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Random walk in random environment with asymptotically zero perturbation

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Abstract. We give criteria for ergodicity, transience and null-recurrence for the random walk in random environment on $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$, with reflection at the origin, where the random environment is subject to a vanishing perturbation. Our results complement existing criteria for random walks in random environments and for Markov chains with asymptotically zero drift, and are significantly different from the previously studied cases. Our method is based on a martingale technique—the method of Lyapunov functions.

Keywords. Random walk in random environment, perturbation of Sinai's regime, recurrence/transience criteria, Lyapunov functions

1. Introduction

In this paper we study a problem with a classical flavour that lies in the intersection of two well-studied problems, those of random walks in one-dimensional random environments and Markov chains with asymptotically small drifts. Separately, these two problems have received considerable attention, but the problem considered in this paper has not been analysed before. Further, our results show that the system studied here exhibits behaviour that is significantly different from that of the previously studied systems.

The random walk in random environment (or RWRE for short) was first studied by Kozlov [12] and Solomon [18], and has since received extensive attention; see for example [16] or [19] for surveys. This paper analyses the behaviour of the RWRE for which the random environment is perturbed by a vanishingly small amount.

The analysis of zero drift random walks in two or more dimensions by the method of Lyapunov functions demonstrated the importance of the investigation of one-dimensional stochastic processes with asymptotically small drifts (see, for example, [2], [13], or [15]). For example, if (Z_t) , with $t = 0, 1, 2, 3, \dots$ time, is a random walk (with zero drift) in the nonnegative quarter plane, analysis of the stochastic process $\|Z_t\|$, where $\|\cdot\|$ denotes

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the Euclidean norm, involves the study of stochastic processes on the half-line with mean drift asymptotically zero.

Early work in this field was done by Lamperti [13, 14]. Criteria for recurrence and transience are given in [15], where the behaviour in the critical regime that Lamperti did not cover was also analysed. Passage-time moments are considered in [2]. In much of this work, Lyapunov functions play a central role.

In this paper we demonstrate the essential difference between a nearest-neighbour random walk in a deterministic environment, perturbed from its critical (null-recurrent) regime, and a nearest-neighbour random walk in a random environment, also perturbed from its critical regime (sometimes called Sinai's regime—see below). Our results quantify the fact that in some sense the random environment is more stable, in that a much larger perturbation is required to disturb the null-recurrent situation. In particular, we give criteria for ergodicity (i.e. positive recurrence), transience and null-recurrence for our perturbed random walk in random environment. We will show that in our (random environment) case the critical magnitude for the perturbation is of the order of $n^{-1/2}$ (see Theorem 6), where n is the distance from the origin (in fact, our more general results are much more precise than this). This compares to a critical magnitude of the order of n^{-1} in the nonrandom environment case (see [15], and Theorem 2 below).

Our method is based upon the theory of Lyapunov functions, a powerful tool in the classification of countable Markov chains (see [6]). Such methods have proven effective in the analysis of random walks in random environments (see e.g. [3]), in addition to Markov chains in nonrandom environments.

Loosely speaking, motivation for our model comes from some one-dimensional physical systems, such as a particle performing a random walk in a homogeneous random one-dimensional field, subject to some vanishing perturbation (such as the presence of another particle). Under what conditions is the perturbation sufficient to alter the character of the random walk?

We now introduce the probabilistic model that we consider. First, we need some notation. We introduce a function χ as follows, which determines our perturbation as described below. Let $\chi : [0, \infty) \rightarrow [0, \infty)$ be a function such that

$$\lim_{x \rightarrow \infty} \chi(x) = 0. \quad (1)$$

As we shall see below, the property (1) means that our perturbation is asymptotically small.

Here, we are interested in the one-dimensional RWRE on the nonnegative integers (we use the notation $\mathbb{Z}^+ := \{0, 1, 2, \dots\}$), with reflection at the origin. One can readily obtain results for the one-dimensional RWRE on the whole of $\mathbb{Z}$ in a similar manner. Formally, we define our RWRE as follows.

We define sequences of random variables ξ_i , $i = 1, 2, \dots$, and Y_i , $i = 1, 2, \dots$, on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the following properties.

Fix ε such that $0 < \varepsilon < 1/2$. Let ξ_i , $i = 1, 2, \dots$, be a sequence of i.i.d. random variables such that

$$\mathbb{P}[\varepsilon \leq \xi_1 \leq 1 - \varepsilon] = 1. \quad (2)$$

The condition (2) is sometimes referred to as *uniform ellipticity*.

Let Y_i , $i = 1, 2, \dots$, be another sequence of i.i.d. random variables, taking values in $[-1, 1]$, on the same probability space as the ξ_i . We allow Y_i to depend on ξ_i , but any collections $(Y_{i_1}, \dots, Y_{i_k})$, $(\xi_{j_1}, \dots, \xi_{j_{k'}})$ are independent if $\{i_1, \dots, i_k\} \cap \{j_1, \dots, j_{k'}\} = \emptyset$.

For a particular realization of the sequences $(\xi_i; i = 1, 2, \dots)$ and $(Y_i; i = 1, 2, \dots)$, we define the quantities p_n and q_n , $n = 1, 2, 3, \dots$, as follows:

$$p_n := \begin{cases} \xi_n + Y_n \chi(n) & \text{if } \varepsilon/2 \leq \xi_n + Y_n \chi(n) \leq 1 - \varepsilon/2, \\ \varepsilon/2 & \text{if } \xi_n + Y_n \chi(n) < \varepsilon/2, \\ 1 - \varepsilon/2 & \text{if } \xi_n + Y_n \chi(n) > 1 - \varepsilon/2, \end{cases} \quad (3)$$

$$q_n := 1 - p_n.$$

We call a particular realization of (p_n, q_n) , $n = 1, 2, \dots$, our *environment*, and we denote it by ω . A given ω is then a realization of our random environment, and is given in terms of the ξ_i and Y_i as in (3).

For a given environment ω , that is, a realization of (p_n, q_n) , $n = 1, 2, \dots$, we consider the Markov chain $(\eta_t(\omega); t \in \mathbb{Z}^+)$ on $\mathbb{Z}^+$, starting at some point in $\mathbb{Z}^+$, defined as follows: $\eta_0(\omega) = r$ for some $r \in \mathbb{Z}^+$, and for $n = 1, 2, \dots$,

$$\begin{aligned} P[\eta_{t+1}(\omega) = n - 1 \mid \eta_t(\omega) = n] &= p_n, \\ P[\eta_{t+1}(\omega) = n + 1 \mid \eta_t(\omega) = n] &= q_n, \end{aligned} \quad (4)$$

and $P[\eta_{t+1}(\omega) = 1 \mid \eta_t(\omega) = 0] = 1/2$, $P[\eta_{t+1}(\omega) = 0 \mid \eta_t(\omega) = 0] = 1/2$. (Here P is the so-called *quenched* probability measure, i.e. for a fixed environment ω .) The given form for the reflection at the origin ensures that the Markov chain is *aperiodic*, which eases some technical complications, but this choice is not special; it can be changed without affecting our results.

Recall that, from (1), $\chi(n) \rightarrow 0$ as $n \rightarrow \infty$. Thus, there exists $n_0 \in (0, \infty)$ such that $\chi(n) < \varepsilon/2$ for all $n \geq n_0$. Hence, under condition (2), for $\mathbb{P}$ -almost every ω we have $\varepsilon/2 < \xi_n + Y_n \chi(n) < 1 - \varepsilon/2$ (since the Y_n are bounded). (For the remainder of the paper we often use ‘a.e. ω ’ as shorthand for ‘ $\mathbb{P}$ -almost every ω ’ when the context is clear.) Thus, for all $n \geq n_0$, (3) implies that, for a.e. ω ,

$$p_n = \xi_n + Y_n \chi(n), \quad q_n = 1 - \xi_n - \chi(n) Y_n, \quad n \geq n_0. \quad (5)$$

Note that our conditions on the variables in (3) ensure that $\varepsilon/2 \leq p_n \leq 1 - \varepsilon/2$ almost surely for all n , so that for a.e. ω , p_n and q_n are true probabilities bounded strictly away from 0 and from 1.

For $n = 1, 2, \dots$, we set

$$\zeta_n := \log \left(\frac{\xi_n}{1 - \xi_n} \right). \quad (6)$$

Write $\mathbb{E}$ for expectation under $\mathbb{P}$.

In our model, by (1), $\chi(n) \rightarrow 0$ as $n \rightarrow \infty$. Thus, from (5), in the limit $n \rightarrow \infty$, we approach the well-known random walk in i.i.d. random environment as studied in [12],

[18] and subsequently. In addition, when $\mathbb{E}[\zeta_1] = 0$, in the limit as $n \rightarrow \infty$ we approach the critical case often referred to as *Sinai's regime* after [17]. Our results show that despite this, the behaviour of our model is, in general, very different from the behaviour of these limiting cases, depending on the nature of the perturbation χ .

