REPRESENTING AND UNDERSTANDING MULTIPLICATION

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In this paper, we examine the importance of representations, in particular with respect to the understanding of multiplication by primary school pupils. We first of all look at the theoretical background to representations in mathematics. In particular, we look at the use of the array representation for reasoning with and understanding multiplication. We then describe some preliminary work that we have carried out, examining Year 4 and Year 6 pupils' use of the array representation for multiplication calculations. Using a novel methodological approach of recording children's workings on a computer, we observed that the array representation can be a powerful tool for supporting work in multiplication. At the same time, we also observed pupils who were unable to access the mathematical meanings of the representation. Further work is needed to understand such difficulties when developing the use of the array as a tool for mathematical understanding.

INTRODUCTION

During primary education, pupils are introduced to a number of big ideas – for example, addition and subtraction in early primary, and multiplication and division in later primary. In teaching addition and subtraction, there has been a clear use of representations such as number squares and number lines, with number lines being the most appropriate representation for demonstrating the characteristics of these operations. In the renewed Primary Framework for mathematics in England and Wales (DfES, 2006), teachers are encouraged to use the array representation for multiplication when introducing pupils to the distributive law. However, how will children be expected to respond to this particular representation? This is the question that we will explore in this paper.

ROLE OF REPRESENTATIONS IN MATHEMATICAL LEARNING

Researchers such as Gray *et al*. (1997), Krutetskii (1976), Carpenter *et al*. (1999), Gravemeijer (1991) and Thompson (1999) have all discussed the importance of representations in developing mathematical competence. Representations in mathematics education refer to both internal and external manifestations of mathematical concepts. Hiebert and Carpenter (1992) state that communicating mathematical ideas requires external representations (so spoken language, written symbols, pictures or physical objects), whereas to think about mathematical ideas requires internal representations. Goldin and Shteingold (2001) also identified computer-based microworlds as external representations, and students' personal symbols, their natural language, visual imagery, spatial representation and problem-solving strategies and heuristics as internal representations.

There are a variety of ways in which external representations can be used in the learning of mathematics. First of all, concrete representations can be used to 'mirror' the structure of a particular concept (Boulton-Lewis, 1998). In turn, they are easier to talk about than language-based or symbol-based procedures, and also allow the teacher to access, to some degree, the cognitive processes of students (Hall, 1998). Representations can be used in problem solving situations to make sense of situations and to structure thinking and approaches to solutions (Fennell and Rowan, 2001).

Goldin and Shteingold's (2001) work on representations suggests that representational systems are important to the learning of mathematics because of the inherent structure contained within each representation. This structure can shape or constrain learning. Furthermore, different representations emphasise different aspects of a concept and so the development of an understanding of a particular concept comes from having a range of representations and being able to move both within and between them.

We identify two important characteristics that are required by a representation, particularly for the young learner:

- **consistency,** both as we move from one operation to another and as the number domain extends through natural numbers, integers and rational numbers.
- **transparency,** so that the characteristics of the operation are clearly available for the learner to visualise and use.

However, Harries and Suggate (2006) suggested that representations are not self-evident, but they need to be worked on so that the essence of the representation is understood. As Mason (2005) has written:

> To get much educational benefit, students need to be active in processing images; they need to work on images, not just look at them. They need to probe beneath surface reactions. Working on and with mental imagery supports this development (p 78).

Kaput (1991) took the view that the way the learner understands notation and representations determines the way in which mathematical thinking can develop. He suggested that mathematical notation acts in a similar way to the architecture of a building in that it constrains and/or supports our experience. He maintained that the ability to see links between different representations (both iconic and symbolic) is a powerful problem-solving tool, and he suggested that linking notational systems helps pupils to extend their reasoning processes from concrete to more abstract systems (Kaput 1994):

> All aspects of a complex idea cannot be adequately represented within a single notation system, and hence require multiple systems for their full expression (p 530).

These ideas from Mason and Kaput suggest that we need to explore ways in which young learners can explore representations and how this exploration may be a way in which understanding is developed.

