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New tests of the pp-wave correspondence

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Abstract

The pp-wave/SYM correspondence is an equivalence relation, $H_{\rm string} = \Delta - J$, between the Hamiltonian $H_{\rm string}$ of string field theory in the pp-wave background and the dilatation operator Δ in $\mathcal{N} = 4$ Super Yang-Mills in the double scaling limit. We calculate matrix elements of these operators in string field theory and in gauge theory. In the string theory Hilbert space we use the natural string basis, and in the gauge theory we use the basis which is isomorphic to it. States in this basis are specific linear combinations of the original BMN operators, and were constructed previously for the case of two scalar impurities. We extend this construction to incorporate BMN operators with vector and mixed impurities. This enables us to verify from the gauge theory perspective two key properties of the three-string interaction vertex of Spradlin and Volovich:
(1) the vanishing of the three-string amplitude for string states with one vector and one scalar impurity; and **(2)** the relative minus sign in the string amplitude involving states with two vector impurities compared to that with two scalar impurities. This implies a spontaneous breaking of the $\mathbb{Z}_2$ symmetry of the string field theory in the pp-wave background. Furthermore, we calculate the gauge theory matrix elements of $\Delta - J$ for states with an arbitrary number of scalar impurities. In all cases we find perfect agreement with the corresponding string amplitudes derived from the three-string vertex.

1 Introduction

The pp-wave/SYM correspondence in its original form [1] emphasises a duality relation between the masses of string states and the anomalous dimensions [2] of the BMN operators in the dual gauge theory. This correspondence was further discussed and extended in [3, 4], where it was expressed in the form [4]

$$\frac{1}{\mu} H_{\text{string}} = \Delta - J \ . \tag{1.1}$$

Here H_{string} is the full string field theory Hamiltonian, μ is the scale parameter of the pp-wave background, and $\Delta - J$ is the gauge theory dilatation operator (which in the radial quantisation formalism plays the rôle of the Hamiltonian H_{SYM}) minus the R-charge. The relation (1.1) is expected to be exact and hold in the double scaling limit $N \sim J^2 \to \infty$ to all orders in the two free parameters of the theory, g_2 and λ'

$$\lambda' = \frac{g_{\text{YM}}^2 N}{J^2} = \frac{1}{(\mu p^+ \alpha')^2} \ , \tag{1.2}$$

$$g_2 = \frac{J^2}{N} = 4\pi\, g_{\text{st}}\, (\mu p^+ \alpha')^2 \ . \tag{1.3}$$

On the gauge theory side, λ' is the effective coupling constant of the BMN sector, and g_2 is the effective genus counting parameter of Feynman diagrams [1, 5, 6]. The right hand sides of (1.2), (1.3) express λ' and g_2 in terms of the pp-wave string theory parameters to the effect that $1/\lambda' \propto \mu$ measures the deviation from the flat background and g_2 is proportional to the string coupling g_{st} in IIB.

Since the two Hamiltonians, H_{string} and Δ, act on the states of two different theories, the duality relation (1.1) requires an isomorphism between the Hilbert spaces of the light-cone gauge pp-wave string field theory and of the BMN sector of the $\mathcal{N} = 4$ gauge theory. More specifically, we need to establish a one-to-one correspondence between the bases of two theories, $\{|s_\alpha\rangle^{\text{string}}\}$ and $\{|s_\alpha\rangle^{\text{SYM}}\}$,

$$|s_\alpha\rangle^{\text{string}} \leftrightarrow |s_\alpha\rangle^{\text{SYM}} \ , \tag{1.4}$$

which preserves the scalar product,

$$^{\text{string}}\langle s_\alpha|s_\beta\rangle^{\text{string}} = {}^{\text{SYM}}\langle s_\alpha|s_\beta\rangle^{\text{SYM}} \ . \tag{1.5}$$

Then the correspondence (1.1) holds at the matrix elements level,

$$^{\text{string}}\langle s_\alpha|\mu^{-1} H_{\text{string}}|s_\beta\rangle^{\text{string}} = {}^{\text{SYM}}\langle s_\alpha|\Delta - J|s_\beta\rangle^{\text{SYM}} \ . \tag{1.6}$$

The string field theory Hilbert space is equipped with a natural basis of multi-string states,

$$\{|s_\alpha\rangle^{\text{string}}\} = |\text{string}_a\rangle \ , \ |\text{string}_b\rangle \otimes |\text{string}_c\rangle \ , \ |\text{string}_d\rangle \otimes |\text{string}_e\rangle \otimes |\text{string}_f\rangle \ , \ldots \tag{1.7}$$

which diagonalises the free string Hamiltonian and is automatically orthonormal. Here $a, b, \ldots$ are the labels of single-string states. This basis does not diagonalise the full string Hamiltonian, $H_{\rm string}$, since free string states in (1.7) can interact (split and join). The splitting and joining of a single string state is described by the three-string interaction, and the corresponding matrix element on the left hand side of (1.6) is

$$\langle {\rm string}_a | H^{\rm int}_{\rm string} | {\rm string}_b \rangle \otimes | {\rm string}_c \rangle \ \equiv\ \langle {\rm string}_a | \langle {\rm string}_b | \langle {\rm string}_c | H_3 \rangle \ . \tag{1.8}$$