In work in preparation, we study the long-run limiting behaviour (as $t \rightarrow \infty$) of our random walk $\eta_t(\omega)$ in terms of its distance from the origin. Of interest are both the almost sure and 'in probability' (see, for example, [17, 4]) behaviour. In Sinai's regime for the RWRE on $\mathbb{Z}^+$, Comets, Menshikov and Popov ([3, Theorem 3.2]) show that, for a.e. ω and any $\varepsilon > 0$,

$$\frac{\eta_t(\omega)}{(\log t)^2} < (\log \log t)^{2+\varepsilon} \quad \text{a.s.}$$

for all but finitely many t (where a.s. stands for P -almost surely). This result (for the RWRE on $\mathbb{Z}$) dates back to Deheuvels and Revész [5]. An exact upper limit result is given in [8]. In work in preparation, we study analogous almost sure results (in both null-recurrent and transient cases) for our perturbed RWRE. For example, in the $\mathbb{P}$ -almost sure transient case of the RWRE perturbed from Sinai's regime (that is, with $\chi(n) = n^{-\alpha}$ for some fixed $0 < \alpha < 1/2$, we have $\mathbb{E}[\zeta_1] = 0$, $\text{Var}[\zeta_1] > 0$ and $\lambda < 0$, where λ is defined at (10)), we find that for a.e. ω and any $\varepsilon > 0$, as $t \rightarrow \infty$,

$$(\log \log \log t)^{-1/\alpha-\varepsilon} < \frac{\eta_t(\omega)}{(\log t)^{1/\alpha}} < (\log \log t)^{2/\alpha+\varepsilon} \quad \text{a.s.}$$

for all but finitely many t . Thus in this case, we see that the random walk, for almost every environment, is contained in a window about $(\log t)^{1/\alpha}$. This aspect of the problem requires additional techniques, however, and we do not discuss this further in the present paper.

In the next section we state our results. Theorems 1–3 are special cases of the model in which some of the random variables ξ_i and Y_i are degenerate (that is, equal to a constant almost surely). In particular, Theorems 1 and 2 include some known results, when our model reduces to previously studied systems. In Theorem 4, the underlying environment is not in the 'critical regime'. Our main results, Theorems 6 and 7, deal with the main case of interest, in which the underlying environment is truly random and is, in a sense to be demonstrated, critical.

2. Main results

Most of our results will be formulated for almost all environments ω (in some sense, for all 'typical' environments), that is, $\mathbb{P}$ -almost surely over $(\Omega, \mathcal{F}, \mathbb{P})$.

If $Y_1 = 0$ $\mathbb{P}$ -a.s., then our model reduces to the standard reflected one-dimensional random walk in an i.i.d. random environment. In this case $p_n = \xi_n$ and $q_n = 1 - \xi_n$, $n = 1, 2, \dots$, and so (with the definition at (6)) $\zeta_n = \log(p_n/q_n)$. Criteria for recurrence of the RWRE $\eta_t(\omega)$ in this case were given by Solomon [18], for the case where $(\xi_i; i = 1, 2, \dots)$ is an i.i.d. sequence, and generalised by Alili [1]. For the case in which larger jumps are permitted, see, for example, [11].

The following well-known result dates back to Solomon [18].

Theorem 1. Let $(\eta_t(\omega); t \in \mathbb{Z}^+)$ be a random walk in i.i.d. random environment, with $\mathbb{P}[Y_1 = 0] = 1$. Suppose $\text{Var}[\zeta_1] > 0$.

- (i) If $\mathbb{E}[\zeta_1] < 0$, then $\eta_t(\omega)$ is transient for a.e. ω .
- (ii) If $\mathbb{E}[\zeta_1] = 0$, then $\eta_t(\omega)$ is null-recurrent for a.e. ω .
- (iii) If $\mathbb{E}[\zeta_1] > 0$, then $\eta_t(\omega)$ is ergodic for a.e. ω .

The critical (null-recurrent) regime $\mathbb{E}[\log(p_1/q_1)] = 0$ is known as *Sinai's regime*, after [17]. This regime has been extensively studied; see, for example, [4, 8, 9, 10]. For an outline proof of Theorem 1 using Lyapunov function methods, similar to those employed in this paper, see Theorem 3.1 of [3]. In this paper we extend the classification criteria of Theorem 1 to encompass the case in which the p_n are not i.i.d. and in which $\mathbb{E}[\log(p_n/q_n)]$ is asymptotically zero as $n \rightarrow \infty$. Our results are, in some sense, a random environment analogue of those for Markov processes with asymptotically zero mean drift given in [15] (see below).

For the remainder of the paper we suppose $\mathbb{P}[Y_1 = 0] < 1$. This includes the interesting case where $Y_1 = b$ $\mathbb{P}$ -a.s., for some $b \in [-1, 1] \setminus \{0\}$. Our techniques do, however, enable us to allow Y_1 to be random.

Although not as famous as the RWRE, another system that has been well studied is the rather classical problem of a Markov chain with asymptotically zero drift. This problem was studied by Lamperti [13, 14]. General criteria for recurrence, transience and ergodicity were given by Menshikov, Asymont, and Iasnogorodskii in [15]. Theorem 2 below is a consequence of their main result, Theorem 3, applied to our problem when $\text{Var}[\zeta_1] = 0$ and $\text{Var}[Y_1] = 0$; that is, the distributions of ξ_1 and Y_1 are both degenerate (i.e. equal to a constant almost surely). In particular, we have a *nonrandom* environment ω . If, on the other hand, ξ_1 is degenerate but Y_1 is not, then we have a random (asymptotically small) perturbation on an underlying nonrandom environment, and we have Theorem 3 below.

We use the notation $\log_1 x := \log x$ and $\log_k x := \log(\log_{k-1} x)$ for $k = 2, 3, \dots$.

Theorem 2. Suppose $\mathbb{P}[Y_1 = b] = 1$ for some $b \in [-1, 0) \cup (0, 1]$. Suppose $\mathbb{P}[\xi_1 = c] = 1$ for some $c \in (0, 1)$.

- (i) If $c < 1/2$, then $\eta_t(\omega)$ is transient.
- (ii) If $c > 1/2$, then $\eta_t(\omega)$ is ergodic.
- (iii) Suppose $c = 1/2$. Suppose there exist $s \in \mathbb{Z}^+$ and $K \in \mathbb{N}$ such that, for all $n \in [K, \infty)$ and some $h > 1$, the following inequality holds:

$$b\chi(n) > \frac{1}{4n} + \frac{1}{4n \log n} + \dots + \frac{h}{4n \prod_{i=1}^s \log_i n}. \quad (7)$$

Then $\eta_t(\omega)$ is ergodic.

- (iv) Suppose $c = 1/2$. Suppose there exist $s, t \in \mathbb{Z}^+$ and $K \in \mathbb{N}$ such that, for all $n \in [K, \infty)$ and some $h < 1$, the following inequality holds:

$$\begin{aligned} -\frac{1}{4n} - \frac{1}{4n \log n} - \dots - \frac{h}{4n \prod_{i=1}^s \log_i n} &\leq b\chi(n) \\ &\leq \frac{1}{4n} + \frac{1}{4n \log n} + \dots + \frac{h}{4n \prod_{i=1}^t \log_i n}. \end{aligned} \quad (8)$$

Then $\eta_t(\omega)$ is null-recurrent.

- (v) Suppose $c = 1/2$. Suppose there exist $s \in \mathbb{Z}^+$ and $K \in \mathbb{N}$ such that, for all $n \in [K, \infty)$ and some $h > 1$, the following inequality holds:

$$b\chi(n) \leq -\frac{1}{4n} - \frac{1}{4n \log n} - \cdots - \frac{h}{4n \prod_{i=1}^s \log_i n}. \quad (9)$$

Then $\eta_t(\omega)$ is transient.

Theorem 2 follows directly by applying Theorem 3 of [15] to our case, with $m(x) = -2\chi(x)$ and $b(x) = 1$.

Remark. In the case $c = 1/2$ the critical case in terms of the recurrence, transience and ergodicity is when the perturbation $\chi(n)$ is, ignoring logarithmic terms, of order n^{-1} ; we say that the ‘critical exponent’ is -1 . This contrasts with our results in the case where $\text{Var}[\xi_1] > 0$ (see Theorems 6 and 7), in which the critical exponent is $-1/2$.

The following result deals with the case in which the distribution of ξ_1 is degenerate, but that of Y_1 is not; in this case we have a homogeneous nonrandom environment subject to an asymptotically small random perturbation. In particular, parts (iii) and (iv) of the theorem deal with the case when the underlying environment is that of the simple random walk. Here, $\stackrel{\mathcal{D}}{=}$ stands for equality in distribution.

Theorem 3. Suppose $\mathbb{P}[\xi_1 = c] = 1$ for some $c \in (0, 1)$, and $\text{Var}[Y_1] > 0$.