UNDERSTANDING, REASONING AND REPRESENTATIONS

The previous discussion touched upon the fact that using representations can develop children's understanding of mathematics. Before we examine how we can represent multiplication, let us consider in more depth the link between representations of mathematical concepts and how we come to understand these concepts. We begin by examining some definitions or explanations of 'understanding' in mathematics. Skemp (1976) identified two types of understanding: relational and instrumental. He described relational understanding as 'knowing both what to do and why' (p 2), and the process of learning relational mathematics as 'building up a conceptual structure' (p 14). Instrumental understanding, on the other hand, was simply described as 'rules without reasons' (p 2).

Nickerson (1985), in examining what understanding is, identified some 'results' of understanding: for example, agreement with experts, being able to see deeper characteristics of a concept, looking for specific information in a situation more quickly, being able to represent situations, and envisioning a situation using mental models. However, he also proposed that 'understanding in everyday life is enhanced by the ability to build bridges between one conceptual domain and another' (p 229). Like Skemp, Nickerson highlighted the importance of knowledge and of relating knowledge:

> The more one knows about a subject, the better one understands it. The richer the conceptual context in which one can embed a new fact, the more one can be said to understand the fact (p 235-236).

Hiebert and Carpenter (1992) specifically defined mathematical understanding as involving the building up of the conceptual 'context' or 'structure' mentioned above.

> The mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and strength of its connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections. (p 67)

Therefore, this idea of understanding being a structure or network of internal mathematical ideas or representations comes out clearly from the literature.

In discussing understanding, we have to be clear as to whether we are referring to understanding as an action or as a result of an action. Sierpinska (1994) clarified this by putting forward three different ways of looking at understanding. First of all, there is the 'act of understanding' which is the mental experience associated with linking what is to be understood with the 'basis' for that understanding. Examples of bases given by her were mental representations, mental models, and memories of past experiences. Secondly, there is 'understanding' which is acquired as a result of the acts of understanding. Thirdly, there are the 'processes of understanding' which involve links being made between acts of understanding through reasoning processes, including developing explanations, learning by example, linking to previous knowledge, linking to figures of speech and carrying out practical and intellectual activities. Sierpinska (1994) saw these processes of understanding as 'cognitive activity that takes place over longer periods of time' (p 2). In making links between understandings of a mathematical concept through reasoning, for example, showing why 12 × 9 gives the same answer as 9 × 12, we further develop our understanding of the concept. This third process is important in that we can make the link between understanding and reasoning. We see reasoning as the process of linking between different representations of a mathematical concept, internally or externally. In the above example, we are using reasoning to link between 12 × 9 and 9 × 12. Thus, we can use the following definitions for understanding and reasoning:

- to understand mathematics is to make connections between mental (internal) representations of a mathematical concept;
- to reason is to make connections between different representations (internal or external) of a mathematical concept, through formal (for example, logic, proof) or informal (for example, through examples) processes.

We can therefore see the close links between understanding and reasoning, and why, in establishing links through reasoning, we might develop understanding of mathematical concepts. We have also established some connection between the internal representations that make up our understanding and the external representations that we use to communicate mathematics. Going back to the ideas of Mason and Kaput earlier, by encouraging children to explore and reason between representations, what we are doing is providing the opportunity for understanding to develop.

USING REPRESENTATIONS TO ILLUSTRATE MULTIPLICATION

In thinking now about visual representations of multiplication, we can use the above ideas about representations and reasoning, and relate these to aspects of multiplication that we would like children to understand.

Nunes and Bryant (1996) stated that a commonly held view of multiplication and division is that they are simply 'different arithmetic operations … taught after they have learned addition and subtraction' (p 144). However, they suggest that this is too limited a view and that in actual fact 'multiplication and division represent a significant qualitative change in children's thinking' (p144). Whilst addition and subtraction can be thought of as the joining of sets, multiplication is about replication. Anghileri (2000) concurs with these ideas and suggests that addition and subtraction are unary operations with each input representing the same kind of element – 3 blocks added to 4 blocks or 3 oranges added to 4 oranges – and the second input being a means of identifying a termination point. However, she suggests that pupils need to view multiplication as a binary operation with two distinctive inputs. The first input represents the size of a set (say, the number of oranges in a particular set) and the second represents the number of replications of that set (how many sets of oranges). In this way the two numbers represent distinct elements of the multiplication process.