Here the $|H_3\rangle$ is the three-string interaction vertex in the light-cone string field theory in the pp-wave background. The expression for this vertex was originally obtained by Spradlin and Volovich [7,8] and further studied and clarified by Pankiewicz and Stefanski in [9, 10] and in other papers including [11, 12]. Its expression is recalled in Appendix A.[1] However, there is a puzzle related to the three-string amplitudes (1.8) built on the Spradlin-Volovich vertex which we would like to clarify in this paper, among other things. The presence of a non-trivial R–R field in the pp-wave background breaks the light-cone Lorentz symmetry $SO(8)$ down to $SO(4) \times SO(4) \times \mathbb{Z}_2$. The two $SO(4)$'s rotate the first and the last four directions among themselves, while the $\mathbb{Z}_2$ symmetry swaps these two groups of four directions. Apparently, the $\mathbb{Z}_2$ part of the bosonic symmetry of the pp-wave background is not respected by the Spradlin-Volovich three-string interactions [13, 14, 8, 11, 15]: there is a relative minus sign in the string amplitude involving states with two oscillators along the first $SO(4)$ compared to that with two oscillators along the second $SO(4)$. An unbroken $\mathbb{Z}_2$-invariance would not allow this to happen. We will argue now that this minus sign implies a spontaneous breaking of the $\mathbb{Z}_2$ symmetry of the string field theory in the pp-wave background.

The ket-vertex $|H_3\rangle$ (A.5), (A.6) of [7,8] is built on the string state $|0\rangle$ which is the ground state of the theory in flat background, but not in the pp-wave background. At the same time, the external string bra-states in (1.8) are built on the true pp-wave ground state $|v\rangle$. It was explained in [11] that these two states, $|0\rangle$ and $|v\rangle$, have an opposite $\mathbb{Z}_2$ parity, i.e. cannot be both invariant under $\mathbb{Z}_2$. Hence, it follows immediately [8, 11] that the amplitude (1.8) is not invariant under the action of $\mathbb{Z}_2$, but changes sign. In the recent paper [12], the result of [7,8], which utilised the vacuum $|0\rangle$, was compared with an alternative formalism of constructing $|H_3\rangle$ starting directly from the true ground-state $|v\rangle$. The two formalisms were found to be identical. Following [12, 7, 8] we choose the $\mathbb{Z}_2$-parity prescription

$$\mathbb{Z}_2 :\ |0\rangle \to |0\rangle \ , \qquad \mathbb{Z}_2 :\ |v\rangle \to -|v\rangle \ . \tag{1.9}$$

This means that the vertex $|H_3\rangle$ built on $|0\rangle$ is invariant under $\mathbb{Z}_2$, but the pp-wave string ground-state $\langle v|$ and, hence, the external states $\langle {\rm string}_a | \langle {\rm string}_b | \langle {\rm string}_c |$ in (1.8), acquire a minus sign. This implies a spontaneous breaking of the $\mathbb{Z}_2$ symmetry of the string field theory in the pp-wave background, which is the physical reason for the minus sign of the matrix element discussed above.

[1] For notational simplicity and in order to distinguish this vertex from other proposals, we will sometimes refer to the vertex of [7, 8, 9, 10] simply as the Spradlin-Volovich vertex.

One of the objectives of this paper is to verify with an independent gauge theory calculation this important minus sign (and hence the spontaneous breaking of $\mathbb{Z}_2$), as well as the related fact that the three-string amplitude (1.8) vanishes for string states with one direction along the first, and one one direction along the second $SO(4)$, i.e. one vector and one scalar impurity in the gauge theory language.

As already mentioned, and following [4, 16, 17], in order to compare (1.8) with matrix elements of the dilatation operator in gauge theory via (1.6), it is important to identify a basis in gauge theory which is isomorphic to the natural string basis (1.7). We discuss this issue in section 2. States in the isomorphic to string basis, $\{|s_\alpha\rangle^{\rm SYM}\}$, are obtained from linear combinations of the original multi-trace BMN [1] operators $\mathcal{O}_\alpha(x)$,

$$|s_\alpha\rangle^{\rm SYM} \;=\; U_{\alpha\beta}\,\mathcal{O}_\beta(x=0)|0\rangle \;, \tag{1.10}$$

where $U_{\alpha\beta}$ is an x-independent matrix. This matrix was determined in [4, 17] by requiring that (1.6) holds, i.e. that the known three-string interaction vertex of the pp-wave light-cone string field theory [7, 8] is reproduced from gauge theory matrix elements of the dilatation operator involving BMN states (operators) with two scalar impurities.

In this paper we will take $U_{\alpha\beta}$ determined in [17], and use it to construct the gauge theory basis (1.10) for an arbitrary number of scalar impurities. With this in hand we can compute generic gauge theory matrix elements on the right hand side of (1.6). The contributions on the left hand side of (1.6) are then computed using (1.8). We will verify (1.6) and hence the Spradlin-Volovich expression for $|H_3\rangle$ for *generic* bosonic impurities. First successful steps in this direction have been already taken in [17, 18] at the level of arbitrary number of identical scalar impurities.[2] However, the inclusion of vector impurities is essential in order to address in the gauge theory the two important properties of the three-string interaction discussed earlier:
(1) the vanishing of the three-string amplitude for string states with one vector and one scalar impurity; and
(2) the relative minus sign in the string amplitude involving states with two vector impurities compared to that with two scalar impurities.