- (i) If $c < 1/2$, then $\eta_t(\omega)$ is transient for a.e. ω .
- (ii) If $c > 1/2$, then $\eta_t(\omega)$ is ergodic for a.e. ω .
- (iii) If $c = 1/2$ and $Y_1 \stackrel{\mathcal{D}}{=} -Y_1$, then $\eta_t(\omega)$ is null-recurrent for a.e. ω .
- (iv) Suppose $c = 1/2$ and $\mathbb{E}[Y_1] \neq 0$. Suppose $\chi(n) = an^{-\beta}$ for $a > 0$ and $\beta > 0$.
 - (a) If $0 < \beta < 1$ and $\mathbb{E}[Y_1] > 0$, then $\eta_t(\omega)$ is ergodic for a.e. ω .
 - (b) If $\beta > 1$, then $\eta_t(\omega)$ is null-recurrent for a.e. ω .
 - (c) If $0 < \beta < 1$ and $\mathbb{E}[Y_1] < 0$, then $\eta_t(\omega)$ is transient for a.e. ω .

We prove Theorem 3 along with our main results in Section 3.

Remarks. Note that in part (iii), $Y_1 \stackrel{\mathcal{D}}{=} -Y_1$ implies that all odd moments of Y_1 are zero. By modifications to the proof of Theorem 3 one can obtain a more refined result, specifically that with $p := \min\{j \in \{1, 3, 5, \dots\} : \mathbb{E}[Y_1^j] \neq 0\}$, for $p > 1$ we have a statement analogous to part (iv) but with $\mathbb{E}[Y_1]$ replaced by $\mathbb{E}[Y_1^p]$ and with the critical value of β being $1/(2(p-1))$ for $p > 1$, rather than 1. We do not go into further details here.

Theorem 3(iv) demonstrates that in the case of a randomly perturbed simple random walk, the critical exponent for the perturbation is -1 , as in the case of the nonrandom perturbation (Theorem 2). It may be possible to refine Theorem 3(iv) to obtain more delicate results analogous to those of Theorem 2.

For the remainder of the paper, we ensure that the underlying environment is *random*, by supposing $\text{Var}[\zeta_1] > 0$. First we consider the case $\mathbb{E}[\zeta_1] \neq 0$. Here we have Theorem 4 below. In this situation, the perturbation introduced by $\chi(n)Y_n$ does not affect the criteria given in (i) and (iii) of Theorem 1.

Theorem 4. Suppose $\text{Var}[\zeta_1] > 0$, $\mathbb{E}[\zeta_1] \neq 0$, and $\mathbb{P}[Y_1 = 0] < 1$.

- (i) If $\mathbb{E}[\zeta_1] < 0$, then $\eta_t(\omega)$ is transient for a.e. ω .
- (ii) If $\mathbb{E}[\zeta_1] > 0$, then $\eta_t(\omega)$ is ergodic for a.e. ω .

The proof of the theorem uses the same methods as employed in the proof of Theorem 3.1 of [3] or later in this paper, but is essentially simpler than for our main results. We can construct a ‘martingale’ (as at (40) below) which is easily shown (by the Law of the Iterated Logarithm, Lemma 3 below) to be bounded or tend to infinity for a.e. ω . Similarly for the stationary measure. The theorem then follows by our Lyapunov function criteria (Lemmas 1 and 2 below). We follow this method in detail, in less straightforward cases, later in the paper, and so do not repeat the argument here.

For the remainder of the paper we consider the more interesting case where $\mathbb{E}[\zeta_1] = 0$, so that we have a random walk in a random environment that is asymptotic to Sinai’s regime. We prove general results about this RWRE with asymptotically zero perturbation that are analogous to Theorem 2, but significantly different.

If $\mathbb{P}[Y_1 = 0] < 1$ (and permitting the case that $\mathbb{P}[Y_1 = c] = 1$ for some c with $0 < |c| \leq 1$) we define

$$\lambda := \mathbb{E}\left[\frac{Y_1}{\xi_1(1 - \xi_1)}\right]. \quad (10)$$

Also, we use the notation

$$\sigma^2 := \text{Var}[\zeta_1]. \quad (11)$$

Note that, under the condition (2), we have $\sigma^2 < \infty$ and, since Y_1 is bounded, $|\lambda| < \infty$. We also draw attention to the fact that, given (2), $\mathbb{P}$ -a.s.,

$$-\frac{1}{\varepsilon^2} \leq \frac{Y_1}{\xi_1(1 - \xi_1)} \leq \frac{1}{\varepsilon^2}, \quad (12)$$

a fact that we shall use later. For what follows, of separate interest are the two cases $\lambda = 0$ and $\lambda \neq 0$. We concentrate on the latter case for most of the results that follow (but see the remark after Theorem 7). However, our first result deals with the case in which $Y_1/\xi_1 \stackrel{\mathcal{D}}{=} -Y_1/(1 - \xi_1)$. This implies $\lambda = 0$ (see (10)), but is a rather special case; Theorem 5 demonstrates that in this case the detailed behaviour of χ is not important: as long as $\chi(n) \rightarrow 0$ as $n \rightarrow \infty$, $\eta_t(\omega)$ is null-recurrent for a.e. ω .

Theorem 5. With σ as defined at (11), suppose that $Y_1/\xi_1 \stackrel{\mathcal{D}}{=} -Y_1/(1 - \xi_1)$, $\mathbb{P}[Y_1 = 0] < 1$, $\mathbb{E}[\zeta_1] = 0$, and $\sigma^2 > 0$. Then $\eta_t(\omega)$ is null-recurrent for a.e. ω .

An example of (Y_1, ξ_1) for which Theorem 5 holds is when Y_1 and ξ_1 are independent uniform random variables on $(-1, 1)$ and $(\varepsilon, 1 - \varepsilon)$ respectively.

Our remaining results deal with the case $\lambda \neq 0$ (but see also the remark after Theorem 7). In our next result (Theorem 6), we give some rather specific conditions on the asymptotic behaviour of the function χ . Theorem 6 is a special case of our general result, Theorem 7.

Theorem 6. *With λ and σ defined at (10) and (11) respectively, suppose that $\lambda \neq 0$, $\mathbb{P}[Y_1 = 0] < 1$, $\mathbb{E}[\xi_1] = 0$, and $\sigma^2 > 0$. Let $c_{\text{crit}} := \sigma 2^{-1/2}$.*

- (i) *If there are constants $c > c_{\text{crit}}$ and $n_0 \in \mathbb{Z}^+$ such that $\lambda \chi(n) \geq cn^{-1/2}(\log \log n)^{1/2}$ for all $n \geq n_0$, then $\eta_t(\omega)$ is ergodic for a.e. ω .*
- (ii) *If there are constants $c \leq c_{\text{crit}}$ and $n_0 \in \mathbb{Z}^+$ such that $|\lambda| \chi(n) \leq cn^{-1/2}(\log \log n)^{1/2}$ for all $n \geq n_0$, then $\eta_t(\omega)$ is null-recurrent for a.e. ω .*
- (iii) *If there are constants $c > c_{\text{crit}}$ and $n_0 \in \mathbb{Z}^+$ such that $\lambda \chi(n) \leq -cn^{-1/2}(\log \log n)^{1/2}$ for all $n \geq n_0$, then $\eta_t(\omega)$ is transient for a.e. ω .*

Remark. Theorem 6 shows that in our case the critical exponent for the perturbation is $-1/2$. This contrasts with the deterministic environment case (as in Theorem 2, and see [15, Theorem 3]), in which the critical exponent is -1 . When the perturbation is smaller than this critical size (as in part (ii)), it is insufficient to change the recurrence/transience characteristics of the Markov chain from those of Sinai's regime. If the perturbation is greater than the critical size, it changes the behaviour of the Markov chain from that of Sinai's regime, making it either transient or ergodic depending on the sign of the perturbation. This feature is present in our most general result, Theorem 7.

Theorem 6 will follow as a corollary to Theorem 7 below. Theorem 7 is more refined than Theorem 6. In order to formulate our deeper result, we need more precise conditions on the behaviour of the perturbation function $\chi(n)$. To achieve this, we define the notions of k -supercritical and k -subcritical below. First, we need some additional notation.

Recall the notation $\log_1(x) := \log(x)$, $\log_k(x) := \log(\log_{k-1}(x))$ for $k = 2, 3, \dots$. Let n_k denote the smallest positive integer such that $\log_{k+1}(n_k) \geq 0$. Let $a_k := 2$ for $k \in \mathbb{N} \setminus \{3\}$ and $a_3 := 3$. For each $k \in \mathbb{N}$ we define the $[0, \infty)$ -valued function φ_k as follows (we use the given form for the φ_k due to the appearance of the Law of the Iterated Logarithm later on). For $x \in [e, \infty)$ and $d \in \mathbb{R}$, let

$$\varphi_1(x; d) := ((2 + d) \log_2 x)^{1/2},$$

and for $k = 2, 3, \dots$, with $x \in [n_k, \infty)$ and $d \in \mathbb{R}$, let

$$\varphi_k(x; d) := \left(\sum_{i=1}^{k-1} a_{i+1} \log_{i+1} x + (a_{k+1} + d) \log_{k+1} x \right)^{1/2}. \quad (13)$$

We shall see that the behaviour of the Markov chain $\eta_t(\omega)$ is determined by the driving function χ . By applying the Law of the Iterated Logarithm, we shall see that the critical form of χ is related to an iterated logarithm expression of the form of φ_k .

