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Phase Transition for the Spherical Hierarchical Model

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Abstract. We present the whole spectrum of the limit theorems for the total magnetization in the hierarchical version of the spherical model in dimensions $\mathsf{dim} > 2$.

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1. Introduction

In this paper we study the spherical model with hierarchical ferromagnetic interactions. We show existence of a phase transition for the total magnetization and give limit laws, for all temperatures and all hierarchical dimensions larger than 2. We show that the fluctuations of the magnetization are Gaussian for temperature larger than critical. A slightly surprising feature is that the critical fluctuations are still Gaussian for $\mathsf{dim} \geq 4$, and non-Gaussian otherwise (but reasonably explicit).

Our study is based on the explicit knowledge of the spectrum of the hierarchical Laplacian and is close in spirit to the paper [5], related to the standard lattice Laplacian on $\mathbf{Z}^d$ and published without proofs. However, the results of our analysis (especially at the critical point) are different from [5]. The Gaussian critical fluctuations for $\mathsf{dim} \geq 4$ might be related to the absence of phase transitions in the spectral theory of the Anderson Hamiltonian on the hierarchical lattice [4].

This work is a first step. We intend to devote future work to random versions of this model, and to non-equilibrium dynamics, for instance aging properties, as those studied for the Spherical Sherrington – Kirkpatrick model in [1].

2. The model and the results

In this section, we introduce first the hierarchical structure and discuss certain analogies between the asymptotic properties of the corresponding random walks with those of the random walks in the Euclidean case. Next, we define the related spherical model on this hierarchical structure and describe the phase transition seen from the limiting distribution of the total magnetization in the thermodynamic limit.

2.1. The hierarchical structure

On the lattice $\mathbf{Z}^d$, $d \geq 1$, we introduce the hierarchical structure in a usual way.

Fix a (lattice) cube $Q_0^{(1)}$ of volume $|Q_0^{(1)}| = \nu > 1$ centered at the origin, consider the tiling T_1 of $\mathbf{Z}^d$ generated by the translates $Q_i^{(1)} \equiv x_i + Q_0^{(1)}$ of $Q_0^{(1)}$,

$$\mathbf{Z}^d = \bigcup_i Q_i^{(1)},$$

and call the subsets $Q_i^{(1)} \subset \mathbf{Z}^d$ *cubes of the* 1^{st} *generation.* Next, consider the (centered at the origin) cube $Q_0^{(2)}$ consisting of ν cubes of the 1^{st} generation. Its translates generate in a similar way the tiling T_2:

$$\mathbf{Z}^d = \bigcup_i Q_i^{(2)}.$$

Clearly, the volume $|Q_i^{(2)}|$ of any cube $Q_i^{(2)}$ of the 2^{nd} generation is ν^2. By repeating this procedure again and again we obtain cubes of the 3^{rd}, $\dots$, i^{th}, $\dots$ generation. Note that each vertex $x \in \mathbf{Z}^d$ belongs to exactly one cube $Q^{(i)}(x) \equiv Q_x^{(i)}$ of the i^{th} generation. In particular, we may put $Q^{(0)}(x) \equiv \{x\}$.

Next, we introduce the hierarchical distance $d_{\mathsf{h}}(\cdot,\cdot)$ between two points x, $y \in \mathbf{Z}^d$ as the minimal rank of generation in which they belong to the same cube,

$$d_{\mathsf{h}}(x,y) = \min\bigl\{r : \exists\, Q^{(r)} \ni x, y\bigr\}. \tag{2.1}$$

Note that $d_{\mathsf{h}}(x,y)$ is exactly half of the graph distance between x and y in the tree representation of the hierarchical structure, see Figure 1 below.

2.1.1. The hierarchical Laplacian and its spectrum

Let α_r be a sequence of positive numbers satisfying the condition

$$\sum_r \alpha_r = 1. \tag{2.2}$$

We define the hierarchical Laplacian Δ_{h} as a (formal) operator in the functional space $L^2(\mathbf{Z}^d)$ via

$$\Delta_{\mathsf{h}}\psi(x) = \sum_{r=1}^{\infty} \frac{\alpha_r}{|Q_0^{(r)}|} \sum_{y\in Q^{(r)}(x)} \big(\psi(y)-\psi(x)\big).$$

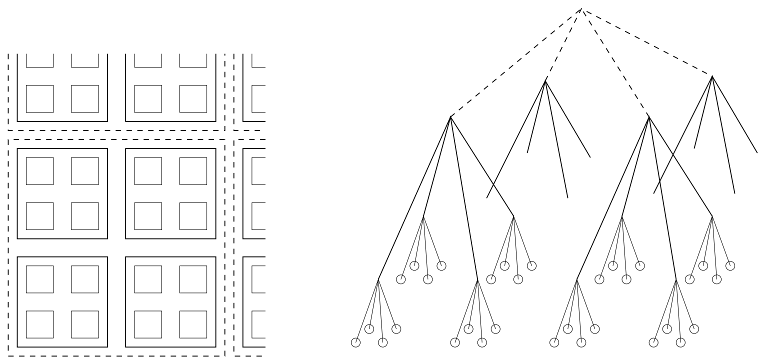

Figure 1. A piece of hierarchical structure and a part of its tree representation.

In what follows we will consider mainly the particular case

$$\alpha_r = pq^{r-1}, \qquad p \equiv 1-q \in (0,1);$$

the aim of this section is to study the spectral properties of the finite-dimensional analogue $\Delta_{\mathsf{h}}^{(R)}$ of Δ_{h} acting in $L^2\big(Q_0^{(R)}\big)$:

$$\begin{aligned}\Delta_{\mathsf{h}}^{(R)}\psi(x) &= \sum_{r=1}^{R} \frac{pq^{r-1}}{\nu^r} \sum_{y\in Q^{(r)}(x)} \big(\psi(y)-\psi(x)\big)\\ &= \sum_{r=1}^{R} \frac{pq^{r-1}}{\nu^r} \sum_{y\in Q^{(r)}(x)} \psi(y) - \big(1-q^R\big)\psi(x).\end{aligned} \tag{2.3}$$

For a fixed R, it is not difficult to find the spectrum of $\Delta_{\mathsf{h}}^{(R)}$ in $V \equiv Q_0^{(R)}$. Define first $\mathcal{L}_1$ as the space of functions from $L^2(V)$ that have vanishing averages on each cube of the 1^{st} generation:

$$\mathcal{L}_1 := \Big\{\psi \in L^2(V) : \forall x, \sum_{y\in Q^{(1)}(x)} \psi(y) = 0\Big\}.$$

Clearly, $\mathcal{L}_1$ is a $(\nu-1)\nu^{R-1}$-dimensional space of eigenfunctions of $\Delta_{\mathsf{h}}^{(R)}$ corresponding to the eigenvalue

$$-\lambda_1 = -\sum_{r=1}^{R}\alpha_r = -\big(1-q^R\big);$$

its orthogonal complement $\mathcal{L}_1^\perp$ in $L^2(V)$ consists of functions that are constant on each cube $Q_i^{(1)}$. Next, define the subspace $\mathcal{L}_2 \subset \mathcal{L}_1^\perp$ via

$$\mathcal{L}_2 := \Big\{\psi \in L^2(V) : \forall x,\ \psi\Big|_{Q^{(1)}(x)} \equiv \psi(x) \ \text{ and } \sum_{y \in Q^{(2)}(x)} \psi(y) = 0\Big\}.$$

It is immediate to check that $\mathcal{L}_2$ is a $(\nu-1)\nu^{R-2}$-dimensional subspace of the eigenfunctions of $\Delta_{\mathsf{h}}^{(R)}$ corresponding to the eigenvalue

$$-\lambda_2 = -\sum_{r=2}^{R} \alpha_r = -\big(q - q^R\big).$$

Continuing further in this way, we define $\mathcal{L}_k$, $k \le R$, as the subspace of functions that are constant on all $Q_i^{(k-1)}$ and have zero average on each cube of the k^{th} generation. This provides us with the eigenvalue

$$-\lambda_k = -\sum_{r=k}^{R} \alpha_r = -\big(q^{k-1} - q^R\big)$$

of multiplicity $(\nu-1)\nu^{R-k}$. Finally, $\mathcal{L}_{R+1}$ is defined as the one-dimensional space of constant functions in V; it corresponds to the simple eigenvalue

$$-\lambda_{R+1} = 0.$$

Summarizing, we obtain the complete spectral description of the operator $\Delta_{\mathsf{h}}^{(R)}$, see Figure 2.

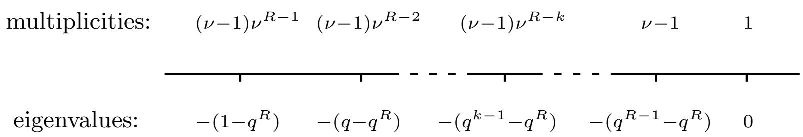

Figure 2. The spectrum of the operator $\Delta_{\mathsf{h}}^{(R)}$.

Finally, since

$$\frac{\log\big(\nu^{-R}|\{j : \lambda_j \ge -(q^{k-1} - q^R)\}|\big)}{\log(q^{k-1} - q^R)} \to \frac{\log \nu}{\log q}$$

as $R \to \infty$, it is natural to define the hierarchical spectral dimension as

$$\mathsf{dim} := \frac{2 \log \nu}{\log 1/q}. \tag{2.4}$$

2.1.2. The analogy with the Euclidean case

To the hierarchical Laplacian Δ_{h} it can be associated a continuous time random walk in the following way. Let at the moment t the random walk be at a site x. After a (random) time $\Delta t \sim \mathsf{Exp}(1)$ the walker chooses the rank r according to the distribution α_r (recall (2.2)) and then jumps uniformly in the associated to x cube $Q_x^{(r)}$ of the r^{th} generation. Then the procedure is repeated again and again.

Alternatively,[1] one can describe this hierarchical random walk as follows. thinking of the lattice sites as of cubes of 0^{th} generation (the lowest level of the tree in Figure 1) the walker waits an $\mathsf{Exp}(1)$ interval of time and then climbs the level r of the tree with probability α_r. After, the walker descends by choosing uniformly between all possible edges on each level. Then the process iterates.

