

Extra dimensions and nonlinear equations

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Solutions of nonlinear multi-component Euler–Monge partial differential equations are constructed in n spatial dimensions by *dimension-doubling*, a method that completely linearizes the problem. Nonlocal structures are an essential feature of the method. The Euler–Monge equations may be interpreted as a boundary theory arising from a linearized bulk system such that all boundary solutions follow from simple limits of those for the bulk. © 2003 American Institute of Physics.

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I. INTRODUCTION

For any theory²² with an infinite number of conservation laws, we may always assemble the conserved currents into a generating function involving a spectral parameter a . If that spectral parameter is independent of any other space–time dimensions in the theory, as is possible in the simplest cases, then effectively the theory possesses an *extra dimension*.²³ Moreover, it is always possible to openly include this extra dimension in some of the dynamical equations, and not just leave it as a *dimension sub rosa*.

For example, suppose a theory is originally expressed in terms of coordinates (x,t) with an infinite number of conserved currents: $\partial_t \rho^{(n)}(x,t) = \partial_x J^{(n)}(x,t)$, $n \in \mathbb{N}$. Then by defining $\rho(x,t,a) \equiv \sum_n (n+1) a^n \rho^{(n)}(x,t)$, as opposed to $\sum_n a^n \rho^{(n)}(x,t)$, and $J(x,t,a) \equiv \sum_n a^{n+1} J^{(n)}(x,t)$, as opposed to $\sum_n a^n J^{(n)}(x,t)$, we have rendered all the conservation laws as a single second-order higher-dimensional partial differential equation (PDE): $\partial_t \rho(x,t,a) = \partial_x \partial_a J(x,t,a)$, as opposed to the first-order $\partial_t \rho(x,t,a) = \partial_x J(x,t,a)$. Hence our choice for the current generating functions has fully exposed an extra dimension in the PDEs satisfied by those generating functions. The extra dimension here does not just ride along as a suppressible label for the currents but it appears explicitly, perhaps even unavoidably, in the dynamical equations. Of course this immediately raises issues about whether the theory requires a to appear explicitly for *all* dynamical equations to be cogently expressed in terms of the original plus extra dimensions, and about covariance properties for the theory in the complete set of dimensions.

In this article we address these issues for a simple but very generally applicable class of nonlinear PDE's:^{10,17} The first order Euler–Monge (E-M) equations $\partial_t \mathbf{u} = (\mathbf{u} \cdot \nabla) \mathbf{u}$. We find the full dynamics of these *nonlinear* theories are elegantly encoded into a higher dimensional set of *linear* “heat” equations obtained through dimension doubling $(\mathbf{x}) \rightarrow (\mathbf{x}, \mathbf{a})$, where for each spatial coordinate x_i there is an *associated coordinate* given by spectral parameter a_i . The original dynamical variables are obtained as spectral parameter boundary limits, $\lim_{\mathbf{a} \rightarrow 0} U_i(\mathbf{x}, t, \mathbf{a}) = u_i(\mathbf{x}, t)$. The fact that the higher dimensional theory is linearized strongly argues that this is the right approach to take. In the linearized theory, the pairs (x_i, a_i) act like “light-cone” variables in the enlarged set of dimensions such that the heat equations for all the dynamical variables are of the form $(\partial/\partial t - \sum_{j=1}^n \partial^2/\partial a_j \partial x_j) U_i(\mathbf{x}, t, \mathbf{a}) = 0$. Thus the extra dimensions appear explicitly and, indeed, unavoidably in these linearized dynamical equations.

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We also find Nambu brackets¹⁸ of the fields, of all orders up to the full Jacobian, as a remarkable feature of the linearizing maps. We know of only one other field theoretic example³ where these brackets appear so naturally. Moreover, the linearizing maps are *nonlocal* in all but the simplest, one component case. The nonlocal structures appropriate for E-M equations with two components in two spatial dimensions are evocative of phase factors in Wilson loops (cf. strings), and when the E-M equations describe n component fields in the original n spatial dimensions these structures extend to higher dimensional constructions involving integrals over $n - 1$ dimensional submanifolds [cf. $(n - 1)$ -branes]. In the one dimensional, one component case, the E-M solution is obtained algebraically from the dimensionally-doubled “bulk” solution for all values of the single spectral parameter. In higher dimensional or multi-component cases the dependence of the solutions on the spectral parameters is more involved. Nevertheless, in all cases the solutions of the E-M equations may be obtained from simple limits of those for the bulk.

II. HISTORICAL OVERVIEW AND OBJECTIVES

The Euler–Monge equations first appeared in 18th and 19th century studies of fluid dynamics¹⁰ and analytic geometry.¹⁷ Riemann took up a study of the equations in the context of gas dynamics, discussing the equations as a theory of invariants²⁰ (for a modern textbook treatment, see Ref. 8). His approach is widely applicable to almost all nonlinear flow problems, although it does not triumph over turbulence. A systematic modern discussion of the E-M equations that synthesizes ideas from both geometry and invariance theory can be found in the review by Dubrovin and Novikov.⁹ Most contemporary texts and reviews stress the universal role played by these nonlinear transport equations in accordance with Whitham’s theory.²¹ Essentially all nonlinear waves, even those in dispersive and dissipative media, involve E-M equations, or simple variants of them, if the nonlinear wavetrains are slowly varying. This makes the equations particularly useful for analyzing the asymptotic behavior of nonlinear solutions. The E-M equations and their conservation laws also serve as a useful starting point in Polyakov’s study of turbulence¹⁹ but without yet leading to a general solution of the Navier–Stokes equations.

