTEX RING-INDEXED LIE ALGEBRAS

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Abstract

Infinite-dimensional Lie algebras are introduced, which are only partially graded, and are specified by indices lying on cyclotomic rings. They may be thought of as generalizations of the Onsager algebra, but unlike it, or its $sl(n)$ generalizations, they are not subalgebras of the loop algebras associated with $sl(n)$. In a particular interesting case associated with $sl(3)$, their indices lie on the Eisenstein integer triangular lattice, and these algebras are expected to underlie vertex operator combinations in CFT, brane physics, and graphite monolayers.

1   The New Algebras

We briefly introduce a class of infinite-dimensional vertex-operator Lie algebras. They have two indices, one of which lacks conventional grading. Instead, its composition motivates placing it on a cyclotomic ring, which thus makes it effectively equivalent to a multiplet of integers. We expect these algebras to feature in CFT and other areas of physics with enhanced symmetry.

Consider the Lie algebras

$$[J_m^a, J_n^b] = J_{m+\omega^a n}^{a+b} - J_{n+\omega^b m}^{a+b} ,$$ (1)

where the indices $a, b, ..., m, n, ...$ and the parameter ω may be arbitrary, in general.

However, as will become evident, the choice of ω as an N-th root of unity, $\omega^N = 1$, hence $1 + \omega + \omega^2 + ... + \omega^{N-1} = 0$, and $a, b, ...$ integers, $m, n, ...$ proportional to integers, yields by far the most interesting family. In that case, the upper indices are only distinct $\mathrm{mod}\, N$, and the lower indices take values in the cyclotomic integer ring $\mathbb{Z}[\omega]$, namely, $r + s\omega + k\omega^2 + ... + j\omega^{N-2}$. Note the grading of the upper indices, but lack of conventional grading for the lower indices, in contrast to conventional maximally graded two-index infinite Lie algebras such as [1, 2, 3]. (The Lie algebras introduced here appear distinct from those based on affine quasicrystals, associated with N-th roots of unity, Coxeter groups, and Penrose pentilings [4] — but some intriguing connection to these algebras should not be excluded.)

This algebra satisfies the Jacobi identity, and possesses the central element $J_0^0 = J_{-\omega^{-a} m}^{-a} J_m^a$. For the cyclotomic family, "Casimir invariants" may be written as

$$J_0^0 = (J_m^a)^N,$$ (2)

provided $m = 0$ if $a = 0$.

In fact, the above Lie algebra might be constructed from the group algebra of associative operators

$$J_m^a J_n^b = J_{m+\omega^a n}^{a+b} \ , \tag{3}$$

which satisfy

$$(J_m^a J_n^b) J_k^c = J_m^a (J_n^b J_k^c). \tag{4}$$

It would be customary in such cases [1] to also consider the anticommutator of these operators, to produce a partner graded Lie algebra,

$$\{J_m^a, J_n^b\} = J_{m+\omega^a n}^{a+b} + J_{n+\omega^b m}^{a+b} \ . \tag{5}$$

A simple operator realization of this algebra is

$$J_m^a = e^{m \exp(x)} \ \omega^{a \partial_x}, \tag{6}$$

as may be checked by virtue of the translation action of $\omega^{\partial_x} f(x) = f(x + \ln \omega) \ \omega^{\partial_x}$. It is easy to see in this realization that the scale of the a, b is fixed, but that of the m, n is labile, as they can be rescaled with no change to the structure of the algebra.

A variant rewriting of this realization results from the simplifying Campbell-Baker-Hausdorff expansion for the particular operators involved,

$$J_m^a = \omega^{a\left(\partial_x + \frac{m}{\omega^a - 1} \exp(x)\right)}. \tag{7}$$

Equivalently, given oscillator operators, $[\alpha, \alpha^\dagger] = 1$, the above realizations may be written in a form evocative of vertex operators,

$$J_m^a = e^{m \ \alpha^\dagger} \ \omega^{a \ \alpha^\dagger \alpha} = \omega^{a\left(\alpha^\dagger \alpha + \frac{m}{\omega^a - 1} \alpha^\dagger\right)}. \tag{8}$$

In the cyclotomic case, $\omega^N = 1$, a, b are equivalent $\mathrm{mod} N$, so $a, b, .. = 1, 2, ..., N - 1$. The $N = 2$ case, $\omega = -1$, is trivial, as the corresponding lower index ring is that of the conventional integers, and the resulting algebra is essentially the Onsager algebra, a subalgebra of the $SL(2)$ loop algebra, discussed in the next section.