To establish the analogy between the hierarchical random walk and the one on the usual lattice, we introduce the "Euclidean" distance $\rho(\cdot,\cdot)$ on the hierarchical structure by requiring the volume of a sphere of radius R to grow as R^{dim}. In other words, we put

$$\rho(x,y) \stackrel{\mathsf{def}}{=} \nu^{d_{\mathsf{h}}(x,y)/\mathsf{dim}} = q^{-d_{\mathsf{h}}(x,y)/2}.$$

Now, using the spectral description of the hierarchical Laplacian Δ_{h}, we obtain

$$p(t,x,y) = \sum e^{\lambda_n t}\psi_n(x)\psi_n(y),$$

$$g(x,y) = \sum \frac{1}{\lambda_n}\psi_n(x)\psi_n(y)$$

with λ_n denoting the eigenvalues and $\psi_n(\cdot)$ — the corresponding eigenfunctions. In particular, the Green function

$$g(x,x) = \frac{\nu-1}{\nu}\sum_{r=0}^{\infty}(\nu q)^{-r}$$

is finite if and only if $\nu q > 1$ (that is $\mathsf{dim} > 2$). Thus, the random walk is recurrent only if $\mathsf{dim} \le 2$ and is transient if $\mathsf{dim} > 2$. Moreover, in the transient case the Green function $g(x,y)$ exhibits the usual "Euclidean" asymptotics

$$g(x,y) \asymp \rho(x,y)^{2-\mathsf{dim}} \qquad \text{as } \rho(x,y) \to \infty.$$

By a direct computation one verifies also that

$$\log p(t,x,y) \asymp -\frac{\rho(x,y)^2}{t}$$

as soon as $\rho(x,y) \to \infty$ in such a way that the right-hand side above remains uniformly bounded.

Some additional analogies will be seen below when studying the phase transition for the magnetization in the related spherical model.

[1] This interpretation will be crucial in studying the random version of the hierarchical structure in the forthcoming paper.

2.2. The spherical model

The spherical model was introduced by Berlin and Kac [2] as a rough, though analytically convenient, approximation to the multidimensional Ising model. To fix the notations we describe it briefly below while referring the reader to the original paper for missing details.

2.2.1. The model

Let a finite set V and a symmetric function $J : V^2 \to \mathbf{R}^1$ be given (that satisfies some additional constraints, see below) and let S_R^V denote the $|V|$-dimensional sphere of radius $\sqrt{R}$,

$$\mathsf{S}_R^V := \Big\{\sigma_V \in \mathbf{R}^{|V|} : \sum_{x\in V} \sigma_x^2 = R\Big\},$$

with a shorthand notation $\mathsf{S}^V \equiv \mathsf{S}_{|V|}^V$. The spherical model in V is defined by assigning to each configuration $\sigma_V \in \mathsf{S}^V$ the energy $\mathcal{H}_V(\sigma_V)$ corresponding to the potential $J(\cdot,\cdot)$:

$$\mathcal{H}_V(\sigma_V) := -\sum_{x,y\in V : x\neq y,} J(x,y)\,\sigma_x\sigma_y$$

and thus introducing the Gibbs measure $\mathsf{P}_V(d\sigma)$ in the usual way,

$$\mathsf{P}_V(d\sigma) := \frac{\exp\big\{-\beta\mathcal{H}_V(\sigma)\big\}}{Z_V^\beta}\,d\sigma, \qquad \sigma \in \mathsf{S}^V; \tag{2.5}$$

here $\beta > 0$ is the inverse temperature and the normalizing constant (the partition function) $Z_V^\beta \equiv Z_V^\beta(|V|)$ is given by

$$Z_V^\beta = \int\limits_{\mathsf{S}^V} \exp\big\{-\beta\mathcal{H}_V(\sigma_V)\big\}\,d\sigma_V.$$

Let E_V denote the operator of the mathematical expectation corresponding to $\mathsf{P}_V(\cdot)$. We will describe below the distribution of the specific magnetization

$$s_V = \frac{1}{|V|}\sum_{x\in V}\sigma_x \tag{2.6}$$

with respect to the Gibbs measure $\mathsf{P}_V(d\sigma)$.

2.2.2. Diagonalizing the partition function

We derive next an analytic expression for the partition function under an additional (innocent in a finite volume) assumption that all the eigenvalues of the quadratic form corresponding to $J(\cdot,\cdot)$ are non-positive.

To this end, let the radius $\sqrt{R} = \sqrt{|V|}$ of the sphere S_R^V vary and consider the Laplace transform $Z_V^\beta(\lambda)$ of the partition function Z_V^β with respect to R:

$$Z_V^\beta(\lambda) = \int_0^\infty Z_V^\beta(R) e^{-\lambda R}\, dR = \int_{\mathrm{R}^{|V|}} \exp\Big\{\beta \sum J(x,y)\sigma_x\sigma_y - \lambda \sum \sigma_x^2\Big\}\, d\sigma, \quad (2.7)$$

using the definition of S_R^V. Diagonalizing the quadratic form in the exponential and applying the classical formula for the Gaussian integral, we obtain:

$$Z_V^\beta(\lambda) = \int_{\mathrm{R}^{|V|}} \prod_{j=1}^{|V|} \exp\{-(\beta\lambda_j + \lambda)\tilde{\sigma}_j^2\}\, d\tilde{\sigma} = \pi^{|V|/2} \prod_{j=1}^{|V|} \frac{1}{\sqrt{\lambda + \beta\lambda_j}},$$

where $-\lambda_j$ denote the (non-positive) eigenvalues of the quadratic form J. As a result, the normalizing constant Z_V^β is expressed via the inverse Laplace transform: for $\lambda_0 > 0$ (i.e., to the right from all eigenvalues of J),

$$Z_V^\beta = \frac{1}{2\pi i} \int_{\lambda_0 - i\infty}^{\lambda_0 + i\infty} \exp\Big\{|V| z - \frac{1}{2} \sum_{j=1}^{|V|} \log\big(z + \beta\lambda_j\big)\Big\}\, dz. \quad (2.8)$$

In the limit of large $|V|$, the integral in the right-hand side of (2.8) is evaluated using the saddle point method.

2.2.3. The characteristic function

We think of $h_V \in \mathbf{R}^{|V|}$ as a linear functional $h_V(\sigma_V)$ acting on the configurations $\sigma_V \in \mathsf{S}^V$ via

$$h_V(\sigma_V) := (h_V, \sigma_V) \equiv \sum_{x \in V} h_x \sigma_x. \quad (2.9)$$

Our next goal is to compute the characteristic function of $h_V(\sigma_V)$ with respect to the Gibbs measure $\mathsf{P}_V(\cdot)$. Note that for $h_V \equiv (1, \dots, 1)/|V|$, the functional $h_V(\sigma_V)$ coincides with the specific magnetization s_V from (2.6).

Denoting

$$Z_V^\beta(R,t) := \int_{\mathrm{S}_R^V} \exp\big\{-\beta \mathcal{H}_V(\sigma_V) + it(h_V, \sigma_V)\big\}\, d\sigma_V,$$

we rewrite the characteristic function $\Psi_{h(\sigma)}(t)$ of $h_V(\sigma_V)$ as (here and below, $Z_V^\beta(R,0) \equiv Z_V^\beta(R)$ in the old notations)

$$\Psi_{h(\sigma)}(t) \equiv \mathsf{E}_V \exp\big\{it(h_V, \sigma_V)\big\} = \frac{Z_V^\beta(|V|, t)}{Z_V^\beta(|V|, 0)} = \frac{Z_V^\beta(t)}{Z_V^\beta}$$

and use the same diagonalization procedure to present $Z_V^\beta(R,t)$ in a form similar to (2.8). Namely, rewriting the Laplace transform $Z_V^\beta(\lambda,t)$ of $Z_V^\beta(R,t)$ as (cf. (2.7))

$$Z_V^\beta(\lambda,t) = \int\limits_{\mathrm{R}^{|V|}} \exp\Bigl\{\beta\sum J(x,y)\sigma_x\sigma_y - \lambda\sum\sigma_x^2 + it\sum h_x\sigma_x\Bigr\}\,d\sigma,$$

we change the variables $\sigma \mapsto \tilde\sigma$, $\tilde\sigma_j = (\sigma_V,\phi_j)$, where ϕ_j is the eigenfunction of J corresponding to the eigenvalue $-\lambda_j$, and obtain

$$\begin{aligned} Z_V^\beta(\lambda,t) &= \int\limits_{\mathrm{R}^{|V|}} \prod_{j=1}^{|V|} \exp\bigl\{-(\beta\lambda_j+\lambda)\tilde\sigma_j^2 + ita_j\tilde\sigma_j\bigr\}\,d\tilde\sigma \\ &= \pi^{|V|/2}\prod_{j=1}^{|V|}\bigl(\sqrt{\lambda+\beta\lambda_j}\,\bigr)^{-1}\exp\Bigl\{-\frac{t^2a_j^2}{4(\lambda+\beta\lambda_j)}\Bigr\} \end{aligned}$$

with $a_j := (h_V,\phi_j)$. A simple application of the inverse Laplace transform leads now to

$$Z_V^\beta(t) = \frac{1}{2\pi i}\int\limits_{\lambda_0-i\infty}^{\lambda_0+i\infty} \exp\Bigl\{|V|z - \frac{t^2}{4}\sum_{j=1}^{|V|}\frac{a_j^2}{z+\beta\lambda_j} - \frac{1}{2}\sum_{j=1}^{|V|}\log\bigl(z+\beta\lambda_j\bigr)\Bigr\}\,dz \tag{2.10}$$

with $\lambda_0 > 0$.

2.3. The hierarchical spherical model

Now we are ready to introduce our model of interest — the hierarchical spherical model.

For a finite box $V = Q_0^{(R)}$, (R being some natural number), define the energy function $\mathcal{H}_{\mathsf{h}}(\sigma)$ via

$$\mathcal{H}_{\mathsf{h}}(\sigma) := -\bigl(\Delta_{\mathsf{h}}^{(R)}\sigma,\sigma\bigr), \qquad \sigma\in\mathsf{S}^V,$$

where $\Delta_{\mathsf{h}}^{(R)}$ is the hierarchical Laplacian from (2.3). Our aim is to describe the limiting distribution of the total magnetization s_V with respect to the Gibbs measure $\mathsf{P}_V^{\mathsf{h}}(\cdot)$ defined as in (2.5) with the hierarchical Hamiltonian $\mathcal{H}_{\mathsf{h}}$. This amounts to consider the functional $h_V(\cdot)$ with $h_V \equiv (1,\dots,1)/|V|$. Since $h_V \in \mathcal{L}_{R+1}$, the last sum in (2.10) simplifies to a single term, the one corresponding to the simple eigenvalue 0. Thus, taking into account the spectral description of the operator $\Delta_{\mathsf{h}}^{(R)}$, we rewrite (2.10) as

$$Z_V^\beta(t) = \frac{1}{2\pi i}\int\limits_{\lambda_0-i\infty}^{\lambda_0+i\infty}\exp\bigl\{\nu^R F_R^t(z)\bigr\}\,dz, \tag{2.11}$$

where

$$F_R^t(z) := z - \frac{t^2}{4z\nu^{2R}} - \frac{1}{2\nu^R}\log z - \frac{\nu-1}{2\nu}\sum_{r=0}^{R-1}\frac{1}{\nu^r}\log\bigl(z+\beta(q^r-q^R)\bigr).$$

Consequently, the question about the limiting behaviour of the specific magnetization s_V is closely related to the asymptotics of the partition function $Z_V^\beta(t)$ as $|V| \equiv \nu^R \to \infty$.