The first order E-M equation $\partial u / \partial t = u \partial u / \partial x$ also gives rise to the Bateman equation⁴ upon substituting $u = (\partial \phi / \partial t) / (\partial \phi / \partial x)$. The resulting second order nonlinear PDE is $0 = \phi_x^2 \phi_{tt} - 2 \phi_x \phi_t \phi_{tx} + \phi_t^2 \phi_{xx}$, and is well known to possess a general implicit solution given by solving $t S_0(\phi) + x S_1(\phi) = \text{const}$, where S_0 and S_1 are arbitrary differentiable functions of $\phi(x, t)$. The structure of this solution incorporates the covariance properties of the PDE: If ϕ is a solution, so is any function of ϕ . In fact, curiously, the generalization of this solution to $n + 1$ functions $S_0(\phi), S_i(\phi)$ of $\phi(\mathbf{x}, t)$, $\mathbf{x} = (x_1, \dots, x_n)$, subject to a single constraint $t S_0(\phi) + \sum x_i S_i(\phi) = 0$, is a “universal solution”¹⁴ to any equation derived from a Lagrangian which is homogeneous of weight one in the first derivatives of ϕ .

Thus the Euler–Monge equations appear widespread across a very broad landscape of physics and applied mathematics problems, and therefore it is important to understand their solutions at as many levels as possible. To that end we shall map all solutions of the E-M equations in arbitrary dimensions into solutions of second-order linear equations. This type of map is reminiscent of the Cole–Hopf^{6,15} transformation (thoroughly reviewed in Ref. 16) used to linearize the Burgers^{5,15} nonlinear diffusion equation, but there are important differences here. The Cole–Hopf transformation only works for curl-free \mathbf{u} , does not use extra dimensions, and fails for $0 = \kappa$ (the diffusivity). The map to follow works for all \mathbf{u} , curl-free or otherwise, does use extra dimensions, but works only for $\kappa = 0$. (We hope to extend the method to $\kappa \neq 0$ and to include the effects of pressure in subsequent studies.)

III. METHOD AND ELEMENTARY RESULTS

We believe it is best to present our results summarily for the simplest examples of fields with one, two, and three components, and then to extend these results to the general case of n compo-

nents. We leave out most details but we do sketch the salient features of the derivations. In the following, \mathcal{M}_n is the n dimensional nonlinear Euler–Monge operator and \mathcal{H}_n is an associated hyperbolic heat operator (introduced in Ref. 19),

$$\begin{aligned} \mathcal{M}_n &\equiv \frac{\partial}{\partial t} - \sum_{j=1}^n u_j \frac{\partial}{\partial x_j} \\ \mathcal{H}_n &\equiv \frac{\partial}{\partial t} - \sum_{j=1}^n \frac{\partial^2}{\partial x_j \partial a_j}. \end{aligned} \tag{1}$$

To begin, however, we will generalize these two definitions to allow for an arbitrary function F in the most elementary results in one spatial dimension. We find that

$$\frac{\partial}{\partial t} U(x, t, a) = F\left(\frac{\partial}{\partial a}\right) \frac{\partial}{\partial x} U(x, t, a) \tag{2}$$

if and only if

$$\frac{\partial}{\partial t} u(x, t) = F(u(x, t)) \frac{\partial}{\partial x} u(x, t) \tag{3}$$

where

$$\begin{aligned} U(x, t, a) &\equiv \frac{e^{au(x, t)} - 1}{a} \\ u(x, t) &= \frac{1}{a} \ln(1 + aU(x, t, a)) \end{aligned} \tag{4}$$

and F is any function with a formal power series. This simple result follows by direct calculation

$$\left(\frac{\partial}{\partial t} - F\left(\frac{\partial}{\partial a}\right) \frac{\partial}{\partial x}\right) \frac{e^{au(x, t)} - 1}{a} = e^{au(x, t)} \left(\frac{\partial}{\partial t} u(x, t) - F(u(x, t)) \frac{\partial}{\partial x} u(x, t)\right). \tag{5}$$

The formal solution for $U(x, t, a)$ in terms of $U(x, t=0, a)$ is now obviously given by

$$(e^{au(x, t)} - 1)/a = e^{t F(\partial/\partial a) \partial/\partial x} ((e^{au(x)} - 1)/a) \tag{6}$$

with $u(x, t=0) = u(x)$.

The bulk solution $U(x, t, a)$ may also be viewed as a simple one-parameter deformation of the boundary data $u(x, t)$, with the extra dimension serving as the deformation parameter. In this exceptional one-component case, we may easily extract $u(x, t)$ from $U(x, t, a)$ for any value of the extra dimension a as given by the logarithmic expression above. But, in particular, we may extract $u(x, t)$ as a limit of the bulk solution $u(x, t) = \lim_{a \rightarrow 0} U(x, t, a)$. This immediately yields the time series solution¹² to the previous E-M equation as a limit:

$$u(x, t) = \lim_{a \rightarrow 0} e^{t F(\partial/\partial a) \partial/\partial x} \left(\frac{e^{au(x)} - 1}{a}\right) = F^{-1} \left[\sum_{j=0}^{\infty} \frac{t^j}{(1+j)!} \frac{d^j}{dx^j} (F[u(x)])^{1+j} \right], \tag{7}$$

where we assume F (locally) invertible in the last step.²⁴ While this time series is an immediate consequence of the previous results, we believe it is neither trivial nor obvious. Similar time series solutions are immediate consequences of all our results. For example, one independent field u in spatial dimensions (x, y_1, \dots, y_n) with dependent “velocity fields” $(u, v_1(u), \dots, v_n(u))$ leads to