The important aspects of multiplication which are discussed by Anghileri (2000) amongst others are:

- replication (rather than joining as in addition/subtraction);
- the binary rather than unary nature of multiplication, and the notion of two distinct and different inputs;
- commutativity for multiplication but not division;
- distributivity.

When considering ways of representing multiplication visually, we need to consider how the representation might both show the above characteristics and also aid the calculation process. Wittmann (1998, 2002, and 2005) in his work on representations considers two aspects. First, he suggests that a representation needs to facilitate the processes of mathematising, exploring, reasoning, communicating. Second, he suggests three criteria that need to be satisfied by a representation for a particular domain of teaching: a limited and compatible set of representations should be used, the representations should capture the fundamental ideas of the domain being considered, and they should be available in both large demonstration format and small personal use format.

Some of the common representations applicable for multiplication are shown below, and to varying degrees they satisfy the aspects considered by Wittmann.

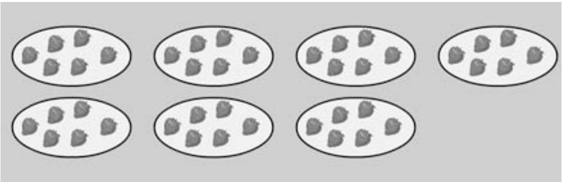


Figure 1: Plates of strawberries representation

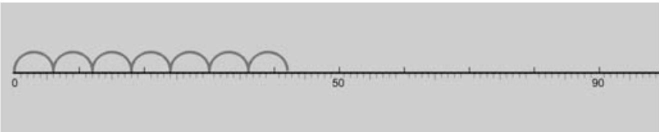


Figure 2: Number line representation

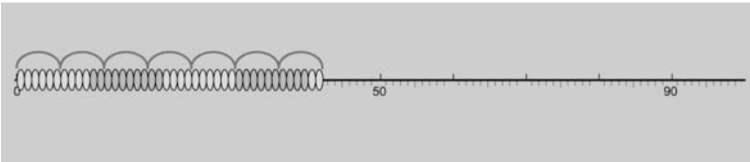


Figure 3: Beads on a string representation

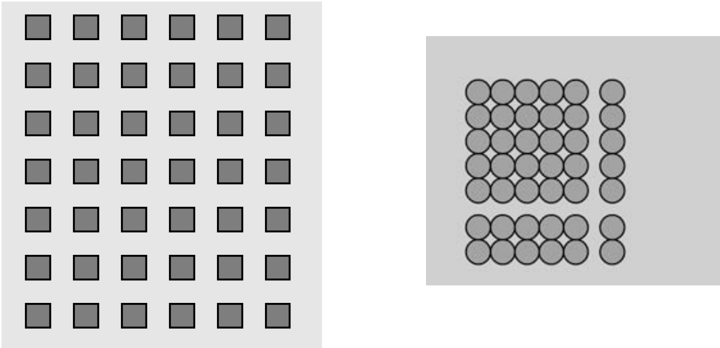


Figure 4: Array representations 1 (left) and 2 (right)

In order to see how well the different representations highlight the important aspects of multiplication, we need to 'interrogate' the representations to see how easily we can in particular reason these aspects, and therefore, we hope, come to understand these aspects as part of multiplication.

INTERROGATING REPRESENTATIONS OF MULTIPLICATION

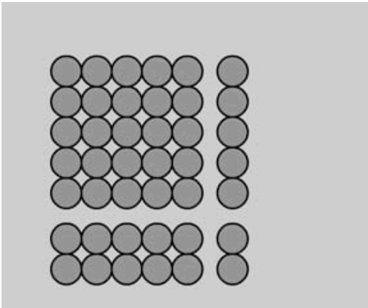
If we explore the nature of these representations, then some points which might be identified are:

- all the representations clearly show the element of replication which is a key aspect of multiplication;
- there are two inputs in each representation – for example, plates and strawberries or rows and columns;
- the groupings in the different representations vary from 'platefuls' or a whole group, to a grouping in which 5 seems to be a key (for example, in array 2);
- both the array representations show the commutative nature of multiplication through a transformation of the representation;
- the distributive law is clearly shown in array 2, and can be shown in array 1;
- all the representations aid calculation but in different ways. All the representations encourage one-to-one counting and counting in groups. Only the arrays can be used for calculations involving the distributive law.