As we shall see in the sequel, the limiting behaviour of s_V depends heavily on the fact whose contribution — the one of the top eigenvalue $-\lambda_{R+1} = 0$ or the one of the rest of the spectrum — wins the infinite volume limit. For this reason, we introduce the function

$$S_R^0(z) := \frac{\nu-1}{2\nu}\sum_{r=0}^{R-1}\frac{1}{\nu^r}\log\bigl(z+\beta(q^r-q^R)\bigr), \qquad z \geq 0, \tag{2.12}$$

containing the total contribution of all negative part of the spectrum of $\Delta_{\mathsf{h}}^{(R)}$ and its derivatives of order $k = 1, 2, \ldots,$

$$S_R^k(z) := \frac{\nu-1}{2\nu}(-1)^{k-1}(k-1)!\sum_{r=0}^{R-1}\frac{1}{\nu^r\bigl(z+\beta(q^r-q^R)\bigr)^k}, \qquad z \geq 0. \tag{2.13}$$

With such notation, the "phase function" $F_R^t(z)$ reads

$$F_R^t(z) := z - \frac{t^2}{4z\nu^{2R}} - \frac{1}{2\nu^R}\log z - S_R^0(z). \tag{2.14}$$

2.4. The results

In the rest of the paper we shall consider the high dimensions $\mathsf{dim} > 2$ (i.e., $\nu q > 1$). In this case the behaviour of our system depends heavily on the value of the inverse temperature β; namely, there is a critical value

$$\beta_{\mathsf{cr}} := \frac{(\nu-1)q}{2(\nu q-1)} \tag{2.15}$$

such that: for $\beta < \beta_{\mathsf{cr}}$ the behaviour of the system is "analytic", in particular, the specific magnetization satisfies the classical central limit theorem, whereas for $\beta > \beta_{\mathsf{cr}}$ the distribution of the specific magnetization approaches certain symmetric Bernoulli law. The behaviour of the model at the critical temperature exhibits certain dependence on the hierarchical dimension dim and, in particular, is asymptotically Gaussian in dimensions $\mathsf{dim} \geq 4$.

More precisely, in the sequel we establish the following results. Recall that $\phi_{s_V}^{\mathsf{h}}(t)$ denotes the characteristic function of the specific magnetization s_V from (2.6).

We start by describing the high-temperature phase.

Theorem 2.1. *Let $0 < \beta < \beta_{\mathsf{cr}}$. For any fixed $t \in \mathbf{R}^1$,*

$$\lim_{R\to\infty} \mathsf{E}_V^{\mathsf{h}}\Bigl\{\frac{it}{\sqrt{|V|}}\sum_{x\in V}\sigma_x\Bigr\} = \exp\Bigl\{-\frac{t^2}{4x_*}\Bigr\},$$

where x_* is the only positive solution to the equation

$$1 - \frac{\nu-1}{2\nu}\sum_{r=0}^{\infty}\frac{1}{\nu^r(x+\beta q^r)} = 0.$$

In other words, the limiting law is the centered Gaussian distribution with variance $1/2x_*$.

As one can expect, the low temperature phase of our system is characterized by the presence of the phase transition resulting in appearing of spontaneous magnetization.

Theorem 2.2. *Let $\beta > \beta_{\mathsf{cr}}$. For any fixed $t \in \mathbf{R}^1$,*

$$\lim_{R\to\infty} \phi^{\mathsf{h}}_{s_V}(t) = \cos\bigl(\sqrt{1-\beta_{\mathsf{cr}}/\beta}\,t\bigr),$$

that is, the law of the specific magnetization s_V tends, as $R \to \infty$, to the symmetric Bernoulli distribution with the atoms at the points $\pm\sqrt{1-\beta_{\mathsf{cr}}/\beta}$.

The next term in the expansion of the characteristic function $\phi^{\mathsf{h}}_{s_V}(t)$ can be also obtained, in particular, a Gaussian correction on the scale $|V|^{-1/2}$ if $\mathsf{dim} > 4$, but since the asymptotics of this correction is analogous to the behaviour of the system at the critical point, we do not do this here.

At the critical temperature, the normalization is dimension dependent and the limiting law is different below and above the critical dimension $\mathsf{dim} = 4$.

Theorem 2.3. *Let $\beta = \beta_{\mathsf{cr}}$.*

For $\mathsf{dim} \geq 4$, the distribution of the (normalized) magnetization

$$|V|^{-(2+\mathsf{dim})/2\mathsf{dim}}\sum_{x\in V}\sigma_x$$

tends, as $R \to \infty$, to the centered Gaussian law with the variance

$$\frac{1}{2\beta_{\mathsf{cr}}} = \frac{\nu q - 1}{(\nu-1)q}.$$

For $2 < \mathsf{dim} < 4$, the distribution of the (normalized) magnetization

$$|V|^{-(2+\mathsf{dim})/2\mathsf{dim}}\sum_{x\in V}\sigma_x$$

tends, as $R \to \infty$, to the law with the characteristic function

$$\int_{\mathrm{R}^1}\exp\bigl\{\Phi^t(s)\bigr\}\,ds \Bigm/ \int_{\mathrm{R}^1}\exp\bigl\{\Phi^0(s)\bigr\}\,ds,$$

where $\Phi^t(s)$ *is given by*

$$\Phi^t(s) := is\alpha_*\beta - \frac{1}{2}\log(1+is) - \frac{t^2}{4\alpha_*\beta(1+is)} + \frac{\nu-1}{2\nu}\sum_{l=1}^{\infty}\nu^l\Big[is\alpha_* q^l - \log\Big(1+\frac{is\alpha_* q^l}{1-(1-\alpha_*)q^l}\Big)\Big],$$

and α_* *denotes the only positive solution to the stationary point equation*

$$\frac{d}{ds}\Phi^0(s)\Big|_{s=0} = 0.$$

We shall see in the sequel (Section 3.3.2) that in the critical dimension $\mathsf{dim} = 4$ the convergence to the limiting Gaussian distribution is in fact very slow.

3. Proofs

We perform here our main analytic task — the asymptotic analysis of the partition function $Z_V^\beta(t)$ from (2.11) with the phase function $F_R^t(z)$ defined in (2.14) thus proving Theorems 2.1–2.3.

Our approach below is based on the saddle point method, the main ingredient of which is the study of the stationary point x_R defined as the (unique) solution to the equation $\frac{d}{dz}F_R^0(x_R) = 0$:

$$1 - \frac{1}{2\nu^R x_R} - S_R^1(x_R) = 0 \tag{3.1}$$

with subsequent expansion of the "phase function" $F_R^t(z)$ in a small neighbourhood of the extremal point x_R. Besides of uniqueness of the solution x_R, the monotonicity of the left-hand side of (3.1) implies also (for any $R > 0$) the following a priori bounds:

$$\frac{1}{2\nu^R} \le x_R \le \frac{1}{2}. \tag{3.2}$$

For $k \ge 1$ define the functions

$$\widetilde{S}_R^k(z) := \frac{\nu-1}{2\nu}(-1)^{k-1}(k-1)!\sum_{r=0}^{R-1}\frac{1}{\nu^r\big(z+\beta q^r\big)^k}. \tag{3.3}$$

We shall see below that in certain interval of values of $z \ge 0$ they give good approximations to the derivatives $S_R^k(z)$ from (2.13). In the sequel we shall also use the following simple properties.

Lemma 3.1. *Put*

$$T_R^k := \sum_{r=0}^{R-1}\frac{1}{(\nu q^{k+1})^r} = \begin{cases} R, & \text{if } \nu q^{k+1} = 1; \\ \dfrac{\nu q^{k+1}}{\nu q^{k+1}-1}\big(1-(\nu q^{k+1})^{-R}\big), & \text{otherwise.} \end{cases} \tag{3.4}$$

Then, uniformly in $x \geq 0$, one has

$$0 \leq S^1_R(x) - \widetilde{S}^1_R(x) \leq \frac{\nu-1}{2\nu\beta}\frac{q^R}{1-q}T^1_R, \tag{3.5}$$

$$\frac{\beta_{\mathsf{cr}}}{\beta}\frac{x}{x+\beta} \leq \widetilde{S}^1_R(0) - \widetilde{S}^1_R(x) \leq \frac{\nu-1}{2\nu\beta^2}\,x\,T^1_R. \tag{3.6}$$

Moreover,

$$S^1_R(0) = \frac{\beta_{\mathsf{cr}}}{\beta} + o_{\mathsf{exp}}(1) \qquad \text{as } R \to \infty. \tag{3.7}$$

Here and below we use $o_{\mathsf{exp}}(1)$ to denote a correction term that vanishes exponentially fast as $R \to \infty$.

For future references, we observe that

$$q^R\,T^1_R = O\big(Rq^R\big) + O\big((\nu q)^{-R}\big) = o_{\mathsf{exp}}(1), \tag{3.8}$$

$$\nu^{-R}\,T^1_R = O\big(R\nu^{-R}\big) + O\big((\nu q)^{-2R}\big) = o_{\mathsf{exp}}(1). \tag{3.9}$$

Proof of Lemma 3.1. We proceed by direct computation. To obtain (3.5), note that uniformly in $x \geq 0$ and $r = 0, 1, \ldots, R-1$,

$$0 \leq \frac{1}{x+\beta(q^r-q^R)} - \frac{1}{x+\beta q^r} \leq \frac{q^R}{\beta q^{2r}(1-q)}$$

and use it in the definitions (2.13)–(3.3). Next, (3.6) follows immediately from the bounds

$$\frac{x}{\beta q^r(x+\beta)} \leq \frac{1}{\beta q^r} - \frac{1}{x+\beta q^r} \leq \frac{x}{\beta^2 q^{2r}}. \tag{3.10}$$

Finally, (3.7) is an implication of the equality

$$\widetilde{S}^1_R(0) = \frac{\nu-1}{2\nu}\sum_{r=0}^{R-1}\frac{1}{\beta(\nu q)^r} = \frac{\beta_{\mathsf{cr}}}{\beta}\big(1-(\nu q)^{-R}\big)$$

together with (3.5) and (3.8). □

Lemma 3.2. *For $k \geq 1$,*

$$\big|S^k_R(0)\big| \leq (k-1)!\,\frac{\nu-1}{4\nu}\Big(\frac{2}{\beta}\Big)^k\Big[T^{k-1}_R + \frac{q^{kR}}{(1-q)^k}T^{2k-1}_R\Big] \tag{3.11}$$

as $R \to \infty$.