In order to formulate our main result we make the following definitions of k -supercritical and k -subcritical.

Definition 1. Recall the definitions of λ and σ at (10) and (11) respectively. Suppose $\lambda \neq 0$. For $k \in \mathbb{N}$, we say χ is k -supercritical if there exist constants $c \in (0, \infty)$ and $n_0 \in \mathbb{Z}^+$ such that, for all $n \geq n_0$,

$$\chi(n) \geq \frac{\sigma}{2|\lambda|} n^{-1/2} \varphi_k(n; c). \quad (14)$$

For $k \in \mathbb{N}$, we say χ is k -subcritical if there exist constants $c \in (0, \infty)$ and $n_0 \in \mathbb{Z}^+$ such that, for all $n \geq n_0$,

$$\chi(n) \leq \frac{\sigma}{2|\lambda|} n^{-1/2} \varphi_k(n; -c). \quad (15)$$

Remarks. Implicit in χ being k -subcritical or k -supercritical is the constant c , a fact that we make repeated use of in the proofs in Section 3. Whenever we consider a k -subcritical or k -supercritical function in what follows, we understand this to imply the existence of such a c , and often refer to the constant c in this context.

Also, observe that if for some $k \in \mathbb{N}$, χ is k -supercritical, with implicit constant $c \in (0, \infty)$, then for any $c' \in (0, c)$ the estimate (14) implies

$$\chi(n) \geq \frac{\sigma}{2|\lambda|} n^{-1/2} \varphi_k(n; c) \geq \frac{\sigma}{2|\lambda|} n^{-1/2} \varphi_k(n; c').$$

Similarly if for some $k \in \mathbb{N}$, χ is k -subcritical, with implicit constant $c \in (0, \infty)$, then for any $c' \in (0, c)$ the estimate (15) implies

$$\chi(n) \leq \frac{\sigma}{2|\lambda|} n^{-1/2} \varphi_k(n; -c) \leq \frac{\sigma}{2|\lambda|} n^{-1/2} \varphi_k(n; -c').$$

Finally, we note that Definition 1 excludes functions that oscillate significantly about the critical region $n^{-1/2}$.

Our most general result is as follows.

Theorem 7. With λ and σ defined at (10) and (11) respectively, suppose that $\lambda \neq 0$, $\mathbb{P}[Y_1 = 0] < 1$, $\mathbb{E}[\xi_1] = 0$ and $\sigma^2 > 0$.

- (i) If, for some $k \in \mathbb{N}$, χ is k -supercritical and $\lambda > 0$, then $\eta_t(\omega)$ is ergodic for a.e. ω .
- (ii) If, for some $k \in \mathbb{N}$, χ is k -subcritical, then $\eta_t(\omega)$ is null-recurrent for a.e. ω .
- (iii) If, for some $k \in \mathbb{N}$, χ is k -supercritical and $\lambda < 0$, then $\eta_t(\omega)$ is transient for a.e. ω .

Remark. In the general case $\lambda = 0$, it turns out that higher moments contribute, and we obtain a slightly more general form of Theorem 7. It is straightforward to modify the proof of Theorem 7 to obtain such a result. Specifically, if for $r \in \mathbb{N}$ we set

$$\lambda_r := \frac{1}{r} \mathbb{E} \left[Y_1^r \left(\frac{1}{(1 - \xi_1)^r} + \frac{(-1)^{r+1}}{\xi_1^r} \right) \right],$$

and $p := \min\{j \in \mathbb{N} : \lambda_j \neq 0\}$, then for $p > 1$ a statement of the form of Theorem 7 holds but with λ replaced by λ_p and the conditions on χ being replaced by conditions on χ^p . We do not pursue the details here.

We will prove Theorem 7 in the next section. The idea behind the proof of the recurrence and transience conditions is to construct a function f of the process $\eta_t(\omega)$ such that $f(\eta_t(\omega))$ is a ‘martingale’ everywhere except in a finite region, and determine the cases in which this function is finite or infinite. The proof of ergodicity relies on the construction of a stationary measure and determining its properties.

3. Proofs of main results

Before embarking upon the proof of Theorem 7, we need some preliminary results. First we present the criteria for classification of countable Markov chains which we will require.

Let $(W_t; t \in \mathbb{Z}^+)$ be a discrete, irreducible, aperiodic, time-homogeneous Markov chain on $\mathbb{Z}^+$. We have the following classification criteria, which are consequences of those given in Chapter 2 of [6]. The following result, which we state without proof, is a consequence of Theorem 2.2.2 of [6], and is slightly more general than Proposition 2.1 of [3].

Lemma 1. *Suppose there exist a function $f : \mathbb{Z}^+ \rightarrow [0, \infty)$ which is uniformly bounded and nonconstant, and a set $A \subset \mathbb{Z}^+$ such that*

$$E[f(W_{t+1}) - f(W_t) | W_t = x] = 0 \quad (16)$$

for all $x \in \mathbb{Z}^+ \setminus A$, and

$$f(x) > \sup_{y \in A} f(y) \quad (17)$$

for at least one $x \in \mathbb{Z}^+ \setminus A$. Then the Markov chain (W_t) is transient.

The following result is contained of Theorem 2.2.1 in [6].

Lemma 2. *Suppose that there exist a function $f : \mathbb{Z}^+ \rightarrow [0, \infty)$ and a finite set $A \subset \mathbb{Z}^+$ such that*

$$E[f(W_{t+1}) - f(W_t) | W_t = x] \leq 0 \quad (18)$$

for all $x \in \mathbb{Z}^+ \setminus A$, and $f(x) \rightarrow \infty$ as $x \rightarrow \infty$. Then the Markov chain (W_t) is recurrent.

We will need Feller’s refined form for the Law of the Iterated Logarithm [7]. The following result is a consequence of Theorem 7 of [7].

Lemma 3. *Let $X_i, i = 1, 2, \dots$, be a sequence of independent random variables with $E[X_i] = 0$ for all i , and $E[X_i^2] = \sigma_i^2 < \infty$ for $i = 1, 2, \dots$. Suppose the X_i are bounded, that is, $P[|X_i| > C] = 0$ for all i and some $0 < C < \infty$. Let*

$$s_n^2 := \sum_{i=1}^n \sigma_i^2. \quad (19)$$

Suppose that $s_n \rightarrow \infty$ as $n \rightarrow \infty$. Let $S_n := \sum_{i=1}^n X_i$. For some $k \in \mathbb{N}$ and $\varepsilon \in (-\infty, \infty)$, define $\varphi_k(n; \varepsilon)$ as at (13). Then

$$P[S_n > s_n \varphi_k(s_n^2; \varepsilon) \text{ i.o.}] = \begin{cases} 1 & \text{if } \varepsilon < 0, \\ 0 & \text{if } \varepsilon > 0. \end{cases} \quad (20)$$

In particular, if the X_i are i.i.d. and bounded random variables with $E[X_1^2] = \sigma^2$, we have

$$P[S_n > \sigma n^{1/2} \varphi_k(n; \varepsilon) \text{ i.o.}] = \begin{cases} 1 & \text{if } \varepsilon < 0, \\ 0 & \text{if } \varepsilon > 0. \end{cases} \quad (21)$$

We will also need the following result. Recall the definition of $\varphi_k(i; d)$ at (13).

Lemma 4. For $k \in \mathbb{N}$, let n_k be the smallest positive integer such that $\log_{k+1} n_k \geq 0$. For any $d \in \mathbb{R}$, we have

$$\sum_{i=n_k}^n i^{-1/2} \varphi_k(i; d) = 2n^{1/2} \varphi_k(n; d) + \alpha_n, \quad (22)$$

where $|\alpha_n| < 6n^{1/2}$ for all n sufficiently large.