In section 3 we will verify **(1)** and **(2)** working at the two-impurity level, and will consider all representations of BMN operators with two vector or scalar impurities, i.e. symmetric traceless, antisymmetric and singlet. By considering BMN operators with vector, scalar and mixed (scalar+vector) impurities we explore and verify the correspondence (1.6) for string states in all the directions of the two $SO(4)$ groups.

In section 5, we will calculate the gauge theory matrix elements of $\Delta - J$ for states with an arbitrary number of scalar impurities. Next we compute the corresponding three-string amplitudes derived from the three-string vertex and compare them to the field theory result, finding perfect agreement.

[2] For further tests of the correspondence in the open-closed string sector, see [19].

Finally, section 4 and section 6 are dedicated to computations of three-point correlators of BMN operators. These results are used earlier in section 3 and 5 for the calculation of matrix elements. More specifically, in section 4 we compute the coefficient of the conformal three-point function of BMN operators with mixed (one scalar and one vector) impurities. In section 6 we generalise this analysis to the case of BMN operators with an arbitrary number of scalar impurities.

2 The dilatation operator in SYM and the natural string basis

As mentioned earlier, the BMN basis in SYM which is isomorphic to the natural string basis in dual string field theory, is a certain linear combination (1.10) of the original BMN operators $\mathcal{O}_\alpha(x)$ proposed in [1]. The states in the natural string basis are not identically equal to the original BMN operators since the former are automatically orthonormal, while the latter are not, and their overlaps depend on the string coupling g_2. In other words, the matrix U in (1.10) is not simply the identity matrix.

Apart from the original BMN basis, there is another distinguished basis of the conformal primary BMN operators $\mathcal{O}_{\Delta_\alpha}(x)$ which are the eigenstates of the dilatation operator Δ in gauge theory. This Δ-BMN basis is again a linear combination of the states from the original BMN basis $\mathcal{O}_\alpha(x)$ with a different x-independent matrix U. For BMN operators with scalar impurities, this basis was constructed in [20] and extended to include vector and mixed impurities in [21]. The Δ-BMN basis is particularly convenient since the two- and three-point correlation functions of Δ-BMN operators can be written in the simple canonical form with a universal x-dependence, guaranteed by conformal invariance of the theory. For conformal primary operators with scalar impurities these canonical correlators are particularly simple and are given by

$$\langle \mathcal{O}^\dagger_{\Delta_\alpha}(x)\mathcal{O}_{\Delta_\beta}(0)\rangle = \frac{\delta_{\alpha\beta}}{(x^2)^{\Delta_\alpha}} \ , \tag{2.1}$$

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}^\dagger_{\Delta_3}(x_3)\rangle = \frac{C_{123}}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}(x_{13}^2)^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}}(x_{23}^2)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}} \ . \tag{2.2}$$

Canonical expression for the correlators involving conformal primary operators with vector impurities appear to be much less illuminating and harder to interpret, however it was noted in [21] that this difficulty is avoided and the correlators for all types of impurities can be expressed in the same form, similar to (2.1) and (2.2), if on the left hand sides of (2.1) and (2.2) we use a different notion of conjugation $\bar{\mathcal{O}}$ instead of $\mathcal{O}^\dagger$ [21]. This different notion of operator conjugation is defined as *hermitian conjugation* followed by an *inversion* of the operator argument $x'_\mu = x_\mu/x^2$. Under inversion a scalar operator

$\mathcal{O}_\Delta(x)$ of conformal dimension Δ transforms as

$$\mathcal{O}^\dagger_\Delta(x) \to \mathcal{O}^{\dagger\prime}_\Delta(x') = x^{2\Delta}\mathcal{O}^\dagger_\Delta(x) \quad , \quad x_\mu \to x'_\mu = \frac{x_\mu}{x^2} \ , \tag{2.3}$$

while a vector or tensor operator (i.e. operator with vector impurities) contains a factor $J_{\mu\nu}(x) = \delta_{\mu\nu} - 2x_\mu x_\nu/x^2$ on the right hand side for each vector index of the operator. $J_{\mu\nu}(x)$ is the usual inversion tensor, in terms of which the Jacobian of the inversion is expressed $\partial x'_\mu/\partial x_\nu = J_{\mu\nu}(x)/x^2$. This prescription is essential in order to make vector Δ-BMN operators orthonormalisable, see section 2 of [21] for more details.

With this prescription, the two-point function (2.1) for vector and for scalar Δ-BMN operators takes the same simple form:

$$\langle \bar{\mathcal{O}}_{\Delta_\alpha}(x)\mathcal{O}_{\Delta_\beta}(0)\rangle = \delta_{\alpha\beta} \ , \tag{2.4}$$

which does not depend on x and hence has the meaning of overlap of the corresponding states in the gauge theory Hilbert space.