$$\frac{\partial}{\partial t} u(x, \mathbf{y}, t) = u(x, \mathbf{y}, t) \frac{\partial}{\partial x} u(x, \mathbf{y}, t) + \sum_{i=1}^n v_i(u(x, \mathbf{y}, t)) \frac{\partial}{\partial y_i} u(x, \mathbf{y}, t) \tag{8}$$

if and only if

$$\int_0^{u(x, \mathbf{y}, t)} du \exp\left(au + \sum_{i=1}^n b_i v_i(u) \right) = e^{t(\partial^2/\partial x \partial a + \sum_{i=1}^n \partial^2/\partial y_i \partial b_i)} \int_0^{u(x, \mathbf{y})} du \exp\left(au + \sum_{i=1}^n b_i v_i(u) \right). \tag{9}$$

Again, this follows by direct calculation, with

$$U(x, \mathbf{y}, t, a, \mathbf{b}) \equiv \int_0^{u(x, \mathbf{y}, t)} du \exp\left(au + \sum_{i=1}^n b_i v_i(u) \right) \tag{10}$$

since

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x \partial a} - \sum_{i=1}^n \frac{\partial^2}{\partial y_i \partial b_i} \right) U(x, \mathbf{y}, t, a, \mathbf{b}) \\ &= \left(\frac{\partial}{\partial t} u(x, \mathbf{y}, t) - u(x, \mathbf{y}, t) \frac{\partial}{\partial x} u(x, \mathbf{y}, t) - \sum_{i=1}^n v_i(u(x, \mathbf{y}, t)) \frac{\partial}{\partial y_i} u(x, \mathbf{y}, t) \right) \\ & \quad \times \exp\left(au(x, \mathbf{y}, t) + \sum_{i=1}^n b_i v_i(u(x, \mathbf{y}, t)) \right). \end{aligned} \tag{11}$$

So, as given, the higher dimensional heat equation is satisfied by the integral form $U(x, \mathbf{y}, t, a, \mathbf{b})$ if and only if the given one-component generalization of the E-M equations holds. The rhs of Eq. (10) is then just the formal solution of the heat equation.²⁵

The last result does not allow for a simple extraction of $u(x, \mathbf{y}, t)$ from the integral form of $U(x, \mathbf{y}, t, a, \mathbf{b})$ for nonvanishing a, \mathbf{b} . However, it does have the simple limit $\lim_{a, \mathbf{b} \rightarrow 0} U(x, \mathbf{y}, t, a, \mathbf{b}) = u(x, \mathbf{y}, t)$, so extraction is trivial on the boundary $a, \mathbf{b} \rightarrow 0$. This is true of all the other heat equation solutions to follow. Also note, $U(x, \mathbf{y}, t, a, \mathbf{b})$ in this one-component case is an integral over the field value. Nevertheless U is still local in all the dimensions, no matter how many.

IV. MULTIPLE COMPONENTS AND NONLOCALITY

Locality in the original spatial dimensions will *not* hold, however, for maps of multi-component fields in higher dimensions. This is first illustrated by the next result,

$$\mathcal{H}_2 U = \mathcal{H}_2 V = 0, \tag{12}$$

if and only if

$$\mathcal{M}_2 u = \mathcal{M}_2 v = 0, \tag{13}$$

where $[\varepsilon(s) \equiv \pm \frac{1}{2} \text{ for } s \geq 0]$

$$\begin{aligned} U(x, y, t, a, b) &\equiv \int_{-\infty}^{\infty} dr \varepsilon(y-r) e^{au(x, r, t) + bv(x, r, t)} \frac{\partial u(x, r, t)}{\partial r} \\ V(x, y, t, a, b) &\equiv \int_{-\infty}^{\infty} dq \varepsilon(x-q) e^{au(q, y, t) + bv(q, y, t)} \frac{\partial v(q, y, t)}{\partial q}. \end{aligned} \tag{14}$$

Once again this is established by direct calculation, assuming u, v , and their derivatives vanish asymptotically in x, y ,

$$\begin{aligned} \mathcal{H}_2 U(x, y, t, a, b) &= e^{au(x, y, t) + bv(x, y, t)} \mathcal{M}_2 u(x, y, t) + b \int_{-\infty}^{\infty} dr \varepsilon(y - r) \\ &\quad \times e^{au(x, r, t) + bv(x, r, t)} \left(\frac{\partial u(x, r, t)}{\partial r} \mathcal{M}_2 v(x, r, t) - \frac{\partial v(x, r, t)}{\partial r} \mathcal{M}_2 u(x, r, t) \right) \end{aligned} \tag{15}$$

$$\begin{aligned} \mathcal{H}_2 V(x, y, t, a, b) &= e^{au(x, y, t) + bv(x, y, t)} \mathcal{M}_2 v(x, y, t) + a \int_{-\infty}^{\infty} dq \varepsilon(x - q) \\ &\quad \times e^{au(q, y, t) + bv(q, y, t)} \left(\frac{\partial v(q, y, t)}{\partial q} \mathcal{M}_2 u(q, y, t) - \frac{\partial u(q, y, t)}{\partial q} \mathcal{M}_2 v(q, y, t) \right). \end{aligned} \tag{16}$$

The converse result then follows by also using the obvious pair of limits $\lim_{a, b \rightarrow 0} \mathcal{H}_2 U(x, y, t, a, b) = \mathcal{M}_2 u(x, y, t)$ and $\lim_{a, b \rightarrow 0} \mathcal{H}_2 V(x, y, t, a, b) = \mathcal{M}_2 v(x, y, t)$.