Proof. We have, for $k \in \mathbb{N}$,

$$\frac{d}{dx} (x^{1/2} \varphi_k(x; d)) = \frac{1}{2} x^{-1/2} \varphi_k(x; d) + x^{1/2} \varphi'_k(x; d),$$

where

$$\varphi'_k(x; d) = \frac{1}{2} (\varphi_k(x; d))^{-1} \cdot \left(\frac{2}{x \log x} + \frac{3}{x \log x \log \log x} + \cdots \right) < \frac{1}{x}$$

for x sufficiently large. Thus, for any $k \in \mathbb{N}$,

$$\begin{aligned} \int_{n_k}^n x^{-1/2} \varphi_k(x; d) dx &= 2[x^{1/2} \varphi_k(x; d)]_{n_k}^n - 2 \int_{n_k}^n x^{1/2} \varphi'_k(x; d) dx \\ &= 2n^{1/2} \varphi_k(n; d) + b_n, \end{aligned} \quad (23)$$

where

$$|b_n| \leq 2 \int_{n_k}^n x^{1/2} \varphi'_k(x; d) dx + 2n_k^{1/2} \varphi_k(n_k; d) \leq C_k + 2 \int_0^n x^{-1/2} dx$$

for some $0 < C_k < \infty$ which depends on k (and d). Thus, for each k , $|b_n| \leq 5n^{1/2}$ for all n sufficiently large. Since $x^{-1/2} \varphi_k(x; d)$ is a decreasing function for all x sufficiently large (depending on k but not d), there exist finite positive constants C'_k and C''_k such that

$$\sum_{i=n_k}^n i^{-1/2} \varphi_k(i; d) + C'_k \geq \int_{n_k}^n x^{-1/2} \varphi_k(x; d) dx \geq \sum_{i=n_k+1}^n i^{-1/2} \varphi_k(i; d) - C''_k.$$

So we have

$$0 \leq \sum_{i=n_k}^n i^{-1/2} \varphi_k(i; d) - \int_{n_k}^n x^{-1/2} \varphi_k(x; d) dx \leq n_k^{-1/2} \varphi_k(n_k; d) + C \quad (24)$$

for some $0 < C < \infty$ that does not depend on n . Then from (24) and (23) we obtain (22). $\square$

For a given realization ω of our random environment, with p_i and q_i , $i = 1, 2, \dots$, defined by (3), let

$$D(\omega) := \sum_{i=1}^{\infty} \frac{1}{q_i} \prod_{j=1}^i \frac{q_j}{p_j} = \frac{1}{p_1} + \frac{q_1}{p_1 p_2} + \frac{q_1 q_2}{p_1 p_2 p_3} + \dots \quad (25)$$

Lemma 5. *If, for a given environment ω , the quantity $D(\omega)$ as defined at (25) is finite, then the Markov chain $\eta_t(\omega)$ is ergodic. On the other hand, if $D(\omega) = \infty$, then the Markov chain $\eta_t(\omega)$ for this ω is not ergodic.*

Proof. For fixed environment ω , i.e., given a configuration of $(p_i; i = 1, 2, \dots)$, $\eta_t(\omega)$ is a reversible Markov chain. For this Markov chain one has the stationary measure $\mu = (\mu_0, \mu_1, \dots)$, where

$$\mu_0 = 2, \quad \mu_1 = \frac{1}{p_1}, \quad \text{and} \quad \mu_n = \frac{1}{p_1} \prod_{i=1}^{n-1} \frac{q_i}{p_{i+1}}, \quad n \geq 2.$$

Then, with the definition of $D(\omega)$ at (25), we have

$$\sum_{i=0}^{\infty} \mu_i = 2 + D(\omega).$$

Thus, if, for this ω , $D(\omega)$ is finite, then the Markov chain $\eta_t(\omega)$ is ergodic, since we can obtain a stationary distribution. On the other hand, if $D(\omega) = \infty$ for this ω , the Markov chain $\eta_t(\omega)$ is not ergodic. $\square$

Our next result, Lemma 6, uses the Law of the Iterated Logarithm to analyse the behaviour of sums of i.i.d. random variables weighted by the function χ .

Lemma 6. *Let Z_i , $i = 1, 2, \dots$, be a sequence of i.i.d. random variables which are bounded (so that $P[|Z_1| > B] = 0$ for some $0 < B < \infty$), such that $E[Z_1] \geq 0$. Let $\chi : [0, \infty) \rightarrow [0, \infty)$ be such that (1) holds. With λ defined at (10), suppose $\lambda \neq 0$.*

(a) *Suppose $E[Z_1] > 0$. Suppose that, for some $k \in \mathbb{N}$, χ is k -subcritical as defined at (15). Then with probability one, for any $\varepsilon > 0$, for all but finitely many n ,*

$$-n^\varepsilon \leq \sum_{i=1}^n Z_i \chi(i) \leq \frac{\sigma E[Z_1]}{|\lambda|} n^{1/2} \varphi_k(n; -c/3). \quad (26)$$

- (b) Suppose $E[Z_1] > 0$. Suppose that, for some $k \in \mathbb{N}$, χ is k -supercritical as defined at (14). Then with probability one, for all but finitely many n ,

$$\sum_{i=1}^n Z_i \chi(i) \geq \frac{\sigma E[Z_1]}{|\lambda|} n^{1/2} \varphi_k(n; c/3). \quad (27)$$

- (c) Suppose $E[Z_1] = 0$. Then for any $\varepsilon > 0$, with probability one, for all but finitely many n ,

$$\sum_{i=1}^n Z_i \chi(i) \leq \varepsilon (n \log \log n)^{1/2}. \quad (28)$$

Remark. When we come to apply Lemma 6 later in the proofs of the theorems, the configuration $(Z_i, i \geq 1)$ that we will use will be specified by the realization of the random environment ω , so that the qualifier ‘with probability one’ in the lemma translates as ‘for a.e. ω ’.

Proof of Lemma 6. Recall the definitions of λ and σ at (10) and (11) respectively. Suppose $\lambda \neq 0$. For the proofs of parts (a) and (b), suppose that $E[Z_1] > 0$. First we prove (a). Suppose that for some $k \in \mathbb{N}$, χ is k -subcritical. Write

$$S_n := \sum_{i=1}^n (Z_i - E[Z_i]) \chi(i). \quad (29)$$

Then

$$\text{Var}[S_n] = \text{Var}[Z_1] \sum_{i=1}^n (\chi(i))^2. \quad (30)$$

Suppose that $\text{Var}[S_n] \rightarrow \infty$ as $n \rightarrow \infty$. Then, by Lemma 3, taking $X_i = Z_i - E[Z_i]$, we see that with probability one the configuration of $(Z_i, i \geq 1)$ is such that

$$S_n > (\text{Var}[S_n])^{1/2} (3 \log \log (\text{Var}[S_n]))^{1/2}$$

for only finitely many n . (The constant 3 appears for the sake of simplicity, any constant strictly greater than 2 will suffice). That is, with probability one, for all but finitely many n ,

$$S_n \leq (\text{Var}[S_n])^{1/2} (3 \log \log (\text{Var}[S_n]))^{1/2} \leq (\text{Var}[S_n])^{1/2} (3 \log \log n)^{1/2},$$

the second inequality following from (30) and (15). Thus, using (30) and (15) once more, we deduce that with probability one, for any $\varepsilon > 0$ and all but finitely many n , $S_n \leq n^\varepsilon$. Thus, with probability one, for all but finitely many n , since $E[Z_1] > 0$ and χ is a nonnegative function,

$$-n^\varepsilon \leq \sum_{i=1}^n Z_i \chi(i) \leq n^\varepsilon + E[Z_1] \sum_{i=1}^n \chi(i). \quad (31)$$

The lower bound in (31) establishes the lower bound in (26). We now need to prove the upper bound. By (15), there exist $c \in (0, \infty)$ and $k \in \mathbb{N}$ such that for all n sufficiently large,

$$\sum_{i=1}^n \chi(i) \leq \frac{\sigma}{2|\lambda|} \sum_{i=1}^n i^{-1/2} \varphi_k(i; -c/2). \quad (32)$$

Then from (32) with (22) we obtain, for all n sufficiently large,

$$\sum_{i=1}^n \chi(i) \leq \frac{\sigma}{|\lambda|} n^{1/2} \varphi_k(n; -c/2) + \frac{3\sigma}{|\lambda|} n^{1/2}. \quad (33)$$

Hence from (33) and the upper bound in (31), we infer that, with probability one, for all but finitely many n ,

$$\sum_{i=1}^n Z_i \chi(i) \leq \frac{\sigma E[Z_1]}{|\lambda|} n^{1/2} \varphi_k(n; -c/2) + \frac{3\sigma E[Z_1]}{|\lambda|} n^{1/2} + n^\varepsilon.$$

Then we can absorb the final two terms on the right hand side to give (26), given that $\text{Var}[S_n] \rightarrow \infty$ as $n \rightarrow \infty$. On the other hand, suppose that $\text{Var}[S_n] \leq C$ for all n and some $C < \infty$. Then, by (30), $\sum_{i=1}^n (\chi(i))^2 < C$ for some $0 < C < \infty$. So, by Jensen's inequality, and the boundedness of the Z_i , for all n ,

$$\sum_{i=1}^n Z_i \chi(i) \leq \sqrt{n \sum_{i=1}^n Z_i^2 (\chi(i))^2} \leq n^{1/2} B \sqrt{\sum_{i=1}^n (\chi(i))^2} \leq C n^{1/2} \quad (34)$$

for some $0 < C < \infty$. Hence we obtain (26) in this case also. This proves part (a).