Note, however, that these Δ-BMN states are the eigenstates of Δ, i.e. the eigenstates of the full interacting string Hamiltonian, so they cannot be identically equal to the states from the natural string basis. The relation between the two bases is again a linear combination

$$|s_\alpha\rangle^{\rm SYM} \ = \ U_{\alpha\beta}\, \mathcal{O}_{\Delta_\beta}(x=0)|0\rangle \ , \tag{2.5}$$

with another constant matrix $U_{\alpha\beta}$. In general, for any basis of operators $\tilde{\mathcal{O}}_\alpha$ such that

$$\tilde{\mathcal{O}}_\alpha \ = \ U_{\alpha\beta}\, \mathcal{O}_{\Delta_\beta} \ , \tag{2.6}$$

the overlap is given by

$$\langle \bar{\tilde{\mathcal{O}}}_\alpha(x)\tilde{\mathcal{O}}_\beta(0)\rangle \ = \ U_{\beta\gamma}U^\dagger_{\gamma\alpha} \ \equiv \ S_{\beta\alpha}. \tag{2.7}$$

The operators $\tilde{\mathcal{O}}_\alpha$ do not anymore have definite scaling dimensions Δ, but since they are expressed as a linear superposition of conformal primary operators which do, there is no problem in performing the inversion required to define $\bar{\tilde{\mathcal{O}}}_\alpha(x)$, and the right hand side of (2.7) follows.

Now we describe a practical way of how to calculate simultaneously the overlaps and the matrix elements of the anomalous dimension operator $\delta = \Delta - \Delta_{\rm cl}$, where $\Delta_{\rm cl}$ is the engineering dimension. Let us define the barred-operator $\bar{\tilde{\mathcal{O}}}(x)$ as the Hermitian conjugation of $\tilde{\mathcal{O}}(x)$ followed by an inversion of the resulting operator, defined as if it was free, i.e. instead of the factor $x^{2\Delta}$ in (2.3) we put $x^{2\Delta_{\rm cl}}$, such that,

$$\bar{\mathcal{O}}_\Delta(x) \ \equiv \ x^{2\Delta_{\rm cl}}\, J\cdot\mathcal{O}^\dagger_\Delta(x) \ , \tag{2.8}$$

where $J_{\mu\nu}(x)$ is the usual inversion tensor for each vector index (each vector impurity) of the operator. Then the two-point function takes the form:

$$\langle\bar{\tilde{\mathcal{O}}}_\alpha(x)\tilde{\mathcal{O}}_\beta(0)\rangle \;=\; U_{\beta\gamma}\, e^{\delta_\gamma \log x^{-2}}\, U^\dagger_{\gamma\alpha} \;=\; S_{\beta\alpha} + T_{\beta\alpha}\log x^{-2} + \mathcal{O}\left((\log x^{-2})^2\right) . \tag{2.9}$$

Here we have expanded the full result in powers of $\log x^{-2}$. The overlap of the two states is defined as the zeroth-order term in the expansion, $S_{\beta\alpha} = U_{\beta\gamma}U^\dagger_{\gamma\alpha}$, and the matrix of anomalous dimensions in this basis is the first order term,

$$T_{\beta\alpha} = U_{\beta\gamma}\delta_\gamma U^\dagger_{\gamma\alpha} . \tag{2.10}$$

We note that (2.9), and hence the definitions of the overlap and the anomalous dimension matrix, are valid to all orders in the gauge coupling, and so can be in principle computed to all orders in λ' and g_2 for any basis $\tilde{\mathcal{O}}_\alpha$.

By initially relating this basis to the Δ-BMN basis we avoided all the problems of removing the 'non-universal' x-dependence on the right hand side of the correlator. Now we can forget about the Δ-BMN basis and follow the simple prescription discussed above: for an arbitrary basis, the overlap matrix $S_{\beta\alpha}$ and the anomalous dimensions matrix $T_{\beta\alpha}$ are the zeroth and the first term in the expansion of (2.9) in powers of $\log x^{-2}$.

We now consider the original BMN basis, for which we have

$$\langle\bar{\mathcal{O}}_\alpha(x)\,\mathcal{O}_\beta\rangle \;=\; S_{\beta\alpha} + T_{\beta\alpha}\log x^{-2} + \cdots , \tag{2.11}$$

and relate this basis to the isomorphic to string basis via (1.10),

$$\mathcal{O}^{\rm string}_\beta = U_{\beta\gamma}\mathcal{O}_\gamma \;\; , \qquad \bar{\mathcal{O}}^{\rm string}_\alpha = \bar{\mathcal{O}}_\delta U^\dagger_{\delta\alpha} . \tag{2.12}$$

In the isomorphic to string basis (which is automatically orthonormal, as explained earlier) we get

$$S^{\rm string} = \mathbb{1} = USU^\dagger , \qquad T^{\rm string} = UTU^\dagger . \tag{2.13}$$

We note that S is a Hermitian, positive matrix (it is a matrix of norms), therefore the matrix $S^{-1/2}$ is well-defined.[3] S is then diagonalised by the matrix $U := S^{-1/2}\cdot V$, where $V^\dagger V = \mathbb{1}$:

$$S \;\longrightarrow\; USU^\dagger = \mathbb{1} \;\; , \tag{2.14}$$

$$T \;\longrightarrow\; UTU^\dagger = V^\dagger(S^{-\frac{1}{2}}TS^{-\frac{1}{2}})V \;\; . \tag{2.15}$$

The arbitrariness contained in V, which is still left at this stage was fixed in [4, 17] by requiring that (1.6) holds and that the known three-string interaction vertex of the pp-wave light-cone string field theory [7,8] is reproduced from gauge theory matrix elements involving BMN states (operators) with two scalar impurities. This condition implies $V = \mathbb{1}$. Hence, the matrix of anomalous dimensions in the string basis is given by

$$\Gamma := T^{\rm string} = S^{-\frac{1}{2}}\,T\,S^{-\frac{1}{2}} . \tag{2.16}$$

[3] We would like to point our that this matrix $S^{-\frac{1}{2}}$ appears also in [16] and [22].