As an aside, a less "asymmetric", albeit more cumbersome rewriting of eqn (3) might be

$$V_m^a \equiv J_{m w^a}^{2a} \ , \tag{9}$$

so that eqn (3) reads

$$V_m^a V_n^b = V_{\omega^{-b} m + \omega^a n}^{a+b} \ . \tag{10}$$

Antisymmetrization leads to the corresponding notation for the Lie algebra (1),

$$[V_m^a, V_n^b] = V_{\omega^{-b} m + \omega^a n}^{a+b} - V_{\omega^b m + \omega^{-a} n}^{a+b} \ . \tag{11}$$

2 $N = 2$ Degenerate Case and the Onsager Algebra

Onsager, in his celebrated solution of the two-dimensional Ising model [5], introduced the integer-indexed infinite-dimensional Lie algebra,

$$[A_m, A_n] = 4G_{m-n}, \qquad [G_m, A_n] = 2(A_{m+n} - A_{n-m}), \qquad [G_m, G_n] = 0. \qquad (12)$$

(Also see [6, 7, 8].) Evidently, $G_{-m} = -G_m$. A potential central element, G_0, is not generated on the r.h.s. of the algebra. Onsager also recognized that his algebra is effectively a subalgebra of the $SL(2)$ loop algebra ($SU(2)$ centerless Kac-Moody algebra in modern conventions). The loop Lie algebra consists of three integer-indexed "towers" of elements, with

$$[K_m^+, K_n^-] = K_{m+n}^0 , \qquad [K_m^0, K_n^\pm] = \pm K_{m+n}^\pm , \qquad [K_m^\pm, K_n^\pm] = [K_m^0, K_n^0] = 0. \qquad (13)$$

Given the linear involutive automorphism of this algebra,

$$K_m^\pm \mapsto K_{-m}^\mp , \qquad\qquad K_m^0 \mapsto -K_{-m}^0 , \qquad (14)$$

the Onsager algebra is identifiable with the (fixed-point) subalgebra [7], ie, the subalgebra invariant under the automorphism, consisting of two "towers",

$$A_m = 2\sqrt{2}(K_m^+ + K_{-m}^-), \qquad\qquad G_m = 2(K_m^0 - K_{-m}^0). \qquad (15)$$

It can be checked that for $N = 2$, thus $a = 0, 1$, the above algebra (1) also contains the Onsager algebra as a subalgebra,

$$A_m = 2J_m^1 , \qquad G_m = J_m^0 - J_{-m}^0 . \qquad (16)$$

It can then be seen that a graded extension of the Onsager algebra of the type (5) is trivial, since

$$H_{m-n} \equiv \{A_m, A_n\} = 4(J_m^0 + J_{-m}^0), \qquad (17)$$

check to be central, ie, they commute with all elements, A_n, G_n.

Thus, $J_m^0 = -J_{-m}^0$ +constant; hence, conversely, requiring a trivial graded extension of the Onsager algebra essentially amounts to (3). (Note from eqn (2) that $A_m A_m$ is not an invariant of the Onsager algebra per se, but only upon this further condition, $A_m A_m = 4J_0^0$.)

The realization (6) reduces here to

$$A_m = 2e^{m \exp(x)}(-)^{\partial_x}, \qquad\qquad G_m = e^{m \exp(x)} - e^{-m \exp(x)}. \qquad (18)$$

In this realization, the potential candidate for a graded extension,

$$H_m = 4(e^{m \exp(x)} + e^{-m \exp(x)}), \qquad (19)$$

manifestly commutes with all elements, A_n, G_n .