Given these identified features of the representations, we contend that in order to encourage pupils to develop their thinking about multiplication as a binary operation, the array representation needs to be used, with rows and columns representing the two inputs. Also, the array representation is advantageous in that it a useful for showing a number of properties concerning multiplication. It is clear that the array representation shows the commutative property of multiplication (Anghileri, 2000). Swapping the '7' and the '6' in Figure 4 simply changes the orientation of the array through a transformation. Also, the representation (especially with the additional spaces inserted) makes clear why the distributive law applies to multiplication. 7×6

is equal to 7 × (5 + 1) or (5 + 2) × (5 + 1). Thus, the 7 times table could be seen as the (5 + 2) times table in this case.

The way in which the distributive law is illustrated through this representation provides a link between what might be called informal methods and a written method of performing the multiplication calculation. The grid method is a natural extension of the method which can be employed using the particular array discussed above.



	5	1
5	25	5
2	10	2

Figure 5: Linking the array representations with the grid method

Thus, this representation has the potential to show the nature of multiplication, illustrate the important features of the operation, and facilitate ways in which the calculation can be implemented both informally and as a written method. By encouraging children to reason the above links based on this external representation, we would hope that understanding can be developed in multiplication. For these reasons we would consider that the array as portrayed above is what Wittmann (2002) calls a Substantial Learning Environment, since:

- it demonstrates and represents the key elements of the concept of multiplication and provides an environment for locating this concept;
- it facilitates the exploration of ideas beyond just multiplication in that it allows the demonstration/exploration of the nature of commutativity and distributivity. Further it is a rich focus for discussion about the nature of and the process for calculating multiplication;
- it is adaptable to classroom use either with or without ICT and it encourages flexible ways of performing calculations;
- it has clear mathematical and pedagogical strengths and so is a rich environment for research

EXAMINING PUPILS USE OF THE ARRAY REPRESENTATION

Having examined some of the theoretical issues involved, let us now look at two case studies looking at children's use of the array representation. In these studies, the focus was on the use of the array for calculating purposes. These case studies came from research that we carried out involving a Year 4 class and a Year 6 class in a primary school in the North East of England. The classroom sessions were based around pairs of pupils using a Macromedia Flash computer program, incorporating the array representation of multiplication. The reason we used a computer program for children’s work is that it enabled us to use a novel methodology for examining children’s mathematics. During the working in pairs and using the Flash program to do problems on the computer, all the actions carried out by the children on the computer were recorded using Camtasia recording software. As children were each wearing a headphones and microphone set, the dialogue of children as they worked on the multiplication problems was also recorded along with their computer actions. Therefore, a rich amount of qualitative audiovisual data, showing the dynamic process of children using the array representation, was obtained. This therefore enabled us to observe pupils as they 'construct their mathematical ways of knowing as they interact with others' (Cobb *et al*, 1992, p 17) whilst they used the array representation for multiplication. We provide here instances from the data that provided some insight into how pupils were using the array representation. For each year group, we make some general comments based on this preliminary study before going on to discuss one specific example in each case.

In the Year 6 group there were twenty four pupils working in pairs. All had experience of multiplication and knew many of their tables.