Now we prove (b). Suppose that for some $k \in \mathbb{N}$, χ is k -supercritical. Again, we use the notation of (29). By (14), $\text{Var}[S_n] \rightarrow \infty$ as $n \rightarrow \infty$. Then, by Lemma 3, taking $X_i = -(Z_i - E[Z_i])$, we see that, with probability one,

$$S_n < -(\text{Var}[S_n])^{1/2} (3 \log \log (\text{Var}[S_n]))^{1/2}$$

for only finitely many n . But $\chi(n) \rightarrow 0$ as $n \rightarrow \infty$, so with probability one there exists a sequence $c_1, c_2, \dots$ such that $c_n \rightarrow \infty$ as $n \rightarrow \infty$ and $\text{Var}[S_n] < n/c_n$ for all n . Thus, with probability one,

$$S_n \geq -n^{1/2} c_n^{-1/2} (3 \log \log n)^{1/2} \quad (35)$$

for all but finitely many n . So, with probability one, for all but finitely many n ,

$$\sum_{i=1}^n Z_i \chi(i) \geq E[Z_1] \sum_{i=1}^n \chi(i) - n^{1/2} c_n^{-1/2} (3 \log \log n)^{1/2}. \quad (36)$$

By (14), there exist $c \in (0, \infty)$ and $k \in \mathbb{N}$ such that for n sufficiently large,

$$\sum_{i=1}^n \chi(i) \geq \frac{\sigma}{2|\lambda|} \sum_{i=1}^n i^{-1/2} \varphi_k(i; c/2). \quad (37)$$

Then from (37) with (22) we obtain, for all n sufficiently large,

$$\sum_{i=1}^n \chi(i) \geq \frac{\sigma}{|\lambda|} n^{1/2} \varphi_k(n; c/2) - \frac{3\sigma}{|\lambda|} n^{1/2}. \quad (38)$$

Hence, with probability one, from (36) and (38) we conclude that, for all but finitely many n ,

$$\sum_{i=1}^n Z_i \chi(i) \geq \frac{\sigma E[Z_1]}{|\lambda|} n^{1/2} \varphi_k(n; c/2) - \frac{3\sigma E[Z_1]}{|\lambda|} n^{1/2} - n^{1/2} c_n^{-1/2} (3 \log \log n)^{1/2},$$

which yields (27). Thus we have proved part (b).

Finally, we prove (c). Suppose now that $E[Z_1] = 0$. Again use the notation of (29). First, suppose that $\text{Var}[S_n] \leq C$ for all n and some $0 < C < \infty$. Then (34) holds. On the other hand, suppose that $\text{Var}[S_n] \rightarrow \infty$ as $n \rightarrow \infty$. But, since $\chi(n) \rightarrow 0$ as $n \rightarrow \infty$, we have $\text{Var}[S_n] = o(n)$. Applying Lemma 3 with $X_i = Z_i \chi(i)$ then yields (28). Thus the proof of the lemma is complete. $\square$

Proof of Theorem 7. First we examine the recurrence and transience criteria for $\eta_t(\omega)$. For the recurrent cases, we proceed in the second part of the proof to analyse the stationary measure given in Lemma 5, in order to distinguish between null-recurrence and ergodicity (positive recurrence). We work for a fixed environment ω , that is, a given realization of p_i and q_i for $i = 1, 2, \dots$, as given by (3).

We aim to apply Lemmas 1 and 2, and so we construct a Lyapunov function f , that is, a function $f : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$ such that $f(\eta_t(\omega))$ behaves as a martingale (with respect to the natural filtration) for $\eta_t(\omega) \neq 0$. To do this, we proceed as follows.

For a given environment ω , set $\Delta_1 := 1$ and for $i = 2, 3, \dots$ let

$$\Delta_i := \prod_{j=1}^{i-1} (p_j/q_j) = \exp \sum_{j=1}^{i-1} \log(p_j/q_j), \quad (39)$$

and then set $f(0) := 0$ and for $n = 1, 2, 3, \dots$ let

$$f(n) := \sum_{i=1}^n \Delta_i. \quad (40)$$

Note that $f(n) \geq 0$. Then, for fixed ω , for $t \in \mathbb{Z}^+$ and $n = 1, 2, \dots$,

$$\begin{aligned} E[f(\eta_{t+1}(\omega)) - f(\eta_t(\omega)) | \eta_t(\omega) = n] &= p_n f(n-1) + q_n f(n+1) - f(n) \\ &= q_n \Delta_{n+1} - p_n \Delta_n = 0, \end{aligned}$$

i.e. $f(\eta_t(\omega))$ is a martingale over $1, 2, 3, \dots$.

We need to examine the behaviour of $f(n)$ as $n \rightarrow \infty$, in order to apply Lemmas 1 and 2. Recall from (5) that there exists $n_0 \in \mathbb{N}$ such that for any $j > n_0$ and almost every realization of the random environment ω , $p_j = \xi_j + Y_j \chi(j)$ and $q_j = 1 - \xi_j - Y_j \chi(j)$. Then, for j sufficiently large and a.e. ω ,

$$\log p_j = \log(\xi_j + Y_j \chi(j)) = \log(\xi_j) + \xi_j^{-1} Y_j \chi(j) + O((\chi(j))^2)$$

and

$$\log q_j = \log(1 - \xi_j - Y_j \chi(j)) = \log(1 - \xi_j) - (1 - \xi_j)^{-1} Y_j \chi(j) + O((\chi(j))^2),$$

so that for j sufficiently large and a.e. ω ,

$$\log(p_j/q_j) = \log\left(\frac{\xi_j}{1 - \xi_j}\right) + \frac{Y_j}{\xi_j(1 - \xi_j)} \chi(j) + O((\chi(j))^2). \quad (41)$$

Note that $\mathbb{E}[\log(p_n/q_n)] = O(\chi(n)) \rightarrow 0$ as $n \rightarrow \infty$, so that in this sense we asymptotically approach Sinai's regime.

Recall from (6) that $\zeta_i = \log(\xi_i/(1 - \xi_i))$ for $i = 1, 2, \dots$. From (40), (39) and (41) we have, for n sufficiently large and a.e. ω ,

$$f(n) = \sum_{i=1}^n \exp \sum_{j=1}^{i-1} \left[\zeta_j + \frac{Y_j}{\xi_j(1 - \xi_j)} \chi(j) + O((\chi(j))^2) \right]. \quad (42)$$

Note that for what follows the $O((\chi(j))^2)$ terms in (42) can be ignored, since when $\lambda \neq 0$ (where λ is given by (10)), the other two terms are dominant. Thus we need to examine the behaviour of the two terms $\sum_{i=1}^n \zeta_i$ and $\sum_{i=1}^n \frac{Y_i}{\xi_i(1 - \xi_i)} \chi(i)$. This behaviour depends upon the sign of λ , and the magnitude of the perturbation χ .

First suppose that for some $k \in \mathbb{N}$, χ is k -subcritical (see (15)). In this case, we show that in (42) the term involving the ζ_j is essentially dominant. We can apply Lemma 6 with $Z_i = Y_i \xi_i^{-1} (1 - \xi_i)^{-1}$ (if $\lambda > 0$) or $Z_i = -Y_i \xi_i^{-1} (1 - \xi_i)^{-1}$ (if $\lambda < 0$), and the boundedness property (12), so that (26) implies that, for any $\varepsilon > 0$, for all but finitely many n and a.e. ω ,

$$-n^\varepsilon \leq \text{sign}(\lambda) \sum_{i=1}^n \frac{Y_i}{\xi_i(1 - \xi_i)} \chi(i) \leq \sigma n^{1/2} \varphi_k(n; -c/3), \quad (43)$$

with $c \in (0, \infty)$ as given in (15). Also, by the Law of the Iterated Logarithm (Lemma 3), for a.e. ω , there are infinitely many values of n for which

$$\sum_{i=1}^n \zeta_i \geq \sigma n^{1/2} \varphi_k(n; -c/4). \quad (44)$$

So by (43) and (44), for a.e. ω , there are infinitely many values of n such that, if $\lambda > 0$,

$$\sum_{i=1}^n \zeta_i + \sum_{i=1}^n \frac{Y_i}{\xi_i(1 - \xi_i)} \chi(i) \geq \sigma n^{1/2} \varphi_k(n; -c/4) - n^\varepsilon,$$

and if $\lambda < 0$,

$$\sum_{i=1}^n \zeta_i + \sum_{i=1}^n \frac{Y_i}{\xi_i(1-\xi_i)} \chi(i) \geq \sigma n^{1/2} (\varphi_k(n; -c/4) - \varphi_k(n; -c/3)).$$

Thus, if we choose ε to be small, then for a.e. ω , there are infinitely many values of n such that

$$\sum_{i=1}^n \zeta_i + \sum_{i=1}^n \frac{Y_i}{\xi_i(1-\xi_i)} \chi(i) \geq C n^{1/2} \quad (45)$$

for some C with $0 < C < \infty$. Thus from (45), (39), and (41), there are, for a.e. ω , infinitely many values of n for which $\Delta_n > 1$, and hence as $n \rightarrow \infty$, $f(n) \rightarrow \infty$ for a.e. ω . Thus, by Lemma 2, $\eta_t(\omega)$ is recurrent for a.e. ω .