As advertised, the two-component map in two spatial dimensions involves a nonlocal transformation between E-M and heat equation solutions: It features line integrals over the original spatial variables. The map is still local in the extra dimensions, however. This nonlocality in the original dimensions persists and is even extended when more components and more spatial dimensions are considered. As a further illustration before giving the generalization to an arbitrary number of dimensions, we have

$$\mathcal{H}_3 U = \mathcal{H}_3 V = \mathcal{H}_3 W = 0, \tag{17}$$

if and only if

$$\mathcal{M}_3 u = \mathcal{M}_3 v = \mathcal{M}_3 w = 0, \tag{18}$$

where

$$\begin{aligned} U(x, y, z, t, a, b, c) &\equiv \int dr \varepsilon(y - r) e^{au + bv + cw} \frac{\partial u(x, r, z, t)}{\partial r} \\ &\quad - c \int \int dr ds \varepsilon(y - r) \varepsilon(z - s) e^{au + bv + cw} \{u, w\}_{rs}(x, r, s, t), \\ V(x, y, z, t, a, b, c) &\equiv \int ds \varepsilon(z - s) e^{au + bv + cw} \frac{\partial v(x, y, s, t)}{\partial s} \\ &\quad - a \int \int dq ds \varepsilon(x - q) \varepsilon(z - s) e^{au + bv + cw} \{v, u\}_{sq}(q, y, s, t), \\ W(x, y, z, t, a, b, c) &\equiv \int dq \varepsilon(x - q) e^{au + bv + cw} \frac{\partial w(q, y, z, t)}{\partial q} \\ &\quad - b \int \int dq dr \varepsilon(x - q) \varepsilon(y - r) e^{au + bv + cw} \{w, v\}_{qr}(q, r, z, t). \end{aligned} \tag{19}$$

There are a few essential new ingredients needed to complete the argument by direct calculation in this case. Define Poisson brackets as usual by

$$\{u, v\}_{rs} = \frac{\partial u}{\partial r} \frac{\partial v}{\partial s} - \frac{\partial u}{\partial s} \frac{\partial v}{\partial r}, \tag{20}$$

where u and v are any two functions of the independent variables r and s . Then it is straightforward to show

$$\begin{aligned} \frac{\partial}{\partial t} \{u, v\}_{zy} - \frac{\partial}{\partial x} (u \{u, v\}_{zy}) - \frac{\partial}{\partial y} (v \{u, v\}_{zy}) - \frac{\partial}{\partial z} (w \{u, v\}_{zy}) &= \{\mathcal{M}_3 u, v\}_{zy} + \{u, \mathcal{M}_3 v\}_{zy}, \\ \frac{\partial}{\partial t} \{u, v\}_{xy} - \frac{\partial}{\partial x} (u \{u, v\}_{xy}) - \frac{\partial}{\partial y} (v \{u, v\}_{xy}) - \frac{\partial}{\partial z} (w \{u, v\}_{xy}) & \\ = \{\mathcal{M}_3 u, v\}_{xy} + \{u, \mathcal{M}_3 v\}_{xy} - \{u, v, w\}_{xyz}, \end{aligned} \tag{21}$$

as well as similar relations obtained by permutation of dependent and independent variables. In the last relation we have introduced the totally antisymmetric Nambu triple bracket (i.e., Jacobian, in this three-dimensional case)

$$\{u, v, w\}_{xyz} = \frac{\partial u}{\partial x} \{v, w\}_{yz} + \frac{\partial u}{\partial y} \{v, w\}_{zx} + \frac{\partial u}{\partial z} \{v, w\}_{xy} = \frac{\partial u}{\partial x} \{v, w\}_{yz} + \frac{\partial v}{\partial x} \{w, u\}_{yz} + \frac{\partial w}{\partial x} \{u, v\}_{yz}. \tag{22}$$

Once equipped with such relations, the complete derivation of the heat equation and E-M equivalence is tedious, perhaps, but not subtle. (See the generalization to follow for additional details.)

The nonlocality appearing in our map for three components in three spatial dimensions is two-dimensional: It features surface integrals over pairs of the original spatial dimensions, perhaps evocative of membrane-based phase factors. Nonetheless, the map is still local in the extra dimensions and the E-M solutions are again trivially given by boundary limits of the bulk constructions.

V. GENERAL RESULTS

The nonlocality is extended to $(n - 1)$ -dimensional integrals when n -component linearizing maps are constructed in n spatial dimensions. This is explicit in the following equations.

$$\mathcal{H}_n U_k(\mathbf{x}, t, \mathbf{a}) = 0 \tag{23}$$

if and only if

$$\mathcal{M}_n u_i(\mathbf{x}, t) = 0 \tag{24}$$

for $i, k \in \{1, \dots, n\}$ where

$$\begin{aligned} U_k(\mathbf{x}, t, \mathbf{a}) \equiv & \int \cdots \int dq_1 \cdots dq_n \delta(q_k - x_k) \left(\frac{e^{a_k u_k} - 1}{a_k} \right) \\ & \times \det \begin{pmatrix} \frac{\partial}{\partial q_1} (\varepsilon(q_1 - x_1) e^{a_1 u_1}) & \cdots & \frac{\partial}{\partial q_n} (\varepsilon(q_1 - x_1) e^{a_1 u_1}) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial q_1} (\varepsilon(q_n - x_n) e^{a_n u_n}) & \cdots & \frac{\partial}{\partial q_n} (\varepsilon(q_n - x_n) e^{a_n u_n}) \end{pmatrix}. \end{aligned} \tag{25}$$

exclude k th row
and k th column

The equivalence is shown as follows. Consider only the first component (*et sic de similibus*).