In the following sections we will show that, with the same choice of $V = \mathbb{1}$, the matrix elements of Γ between BMN operators with

- two vector impurities,
- one vector and one scalar impurity and, finally,
- an arbitrary number of scalar impurities,

precisely agree with the corresponding matrix elements of the interacting string Hamiltonian. We will consider all representations of BMN operators with vector or scalar impurities, i.e. symmetric traceless, antisymmetric and singlet. The inclusion of vector, mixed (scalar-vector) and scalar BMN operators allows us to study the correspondence for string states in all of the pp-wave light-cone directions.

Other studies of the dilatation operator in gauge theory and its interpretation in quantum mechanical models, which we do not pursue here, can be found in the recent papers [23, 24, 22, 25].

3 Tests of the correspondence in the two-impurity sector: scalar, mixed, and vector states

In this and the following sections we will need the expressions for the single-trace original BMN operators:

$$\mathcal{O}^J_{\rm vac} = \frac{1}{\sqrt{JN_0^J}}{\rm Tr}Z^J \ , \tag{3.1}$$

$$\mathcal{O}^J_{ij,m} = \mathcal{C}\left(\sum_{l=0}^{J} e^{\frac{2\pi i m l}{J}}{\rm Tr}\left(\phi_i Z^l \phi_j Z^{J-l}\right)\right) \ , \tag{3.2}$$

$$\mathcal{O}^J_{\mu\nu,m} = \frac{\mathcal{C}}{2}\left(\sum_{l=0}^{J} e^{\frac{2\pi i m l}{J}}{\rm Tr}\left[(D_\mu Z)Z^l(D_\nu Z)Z^{J-l}\right] + {\rm Tr}\left[(D_\mu D_\nu Z)Z^{J+1}\right]\right) \ , \tag{3.3}$$

$$\mathcal{O}^J_{i\mu,m} = \frac{\mathcal{C}}{\sqrt{2}}\left(\sum_{l=0}^{J} e^{\frac{2\pi i m l}{J}}{\rm Tr}\left[\phi_i Z^l(D_\mu Z)Z^{J-l}\right] + {\rm Tr}\left[(D_\mu \phi_i)Z^{J+1}\right]\right) \ , \tag{3.4}$$

where $i, j = 1, \ldots, 4$, $\mu, \nu = 1, \ldots, 4$ label the scalar and the vector impurities. Note that in writing $\mathcal{O}^J_{ij,m}$ and $\mathcal{O}^J_{\mu\nu,m}$ we have taken $i \neq j$ and $\mu \neq \nu$, where the above expressions take the simple form (3.2) and (3.3). We also defined

$$\mathcal{C} := \frac{1}{\sqrt{JN_0^{J+2}}} \ , \qquad N_0 := \frac{g^2}{2}\frac{N}{4\pi^2} \ . \tag{3.5}$$

The normalisation of the operators is such that their two-point functions take the canonical form in the planar limit. We also note that expressions for $\mathcal{O}^J_{\mu\nu,n}$ and $\mathcal{O}^J_{i\mu,m}$ contain appropriate compensating terms [26,27]. These terms are required in order for the corresponding operator to be conformal primaries in the BMN limit.

The operators in (3.2)–(3.4) are the original BMN operators. They are related to each other by supersymmetry transformations [27]. In order to test the correspondence, we need to use a different basis of operators which is isomorphic to string states, as discussed earlier. Importantly, the isomorphic to string operators $\tilde{\mathcal{O}}^J_{i\mu,m}$, $\tilde{\mathcal{O}}^J_{\mu\nu,m}$ are related to the $\tilde{\mathcal{O}}^J_{ij,m}$ in exactly the same way as the original BMN operators are. This is because the matrix U in (2.6) is a numerical matrix, i.e. it does not contain any fields and does not transform under supersymmetry. Hence, U is the same for scalar, vector and mixed impurity BMN operators.

We will also need the expressions for the double-trace operators

$$\mathcal{T}^{J,y}_{ij,m} = \ :\mathcal{O}^{y\cdot J}_{ij,m}: :O^{(1-y)\cdot J}_{\rm vac}: , \tag{3.6}$$

$$\mathcal{T}^{J,y}_{\mu\nu,m} = \ :\mathcal{O}^{y\cdot J}_{\mu\nu,m}: :O^{(1-y)\cdot J}_{\rm vac}: , \tag{3.7}$$

$$\mathcal{T}^{J,y}_{i\mu,m} = \ :\mathcal{O}^{y\cdot J}_{i\mu,m}: :O^{(1-y)\cdot J}_{\rm vac}: , \tag{3.8}$$

where $y \in (0,1)$.