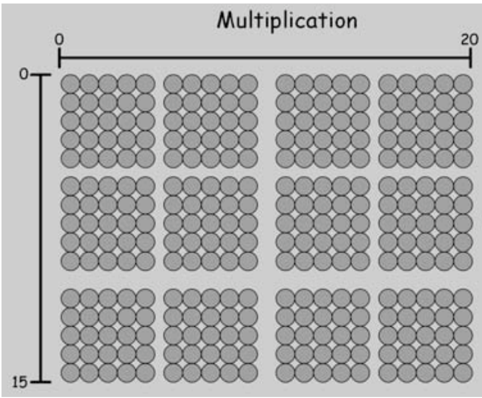


Figure 6: The array showing all multiplications up to 20 × 15

- For all pupils the diagram above did not pose any problems and they were all able to identify multiplication as ‘rows’ by ‘columns’.
- Even when they knew the answer from memory they still seemed to take an interest in devising methods to calculate the total number of coloured circles.
- All pupils used the structure of the diagram to count in 25s, 5s and 1s as a mechanism for working out the answer.
- Some pupils used a method of mentally moving rows of 5 in order to complete blocks of 25.
- There was constant discussion about what they were doing and why. For example, in the calculation 18 × 9, the pupils discussed how they could start from 18 × 10 and then take away the bottom row, that is, take away 5, 5, 5 and 3.

- There was evidence of pupils using the commutative law in order that they could show 14 × 18 on the diagram (which would not fit as 18 × 14).
- There was some evidence (see below) of pupils being able to use the distributive law.

The points made above are illustrated by the specific example below concerning the way in which James and Tom worked with the programme.

James and Tom (Year 6)

James and Tom were using a programme which asked them to do a multiplication calculation using the array. The numbers were entered as rows and columns, and then we were interested in how they would do the calculation. The two diagrams below illustrate the strategies used.

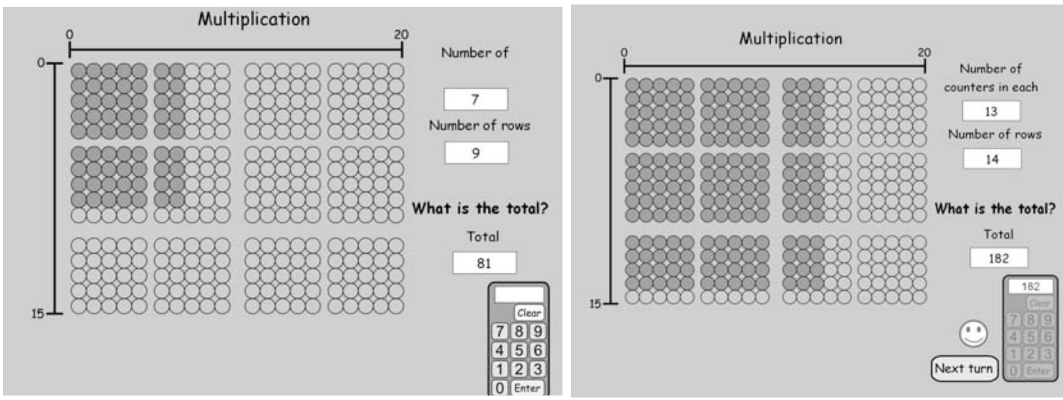


Figure 7: Screenshots from James and Tom’s use of the array

First of all, with encouragement from the teacher, the pupils focussed on the 25 block and used that as a starting point for counting, using the language ‘that’s five 5s’, and sometimes using the pointer to indicate the block. Again with strong teacher input, they filled in blank counters to make up another block of 25. For example, in the left-hand figure above, by moving one line of 5 down to the bottom of the array. Finally they counted the rest as a 5 and four 2s. However in the second example, where there was no teacher input, we saw what could be viewed as a more sophisticated and extendable strategy which used the distributive law. This time, the conversation went as follows:

> Got 100 … 50, 50, 100. And then 4 times 5 equals 20. And another. 3 times 5 equals 15; and 3 times 4 is 12; so two 15s, 12, and two 20s, and 100. So that’s 182.

This time, not necessarily intentionally, the pupils had calculated (5+5+4) × (5+5+3) by taking appropriate sections of the array. Hence they were demonstrating the use of the distributive law.

Thus for the Year 6 pupils, seeing multiplication within this array was not a problem. They used their prior knowledge to identify the calculation required and used the array structure to calculate the answer. It was an environment which encouraged them to talk about what they were doing and why they were doing it.

In the Year 4 group, there were again 24 pupils working in pairs but they had more limited experience of multiplication and there was more variation in the way in which they responded to the questions asked. For them:

- The requirement to mathematise the idea of multiplication within the array representation did sometimes cause difficulty, especially when they knew the answer and so were trying to represent this ‘answer’ in terms of 10s and units, rather than ‘so many equal groups of …’.
- The environment again encouraged discussion about how the representation worked and the pupils explored ways in which the total number of coloured counters could be efficiently counted.
- There was limited evidence of the use of the commutative law or the distributive law.