$$\begin{aligned}
 U_1(\mathbf{x}, t, \mathbf{a}) &= \int \cdots \int dq_1 \cdots dq_n \delta(q_1 - x_1) \left(\frac{e^{a_1 u_1} - 1}{a_1} \right) \\
 &\quad \times \det \begin{pmatrix} \frac{\partial}{\partial q_2} (\varepsilon(q_2 - x_2) e^{a_2 u_2}) & \cdots & \frac{\partial}{\partial q_n} (\varepsilon(q_2 - x_2) e^{a_2 u_2}) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial q_2} (\varepsilon(q_n - x_n) e^{a_n u_n}) & \cdots & \frac{\partial}{\partial q_n} (\varepsilon(q_n - x_n) e^{a_n u_n}) \end{pmatrix} \\
 &= \int \cdots \int dq_2 \cdots dq_n \left(\frac{e^{a_1 u_1} - 1}{a_1} \right) \\
 &\quad \times \det \begin{pmatrix} \frac{\partial}{\partial q_2} (\varepsilon(q_2 - x_2) e^{a_2 u_2}) & \cdots & \frac{\partial}{\partial q_n} (\varepsilon(q_2 - x_2) e^{a_2 u_2}) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial q_2} (\varepsilon(q_n - x_n) e^{a_n u_n}) & \cdots & \frac{\partial}{\partial q_n} (\varepsilon(q_n - x_n) e^{a_n u_n}) \end{pmatrix} (x_1, q_2, \dots, q_n, t) \\
 &= \int \cdots \int dq_2 \cdots dq_n \left(\frac{e^{a_1 u_1} - 1}{a_1} \right) \\
 &\quad \times \varepsilon_{i_2 \cdots i_n} \frac{\partial}{\partial q_{i_2}} (\varepsilon(q_2 - x_2) e^{a_2 u_2}) \frac{\partial}{\partial q_{i_3}} (\varepsilon(q_3 - x_3) e^{a_3 u_3}) \cdots \frac{\partial}{\partial q_{i_n}} (\varepsilon(q_n - x_n) e^{a_n u_n}),
 \end{aligned}$$

where in the last expression the i_k dummy indices, $k \in \{2, \dots, n\}$, are summed from 2 to n , i.e., 1 is excluded. Now we integrate by parts assuming all fields and their derivatives vanish as $x \rightarrow \infty$. To do this, there are clearly $n - 1$ equivalent choices. We elect to integrate $\partial/\partial q_{i_2}$ by parts to obtain

$$\begin{aligned}
 U_1(\mathbf{x}, t, \mathbf{a}) &= -\varepsilon_{i_2 \cdots i_n} \int \cdots \int dq_2 \cdots dq_n \varepsilon(q_2 - x_2) e^{a_2 u_2} \frac{\partial}{\partial q_{i_2}} \left(\frac{e^{a_1 u_1} - 1}{a_1} \right) \\
 &\quad \times \frac{\partial}{\partial q_{i_3}} (\varepsilon(q_3 - x_3) e^{a_3 u_3}) \cdots \frac{\partial}{\partial q_{i_n}} (\varepsilon(q_n - x_n) e^{a_n u_n}) \\
 &= -\varepsilon_{i_2 \cdots i_n} \int \cdots \int dq_2 \cdots dq_n \varepsilon(q_2 - x_2) \frac{\partial u_1}{\partial q_{i_2}} \\
 &\quad \times \left(\delta_{i_3} \delta(q_3 - x_3) + a_3 \varepsilon(q_3 - x_3) \frac{\partial u_3}{\partial q_{i_3}} \right) \cdots \left(\delta_{i_n} \delta(q_n - x_n) + a_n \varepsilon(q_n - x_n) \frac{\partial u_n}{\partial q_{i_n}} \right) e^{\mathbf{a} \cdot \mathbf{u}}.
 \end{aligned}$$

Expanding out the products of the various paired terms in parentheses in the last line gives

$$\begin{aligned}
 U_1(\mathbf{x}, t, \mathbf{a}) = & -\varepsilon_{i_2 \dots i_n} a_3 \dots a_n \int \dots \int dq_2 dq_3 \dots dq_n \varepsilon(q_2 - x_2) \\
 & \times \varepsilon(q_3 - x_3) \dots \varepsilon(q_n - x_n) \frac{\partial u_1}{\partial q_{i_2}} \frac{\partial u_3}{\partial q_{i_3}} \dots \frac{\partial u_n}{\partial q_{i_n}} e^{\mathbf{a} \cdot \mathbf{u}} \\
 & - \varepsilon_{i_2 \dots i_n} \sum_{j=3}^n \int dq_2 \varepsilon(q_2 - x_2) \delta_{ji_j} \\
 & \times \left(\prod_{\substack{k=3 \\ k \neq j}}^n \left(a_k \int dq_k \varepsilon(q_k - x_k) \frac{\partial u_k}{\partial q_{i_k}} \right) \right) \frac{\partial u_1}{\partial q_{i_2}} e^{\mathbf{a} \cdot \mathbf{u}} \\
 & - \varepsilon_{i_2 \dots i_n} \sum_{j=3}^n \sum_{\substack{k=4 \\ k > j}}^n \int dq_2 \varepsilon(q_2 - x_2) \delta_{ji_j} \delta_{ki_k} \\
 & \times \left(\prod_{\substack{m=3 \\ m \neq j, k}}^n \left(a_m \int dq_m \varepsilon(q_m - x_m) \frac{\partial u_m}{\partial q_{i_m}} \right) \right) \frac{\partial u_1}{\partial q_{i_2}} e^{\mathbf{a} \cdot \mathbf{u}} \dots \\
 & - \sum_{j=3}^n a_j \int dq_2 \varepsilon(q_2 - x_2) \int dq_j \varepsilon(q_j - x_j) \left(\frac{\partial u_1}{\partial q_2} \frac{\partial u_j}{\partial q_j} - \frac{\partial u_1}{\partial q_j} \frac{\partial u_j}{\partial q_2} \right) e^{\mathbf{a} \cdot \mathbf{u}} \\
 & - \int dq_2 \varepsilon(q_2 - x_2) \frac{\partial u_1}{\partial q_2} e^{\mathbf{a} \cdot \mathbf{u}}.
 \end{aligned}$$