From the three-string vertex of [7,8] one extracts the following matrix elements of the string Hamiltonian in the large-μ limit:

$$\frac{1}{\mu}\langle\mathcal{O}^J_{ij,m}|H_{\rm string}|\mathcal{T}^{J,y}_{ij,n}\rangle = -C_{\rm norm}\,\frac{\lambda'}{\pi^2 y}\,\sin^2(\pi m y) , \tag{3.9}$$

$$\frac{1}{\mu}\langle\mathcal{O}^J_{i\mu,m}|H_{\rm string}|\mathcal{T}^{J,y}_{i\mu,n}\rangle = 0 , \tag{3.10}$$

$$\frac{1}{\mu}\langle\mathcal{O}^J_{\mu\nu,m}|H_{\rm string}|\mathcal{T}^{J,y}_{\mu\nu,n}\rangle = C_{\rm norm}\,\frac{\lambda'}{\pi^2 y}\,\sin^2(\pi m y) , \tag{3.11}$$

for $\mu \neq \nu$ and $i \neq j$. The overall normalisation $C_{\rm norm}$ is left undetermined in string field theory. In order to get agreement with the field theory result we will set here[4]

$$C_{\rm norm} = -\frac{g_2}{2}\,\frac{\sqrt{y(1-y)}}{\sqrt{J}} . \tag{3.12}$$

Using this normalisation, (3.9) and (3.11) become

$$\frac{1}{\mu}\langle\mathcal{O}^J_{ij,m}|H_{\rm string}|\mathcal{T}^{J,y}_{ij,n}\rangle = -\frac{1}{\mu}\langle\mathcal{O}^J_{\mu\nu,m}|H_{\rm string}|\mathcal{T}^{J,y}_{\mu\nu,n}\rangle = \lambda'\,\frac{g_2}{\sqrt{J}}\,\frac{\sqrt{(1-y)/y}\,\sin^2(\pi m y)}{2\,\pi^2} . \tag{3.13}$$

[4] $C_{\rm norm}$ is further discussed in section 5.2, where we consider the case of arbitrary many impurities, see (5.24).

As mentioned earlier, the agreement of (3.9) with the corresponding gauge theory matrix elements was found in [17]. We will show that agreement with gauge theory holds also for (3.10) and (3.11).

We now need the explicit form of the matrices S and T in the original BMN basis. Both S and T have an expansion in powers of g_2, but in our analysis we will need their expressions only up to and including $\mathcal{O}(g_2)$ terms. We will also work at one loop in the Yang-Mills effective coupling λ', where the matrix T is of $\mathcal{O}(\lambda')$, whereas S is of $\mathcal{O}(1)$. In this case, (2.9) is simply

$$\langle \mathcal{O}_\alpha(0)\bar{\mathcal{O}}_\beta(x)\rangle = S_{\alpha\beta} + T_{\alpha\beta}\log(x\Lambda)^{-2} \ . \tag{3.14}$$

The pleasant fact is that expressions for S and T are closely related and can be obtained from the coefficients of the three-point functions, which were derived in [20,21] for BMN operators with two scalar and two vector impurities, respectively. We also need to know S and T in the case of mixed (i.e. one scalar and one vector) impurities. The three-point functions of such BMN operators were not considered previously, and they will be calculated in section 4.

The diagonal elements of S and T can be immediately obtained from

$$\langle \mathcal{O}^J_{AB,m}(0)\,\bar{\mathcal{O}}^J_{AB,n}(x)\rangle \;=\; \delta_{mn}\,\left(1+\lambda' m^2\log(\Lambda x)^{-2}\right)\ , \tag{3.15}$$

$$\langle \mathcal{T}^{J,y}_{AB,m}(0)\,\bar{\mathcal{T}}^{J,z}_{AB,n}(x)\rangle \;=\; \delta_{mn}\delta_{yz}\,\left(1+\lambda'(m^2/y^2)\log(\Lambda x)^{-2}\right)\ . \tag{3.16}$$

The previous expressions are valid up to $\mathcal{O}(\lambda')$ and $\mathcal{O}(g_2)$, and were derived originally in [6,20] for the scalar case, and in [26,29,30] for the mixed and vector case.

To determine the off-diagonal elements, we need to compute the two-point correlators $\langle \mathcal{T}^{J,y}_{AB,n}(0)\,\bar{\mathcal{O}}^J_{AB,n}(x)\rangle$. To this end, let us momentarily focus on the following class of *three*-point correlators,

$$G(x_1,x_2,x_3) = \langle \mathcal{O}^{y\cdot J}_{AB,n}(x_1)\mathcal{O}^{(1-y)\cdot J}_{\rm vac}(x_2)\bar{\mathcal{O}}^J_{AB,m}(x_3)\rangle\ , \tag{3.17}$$

where $A=(i,\mu)$ and $A\neq B$. On general grounds, these three-point function have the form [6,31,20,32]

$$G(x_1,x_2,x_3) = g_2 C_{m,ny}\left[1-\lambda'\left(a_{m,ny}\log(x_{31}\Lambda)^2 + b_{m,ny}\log(x_{32}x_{31}\Lambda/x_{12})\right)\right]\ , \tag{3.18}$$

where $g_2 C_{m,ny}$ is the tree-level contribution, with

$$C_{m,ny} := \frac{\sqrt{(1-y)/y}\,\sin^2(\pi m y)}{\sqrt{J}\,\pi^2(m-n/y)^2}\ , \tag{3.19}$$

and the coefficients $a_{m,ny}$ and $b_{m,ny}$ must be calculated in perturbation theory at $\mathcal{O}(\lambda')$. The *two*-point function $\langle \mathcal{T}^{J,y}_{AB,n}(0)\,\bar{\mathcal{O}}^J_{AB,m}(x)\rangle$ can easily be deduced from (3.17) by setting $x_{13}=x_{23}=x$ and $x_{12}=\Lambda^{-1}$ [32],

$$\langle \mathcal{T}^{J,y}_{AB,n}(0)\,\bar{\mathcal{O}}^J_{AB,m}(x)\rangle = g_2 C_{m,ny}\left[1+\lambda'\,(a_{m,ny}+b_{m,ny})\log(x\Lambda)^{-2}\right]\ . \tag{3.20}$$