Proof. Using the simple bound

$$\frac{1}{q^r-q^R} \leq \frac{1}{q^r} + \frac{q^R}{(1-q)q^{2r}}, \qquad r = 0, 1, \ldots, R-1,$$

and the Cauchy inequality, we get

$$\begin{aligned}\left|S_R^k(0)\right| &\le \frac{\nu-1}{2\nu\beta^k}(k-1)!\sum_{r=0}^{R-1}\frac{1}{\nu^r}\Big[\frac{1}{q^r}+\frac{q^R}{(1-q)q^{2r}}\Big]^k\\ &\le 2^{k-2}\frac{\nu-1}{\nu\beta^k}(k-1)!\sum_{r=0}^{R-1}\Big[\frac{1}{(\nu q^k)^r}+\frac{q^{kR}}{(1-q)^k(\nu q^{2k})^r}\Big]\\ &=(k-1)!\frac{\nu-1}{4\nu}\Big(\frac{2}{\beta}\Big)^k\Big[T_R^{k-1}+\frac{q^{kR}}{(1-q)^k}T_R^{2k-1}\Big].\end{aligned}$$

□

3.1. The high-temperature region

First, we consider the high-temperature phase of our model, i.e.,

$$\nu q>1 \quad \text{and} \quad 0<\beta<\beta_{\mathsf{cr}}\equiv\frac{(\nu-1)q}{2(\nu q-1)}$$

and begin by investigating the unique positive solution to the stationary point equation (3.1),

$$1-\frac{1}{2\nu^R x_R}-\frac{\nu-1}{2\nu}\sum_{r=0}^{R-1}\frac{\nu^{-r}}{\beta(q^r-q^R)+x_R}=0.$$

Lemma 3.3. *Let $\nu q>1$ and $0<\beta<\beta_{\mathsf{cr}}$. Then there exist two positive constants $c_i=c_i(\beta,\nu,q)$, $i=1,2$, such that the solution x_* to the equation*

$$1-\frac{\nu-1}{2\nu}\sum_{r=0}^{\infty}\frac{1}{\nu^r(x+\beta q^r)}=0 \tag{3.12}$$

satisfies the inequality $0<c_1\le x_*\le c_2$.

Proof. Using the estimate (3.10), we obtain, for any $x>0$,

$$\sum_{r\ge 0}\frac{1}{\nu^r(x+\beta q^r)}\le\frac{\nu q}{\beta(\nu q-1)}-\frac{\nu q}{\beta(\nu q-1)}\frac{x}{x+\beta},$$

thus bounding below the left-hand side of (3.12) by $1-\beta_{\mathsf{cr}}/(x_*+\beta)$; as a result,

$$x_*\le\beta_{\mathsf{cr}}-\beta=:c_2.$$

For the lower bound, find a finite $\bar R$ such that (recall (3.3))

$$\tilde S_{\bar R}^1(0)=\Big(1-\frac{1}{(\nu q)^{\bar R}}\Big)\frac{\beta_{\mathsf{cr}}}{\beta}>1.$$

Now the left-hand side of (3.12) is smaller than $1-\tilde S_{\bar R}^1(x_*)$ and it remains to observe that $\tilde S_{\bar R}^1(x)$, $x\ge 0$, is a strictly convex decreasing function with finite derivative at zero. □

We show next that for large R the true stationary point x_R (i.e., the unique solution to (3.1)) is a small perturbation of x_*.

Lemma 3.4. *Under conditions of Lemma* 3.3*, the stationary point x_R satisfies, as $R \to \infty$, the relation*

$$x_R = x_* + o_{\mathsf{exp}}(1), \tag{3.13}$$

x_* being the unique positive solution to (3.12).

Proof. We show first that for large R the solution x_R is uniformly positive. Indeed, bounding the left-hand side of (3.1) above by $1 - \widetilde{S}^1_R(x_R)$, we use (3.12) to get

$$\widetilde{S}^1_R(x_R) - \widetilde{S}^1_R(x_*) \le \frac{\nu-1}{2\nu}\sum_{r=R}^{\infty}\frac{1}{\nu^r(x_*+\beta q^r)} \le \frac{1}{2x_*\nu^R}.$$

By convexity, $\widetilde{S}^1_R(x_R) - \widetilde{S}^1_R(x_*) \ge \widetilde{S}^2_R(x_*)(x_R - x_*)$, where

$$\widetilde{S}^2_R(x_*) = -\frac{\nu-1}{2\nu}\sum_{r=0}^{R-1}\frac{1}{\nu^r(x_*+\beta q^r)^2} \le -\frac{1-\nu^{-R}}{2(x_*+\beta)^2},$$

and therefore (only the case $x_R \le x_*$ needs our attention),

$$\frac{1-\nu^{-R}}{2(x_*+\beta)^2}(x_* - x_R) \le \frac{1}{2x_*\nu^R}.$$

Thus, $x_R \ge x_* - O(\nu^{-R}) \ge c_1/2$ for all R large enough and as a result the stationary point equation (3.1) can be rewritten as (recall (3.3))

$$1 - \widetilde{S}^1_R(x_R) = o_{\mathsf{exp}}(1).$$

Noting that

$$0 \le 1 - \widetilde{S}^1_R(x_*) = \frac{\nu-1}{2\nu}\sum_{r=R}^{\infty}\frac{1}{\nu^r(x_*+\beta q^r)} \le \frac{1}{2x_*\nu^R},$$

we obtain $\widetilde{S}^1_R(x_*) - \widetilde{S}^1_R(x_R) = o_{\mathsf{exp}}(1)$ as $R \to \infty$; finally, (3.13) follows in view of Lemma 3.3 and the analyticity properties of $\widetilde{S}^1_R(\cdot)$. □

Our next goal is to study the phase function $F^t_R(z)$ in a neighbourhood of the stationary point x_R. Being in the high-temperature region, we expect the validity of the central limit theorem for the total magnetization, i.e., that the law of (cf. (2.6))

$$\sqrt{|V|}s_V \equiv \frac{1}{\sqrt{|V|}}\sum_{x\in V}\sigma_x$$

converges to a Gaussian distribution. For this reason, we change t to $t\sqrt{|V|}$ in $F^t_R(z)$ or simply replace (2.14) by

$$F^t_R(z) := z - \frac{1}{2\nu^R}\log z - \frac{t^2}{4z\nu^R} - \frac{\nu-1}{2\nu}\sum_{r=0}^{R-1}\frac{1}{\nu^r}\log\bigl(z+\beta(q^r-q^R)\bigr).$$

Lemma 3.5. *Take* $z = x_r + isy_R \equiv x_R(1+is/\sqrt{|V|})$*. Then uniformly in* s *from any bounded subset of* $\mathbf{R}^1$ *we have*

$$\nu^R\big[F_R^t(z) - F_R^0(x_R)\big] = -\frac{t^2}{4x_R} - \frac{\nu-1}{4\nu}\sum_{r=0}^{R-1}\frac{x_R^2 s^2}{\nu^r(x_R+\beta q^r)^2} + o_{\mathsf{exp}}(1).$$

Proof. We proceed by a direct computation. First, using the notations from (2.13) and (2.12), we rewrite the Taylor formula as

$$\nu^R\big[S_R^0(z) - S_R^0(x_R)\big] = \sum_{k=1}^{\infty}\frac{1}{k!}S_R^k(x_R)\big(isy_R\big)^k|V|.$$

However, for $x > 0$,

$$\big|S_R^k(x)\big| \le \frac{(\nu-1)}{2\nu}(k-1)!\sum_{r=0}^{R-1}\frac{1}{\nu^r x^k} \le \frac{(k-1)!}{2x^k}$$

and therefore

$$\Big|\sum_{k=3}^{\infty}\frac{1}{k!}S_R^k(x_R)\big(isy_R\big)^k\Big| \le \sum_{k=3}^{\infty}\frac{1}{2k}\Big(\frac{s}{\sqrt{|V|}}\Big)^k = O\big(|V|^{-3/2}\big)$$

for all R large enough.

On the other hand, for any $x > 0$,

$$\frac{\nu}{\nu-1}\Big(1-\frac{1}{\nu^R}\Big)\frac{1}{(x+\beta)^2} \le \sum_{r=0}^{R-1}\frac{1}{\nu^r(x+\beta q^r)^2} \le \frac{\nu}{\nu-1}\frac{1}{x^2};$$

consequently,

$$\frac{|V|}{2}S_R^2(x_R)\big(isy_R\big)^2 = -\frac{D_R}{2}s^2 + o_{\mathsf{exp}}(1) \tag{3.14}$$

with

$$D_R := \frac{\nu-1}{2\nu}\sum_{r=0}^{R-1}\frac{x_R^2}{\nu^r(x_R+\beta q^r)^2} \in \Big[\frac{1-\nu^{-R}}{2}\Big(\frac{x_R}{x_R+\beta}\Big)^2, \frac{1}{2}\Big].$$

Next, in the region under consideration,

$$\frac{1}{2}\log\Big(1+\frac{is}{\sqrt{|V|}}\Big) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k}\Big(\frac{is}{\sqrt{|V|}}\Big)^k = \frac{is}{2\sqrt{|V|}} + O\Big(\frac{1}{|V|}\Big)$$

and

$$\frac{t^2}{4z} = \frac{t^2}{4x_R(1+is/\sqrt{|V|})} = \frac{t^2}{4x_R} + O\Big(\frac{1}{\sqrt{|V|}}\Big).$$

Finally, using the estimates above together with the stationary point equation (3.1) we finish the proof. □

Now we are ready to verify Theorem 2.1 the proof of which is a standard application of the Laplace method [6]; it is based upon the uniform estimate from Lemma 3.5, relation (3.14), and on the following result.