An alternate realization in terms of Pauli matrices is

$$A_m = 2e^{m\sigma_3}\sigma_1 , \qquad\qquad G_m = (e^m - e^{-m})\sigma_3 , \qquad (20)$$

similarly illustrating the triviality of $H_m \propto \mathbf{1}$.

3 $N = 3$ and the Eisenstein Integer Lattice

For $N = 3$, the resulting algebra appears to be new, since, for $\omega = e^{2\pi i/3} = -1 - \omega^2$, the lower indices are of the form $m \equiv k + j\omega$ (with integer k, j), closing under addition, subtraction, and multiplication. These comprise the Euclidean ring $\mathbb{Z}[\omega]$ of Eisenstein-Jacobi integers [9], which define a triangular 2-d lattice with hexagonal rotational symmetry: there are three lines at 60° to each other going through each such integer and connecting it to its six nearest neighbors, forming honeycomb hexagons.
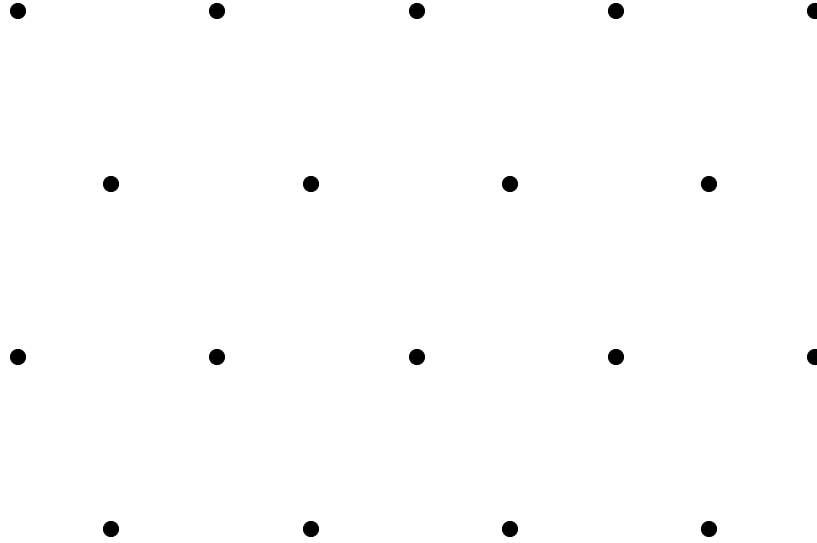
This lattice is of utility in cohesive energy calculations for monolayer graphite [10], 3-state-Potts models associated with WZW CFT models [13], and, perhaps more provocatively, complexifies [12] to define the complex Leech lattice, of significance in string theory, and $\mathbb{Z}_3$ orbifolds in CFT [11].

Each point on the lattice may be connected to the origin by shifts along the ω root and along the x-axis. A 60⁰ rotation ωm, on $m \equiv k + j\omega$, for integer coordinates k, j, may be represented by

$$\boldsymbol{\Omega} \begin{pmatrix} k \\ j \end{pmatrix} \equiv \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} k \\ j \end{pmatrix}, \tag{21}$$

for $\boldsymbol{\Omega}^3 = \mathbb{1}$, and $\boldsymbol{\Omega}^2 = -\mathbb{1} - \boldsymbol{\Omega}$. Thus, the lower indices of the algebra may be considered as a doublet of integers composing through this rule.

We care to illustrate this case explicitly to stress the differences from conventional loop algebras and $sl(3)$ generalizations of the Onsager algebra. Instead of the differential realization (6), consider a faithful representation in terms of 3×3 matrices. Sylvester's "nonion" basis for $GL(3)$ groups [14], is built out of his standard clock and shift unitary unimodular matrices,

$$Q \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad P \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{22}$$

so that $Q^3 = P^3 = \mathbb{1}$. These obey the braiding identity $PQ = \omega\, QP$ [14, 15]. For integer indices adding mod 3, the complete set of nine unitary unimodular 3×3 matrices