Now suppose that for some $k \in \mathbb{N}$, χ is k -supercritical (see (14)). In this case, we show that the term in (42) involving $Y_j \xi_j^{-1} (1 - \xi_j)^{-1}$ is essentially dominant, and thus the sign of λ determines the behaviour. This time, from Lemma 6 with $Z_i = Y_i \xi_i^{-1} (1 - \xi_i)^{-1}$ (if $\lambda > 0$) or $Z_i = -Y_i \xi_i^{-1} (1 - \xi_i)^{-1}$ (if $\lambda < 0$), and the boundedness property (12), we find that (27) implies that, for a.e. ω and all but finitely many n ,

$$\text{sign}(\lambda) \sum_{i=1}^n \frac{Y_i}{\xi_i(1-\xi_i)} \chi(i) \geq \sigma n^{1/2} \varphi_k(n; c/3). \quad (46)$$

Also, by the Law of the Iterated Logarithm (Lemma 3), for a.e. ω , there are only finitely many n such that

$$\sum_{i=1}^n \zeta_i \geq \sigma n^{1/2} \varphi_k(n; c/4). \quad (47)$$

If $\lambda < 0$, from (46) and (47) we see that, for a.e. ω , there are only finitely many n such that

$$\sum_{i=1}^n \zeta_i + \sum_{i=1}^n \frac{Y_i}{\xi_i(1-\xi_i)} \chi(i) \geq \sigma n^{1/2} (\varphi_k(n; c/4) - \varphi_k(n; c/3)). \quad (48)$$

So if $\lambda < 0$, from (48), (39), and (41) we infer that for a.e. ω there are only finitely many values of n for which

$$\Delta_n \geq \exp(-C_1 n^{1/2})$$

for some C_1 , not depending on ω , with $0 < C_1 < \infty$. Thus for a.e. ω there exists a constant C_2 (depending on ω) with $0 < C_2 < \infty$ such that

$$f(n) \leq C_2 + \sum_{i=1}^{\infty} \exp(-C_1 i^{1/2}),$$

which is bounded. So in this case, by Lemma 1, $\eta_t(\omega)$ is transient for a.e. ω .

On the other hand, if $\lambda > 0$ then Lemma 3 with (46) implies that for a.e. ω there are infinitely many values of n for which

$$\sum_{i=1}^n \zeta_i + \sum_{i=1}^n \frac{Y_i}{\xi_i(1-\xi_i)} \chi(i) \geq \sigma n^{1/2} (\varphi_k(n; c/3) - \varphi_k(n; c/4)) \geq C_1 n^{1/2} \quad (49)$$

for some C_1 , not depending on ω , with $0 < C_1 < \infty$. So if $\lambda > 0$, by (49), (39), and (41) for a.e. ω there are infinitely many values of n for which

$$\Delta_n \geq \exp(C_1 n^{1/2}).$$

Thus $f(n) \rightarrow \infty$ $\mathbb{P}$ -a.s., and in this case $\eta_t(\omega)$ is recurrent for a.e. ω , by Lemma 2.

We now classify the recurrent cases further into ergodic (positive recurrent) and null-recurrent. To determine ergodicity, we apply Lemma 5. Given ω , and with $D(\omega)$ as defined at (25), we have

$$D(\omega) = \sum_{i=1}^{\infty} \frac{1}{q_i} \exp\left(-\sum_{j=1}^i \log(p_j/q_j)\right) = \sum_{i=1}^{\infty} \frac{1}{\Delta_{i+1} q_i},$$

where Δ_i is as defined at (39). By a similar argument to (41), for n sufficiently large and a.e. ω ,

$$\frac{1}{\Delta_n} = \exp\left(-\sum_{i=1}^{n-1} \zeta_i - \sum_{i=1}^{n-1} \frac{Y_i}{\xi_i(1-\xi_i)} \chi(i) + O\left(\sum_{i=1}^{n-1} (\chi(i))^2\right)\right).$$

We use similar arguments as in the proof of recurrence and transience to analyse $D(\omega)$. First suppose that for some $k \in \mathbb{N}$, χ is k -subcritical. Then, by a similar argument to (45), for a.e. ω there are infinitely many values of i for which

$$-\sum_{i=1}^n \zeta_i - \sum_{i=1}^n \frac{Y_i}{\xi_i(1-\xi_i)} \chi(i) \geq C n^{1/2}$$

for $0 < C < \infty$. Thus for a.e. ω there are infinitely many values of n for which $1/\Delta_{n+1} > 1$ and $1/(\Delta_{n+1} q_n) > 1$. Hence $D(\omega) = \infty$ for a.e. ω . So, for a.e. ω , by Lemma 5, $\eta_t(\omega)$ is not ergodic.

Now suppose that for some $k \in \mathbb{N}$, χ is k -supercritical. If $\lambda > 0$, using similar arguments to those before, we deduce that for a.e. ω there are only finitely many n for which

$$-\sum_{i=1}^n \zeta_i - \sum_{i=1}^n \frac{Y_i}{\xi_i(1-\xi_i)} \chi(i) \geq \sigma n^{1/2} (\varphi_k(n; c/4) - \varphi_k(n; c/3)).$$

So for a.e. ω there are only finitely many values of n for which

$$1/\Delta_n \geq \exp(-C_1 n^{1/2})$$

for some $0 < C_1 < \infty$. Thus for a.e. ω there exists a constant C_2 (depending on ω) with $0 < C_2 < \infty$ such that

$$D(\omega) \leq C_2 + \sum_{i=1}^{\infty} \exp(-C_1 i^{1/2}),$$

which is bounded. So in this case, for a.e. ω , by Lemma 5, $\eta_t(\omega)$ is ergodic.

This completes the proof of Theorem 7. $\square$

Proof of Theorem 6. First we prove parts (i) and (iii). Suppose that, for all n sufficiently large, $\lambda\chi(n) \geq cn^{-1/2}(\log \log n)^{1/2}$ for some $c > c_{\text{crit}}$ where $c_{\text{crit}} = \sigma 2^{-1/2}$. Then we see that χ is k -supercritical for $k = 2, 3, \dots$, since, for example,

$$\begin{aligned} \frac{c}{|\lambda|} n^{-1/2} (\log \log n)^{1/2} &= \frac{c}{c_{\text{crit}}} \frac{\sigma}{2|\lambda|} n^{-1/2} (2 \log \log n)^{1/2} \\ &\geq \frac{\sigma}{2|\lambda|} n^{-1/2} (2 \log \log n + 4 \log \log \log n)^{1/2} \end{aligned}$$

for n sufficiently large and $c > c_{\text{crit}}$. Hence (i) follows from Theorem 7(i). Similarly, (iii) follows from Theorem 7(iii).

For part (ii), suppose that $|\lambda|\chi(n) \leq cn^{-1/2}(\log \log n)^{1/2}$ for all n sufficiently large, with $c \leq c_{\text{crit}}$. Then we see that χ is k -subcritical for $k = 2, 3, \dots$, since, for example,

$$\begin{aligned} \frac{c}{|\lambda|} n^{-1/2} (\log \log n)^{1/2} &\leq \frac{\sigma}{2|\lambda|} n^{-1/2} (2 \log \log n)^{1/2} \\ &\leq \frac{\sigma}{2|\lambda|} n^{-1/2} (2 \log \log n + 2 \log \log \log n)^{1/2} \end{aligned}$$

for n sufficiently large. Then Theorem 7(ii) gives Theorem 6(ii), and the proof of Theorem 6 is complete. $\square$

Proof of Theorem 5. By Lemma 3, for a.e. ω there are infinitely many values of n for which

$$\sum_{i=1}^n \zeta_i \geq \sigma n^{1/2} (\log \log n)^{1/2}. \quad (50)$$

By a similar argument to (41), but keeping track of higher order terms in the Taylor series, we now have

$$\log(p_i/q_i) = \zeta_i + \sum_{r=1}^{\infty} \frac{1}{r} Y_i^r \left(\frac{1}{(1-\xi_i)^r} + \frac{(-1)^{r+1}}{\xi_i^r} \right) (\chi(i))^r. \quad (51)$$

By the condition $Y_1/\xi_1 \stackrel{\mathcal{D}}{=} -Y_1/(1-\xi_1)$, the expectation of the sum on the right of (51) is zero. Hence we can apply Lemma 6(c) with

$$Z_i = \sum_{r=1}^{\infty} \frac{1}{r} Y_i^r \left(\frac{1}{(1-\xi_i)^r} + \frac{(-1)^{r+1}}{\xi_i^r} \right) (\chi(i))^{r-1} \quad (52)$$

to deduce that for all but finitely many n and a.e. ω ,

$$\sum_{i=1}^n \sum_{r=1}^{\infty} \frac{1}{r} Y_i^r \left(\frac{1}{(1-\xi_i)^r} + \frac{(-1)^{r+1}}{\xi_i^r} \right) (\chi(i))^r \geq -\varepsilon n^{1/2} (\log \log n)^{1/2}, \quad (53)$$

and by choosing ε sufficiently small we conclude from (51), (50) and (53) that for a.e. ω there are infinitely many values of n for which

$$\sum_{i=1}^n \log(p_i/q_i) \geq C n^{1/2} (\log \log n)^{1/2}$$

for some $0 < C < \infty$. Thus with Δ_n defined at (39), for a.e. ω there are infinitely many values of n for which

$$\Delta_n \geq \exp(C n^{1/2} (\log \log n)^{1/2}),$$

and hence $f(n) \rightarrow \infty$ $\mathbb{P}$ -a.s., and so, by Lemma 2, $\eta_t(\omega)$ is recurrent for a.e. ω .