Lemma 3.6. *Let $z = x_R + isy_R$ with $y_R \to 0$ (as $R \to \infty$) in such a way that the inequality*

$$\nu^R y_R^2 \, |S_R^2(x_R)| \geq c > 0 \tag{3.15}$$

holds for all R large enough. Then there exists a function $\varepsilon(A)$ of $A \geq 0$, $\varepsilon(A) \downarrow 0$ as $A \uparrow \infty$, such that uniformly in $t \in \mathbf{R}^1$ and all R large enough one has

$$\Big| \int\limits_{|s| \geq A} \exp\big\{|V| \big[F_R^t(z) - F_R^0(x_R)\big]\big\} \, ds \Big| \leq \varepsilon(A).$$

Proof. By a direct inspection we verify that the real part of

$$|V| \big[F_R^t(z) - F_R^0(x_R)\big]$$

is bounded above (uniformly in $t \in \mathbf{R}^1$) by $-L_R(s)$, where

$$L_R(s) := \frac{\nu - 1}{4\nu} \sum_{r=0}^{R-1} \nu^{R-r} \log\Big(1 + \Big(\frac{y_R}{x_R + \beta q^r - \beta q^R}\Big)^2 s^2\Big).$$

Let first $|s| y_R \leq x_R + \beta$; then

$$\frac{y_R |s|}{x_R + \beta q^r - \beta q^R} \leq \frac{x_R + \beta}{x_R + \beta q^r - \beta q^R} \leq \frac{1 + 2\beta}{1 + 2\beta(q^r - q^R)} \leq 1 + 2\beta,$$

where the a priori upper bound (3.2) was used. Therefore, in view of the elementary inequality

$$\log(1 + w) \geq \frac{w}{a} \log(1 + a),$$

if only $w \in [0, a]$, we obtain (recall (3.15))

$$L_R(s) \geq \frac{\log\big(1 + (1 + 2\beta)^2\big)}{2(1 + 2\beta)^2} \nu^R y_R^2 \, |S_R^2(x_R)| \, s^2 \geq \tilde{c} s^2$$

with some positive constant $\tilde{c}$. As a result,

$$\int\limits_A^{(\beta + x_R)/y_R} \exp\big\{-L_R(s)\big\} \, ds \leq \int\limits_A^{\infty} e^{-\tilde{c}s^2} \, ds =: \frac{\varepsilon(A)}{4}.$$

On the other hand, for $|s| \geq (x_R + \beta)/y_R$ we easily get the inequality

$$\begin{aligned} L_R(s) &\geq \frac{\nu - 1}{2\nu} \sum_{r=0}^{R-1} \nu^{R-r} \log \frac{y_R |s|}{x_R + \beta q^r - \beta q^R} \\ &= \frac{\nu^R - 1}{2} \log \frac{y_R |s|}{x_R + \beta} + \frac{\nu - 1}{2\nu} \sum_{r=0}^{R-1} \nu^{R-r} \log \frac{x_R + \beta}{x_R + \beta q^r - \beta q^R}. \end{aligned}$$

However, for any $r \geq 1$

$$\frac{x_R + \beta}{x_R + \beta q^r - \beta q^R} \geq \frac{x_R + \beta}{x_R + \beta q} \geq \frac{1 + 2\beta}{1 + 2\beta q}$$

and therefore

$$\frac{\nu - 1}{2\nu} \sum_{r=0}^{R-1} \nu^{-r} \log \frac{x_R + \beta}{x_R + \beta q^r - \beta q^R} \geq \frac{1}{4\nu} \log \frac{1 + 2\beta}{1 + 2\beta q} =: \eta > 0$$

for all R large enough. Consequently,

$$\begin{aligned}\int\limits_{(x_R+\beta)/y_R}^{\infty} \exp\bigl\{-L_R(s)\bigr\}\, ds &\leq \int\limits_{(x_R+\beta)/y_R}^{\infty} \Bigl(\frac{y_R s}{x_R + \beta}\Bigr)^{-(\nu^R - 1)/2} e^{-\eta\nu^R}\, ds \\ &= \frac{2e^{-\eta\nu^R}}{\nu^R - 3} \frac{x_R + \beta}{y_R} \leq C|S_R^2(x_R)| e^{-\eta\nu^R}\end{aligned}$$

and in view of monotonicity of $|S_R^2(\cdot)|$, relations (3.11) and (3.4), the last expression is bounded above by

$$C \exp\{c_1 R - \eta\nu^R\} \leq \frac{\varepsilon(A)}{4}$$

provided R is sufficiently large. □

3.2. The low-temperature region

We describe first the stationary point asymptotics.

Lemma 3.7. *Let $\nu q > 1$ and $\beta > \beta_{\mathsf{cr}}$. Then, as $R \to \infty$,*

$$x_R = \frac{\beta}{2(\beta - \beta_{\mathsf{cr}})\nu^R}\bigl(1 + o_{\mathsf{exp}}(1)\bigr). \tag{3.16}$$

Proof. We start by observing that for any fixed $\eta > 0$ and all R large enough

$$x_R \nu^R \leq \frac{\beta}{2(\beta - \beta_{\mathsf{cr}})}\bigl(1 + \eta\bigr).$$

Indeed, for x not satisfying this condition we get $(2\nu^R x)^{-1} < \dfrac{\beta - \beta_{\mathsf{cr}}}{\beta(1 + \eta)}$; this together with the uniform estimate (see (3.7))

$$S_R^1(x) \leq S_R^1(0) = \frac{\beta_{\mathsf{cr}}}{\beta} + o_{\mathsf{exp}}(1)$$

renders the left-hand side of (3.1) positive.

Next, uniformly in x satisfying $0 \le x\nu^R \le \dfrac{\beta(1+\eta)}{2(\beta-\beta_{\mathsf{cr}})}$, we have (using (3.9))

$$S_R^1(x) = S_R^1(0) + O(x\,T_R^1) + o_{\mathsf{exp}}(1) = \frac{\beta_{\mathsf{cr}}}{\beta} + o_{\mathsf{exp}}(1) \tag{3.17}$$

and therefore (recall (3.16))

$$1 - \frac{1}{2\nu^R x_R} - \frac{\beta_{\mathsf{cr}}}{\beta} + o_{\mathsf{exp}}(1) = 0.$$

□

Now, we verify that in the region under consideration the correction $S_R^0(z)$ is small, i.e., the function $F_R^t(z)$ in (2.14) is reduced essentially to the first three terms only.

Lemma 3.8. *Let $z = x_R(1+is)$. Then, uniformly in s from any bounded subset of $\mathbf{R}^1$, one has*

$$\nu^R\big[S_R^0(z) - S_R^0(x_R)\big] = \frac{i\beta_{\mathsf{cr}}}{2(\beta-\beta_{\mathsf{cr}})}s + o_{\mathsf{exp}}(1) \qquad \text{as } R \to \infty.$$

Proof. In view of the properties (3.16), (3.11), and the simple inequality

$$\big|S_R^k(z)\big| \le \big|S_R^k(\Re z)\big| \le \big|S_R^k(0)\big|$$

valid for any complex z with $\Re z \ge 0$, we obtain

$$\begin{aligned}\Big|\sum_{k=2}^{\infty} \frac{S_R^k(x_R)\big(ix_R s\big)^k}{k!}\Big| &\le \frac{\nu-1}{4\nu}\sum_{k=2}^{\infty}\frac{1}{k}\Big|\frac{2x_R s}{\beta}\Big|^k \Big[T_R^{k-1} + \Big(\frac{q^R}{1-q}\Big)^k T_R^{2k-1}\Big]\\ &= o_{\mathsf{exp}}(1)\end{aligned}$$

uniformly in s under consideration (and all R large enough). Using next the relation (3.17) for $S_R^1(x_R)$ and the Taylor formula together with the asymptotics (3.16), we finish the proof. □

It is immediate now to deduce the following fact.

Corollary 3.1. *Fix any $t \in \mathbf{R}^1$ and denote (cf. (3.16))*

$$\alpha := \frac{\beta}{2(\beta-\beta_{\mathsf{cr}})} > 0. \tag{3.18}$$

Then, uniformly in s from any bounded subset of $\mathbf{R}^1$, one has

$$\nu^R\big[F_R^t(z) - F_R^0(x_R)\big] = \frac{is}{2} - \frac{1}{2}\log(1+is) - \frac{t^2}{4\alpha(1+is)} + o_{\mathsf{exp}}(1),$$

as $R \to \infty$.

Our next goal is to establish the corresponding limit result, Theorem 2.2. Take $h_x \equiv 1/|V|$, $x \in V$, in (2.9) so that $h_V(\sigma_V)$ is the specific magnetization s_V from (2.6) and denote by $\phi^{\mathsf{h}}_{s_V}(t)$ the characteristic function of s_V,

$$\phi^{\mathsf{h}}_{s_V}(t) := \mathsf{E}^{\mathsf{h}}_V \exp\bigl\{its_V\bigr\}, \qquad t \in \mathbf{R}^1,$$

$\mathsf{E}^{\mathsf{h}}_V$ being the operator of mathematical expectation corresponding to the Gibbs measure $\mathsf{P}^{\mathsf{h}}_V(\cdot)$ with the hierarchical spherical Hamiltonian $\mathcal{H}_{\mathsf{h}}$.

Though the proof of Theorem 2.2 can be established by applying the standard Laplace method (giving even the asymptotic expansion to higher order terms), we shall reduce the computational routine by using the following simple corollary of the inversion formula for the gamma distribution: for any $m \in \mathbf{N}$,

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\exp\bigl\{is/2\bigr\}}{(1+is)^{m+1/2}}\,ds = -\sqrt{\frac{2}{e}}\frac{2^{-m}}{\Gamma(m+1/2)} = -\sqrt{\frac{2}{\pi e}}\frac{1}{(2m-1)!!} \tag{3.19}$$

with $\Gamma(\cdot)$ denoting the usual gamma function.

In addition, we shall need two inequalities given in the next lemma.

Lemma 3.9. *Fix any* $t \in \mathbf{R}^1$. *There exists a function* $\varepsilon(K)$ *of* $K \geq 0$, $\varepsilon(K) \downarrow 0$ *as* $K \uparrow \infty$, *such that* (*uniformly in large* R)

$$\int_{|s|>K} \frac{e^{is/2}}{\sqrt{1+is}} e^{-t^2/4\alpha(1+is)}\,ds \leq \varepsilon(K), \tag{3.20}$$

$$\int_{|s|>K} \exp\bigl\{\nu^R\bigl[F^t_R\bigl(x_R(1+is)\bigr) - F^0_R(x_R)\bigr]\bigr\}\,ds \leq \varepsilon(K). \tag{3.21}$$

Proof of Theorem 2.2. Fix any $t \in \mathbf{R}^1$. Using the Taylor formula, the definition (3.18), and the relations (3.19), we obtain

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{e^{is/2}}{\sqrt{1+is}} e^{-t^2/4\alpha(1+is)}\,ds &= \sum_{k=0}^{\infty} \frac{(-t^2)^k}{(4\alpha)^k k!}\int_{-\infty}^{+\infty} \frac{e^{is/2}}{(1+is)^{k+1/2}}\,ds \\ &= -\sqrt{\frac{2}{\pi e}}\sum_{k=0}^{\infty} \frac{(-t^2)^k}{(2\alpha)^k(2k)!} \\ &= -\sqrt{\frac{2}{\pi e}}\cos\Bigl(\frac{t}{\sqrt{2\alpha}}\Bigr).\end{aligned}$$

On the other hand, applying the Laplace method [6] to the integral (2.11) in combination with Corollary 3.1 and Lemma 3.9, we get

$$Z^\beta_V(t) = -\sqrt{\frac{2}{\pi e}} x_R \exp\bigl\{\nu^R F^0_R(x_R)\bigr\}\cos\bigl(\sqrt{1-\beta_{\mathsf{cr}}/\beta}\,t\bigr)\bigl(1+o(1)\bigr),$$

thus finishing the proof. □

Proof of Lemma 3.9. Our argument is based upon the following fact (which follows from the well-known Dirichlet criterion [3, p. 586]).