Some of these points are illustrated by the work of Anna and Beth below.

Anna and Beth (Year 4 pupils)

Anna and Beth were using a program where we had asked them to use an ‘L-shape’ with an array (see Figure 8 below) to do multiplication problems. The idea was to move the bars at the right and bottom of the array until the L-shape ‘captured’ the required calculation.

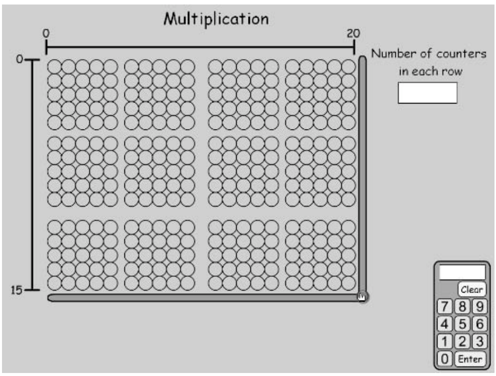


Figure 8: Screenshots from Anna and Beth’s use of the array

First of all, they were asked to do the problem 11 × 6. They knew the answer to be 66. However, they then struggled for a few minutes in moving the L-shape so that only 66 squares were enclosed. They did not associate the dimensions of ‘11’ and ‘6’ with the dimensions of the required array. This was despite the fact that they had used a similar program in a previous session, where they typed in the two numbers being multiplied to produce an array of highlighted counters (see the Year 6 example). Eventually, by accident, they placed the L-shape in the correct position and complete the question by writing ‘11’ for the number of rows, ‘6’ for the number of columns, then ‘66’ for the answer. They then tried 14 × 7. They began by trying to calculate the answer first (incorrectly getting 104), then tried to show 104 on the array. They again made no association between the array and the multiplication sum. After about eight minutes of moving the L-shape around the array, one of the researchers intervened.

Here Anna and Beth are clearly struggling to use the environment to explore what the calculation 11×6 would look like in this representation. The difficulty they have is mathematising within this structure which requires them to think in terms of rows and columns, rather than the answer in terms of tens and units. Nonetheless it provides a clear opportunity for discussion about multiplication.

CONCLUSIONS

On the basis of this small preliminary study, the following points can be made. Firstly, the representation can provide a focus for discussion and reasoning of the calculation being considered. Secondly, the array can provide a representation of multiplication that not only emphasises the replication element of multiplication, but can also be used to aid the calculation of the total. Thirdly, the pupils could be imaginative in their use of the array to calculate the total number of counters. They seemed to see it as a counting exercise and sought ways to make this exercise efficient, by using the 5 by 5 block as a kind of benchmark from which to work. There is a sense, of course, in which this could be seen as a distraction since the calculating exercise then loses the element of replication. Fourthly, the distributive element of multiplication can be used to perform the calculation and this could be an important developmental point.

However, we do need to be careful to examine how the pupils see the representation themselves. As in the case of the Year 4 pupils, we cannot take for granted how the children will interpret the representation. Further work now needs to be undertaken to gain a better understanding of how pupils reason with a representation and in particular how they make sense of the array in the context of multiplication: whether it encourages the development of informal strategies, whether it provides an important link to a more formal method such as the grid method, and whether it can be used to represent a range of word problems. Also, more work will be required in trying to develop a clear sense of progression within this field of representations and multiplication.

In our discussion above, we have also examined the potential of reasoning with the array representation. From our observations, we conclude that this particular form of the array has the potential for supporting reasoning in multiplication, but we have to be aware of the problems as well. For example, the Year 4 pupils needed to realise that the representation was not just showing a particular number but an operation as well. This therefore placed a restriction on the way in which the representation should have been reasoned with. Furthermore, the Year 6 pupils would need to be encouraged to think about methods that were extendable. More work now needs to be done to explore the possibility of developing a pedagogy based on the idea of understanding through reasoning, using representation as the medium through which the reasoning can be focused. However, this paper has shown how we can use the array to reason, and therefore how we can possibly use it to develop understanding in multiplication.