That is to say, the result is given in terms of Nambu brackets¹⁸ of all ranks from $n - 1$ down to 2 (i.e., Poisson), as well as a final single derivative term. Thus

$$\begin{aligned}
 U_1(\mathbf{x}, t, \mathbf{a}) = & -a_3 \dots a_n \int \dots \int dq_2 dq_3 \dots dq_n \varepsilon(q_2 - x_2) \varepsilon(q_3 - x_3) \dots \varepsilon(q_n - x_n) \\
 & \times \{u_1, u_3, \dots, u_n\}_{23 \dots n} e^{\mathbf{a} \cdot \mathbf{u}} \\
 & - \sum_{j=3}^n \int dq_2 \varepsilon(q_2 - x_2) \left(\prod_{\substack{k=3 \\ k \neq j}}^n \left(a_k \int dq_k \varepsilon(q_k - x_k) \right) \right) \\
 & \times \{u_1, u_3, \dots, u_{j-1}, u_{j+1}, \dots, u_n\}_{23 \dots j-1 j+1 \dots n} e^{\mathbf{a} \cdot \mathbf{u}} \\
 & \dots - \sum_{j=3}^n \sum_{\substack{k=4 \\ k > j}}^n a_j a_k \int dq_2 \varepsilon(q_2 - x_2) \int dq_j \varepsilon(q_j - x_j) \int dq_k \varepsilon(q_k - x_k) \\
 & \times \{u_1, u_j, u_k\}_{2jk} e^{\mathbf{a} \cdot \mathbf{u}} \\
 & - \sum_{j=3}^n a_j \int dq_2 \varepsilon(q_2 - x_2) \int dq_j \varepsilon(q_j - x_j) \{u_1, u_j\}_{2j} e^{\mathbf{a} \cdot \mathbf{u}} \\
 & - \int dq_2 \varepsilon(q_2 - x_2) \frac{\partial u_1}{\partial q_2} e^{\mathbf{a} \cdot \mathbf{u}}.
 \end{aligned}$$

In the preceding equation, it is to be understood that the sum $\sum_{j=3}^n$ in the second term begins at its lower limit with

$$-a_4 \dots a_n \int \dots \int dq_2 dq_4 \dots dq_n \varepsilon(q_2 - x_2) \varepsilon(q_4 - x_4) \dots \varepsilon(q_n - x_n) \{u_1, u_4, \dots, u_n\}_{24 \dots n} e^{\mathbf{a} \cdot \mathbf{u}}$$

and terminates at its upper limit with

$$-a_3 \cdots a_{n-1} \int \cdots \int dq_2 dq_3 \cdots dq_{n-1} \varepsilon(q_2 - x_2) \varepsilon(q_3 - x_3) \cdots \varepsilon(q_{n-1} - x_{n-1}) \times \{u_1, u_3, \dots, u_{n-1}\}_{23 \cdots n-1} e^{\mathbf{a} \cdot \mathbf{u}}.$$

Next we act with the heat operator on $U_1(\mathbf{x}, t, \mathbf{a})$. The ε 's permit the appropriate "outside" (i.e., x) partials to be converted, through integration by parts, into "inside" (i.e., q) partials. Also, factors of a_i outside the exponentials produce some extra terms from the cross-partial $\partial^2/\partial x_i \partial a_i$ in \mathcal{H}_n . We obtain