The matrices S and T are then given, up to $\mathcal{O}(g_2)$, by

$$S = \begin{pmatrix} \delta_{mn} & g_2\, C_{m,qz} \\ g_2\, C_{py,n} & \delta_{pq} \end{pmatrix} + \mathcal{O}(g_2^2) = \mathbb{1} + g_2 s + \mathcal{O}(g_2^2)\ , \tag{3.21}$$

$$(3.22)$$

$$\begin{aligned} T &= \lambda' \begin{pmatrix} m^2\, \delta_{mn} & g_2\, C_{m,ny}\, (a+b)_{m,qz} \\ g_2\, C_{py,n}\, (a+b)_{py,n} & (p^2/y^2)\, \delta_{pq}\delta_{yz} \end{pmatrix} + \mathcal{O}(g_2^2) \qquad (3.23) \\ &\equiv d + g_2 t + \mathcal{O}(g_2^2)\ , \end{aligned}$$

with

$$d = \lambda' \begin{pmatrix} m^2\, \delta_{mn} & 0 \\ 0 & (p^2/y^2)\, \delta_{pq}\delta_{yz} \end{pmatrix}\ , \tag{3.24}$$

$$t = \lambda' \begin{pmatrix} 0 & C_{m,ny}\, (a+b)_{m,qz} \\ C_{py,n}\, (a+b)_{py,n} & 0 \end{pmatrix}\ . \tag{3.25}$$

It then follows that

$$S^{-1/2} = \mathbb{1} - g_2(s/2) + \mathcal{O}(g_2^2) \tag{3.26}$$

diagonalises S at $\mathcal{O}(g_2)$.

We now need to compute the explicit expressions for a^y_{mn} and b^y_{mn}, in the scalar case, mixed (scalar-vector) case, and finally in the vector case.

It is easy to compute at $\mathcal{O}(\lambda')$ the coefficient a^y_{mn} in planar perturbation theory, which turns out to be

$$a_{m,ny} = \frac{n^2}{y^2}\ , \tag{3.27}$$

independently of the type of impurity considered. Notice that this is exactly the $\mathcal{O}(\lambda')$ anomalous dimension[5] of the "small" BMN operator at x_1.

We will now explain how $b_{m,ny}$ is determined from the coefficients of the conformal three-point function. First we note that the correlator (3.17) does not take the conformal form (2.2) since the original BMN operators in (3.18) are not conformal primaries for $g_2 \neq 0$ due to operator mixing [33, 20]. However, at leading order in g_2, the only mixing effect which contributes to (2.2) is the presence of the double-trace corrections in the

[5]It is immediate to convince oneself that the Feynman diagrams contributing to the $\log x_{31}^2$ part of $\langle \mathcal{O}^{y\cdot J}_{AB,n}(x_1)\mathcal{O}^{(1-y)\cdot J}_{\rm vac}(x_2)\bar{\mathcal{O}}^{J}_{AB,m}(x_3)\rangle$ are those where the operator $\mathcal{O}^{(1-y)\cdot J}_{\rm vac}(x_2)$ does not participate in the interaction, i.e. they are precisely the Feynman diagrams contributing to the anomalous dimension of $\mathcal{O}^{y\cdot J}_{AB,n}(x_1)$ - embedded in a three-point function.

expression for the conjugate Δ-BMN operator.[6] Importantly, [28,21], these mixing effects cannot affect the remaining logarithm, $\lambda' \log x_{12}^2$, which can then be computed without taking into account mixing altogether. Hence, we can use the right hand side of the conformal expression (2.2) in order to compute the coefficient $b_{m,ny}$ in (3.18). Expanding the right-hand side of (2.2) to $\mathcal{O}(\lambda')$, and equating the coefficient of the $\log x_{12}^2$ to the corresponding term in (3.17), we obtain

$$g_2 C_{m,ny}\, b_{m,ny} = (m^2 - n^2/y^2)\; C({\scriptstyle A_n B_{-n}},\, \mathrm{vac}|\, {\scriptstyle A_m B_{-m}}) \;, \tag{3.28}$$

where $C({\scriptstyle C_n D_{-n}},\, \mathrm{vac}|\, {\scriptstyle A_m B_{-m}})$ is the coefficient C_{123} of the conformal three-point function $\langle \mathcal{O}^{J_1}_{CD,n}(x_1)\mathcal{O}^{J_2}_{\mathrm{vac}}(x_2)\bar{\mathcal{O}}^{J}_{AB,m}(x_3)\rangle$. We used $\Delta_1 = J_1 + 2 + \lambda' n^2/y^2$, $\Delta_2 = J$, and $\Delta_3 = J + 2 + \lambda' m^2$.