$$M_{(m_1, m_2)} \equiv \omega^{m_1 m_2/2}\, Q^{m_1} P^{m_2}, \tag{23}$$

where $M^\dagger_{(m_1,m_2)} = M_{(-m_1,-m_2)}$, and $\text{Tr}M_{(m_1,m_2)} = 0$, except for $m_1 = m_2 = 0$ mod3, suffice to span the group algebra of $GL(3)$. Since

$$M_\mathbf{m} M_\mathbf{n} = \omega^{\mathbf{n} \times \mathbf{m}/2} M_{\mathbf{m}+\mathbf{n}} \ , \tag{24}$$

where $\mathbf{m} \times \mathbf{n} \equiv m_1 n_2 - m_2 n_1$, they further satisfy the Lie algebra of $su(3)$ [1],

$$[M_\mathbf{m}, M_\mathbf{n}] = -2i \ \sin\left(\frac{\pi}{3}\mathbf{m} \times \mathbf{n}\right) \ M_{\mathbf{m}+\mathbf{n}} \ . \tag{25}$$

It is then simple to realize (1,3) in the unimodular 3×3 matrix representation,

$$J^a_m = e^{mQ} \ P^a, \tag{26}$$

ie, the three "towers",

$$J^1_m = \begin{pmatrix} 0 & e^m & 0 \\ 0 & 0 & e^{m\omega} \\ e^{m\omega^2} & 0 & 0 \end{pmatrix}, \qquad J^2_m = \begin{pmatrix} 0 & 0 & e^m \\ e^{m\omega} & 0 & 0 \\ 0 & e^{m\omega^2} & 0 \end{pmatrix}, \qquad J^0_m = \begin{pmatrix} e^m & 0 & 0 \\ 0 & e^{m\omega} & 0 \\ 0 & 0 & e^{m\omega^2} \end{pmatrix}. \tag{27}$$

One may contrast this Lie algebra to not only $su(3)$ loop algebra, but also to its subalgebras, such as the the $sl(3)$ generalization of the Onsager algebra, introduced by ref [7] and consisting of five towers. Specifically, the relevant involutive automorphism of $su(3)$ loop algebra, in standard Chevalley notation, is

$$H^{1,2}_m \mapsto -H^{1,2}_{-m} \ , \qquad E^{\pm 1}_m \mapsto E^{\mp 1}_{-m} \ , \qquad E^{\pm 2}_m \mapsto E^{\mp 2}_{-m} \ , \qquad E^{\pm 3}_m \mapsto -E^{\mp 3}_{-m} \ . \tag{28}$$

The subalgebra left invariant under this automorphism consists of the five towers [7],

$$H^{1,2}_m - H^{1,2}_{-m} \ , \qquad E^1_m + E^{-1}_{-m} \ , \qquad E^2_m + E^{-2}_{-m} \ , \qquad E^3_m - E^{-3}_{-m} \ , \tag{29}$$

or, explicitly,

$$h^1_m = \frac{1}{\sqrt{6}}\begin{pmatrix} e^m - e^{-m} & 0 & 0 \\ 0 & e^{-m} - e^m & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h^2_m = \frac{1}{3\sqrt{2}}\begin{pmatrix} e^m - e^{-m} & 0 & 0 \\ 0 & e^m - e^{-m} & 0 \\ 0 & 0 & 2e^{-m} - 2e^m \end{pmatrix}, \tag{30}$$

$$e^1_m = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & e^m & 0 \\ e^{-m} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e^2_m = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & e^m \\ 0 & 0 & 0 \\ e^{-m} & 0 & 0 \end{pmatrix}, \quad e^3_m = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^m \\ 0 & -e^{-m} & 0 \end{pmatrix}. \tag{31}$$

4   General Case: $N > 3$, and Quasicrystals

For higher N, the cyclotomic integer rings $\mathbb{Z}[\omega]$ are less compelling, and are only linked to quasicrystals. Specifically, the 2-dimensional complex plane quasilattice fills up densely with the set of indices, which fail to close to a "sparse" periodic structure analogous to the Eisenstein lattice. A quasicrystal is a higher-dimensional deterministic discrete periodic structure whose projection to an embedded

"external space" (in our case, the complex plane) yields nonperiodic structures of enhanced regularity [16].