$$\begin{aligned} \mathcal{H}_n U_1(\mathbf{x}, t, \mathbf{a}) = & -a_3 \cdots a_n \int \cdots \int dq_2 dq_3 \cdots dq_n \varepsilon(q_2 - x_2) \varepsilon(q_3 - x_3) \cdots \varepsilon(q_n - x_n) \\ & \times \mathcal{H}_n(\{u_1, u_3, \dots, u_n\}_{23 \cdots n} e^{\mathbf{a} \cdot \mathbf{u}}) \\ & + \sum_{i=3}^n \frac{\partial}{\partial a_i} (a_3 \cdots a_n) \int \cdots \int dq_2 dq_3 \cdots dq_n \frac{\partial}{\partial x_i} [\varepsilon(q_2 - x_2) \varepsilon(q_3 - x_3) \cdots \varepsilon(q_n - x_n)] \\ & \times \{u_1, u_3, \dots, u_n\}_{23 \cdots n} e^{\mathbf{a} \cdot \mathbf{u}} \\ & - \sum_{j=3}^n \int dq_2 \varepsilon(q_2 - x_2) \left(\prod_{\substack{k=3 \\ k \neq j}}^n \left(a_k \int dq_k \varepsilon(q_k - x_k) \right) \right) \\ & \times \mathcal{H}_n(\{u_1, u_3, \dots, u_{j-1}, u_{j+1}, \dots, u_n\}_{23 \cdots j-1j+1 \cdots n} e^{\mathbf{a} \cdot \mathbf{u}}) \\ & + \sum_{j=3}^n \int dq_2 \varepsilon(q_2 - x_2) \sum_{i=3}^n \frac{\partial}{\partial a_i} \frac{\partial}{\partial x_i} \left(\prod_{\substack{k=3 \\ k \neq j}}^n \left(a_k \int dq_k \varepsilon(q_k - x_k) \right) \right) \\ & \times \{u_1, u_3, \dots, u_{j-1}, u_{j+1}, \dots, u_n\}_{23 \cdots j-1j+1 \cdots n} e^{\mathbf{a} \cdot \mathbf{u}} \\ & - + \cdots - \sum_{j=3}^n a_j \int dq_2 \varepsilon(q_2 - x_2) \int dq_j \varepsilon(q_j - x_j) \mathcal{H}_n(\{u_1, u_j\}_{2j} e^{\mathbf{a} \cdot \mathbf{u}}) \\ & + \sum_{i=3}^n \frac{\partial}{\partial a_i} \frac{\partial}{\partial x_i} \left(\sum_{j=3}^n a_j \int dq_2 \varepsilon(q_2 - x_2) \int dq_j \varepsilon(q_j - x_j) \right) \{u_1, u_j\}_{2j} e^{\mathbf{a} \cdot \mathbf{u}} \\ & - \int dq_2 \varepsilon(q_2 - x_2) \mathcal{H}_n \left(\frac{\partial u_1}{\partial q_2} e^{\mathbf{a} \cdot \mathbf{u}} \right). \end{aligned}$$

The first term (first two lines) on the rhs of $\mathcal{H}_n U_1$ reduces to terms linear in the E-M equations for the u 's. The second term (third and fourth lines) and third term (fifth and sixth lines) on the rhs combine to give similar terms linear in the E-M equations. And so it goes with subsequent pairs of terms on the rhs of $\mathcal{H}_n U_1$, until finally the last two terms (last two lines) on the rhs combine to give terms linear in the E-M equations.

To establish these statements, one needs to use several identities involving the action of the heat operator on exponentially weighted derivatives of the component fields, in particular on so-weighted Nambu brackets. For example, these identities range from the simplest for the full Jacobian

$$\begin{aligned} \mathcal{H}_n(e^{\mathbf{a} \cdot \mathbf{u}} \{u_1, u_2, \dots, u_n\}_{12 \cdots n}) = & e^{\mathbf{a} \cdot \mathbf{u}} (\mathbf{a} \cdot \mathcal{M}_n \mathbf{u}) \{u_1, u_2, \dots, u_n\}_{12 \cdots n} + e^{\mathbf{a} \cdot \mathbf{u}} (\{\mathcal{M}_n u_1, u_2, \dots, u_n\}_{12 \cdots n} \\ & + \{u_1, \mathcal{M}_n u_2, \dots, u_n\}_{12 \cdots n} + \cdots + \{u_1, u_2, \dots, \mathcal{M}_n u_n\}_{12 \cdots n}) \end{aligned}$$

to those involving lower rank Nambu brackets such as

$$\begin{aligned} \mathcal{H}_n(e^{\mathbf{a}\cdot\mathbf{u}}\{u_2, u_3, \dots, u_n\}_{23\dots n}) &= e^{\mathbf{a}\cdot\mathbf{u}}(-\{u_1, u_2, u_3, \dots, u_n\}_{123\dots n} + (\mathbf{a}\cdot\mathcal{M}_n\mathbf{u})\{u_2, u_3, \dots, u_n\}_{23\dots n}) \\ &\quad + e^{\mathbf{a}\cdot\mathbf{u}}(\{\mathcal{M}_n u_2, u_3, \dots, u_n\}_{23\dots n} + \{u_2, \mathcal{M}_n u_3, \dots, u_n\}_{23\dots n} + \dots \\ &\quad + \{u_2, u_3, \dots, \mathcal{M}_n u_n\}_{23\dots n}) \end{aligned}$$

including that needed to deal with the first rhs line of $\mathcal{H}_n U_1$

$$\begin{aligned} \mathcal{H}_n(e^{\mathbf{a}\cdot\mathbf{u}}\{u_1, u_3, \dots, u_n\}_{23\dots n}) &= e^{\mathbf{a}\cdot\mathbf{u}}(\mathbf{a}\cdot\mathcal{M}_n\mathbf{u})\{u_1, u_3, \dots, u_n\}_{23\dots n} + e^{\mathbf{a}\cdot\mathbf{u}}(\{\mathcal{M}_n u_1, u_3, \dots, u_n\}_{23\dots n} \\ &\quad + \{u_1, \mathcal{M}_n u_3, \dots, u_n\}_{23\dots n} + \dots + \{u_1, u_3, \dots, \mathcal{M}_n u_n\}_{23\dots n}) \end{aligned}$$

as well as other relations obtained by permutations of the indices of these, etc., all the way down to the final

$$\mathcal{H}_n(e^{\mathbf{a}\cdot\mathbf{u}}\partial_j u_k(\mathbf{x}, t)) = e^{\mathbf{a}\cdot\mathbf{u}}\left(\partial_j(\mathcal{M}_n u_k) + (\mathbf{a}\cdot\mathcal{M}_n\mathbf{u})\partial_j u_k - \sum_i \{u_k, u_i\}_{ji}\right)$$

as needed to deal with the last two rhs lines in $\mathcal{H}_n U_1$. All such identities are straightforward to substantiate by direct calculation.