REFERENCES

Anghileri, J. (2000) *Teaching Number Sense* (London, Continuum).

Boulton-Lewis, G. M. (1998) Children’s strategy use and interpretations of mathematical representations’, *Journal of Mathematical Behaviour*, 17(2), 219-237.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L. & Empson, S. B. (1999) *Children’s mathematics: Cognitively Guided Instruction* (Portsmouth, NH, Heinemann).

Cobb, P., Yackel, E. & Wood, T. (1992) A constructivist alternative to the representational view of mind in mathematics education, *Journal of Research in Mathematics Education*, 23(1), 2-33.

Department for Education and Skills (2006) *Primary Framework for literacy and mathematics: core position papers underpinning the renewal of guidance for teaching literacy and mathematics* (Norwich, DfES).

Fennell, F. & Rowan, T. (2001) Representation: an important process for teaching and learning mathematics, *Teaching Children Mathematics*, 7(4), 288-292.

Goldin, G. A. & Shteingold, N. (2001) Systems of representation and the development of mathematical concepts, in: A. A. Cuoco & F. R. Curcio (Eds) *The roles of representation in school mathematics* (Reston, VA, National Council of Teachers of Mathematics).

Gravemeijer, K. P. E. (1991) An instruction-theoretical reflection on the use of manipulatives, in: L. Streefland (Ed) *Realistic Mathematics Education in Primary School* (Utrecht, Cdß-Press).

Gray, E., Pitta, D. & Tall, D. O. (1997) The nature of the object as an integral component of numerical processes, in: E. Pehkonen (Ed) *Proceedings of the 21st International Conference for the Psychology of Mathematics Education* (Lahti, Finland), Vol. 1, 115–130.

Hall, N. (1998) Concrete representations and the procedural analogy theory, *Journal of Mathematical Behaviour*, 17(1), 33-51.

Harries, T. & Suggate, J. (2006) Exploring links across representations of numbers with young children, *International Journal for Technology in Mathematics Education*, 13(2) (online journal).

Hiebert, J. & Carpenter, T. P. (1992) Learning and teaching with understanding, in: D. A. Grouws (Ed) *Handbook of Research on Mathematics Teaching and Learning* (New York, NCTM).

Kaput, J. (1991) Notations and representations as mediators of constructive processes, in E. von Glasersfeld (Ed) *Constructivism and mathematics education* (Dordrecht, Netherlands, Kluwer).

Kaput, J. (1994) Technology and Mathematics Education, in: D. A. Grouws (Ed) *Handbook of Research on Mathematics Teaching and Learning* (New York, NCTM).

Krutetskii, V. (1976) *The psychology of mathematical abilities in school children* (Chicago, University of Chicago Press).

Mason, J. (2005) Micro-structure of attention in the teaching and learning of mathematics, in: *Mathematics Teaching 2005 Conference Report*, (Edinburgh Centre for Mathematical Education), 36-41.

Nickerson, R. S. (1985) Understanding understanding, *American Journal of Education*, 93(2), 201-239.

Nunes, T. & Bryant P. (1996) *Children doing Mathematics* (Oxford, Blackwell Publishers).

Sierpinska, A. (1994) *Understanding in Mathematics* (London, The Falmer Press).

Skemp, R. (1976) Relational understanding and instrumental understanding, *Mathematics Teaching*, 77, 20-26.

Thompson, I. (1999) *Issues in Teaching Numeracy in Primary Schools* (Oxford, Open University Press).

Wittmann, E. Ch. (1998) Standard number representations in teaching arithmetic, *Journal fur Mathematik-Didaltik*, 19(2), 149-178.

Wittmann, E. Ch. (2002) Developing mathematics education in a systemic process, *Educational Studies in Mathematics*, 48(1), 1-20.

Wittmann, E. Ch. (2005) Mathematics as the science of patterns – a guideline for developing mathematics education from early childhood to adulthood, *Plenary Lecture at International Colloquium 'Mathematical learning from Early Childhood to Adulthood* (Belgium, Mons).