Proposition 3.1. *If a (continuous) positive function $f(x)$ decreases on $[A,\infty)$ and vanishes asymptotically as $x \to \infty$, then the integral*

$$\int_A^\infty f(x)e^{ix}\,dx \tag{3.22}$$

converges; in particular,

$$\int_{x\geq K} f(x)e^{ix}\,dx \to 0$$

as $K \to \infty$.

In addition, we shall use the two properties below, which are easy to verify:

1) Take any $b > 0$ and $A \geq 0$. Then the function

$$bs - \arctan(bs) + \frac{As}{1+s^2} \tag{3.23}$$

is monotonically increasing with uniformly positive derivative for all $s \geq 0$ sufficiently large.

2) Let $A \geq 0$. The function

$$\frac{1}{\sqrt[4]{1+s^2}}\exp\Bigl\{-\frac{A}{1+s^2}\Bigr\} \tag{3.24}$$

decays to zero, as $s \to \infty$, monotonically for all $s \geq 0$ sufficiently large.

First, let us verify (3.20). To this end, we rewrite the integrand above as $\psi(s)\exp\{i\varphi(s)/2\}$, where

$$\psi(s) := \frac{1}{\sqrt[4]{1+s^2}}\exp\Bigl\{-\frac{t^2}{4\alpha(1+s^2)}\Bigr\},$$

$$\varphi(s) := s - \arctan s + \frac{t^2 s}{2\alpha(1+s^2)}.$$

For $s \geq K$ with K large enough,[2] the function $\psi(\cdot)$ is monotonically decreasing to zero and $\varphi(\cdot)$ is monotonically increasing to infinity (with uniformly positive derivative). Thus, the change of variables $x = \varphi(s)/2$ gives

$$\int_A^\infty \psi(s)\exp\bigl\{i\varphi(s)/2\bigr\}\,ds = \int_{\varphi(A)/2}^\infty \frac{2\psi\bigl(\varphi^{-1}(2x)\bigr)}{\varphi'\bigl(\varphi^{-1}(2x)\bigr)}\exp\bigl\{ix\bigr\}\,dx,$$

[2] Because of symmetry, the case of negative values of s is analogous.

where the positive function $\psi\bigl(\varphi^{-1}(2x)\bigr)/\varphi'\bigl(\varphi^{-1}(2x)\bigr)$ decays monotonically to zero as $x \to \infty$ (for all x large enough). It remains to apply (3.22).

Next, we check (3.21). Separating the real and the imaginary part in the exponent we rewrite the integrand of (3.21) as $\tilde{\psi}(s)\exp\{i\tilde{\varphi}(s)\}$, where $\tilde{\psi}(s) := \psi(s)\psi_1(s)$,

$$\begin{aligned}\psi_1(s) :=& \exp\Bigl\{-\frac{\nu-1}{4\nu}\sum_{r=0}^{R-1}\frac{\nu^R}{\nu^r}\log\Bigl(1+\Bigl[\frac{x_R}{x_R+\beta(q^r-q^R)}\Bigr]^2 s^2\Bigr)\Bigr\},\\ \tilde{\varphi}(s) :=& \alpha_R s - \frac{1}{2}\arctan s + \frac{t^2 s}{4\alpha_R(1+s^2)}\\ &-\frac{\nu-1}{2\nu}\sum_{r=0}^{R-1}\frac{\nu^R}{\nu^r}\arctan\frac{s x_R}{x_R+\beta(q^r-q^R)}\end{aligned}$$

with the notation (cf. (3.18))

$$\alpha_R := x_R\nu^R = \frac{\beta}{2(\beta-\beta_{\mathsf{cr}})}\bigl(1+o_{\mathsf{exp}}(1)\bigr).$$

Since the function $\psi_1(s)$ is monotonically decreasing (to zero), we need only to check the monotonicity properties of $\tilde{\varphi}(\cdot)$. To this end, we rewrite

$$\begin{aligned}\tilde{\varphi}(s) :=& \frac{s}{2} - \frac{1}{2}\arctan s + \frac{t^2 s}{4\alpha_R(1+s^2)}\\ &+\frac{\nu-1}{2\nu}\sum_{r=0}^{R-1}\frac{\nu^R}{\nu^r}\Bigl[\frac{s x_R}{x_R+\beta(q^r-q^R)} - \arctan\frac{s x_R}{x_R+\beta(q^r-q^R)}\Bigr]\\ &+\alpha_R s\Bigl[1-\frac{1}{2x_R\nu^R}-\frac{\nu-1}{2\nu}\sum_{r=0}^{R-1}\frac{1}{\nu^r(x_R+\beta(q^r-q^R))}\Bigr],\end{aligned}$$

where the first two lines give us a function with the desired properties and the last line vanishes identically due to the very definition (3.1) of x_R. Now, change variables $x = \tilde{\varphi}(s)$ and proceed as before. □

3.3. At the critical temperature

The limiting behaviour of our model at the critical temperature

$$\beta = \beta_{\mathsf{cr}} = \frac{(\nu-1)q}{2(\nu q-1)} > \frac{1}{2} \tag{3.25}$$

exhibits certain dependence on the hierarchical dimension dim. We describe first the asymptotics of the stationary point x_R, i.e., the unique solution to equation (3.1) (recall also the definition (2.13))

$$1-\frac{1}{2\nu^R x_R} - S_R^1(x_R) = 0.$$

It is easy to obtain an a priori upper bound for x_R; noting that the left-hand side of the stationary point equation is strictly increasing, we get

$$1 - \frac{1}{2\nu^R x} - S_R^1(x) \geq 1 - \frac{1}{2\beta\nu^R q^R} - S_R^1(\beta q^R) = \Bigl[1 - \frac{1}{2\beta}\Bigr]\bigl(\nu q\bigr)^{-R} > 0$$

as soon as $x \geq \beta q^R$; consequently, the unique solution x_R to equation (3.1) satisfies $0 < x_R < \beta q^R$. Next, in the last region we have

$$S_R^1(x) \geq \frac{\nu - 1}{2\nu\beta} \sum_{r=0}^{R} (\nu q)^{-r} = 1 - (\nu q)^{-R}$$

and hence the lower bound $x_R \geq q^R/2$. As a result,

$$\alpha_R := \frac{x_R}{\beta q^R} \in \Bigl(\frac{1}{2\beta}, 1\Bigr). \tag{3.26}$$

A more precise information about x_R is given by the following claim.

Lemma 3.10. *Let $\nu q > 1$, $\beta = \beta_{\mathsf{cr}}$, and let $\alpha_* \in \bigl(1/(2\beta), 1\bigr)$ denote the only positive solution to the equation*

$$1 - \frac{1}{2\beta_{\mathsf{cr}}\alpha_*} = \frac{\nu - 1}{2\nu\beta_{\mathsf{cr}}} \sum_{k\geq 1} \bigl(1 - \alpha_*\bigr)^k \frac{\nu q^{k+1}}{1 - \nu q^{k+1}}. \tag{3.27}$$

Then, as $R \to \infty$,

$$\frac{x_R}{\beta q^R} = \begin{cases} \Bigl[1 - (2\beta - 1)\dfrac{\nu q^2 - 1}{\nu q^2 - q^2}(\nu q^2)^{-R}\bigl(1 + o_{\mathsf{exp}}(1)\bigr)\Bigr], & \text{if } \nu q^2 > 1; \\ \Bigl[1 - (2\beta - 1)\dfrac{\nu}{\nu - 1} R^{-1}\bigl(1 + O(R^{-1})\bigr)\Bigr], & \text{if } \nu q^2 = 1; \\ \alpha_*\bigl(1 + o_{\mathsf{exp}}(1)\bigr), & \text{if } \nu q^2 < 1. \end{cases} \tag{3.28}$$

Proof. Denoting $y = x/(\beta q^R)$ and using the notation (3.4), we rewrite $S_R^1(y\beta q^R)$ in the region $y \in (0, 1)$ as follows:

$$\begin{aligned} S_R^1(y\beta q^R) &= \frac{\nu - 1}{2\nu\beta} \sum_{r=0}^{R-1} \frac{1}{\nu^r\bigl(q^r - (1 - y)q^R\bigr)} \\ &= \frac{\nu - 1}{2\nu\beta} \sum_{k\geq 0} (1 - y)^k q^{kR} \sum_{r=0}^{R-1} \frac{1}{(\nu q^{k+1})^r} \\ &= 1 - \frac{1}{(\nu q)^R} + \frac{\nu - 1}{2\nu\beta} \sum_{k\geq 1} (1 - y)^k q^{kR} T_R^k. \end{aligned}$$

Recalling now the simple property

$$(\nu q^{k+1})^R T_R^k = \begin{cases} \dfrac{\nu q^{k+1}}{\nu q^{k+1}-1}(\nu q^{k+1})^R\bigl(1+o_{\mathsf{exp}}(1)\bigr), & \text{if } \nu q^{k+1}>1; \\ R, & \text{if } \nu q^{k+1}=1; \\ \dfrac{\nu q^{k+1}}{1-\nu q^{k+1}}\bigl(1+o_{\mathsf{exp}}(1)\bigr), & \text{if } \nu q^{k+1}<1, \end{cases} \tag{3.29}$$

we observe that behaviour of the only solution $y=\alpha_R=x_R/(\beta q^R)$ to the stationary point equation

$$1-\frac{1}{2\beta y}=\frac{\nu-1}{2\nu\beta}\sum_{k\ge1}(1-y)^k(\nu q^{k+1})^R T_R^k \tag{3.30}$$

depends on the hierarchical dimension dim.

Case 1. Let first $\nu q^2<1$ (i.e., $\mathsf{dim}<4$). Then all terms in the sum in (3.30) are of order 1 and the sum itself is finite for any $y\in(0,1)$. Taking the limit $R\to\infty$, we obtain (3.27) and the corresponding asymptotics of the stationary point x_R.

Case 2. Let now $\nu q^2=1$ (i.e., $\mathsf{dim}=4$). Then the stationary point equation (3.30) reads

$$1-\frac{1}{2\beta y}=\frac{\nu-1}{2\nu\beta}\Bigl[(1-y)R+\sum_{k\ge2}(1-y)^k(\nu q^{k+1})^R T_R^k\Bigr],$$

and since the right-hand side equals $\dfrac{\nu-1}{2\nu\beta}\bigl((1-y)R+O(1)\bigr)$, we immediately deduce

$$1-y=\frac{2\nu\beta}{\nu-1}\Bigl[\frac{2\beta-1}{2\beta R}+O(R^{-2})\Bigr]$$

and the second line in (3.28).