Thus, given the E-M equations for the u 's, the heat equation for U_1 follows. Moreover, the only terms on the rhs of $\mathcal{H}_n U_1$ which survive in the limit of vanishing spectral parameters are the last two lines, which give

$$\lim_{\mathbf{a}\rightarrow\mathbf{0}} \mathcal{H}_n U_1(\mathbf{x}, t, \mathbf{a}) = \mathcal{M}_n u_1(\mathbf{x}, t). \tag{26}$$

Thus, given the heat equation for U_1 , the E-M equation for u_1 follows. Similar results obtain for all the other components, so that $\mathcal{H}_n U_k = 0$ iff $\mathcal{M}_n u_j = 0$.

Formally, time evolution in the bulk is once more given by a simple exponentiation

$$\mathbf{U}(\mathbf{x}, t, \mathbf{a}) = e^{t\sum_{j=1}^n \partial^2/\partial x_j \partial a_j} \mathbf{U}(\mathbf{x}, t=0, \mathbf{a}). \tag{27}$$

This gives a time-series solution on the boundary upon taking the limit $\mathbf{a}\rightarrow\mathbf{0}$.

$$\mathbf{u}(\mathbf{x}, t) = \lim_{\mathbf{a}\rightarrow\mathbf{0}} e^{t\sum_{j=1}^n \partial^2/\partial x_j \partial a_j} \mathbf{U}(\mathbf{x}, t=0, \mathbf{a}) \tag{28}$$

with initial boundary data $\mathbf{u}(\mathbf{x}) = \lim_{\mathbf{a}\rightarrow\mathbf{0}} \mathbf{U}(\mathbf{x}, t=0, \mathbf{a})$. Moreover, the n -fold infinite sequences of conservation laws for the E-M equations in n spatial dimensions are directly encoded into the bulk solutions.

$$\frac{\partial}{\partial t} U_k(\mathbf{x}, t, \mathbf{a}) = \nabla \cdot \mathbf{J}_k(\mathbf{x}, t, \mathbf{a}), \quad \mathbf{J}_k(\mathbf{x}, t, \mathbf{a}) = \nabla_{\mathbf{a}} U_k(\mathbf{x}, t, \mathbf{a}), \quad k \in \{1, 2, \dots, n\}. \tag{29}$$

Explicit sequences of charge and current densities on the boundary follow immediately from power series expansions in the a_j . These conservation laws are elementary consequences of the heat equation obeyed by \mathbf{U} .

VI. CONCLUSION

This is as far as we have completed the application of the extra dimensional approach to classical nonlinear PDEs. It remains to apply this approach to other types of nonlinear PDEs, in particular to those higher-order extensions of the E-M equations involving dispersion, such as the Korteweg–deVries equation, and to those involving diffusion, such as the Burgers and Navier–Stokes equations. Another immediately obvious challenge is to carry the method over to quantum field theories (QFTs). This will not be done here. However, we suspect that the implementation of

these ideas in QFT will involve the use of quantum Nambu brackets (QNBs), given that the classical versions of these appear above. QNBs have a long-standing notoriety, but recently⁷ it has been shown that theirs is an undeserved bad reputation. QNBs can be defined in terms of operators (or in terms of noncommutative geometry) so as to fulfill their expected roles in the quantum evolution of dynamical systems. Perhaps these developments will be useful to meet the challenge of quantizing the E-M equations as well as their higher-order generalizations.

As emphasized previously, the Euler-Monge equations appear widespread throughout physics and the mathematics of nonlinear partial differential equations. Based on the maps we have presented to linearize these equations, we have come to the conclusion that extra dimensions and nonlocal structures are ubiquitous features to be found upon analyzing solutions of such nonlinear partial differential equations, and are quite natural constructs in many physical theories.

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- ²²We dedicate this paper to Peter Freund on the occasion of his becoming Professor Emeritus at the University of Chicago, and thereby begin with general remarks about the origin of extra dimensions, allowing for the possibility that these are similar to but not necessarily on the exact same footing as the original dimensions. For related points of view within the Kaluza–Klein physics framework,¹ see Ref. 2.
- ²³More precisely, an extra *bosonic* dimension. A *finite* number of conservation laws evokes extra *fermionic* or *anyonic* dimensions, θ , involving k th order superspace or Grassman variables. This and other noncommutative geometries will not be discussed further here.
- ²⁴Given the close relation of the Monge and Bateman equations, it might be expected that the latter also admits a power series solution of simple form. Indeed this is so. Treating the equation as hyperbolic with initial conditions $\phi(x,0)$

$=f(x)$, $\partial\phi(x,0)/\partial t=g(x)$, the time series solution of the Bateman equation is $\phi(x,t)=f(x)+tg(x)+\sum_{j=1}^{\infty}t^{1+j}/(1+j)!d^j/dx^j(g(x)^{1+j}/(df(x)/dx)^j)$.

²⁵As is true for the Bateman equation and the one-component Monge equation in one spatial dimension, there is a corresponding second order equation for the case of one component in $n+1$ spatial dimensions which our solution satisfies. It is the so-called "Universal Field Equation" which may be obtained by elimination of u from the first order equations.^{11,12}