To prove null-recurrence, it remains to show that the Markov chain is not ergodic. Consider $D(\omega)$ as defined at (25) again. By Lemma 3, for a.e. ω there are infinitely many values of n for which

$$-\sum_{i=1}^n \zeta_i \geq \sigma n^{1/2} (\log \log n)^{1/2}.$$

From Lemma 6(c) with Z_i as at (52) we see that for all but finitely many n and a.e. ω ,

$$-\sum_{i=1}^n Z_i \chi(i) \geq -\varepsilon n^{1/2} (\log \log n)^{1/2},$$

and by choosing ε sufficiently small we conclude that for a.e. ω there are infinitely many values of n for which

$$1/\Delta_n \geq \exp(C n^{1/2} (\log \log n)^{1/2})$$

for some $0 < C < \infty$, and so $D(\omega) = \infty$ $\mathbb{P}$ -a.s. Thus, by Lemma 5, the Markov chain is $\mathbb{P}$ -a.s. not ergodic. Thus, for a.e. ω , $\eta_t(\omega)$ is null-recurrent. $\square$

Proof of Theorem 3. Parts (i) and (ii) follow easily with the methods used in the proof of Theorem 7. We prove part (iii). By a similar argument to (41), we now have

$$\log(p_i/q_i) = \sum_{r=1}^{\infty} \frac{4^r}{2r-1} Y_i^{2r-1} (\chi(i))^{2r-1} = 4Y_i \chi(i) + O((\chi(i))^3). \quad (54)$$

Since $Y_1 \stackrel{\mathcal{D}}{=} -Y_1$, all odd powers of Y_1 have zero expectation, so that the expectation of the right hand side of (54) is zero. Thus it is clear that for a.e. ω there are infinitely many values of n for which $\sum_{i=1}^n \log(p_i/q_i) \geq 0$, and hence $\Delta_n \geq 1$, and so $f(n) \rightarrow \infty$ for a.e. ω , and we have $\mathbb{P}$ -a.s. recurrence, by Lemma 2.

To prove null-recurrence, it remains to show that the Markov chain is not ergodic. Once more, consider $D(\omega)$ as defined at (25). By a similar argument to the above, for a.e. ω there are infinitely many values of n for which $\sum_{i=1}^n \log(p_i/q_i) \leq 0$ and hence

$1/\Delta_n \geq 1$, and so $D(\omega) = \infty$ for a.e. ω . Thus, by Lemma 5, the Markov chain is $\mathbb{P}$ -a.s. not ergodic. This completes the proof of part (iii).

We now prove (iv). Once again we analyse the properties of the expression (54). Suppose that $\chi(n) = an^{-\beta}$ for some $a, \beta > 0$. Now suppose that $0 < \beta < 1$ and that $\mathbb{E}[Y_1] < 0$. Then from (54), there exist $0 < C_1, C_2 < \infty$ such that

$$-C_1 n^{1-\beta} \leq \mathbb{E} \sum_{i=1}^n \log(p_i/q_i) \leq -C_2 n^{1-\beta}.$$

If $\beta > 1/2$, then, by the boundedness of Y_1 , we have

$$\sup_n \mathbb{E} \left| \sum_{i=1}^n \log(p_i/q_i) - \mathbb{E} \sum_{i=1}^n \log(p_i/q_i) \right|^k < \infty$$

for all $k \in \mathbb{N}$, so that $\mathbb{P}$ -a.s.,

$$\left| \sum_{i=1}^n \log(p_i/q_i) - \mathbb{E} \sum_{i=1}^n \log(p_i/q_i) \right| < n^\varepsilon$$

for all but finitely many n , and any $\varepsilon > 0$. So, for all but finitely many n and a.e. ω ,

$$\Delta_n \leq \exp(-Cn^{1-\beta} + n^\varepsilon)$$

for some C with $0 < C < \infty$, so that, for ε small enough, $f(n) = \sum_{i=1}^n \Delta_i$ is bounded for a.e. ω , which implies that $\eta_t(\omega)$ is $\mathbb{P}$ -a.s. transient, by Lemma 1.

Also, if $\beta = 1/2$ we infer from (54) that there exist $0 < C_1, C_2 < \infty$ such that

$$C_1 \log n \geq \text{Var} \sum_{i=1}^n \log(p_i/q_i) \geq C_2 \log n \rightarrow \infty$$

as $n \rightarrow \infty$, and then we can apply Lemma 3 to obtain, for a.e. ω ,

$$\sum_{i=1}^n \log(p_i/q_i) \leq -C_1 n^{1/2} + C_2 (\log n)^{1/2} (\log \log n)^{1/2}$$

for some constants $0 < C_1, C_2 < \infty$ (depending on ω) and all but finitely many n . Thus $f(n)$ is $\mathbb{P}$ -a.s. bounded, and so we have $\mathbb{P}$ -a.s. transience by Lemma 1.

Finally, if $0 < \beta < 1/2$, by (54), there exist $0 < C_1, C_2 < \infty$ such that

$$C_1 n^{1-2\beta} \geq \text{Var} \sum_{i=1}^n \log(p_i/q_i) \geq C_2 n^{1-2\beta} \rightarrow \infty$$

as $n \rightarrow \infty$, and then by Lemma 3 we obtain, for a.e. ω ,

$$\sum_{i=1}^n \log(p_i/q_i) \leq -C_1 n^{1-\beta} + C_2 n^{1/2-\beta} (\log \log n)^{1/2}$$

for some constants $0 < C_1, C_2 < \infty$ (depending on ω) and all but finitely many n . So once again $f(n)$ is $\mathbb{P}$ -a.s. bounded, and we have $\mathbb{P}$ -a.s. transience by Lemma 1. This proves part (c).

To prove (a), we apply Lemma 5. Suppose that $\mathbb{E}[Y_1] > 0$. By similar arguments to those above, this time for a.e. ω we have

$$\sum_{i=1}^n \log(p_i/q_i) \geq Cn^{1-\beta}$$

for some $0 < C < \infty$ and all but finitely many n . Thus, for a.e. ω and all but finitely many n ,

$$\frac{1}{\Delta_n} = \exp\left(-\sum_{i=1}^{n-1} \log(p_i/q_i)\right) \leq \exp(-Cn^{1-\beta}),$$

and so, for $D(\omega)$ as defined at (25), $D(\omega) < \infty$ $\mathbb{P}$ -a.s.; hence, by Lemma 5, the Markov chain is $\mathbb{P}$ -a.s. ergodic, proving part (a).

Finally, we prove (b). Suppose that $\beta > 1$. Now, since $-1 \leq Y_i \leq 1$ and $\chi(n) = an^{-\beta}$, we see from (54) that there exists a constant C_1 (not depending on ω) with $0 < C_1 < \infty$ such that, for a.e. ω ,

$$\left| \sum_{i=1}^n \log(p_i/q_i) \right| \leq C_1 \sum_{i=1}^n i^{-\beta} \leq C_2$$

for some finite positive C_2 , not depending on ω or n , this last inequality following since $\beta > 1$. Thus for a.e. ω and each n ,

$$0 < \exp(-C_2) \leq \exp\left(\sum_{i=1}^n \log(p_i/q_i)\right) \leq \exp(C_2) < \infty,$$

so that for each n , Δ_n and $1/\Delta_n$ are each bounded strictly away from 0 and from ∞ , so that $\mathbb{P}$ -a.s. $f(n) \rightarrow \infty$ as $n \rightarrow \infty$, and $D(\omega) = \infty$ $\mathbb{P}$ -a.s. Thus by Lemma 1 the Markov chain is $\mathbb{P}$ -a.s. recurrent, and by Lemma 5, it is $\mathbb{P}$ -a.s. not ergodic. Thus, for a.e. ω , $\eta_t(\omega)$ is null-recurrent. This completes the proof of Theorem 3. $\square$

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