Case 3. Finally, consider the case of large dimensions $\mathsf{dim}>4$ (or $\nu q^2>1$). Then the limiting behaviour of the right-hand side of (3.30) is governed by the first term,

$$\frac{\nu-1}{2\nu\beta}(1-y)(\nu q^2)^R T_R^1=(1-y)\frac{(\nu q-1)q}{\nu q^2-1}(\nu q^2)^R\bigl(1+o_{\mathsf{exp}}(1)\bigr)$$

thus leading to the solution

$$1-y=\frac{2\beta-1}{2\beta}\,\frac{\nu q^2-1}{(\nu q-1)q}(\nu q^2)^{-R}\bigl(1+o_{\mathsf{exp}}(1)\bigr)$$

and to the first line in (3.28). □

Our next goal is to describe the limiting behaviour of the (scaled) specific magnetization (recall (2.4))

$$\frac{(\nu q)^{R/2}}{\nu^R}S_V=\Bigl(\frac{q}{\nu}\Bigr)^{R/2}\sum_{x\in V}\sigma_x=|V|^{-(2+\mathsf{dim})/2\mathsf{dim}}\sum_{x\in V}\sigma_x$$

and the corresponding phase function (cf. (2.14))

$$F_R^t(z) := z - \frac{1}{2\nu^R}\log z - \frac{q^R t^2}{4z\nu^R} - \frac{\nu-1}{2\nu}\sum_{r=0}^{R-1}\frac{1}{\nu^r}\log\bigl(z+\beta(q^r-q^R)\bigr). \quad (3.31)$$

As one might have already guessed, the result depends on the hierarchical dimension dim.

3.3.1. High dimensions

We consider first the case $\mathsf{dim} > 4$ (i.e., $\nu q^2 > 1$).

Lemma 3.11. *Fix any $K > 0$ and put $z = x_R + isy_R = x_R\bigl(1 + is(\nu q^2)^{-R/2}\bigr)$. Then, as $R \to \infty$, one obtains*

$$\nu^R\bigl[isy_R - S_R(z) + S_R(x_R)\bigr] = -\frac{(\nu-1)q^2}{4(\nu q^2-1)}s^2 + o_{\mathsf{exp}}(1)$$

uniformly in $|s| \le K$.

Proof. Using the Taylor formula, we rewrite (recall (2.13))

$$S_R(z) - S_R(x_R) = \sum_{k=1}^{\infty}\frac{1}{k!}S_R^k(x_R)\bigl(isy_R\bigr)^k,$$

where $y_R = x_R(\nu q^2)^{-R/2} = \alpha_R\beta\nu^{-R/2}$. Next, for positive x, we have $|S_R^k(x)| \le |S_R^k(0)|$, the latter quantity being bounded above in (3.11). Therefore,

$$\begin{aligned}
&\Bigl|\nu^R\sum_{k=3}^{\infty}\frac{1}{k!}S_R^k(x_R)\bigl(isy_R\bigr)^k\Bigr| \\
&\qquad \le \frac{\nu-1}{4\nu}\sum_{k\ge3}\frac{1}{k}\Bigl(\frac{2y_R|s|}{\beta q^R}\Bigr)^k\Bigl[(\nu q^k)^R T_R^{k-1} + \frac{(\nu q^{2k})^R}{(1-q)^k}T_R^{2k-1}\Bigr] \qquad (3.32)\\
&\qquad = \frac{\nu-1}{4\nu}\sum_{k\ge3}\frac{1}{k}(2\alpha_R K)^k\Bigl[O\bigl(\nu^{R(1-k/2)}\bigr) + O\Bigl(\frac{q^{kR}\nu^{R(1-k/2)}}{(1-q)^k}\Bigr)\Bigr],
\end{aligned}$$

the latter sum being of order $o_{\mathsf{exp}}(1)$ for R sufficiently large (uniformly in s under consideration).

On the other hand,

$$\begin{aligned}
\frac{\nu^R}{2}\bigl(isy_R\bigr)^2 S_R^2(x_R) &= \frac{\nu-1}{4\nu}(s\alpha_R)^2\sum_{r=0}^{R-1}\frac{1}{\nu^r q^{2r}}\frac{1}{\bigl(1-(1-\alpha_R)q^{R-r}\bigr)^2}\\
&= \frac{\nu-1}{4\nu}(s\alpha_R)^2\Bigl[T_R^1 + \sum_{k\ge1}(k+1)(1-\alpha_R)^k q^{kR}T_R^{k+1}\Bigr].
\end{aligned} \quad (3.33)$$

Recalling the asymptotics (3.28), we deduce that the last sum is of order

$$\sum_{k\geq 1}(k+1)\Big(\frac{C}{(\nu q)^R}\Big)^k T_R^{k+1} = \sum_{k\geq 1}(k+1)\,O\Big(\frac{C^k}{(\nu q^2)^{(k+1)R}}\Big) = o_{\mathsf{exp}}(1).$$

On the other hand, applying (3.4) to the first term, we obtain

$$\frac{\nu^R}{2}\big(isy_R\big)^2 S_R^2(x_R) = \frac{(\nu-1)q^2}{4(\nu q^2-1)}s^2 + o_{\mathsf{exp}}(1). \tag{3.34}$$

Finally, we rewrite

$$\begin{aligned} 1-S_R^1(x_R) &= 1-\frac{\nu-1}{2\nu\beta}\sum_{r=0}^{R-1}\frac{1}{\nu^r q^r(1-(1-\alpha_R)q^{R-r})} \\ &= 1-\frac{\nu-1}{2\nu\beta}\sum_{k\geq 0}(1-\alpha_R)^k q^{kR}T_R^k \\ &= \frac{1}{(\nu q)^R}-\frac{\nu-1}{2\nu\beta}\sum_{k\geq 1}(1-\alpha_R)^k q^{kR}T_R^k \end{aligned} \tag{3.35}$$

and thus, using again (3.28), (3.29), and the definition of y_R we get

$$\nu^R isy_R\big(1-S_R^1(x_R)\big) = o_{\mathsf{exp}}(1).$$

□

Corollary 3.2. *Under conditions of the lemma above,*

$$\nu^R\big[F_R^t(z)-F_R^0(x_R)\big] = -\frac{(\nu-1)q^2}{4(\nu q^2-1)}s^2 - \frac{t^2}{4\beta} + o_{\mathsf{exp}}(1)$$

uniformly in $|s|\leq K$.

Recall relation (3.34); by a standard application of Lemma 3.6 and Corollary 3.2 we deduce the CLT part of Theorem 2.3 in $\mathsf{dim}>4$.

3.3.2. The critical dimension $\mathsf{dim}=4$

We start by verifying an analogue of Lemma 3.11.

Lemma 3.12. *Fix any* $K>0$ *and put* $z = x_R+isy_R = \alpha_R\beta q^R\big(1+is/\sqrt{R}\big)$. *Then, as* $R\to\infty$, *one gets*

$$\nu^R\big[isy_R - S_R(z)+S_R(x_R)\big] = -\frac{\nu-1}{4\nu}s^2 + O\big(R^{-1/2}\big)$$

uniformly in $|s|\leq K$.

Proof. We follow the same scenario as the one of Lemma 3.11, by using the Taylor formula and establishing analogues of (3.32), (3.33), and (3.35).

First (recall (3.29)),

$$\begin{aligned}
&\Big|\nu^R \sum_{k=3}^{\infty} \frac{1}{k!} S_R^k(x_R)\big(isy_R\big)^k\Big| \\
&\qquad \le \frac{\nu-1}{4\nu} \sum_{k\ge 3} \frac{1}{k}\Big(\frac{2\alpha_R K}{\sqrt{R}}\Big)^k \Big[O(q^k) + O\Big(\frac{q^{2k}}{(1-q)^k}\Big)\Big] = O\big(R^{-3/2}\big)
\end{aligned}$$

if only R is large enough.

Next, taking into account the bound $1-\alpha_R \le C/R$, we obtain

$$\begin{aligned}
\frac{\nu^R}{2}\big(isy_R\big)^2 S_R^2(x_R) =& \frac{\nu-1}{4\nu}\frac{(s\alpha_R)^2}{R}\Big[(\nu q^2)^R T_R^1 \\
&\qquad + \sum_{k\ge 1}(k+1)(1-\alpha_R)^k(\nu q^{k+2})^R T_R^{k+1}\Big] \\
=& \frac{\nu-1}{4\nu}s^2 + O\big(R^{-1}\big).
\end{aligned} \tag{3.36}$$

Finally,

$$\begin{aligned}
\nu^R isy_R\big[1 - S_R^1(x_R)\big] &= \frac{is\alpha_R\beta(\nu q)^R}{\sqrt{R}}\Big[\frac{1}{(\nu q)^R} - \frac{\nu-1}{2\nu\beta}\sum_{k\ge 1}(1-\alpha_R)^k q^{kR} T_R^k\Big] \\
&= O(R^{-1/2})
\end{aligned}$$

as soon as R is large enough. $\square$

As a simple corollary we get, under conditions of Lemma 3.12, the asymptotics

$$\nu^R\big[F_R^t(z) - F_R^0(x_R)\big] = -\frac{\nu-1}{4\nu}s^2 - \frac{t^2}{4\beta} + O\big(R^{-1/2}\big)$$

uniformly in $|s| \le K$. This, together with relation (3.36), implies the CLT statement of Theorem 2.3 for $\mathsf{dim} = 4$.

3.3.3. Low dimensions

It remains to study the case of low dimensions, $2 < \mathsf{dim} < 4$ (i.e., $q < \nu q^2 < 1$). As usual, we start by describing the asymptotics of the difference (recall (3.31))

$$\nu^R\big[F_R^t(z) - F_R^0(x_R)\big].$$

Denote

$$\begin{aligned}
\Phi^t(s) :=& is\alpha_*\beta - \frac{1}{2}\log(1+is) - \frac{t^2}{4\alpha_*\beta(1+is)} \\
&+ \frac{\nu-1}{2\nu}\sum_{l=1}^{\infty}\nu^l\Big[is\alpha_* q^l - \log\Big(1 + \frac{is\alpha_* q^l}{1-(1-\alpha_*)q^l}\Big)\Big]
\end{aligned} \tag{3.37}$$

with α_* being the only positive solution to equation (3.27).

Lemma 3.13. *Fix any $K > 0$ and put $z = x_R(1 + is)$. Then, as $R \to \infty$, one obtains* (*recall* (3.37))

$$\nu^R\big[F_R^t(z) - F_R^0(x_R)\big] = \Phi^t(s) + o(1)$$

uniformly in $|s| \le K$.