For example, for $N = 5$, motions are symmetric on a 4-dimensional periodic lattice, $\boldsymbol{\Omega}^5 = \mathbb{1}$, and $\boldsymbol{\Omega}^4 = -\mathbb{1} - \boldsymbol{\Omega} - \boldsymbol{\Omega}^2 - \boldsymbol{\Omega}^3$, with

$$\boldsymbol{\Omega} \equiv \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \tag{32}$$

so lower indices may be effectively regarded as a quartet of integers—and, likewise, an $N-1$-tuplet of integers for higher N. However, projected on the actual complex plane, nearby numbers are not necessarily represented by contiguous points on the 4-d lattice.

As indicated at the beginning, there may be links between the present algebras over cyclotomic fields and those on quasicrystals which exhibit a five-fold symmetry [4]. For $\omega^5 = 1$ and the golden ratio, $\tau \equiv \frac{1}{2}(1 + \sqrt{5})$, which satisfies $\tau^2 = 1 + \tau$, one sees that $\tau = -\omega^2 - \omega^3$, since then $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$ follows. There is considerable work [4] on algebras defined over such quadratic number fields, $\mathbb{Z}[\tau] = \mathbb{Z} + \mathbb{Z}\tau$, while the associated geometric constructions of quasicrystal lattices are available in textbooks [16]. Possibly, detailed investigations of the connection with algebras defined over the cyclotomic fields will be a fruitful source of insight. Given the vertex operator realization of the Lie algebras introduced here and its evocation of coherent states, useful applications in CFT and brane physics appear likely.

References

[1] D Fairlie, P Fletcher, and C Zachos, Phys Lett **B218** (1989) 203;
    D Fairlie and C Zachos, Phys Lett **B224** (1989) 101-107

[2] J Hoppe, Phys Lett **215B** (1988) 706-710;
    Int J Mod Phys **A4** (1989) 5235-5248

[3] M Saveliev and A Vershik, Phys Lett **143A** (1990) 121-128.

[4] J Patera and R Twarock, J Phys **A 35** (2002) 1551-1574;
    V Mazorchuk and R Twarock, J Phys **A36** (2003) 4363-4373;
    J Patera, E Pelantová, and R Twarock, Phys Lett **246A** (1998) 209-213; H C Jeong, E Kim, and C Lee, J Phys **A34** (2001) R1-R19

[5] L Onsager, Phys Rev **65** (1944) 117-149.

[6] B Davies, J Phys **A23** (1990) 2245-2261

[7] D Uglov, I Ivanov, J Stat Phys **82** (1996) 87-113, [hep-th/9502068]

[8] E Date and S-s Roan, J Phys **A33** (2000) 3275-3296

[9] E Weisstein, "Eisenstein Integer", in MathWorld—A Wolfram Web Resource,
http://mathworld.wolfram.com/EisensteinInteger.html

[10] Chen Nanxian, Chen Zhaodou, Liu Shaojun, Shen Yanan and Ge Xijin, J Phys **A29** (1996)
5591-5603

[11] P Montague, [hep-th/9307076];
L Dolan, P Goddard, and P Montague, Commun Math Phys **179** (1996) 61-120

[12] J Conway and N Sloane, *Sphere Packings, Lattices, and Groups* (3rd edition, Springer, 1999).

[13] I Affleck, M Oshikawa and H Saleur, Nucl Phys **B594** (2001) 535-606

[14] J Sylvester, *Johns Hopkins University Circulars* **I** (1882) 241-242; *ibid* **II** (1883) 46; *ibid* **III**
(1884) 7-9. Summarized in *The Collected Mathematics Papers of James Joseph Sylvester* (Cambridge University Press, 1909) v III

[15] H Weyl, *Z Phys* **46** (1927) 1-33; H Weyl, *The Theory of Groups and Quantum Mechanics* (Dover,
New York, 1931)

[16] M Senechal, *Quasicrystals and Geometry* (Cambridge Univ Press, Cambridge, UK, 1995)