Proof. Using relation (3.25), definition (3.26), and a simple summation, we rewrite

$$\nu^R\big[F_R^t(z) - F_R^0(x_R)\big] = is\alpha_R\beta - \frac{1}{2}\log(1+is) - \frac{t^2}{4\alpha_R\beta(1+is)} + \frac{\nu-1}{2\nu}\sum_{l=1}^{R}\nu^l\Big[is\alpha_R q^l - \log\Big(1+\frac{is\alpha_R q^l}{1-(1-\alpha_R)q^l}\Big)\Big].$$

In view of the last line in (3.28), our task is reduced to establishing the convergence of

$$\sum_{l=1}^{R}\nu^l\Big[is\alpha_R q^l - \log\Big(1+\frac{is\alpha_R q^l}{1-(1-\alpha_R)q^l}\Big)\Big] = \sum_{l=1}^{\infty}\nu^l\Big[is\alpha_* q^l - \log\Big(1+\frac{is\alpha_* q^l}{1-(1-\alpha_*)q^l}\Big)\Big] + o(1)$$

uniformly in $|s| \le K$. We deduce it in an obvious way from the two properties below:

1) There exists a function $\varepsilon(M)$ of $M \ge 1$, $\varepsilon(M) \downarrow 0$ as $M \to \infty$, such that (uniformly in $|s| \le K$ and all R large enough)

$$\Big|\sum_{l=M}^{R}\nu^l\Big[is\alpha_R q^l - \log\Big(1+\frac{is\alpha_R q^l}{1-(1-\alpha_R)q^l}\Big)\Big]\Big| \le \varepsilon(M), \tag{3.38}$$

$$\Big|\sum_{l=M}^{\infty}\nu^l\Big[is\alpha_* q^l - \log\Big(1+\frac{is\alpha_* q^l}{1-(1-\alpha_*)q^l}\Big)\Big]\Big| \le \varepsilon(M). \tag{3.39}$$

2) For any fixed $M \ge 1$ (and uniformly in $|s| \le K$), the sum

$$\sum_{l=1}^{M-1}\nu^l\Big[is(\alpha_R-\alpha_*)q^l - \log\Big(\frac{1-(1-\alpha_R-is\alpha_R)q^l}{1-(1-\alpha_*-is\alpha_*)q^l}\cdot\frac{1-(1-\alpha_*)q^l}{1-(1-\alpha_R)q^l}\Big)\Big]$$

vanishes asymptotically as $R \to \infty$.

Since the latter property is an immediate corollary of the last line in (3.28), we concentrate ourselves on the proof of the former one. We start by the following simple observation: for any $\rho \in (0,1)$ there exists a finite constant $C_\rho > 0$ such that uniformly in $|w| \le \rho$ one has

$$\bigl|iw - \log(1+iw)\bigr| \le C_\rho |w|^2. \tag{3.40}$$

Let us first verify (3.38). Since $0 < \alpha_R \le 1$, for any $l \ge M$ we obtain

$$\Bigl|\frac{is\alpha_R q^l}{1-(1-\alpha_R)q^l}\Bigr| \le \frac{Kq^l}{1-q^l} \le \frac{Kq^M}{1-q^M} \le \rho$$

and therefore (3.40) implies

$$\begin{aligned}\Bigl|\sum_{l=M}^{R} \nu^l \Bigl[\frac{is\alpha_R q^l}{1-(1-\alpha_R)q^l} - \log\Bigl(1+\frac{is\alpha_R q^l}{1-(1-\alpha_R)q^l}\Bigr)\Bigr]\Bigr|& \\ \le \frac{C_\rho K^2}{(1-q^M)^2} \sum_{l=M}^{\infty} (\nu q^2)^l \le \frac{C_\rho (K+\rho)^2}{1-\nu q^2} (\nu q^2)^M&\end{aligned}$$

uniformly in $R \ge M$ and s under consideration. On the other hand,

$$\begin{aligned}\Bigl|\sum_{l=M}^{R} \nu^l is\alpha_R \Bigl[\frac{q^l}{1-(1-\alpha_R)q^l} - q^l\Bigr]\Bigr| &\le \frac{K}{4(1-q^M)} \sum_{l=M}^{\infty} (\nu q^2)^l \\ &\le \frac{K+\rho}{4(1-\nu q^2)} (\nu q^2)^M\end{aligned}$$

and (3.38) follows with

$$\varepsilon(M) = 4 \max\Bigl(\frac{C_\rho (K+\rho)^2}{1-\nu q^2}, \frac{K+\rho}{4(1-\nu q^2)}\Bigr) (\nu q^2)^M.$$

A simple check shows that (a minor modification of) the argument above proves (3.39) as well. □

Next, we verify the following uniform integrability property (cf. Lemma 3.9).

Lemma 3.14. *Fix any $t \in \mathbf{R}^1$. There exists a function $\varepsilon(K)$ of $K \ge 0$, $\varepsilon(K) \downarrow 0$ as $K \uparrow \infty$, such that (uniformly in large R)*

$$\int_{|s|>K} \exp\bigl\{\Phi^t(s)\bigr\}\, ds \le \varepsilon(K), \tag{3.41}$$

$$\int_{|s|>K} \exp\bigl\{\nu^R \bigl[F_R^t\bigl(x_R(1+is)\bigr) - F_R^0(x_R)\bigr]\bigr\}\, ds \le \varepsilon(K). \tag{3.42}$$

Proof. As in the proof of Lemma 3.9, our argument is based upon the fact (3.22) combined with the monotonicity properties (3.23) and (3.24).

We start by establishing (3.41). To this end, we rewrite the integrand as

$$\exp\bigl\{\Re\Phi^t(s)\bigr\}\exp\bigl\{i\Im\Phi^t(s)\bigr\},$$

where

$$\begin{aligned}\Re\Phi^t(s) = &-\frac{1}{4}\log(1+s^2) - \frac{t^2}{4\alpha_*\beta(1+s^2)}\\ &-\frac{\nu-1}{4\nu}\sum_{l=0}^{\infty}\nu^l\log\Bigl(1+\Bigl[\frac{\alpha_* q^l}{1-(1-\alpha_*)q^l}\Bigr]^2 s^2\Bigr)\end{aligned}$$

and

$$\begin{aligned}\Im\Phi^t(s) =&\frac{s}{2}-\frac{1}{2}\arctan s+\frac{t^2 s}{4\alpha_*\beta(1+s^2)}\\ &+\frac{\nu-1}{2\nu}\sum_{l=0}^{\infty}\nu^l\Bigl[\frac{s\alpha_* q^l}{1-(1-\alpha_*)q^l}-\arctan\frac{s\alpha_* q^l}{1-(1-\alpha_*)q^l}\Bigr]\\ &+s\alpha_*\beta\Bigl[1-\frac{1}{2\alpha_*\beta}+\frac{\nu-1}{2\nu\beta}\sum_{l=0}^{\infty}(\nu q)^l\Bigl(1-\frac{1}{1-(1-\alpha_*)q^l}\Bigr)\Bigr].\end{aligned}$$

Clearly, for all s large enough, $\exp\bigl\{\Re\Phi^t(s)\bigr\}\downarrow 0$ as $s\uparrow\infty$; also, the first two lines in the representation of $\Im\Phi^t(s)$ give a function increasing monotonically to ∞ (as $s\uparrow\infty$). We are going to verify below that the last line in this representation vanishes identically (and then (3.41) follows by change of variables $x=\Im\Phi^t(s)$; for details, see the proof of Lemma 3.9 above). But the latter is an easy task: since $\nu q^{k+1}\le \nu q^2<1$ for $k\ge 1$,

$$\begin{aligned}-\sum_{l=1}^{\infty}(\nu q)^l\Bigl[1-\frac{1}{1-(1-\alpha_*)q^l}\Bigr] &= \sum_{l\ge 1}(\nu q)^l\sum_{k\ge 1}(1-\alpha_*)^k q^{kl}\\ &=\sum_{k\ge 1}(1-al_*)^k\frac{\nu q^{k+1}}{1-\nu q^{k+1}}\\ &=\Bigl(1-\frac{1}{2\alpha_*\beta}\Bigr)\frac{2\nu\beta}{\nu-1},\end{aligned}$$

where in the last equality we used the very definition of α_*. Thus, (3.41) follows directly.

Since the proof of (3.42) follows the same scenario, we check only the needed monotonicity property of the imaginary part,

$$\begin{aligned}&\Im\nu^R\bigl[F_R^t\bigl(x_R(1+is)\bigr)-F_R^0(x_R)\bigr]\\ &\quad=\frac{s}{2}-\frac{1}{2}\arctan s+\frac{t^2 s}{4\alpha_R\beta(1+s^2)}\end{aligned}$$

$$
\begin{aligned}
&+\frac{\nu-1}{2\nu}\sum_{l=0}^{R}\nu^l\Big[\frac{s\alpha_R q^l}{1-(1-\alpha_R)q^l}-\arctan\frac{s\alpha_R q^l}{1-(1-\alpha_R)q^l}\Big]\\
&+s\alpha_R\beta\Big[(\nu q)^R-\frac{1}{2\alpha_R\beta}-\frac{\nu-1}{2\nu\beta}\sum_{l=0}^{R}\frac{(\nu q)^l}{1-(1-\alpha_R)q^l}\Big].
\end{aligned}
$$

As before, only the last line needs our attention; we get

$$
\begin{aligned}
\sum_{l=0}^{R}\frac{(\nu q)^l}{1-(1-\alpha_R)q^l}&=\sum_{k\geq 0}(1-\alpha_R)^k\sum_{l=1}^{R}(\nu q^{k+1})^l\\
&=(\nu q)^R T_R^0+\sum_{k\geq 1}(1-\alpha_R)^k(\nu q^{k+1})^R T_R^k\\
&=\frac{\nu q}{\nu q-1}\big((\nu q)^R-1\big)+\frac{2\nu\beta}{\nu-1}\Big(1-\frac{1}{2\alpha_R\beta}\Big),
\end{aligned}
$$

where the stationary point equation (3.30) was used in the last equality. As a result,

$$
\begin{aligned}
&(\nu q)^R-\frac{1}{2\alpha_R\beta}-\frac{\nu-1}{2\nu\beta}\sum_{l=0}^{R}\frac{(\nu q)^l}{1-(1-\alpha_R)q^l}\\
&\quad=(\nu q)^R-\frac{1}{2\alpha_R\beta}-\frac{\nu-1}{2\nu\beta}\frac{\nu q}{\nu q-1}\big((\nu q)^R-1\big)-\Big(1-\frac{1}{2\alpha_R\beta}\Big)=0.
\end{aligned}
$$

The proof is finished. □

The non-Gaussian asymptotics in Theorem 2.3 now follows in a standard way by using estimates of Lemmas 3.13 and 3.14.

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