Equation (3.28) determines $b_{m,ny}$ in terms of the coefficients $C({\scriptstyle C_n D_{-n}},\, \mathrm{vac}|\, {\scriptstyle A_m B_{-m}})$ of the three-point function. These coefficients for BMN operators with two scalar impurities, one scalar and one vector impurity, and two vector impurities are given by:

$$C(k_n l_{-n},\, \mathrm{vac}|\, i_m j_{-m}) \;=\; C^{\mathrm{vac}}_{123}\frac{2\,\sin^2(\pi m y)}{y\,\pi^2(m^2-\frac{n^2}{y^2})^2}\left(\delta_{i(k}\delta_{l)j}\; m^2 + \delta_{i[k}\delta_{l]j}\,\frac{mn}{y} + \tfrac{1}{4}\delta_{ij}\delta_{kl}\,\frac{n^2}{y^2}\right) \tag{3.29}$$

$$C(j_n \nu_{-n},\, \mathrm{vac}|\, i_m \mu_{-m}) \;=\; C^{\mathrm{vac}}_{123}\frac{2\,\sin^2(\pi m y)}{y\,\pi^2(m^2-\frac{n^2}{y^2})^2}\delta_{ij}\delta_{\mu\nu}\,\frac{1}{4}\left(m+\frac{n}{y}\right)^2 \;, \tag{3.30}$$

$$C(\rho_n \sigma_{-n},\, \mathrm{vac}|\, \mu_m \nu_{-m}) \;=\; C^{\mathrm{vac}}_{123}\frac{2\,\sin^2(\pi m y)}{y\,\pi^2(m^2-\frac{n^2}{y^2})^2}\left(\delta_{\mu(\rho}\delta_{\sigma)\nu}\,\frac{n^2}{y^2} + \delta_{\mu[\rho}\delta_{\sigma]\nu}\,\frac{mn}{y} + \tfrac{1}{4}\delta_{\mu\nu}\delta_{\rho\sigma}m^2\right) \;, \tag{3.31}$$

where $C^{\mathrm{vac}}_{123} = \sqrt{JJ_1J_2}/N = (g_2/\sqrt{J})\sqrt{y(1-y)}$, and the symmetric traceless and anti-symmetric traceless combinations of two Kronecker deltas are defined as

$$\delta_{i(k}\delta_{l)j} = \tfrac{1}{2}(\delta_{ik}\delta_{lj} + \delta_{il}\delta_{kj}) - \tfrac{1}{4}\delta_{ij}\delta_{kl} \;, \quad \delta_{i[k}\delta_{l]j} = \tfrac{1}{2}(\delta_{ik}\delta_{lj} - \delta_{il}\delta_{kj}) \;. \tag{3.32}$$

The three-point function coefficient for scalars (3.29) was derived in [20] (the simple case $n = 0$ was first obtained in [31]), whereas that for the vectors, (3.31), was recently obtained in [21]. The three-point function coefficient (3.30) for the case of mixed scalar-vector impurities is a new result, and its derivation is presented in section 4 of this paper.

From (3.29)–(3.31) and (3.28), it is then immediate to derive the coefficients $b_{m,ny}$

[6] This is because the double-trace corrections to the single-trace expression for an original BMN operator is of $\mathcal{O}(g_2)$, i.e. suppressed with $1/N$. This can be compensated by factorising the three-point function into a product of two two-point functions, which is possible only for the double-trace mixing in the operator $\bar{\mathcal{O}}$.

which correspond to considering scalar, mixed, or vector BMN operators in (3.17):

$$[b_{m,ny}]_{\text{scalar}} = m^2 - \frac{mn}{y} , \tag{3.33}$$

$$[b_{m,ny}]_{\text{scalar}-\text{vector}} = \frac{1}{2}\left(m^2 - \frac{n^2}{y^2}\right) , \tag{3.34}$$

$$[b_{m,ny}]_{\text{vector}} = -\frac{n^2}{y^2} + \frac{mn}{y} . \tag{3.35}$$

In conclusion, using (3.20) we get, up to $\mathcal{O}(g_2)$,

$$\langle \mathcal{T}^{J,y}_{ij,n}(0)\,\bar{\mathcal{O}}^J_{ij,m}(x)\rangle = g_2 C_{m,ny}\left[1 + \lambda'\left(m^2 - \frac{mn}{y} + \frac{n^2}{y^2}\right)\log(x\Lambda)^{-2}\right] , \tag{3.36}$$

$$\langle \mathcal{T}^{J,y}_{j\nu,n}(0)\,\bar{\mathcal{O}}^J_{i\mu,m}(x)\rangle = g_2 C_{m,ny}\left[1 + \frac{\lambda'}{2}\left(m^2 + \frac{n^2}{y^2}\right)\log(x\Lambda)^{-2}\right]\delta_{ij}\delta_{\mu\nu} , \tag{3.37}$$

$$\langle \mathcal{T}^{J,y}_{\mu\nu,n}(0)\,\bar{\mathcal{O}}^J_{\mu\nu,m}(x)\rangle = g_2 C_{m,ny}\left[1 + \lambda'\left(\frac{mn}{y}\right)\log(x\Lambda)^{-2}\right] . \tag{3.38}$$

We will now make use of the expressions for these three correlators to construct the matrix T, and therefore the matrix Γ dual to $H^{\text{int}}_{\text{string}}$, in the three cases of BMN states with (i) two scalar, (ii) one scalar and one vector, and finally (iii) two vector impurities. These three cases are addressed separately below.

3.1 Matrix elements with scalar BMN states

This case was first analysed in [17], and we review it here for completeness.

Substituting (3.27) and (3.33) in (3.23), we find that the matrix T_{scalar} is given, at $\mathcal{O}(g_2)$, by[7]

$$\begin{aligned} T_{\text{scalar}} &= \lambda' \begin{pmatrix} m^2 & g_2 C_{m,ny}\,(m^2 - mn/y + n^2/y^2) \\ g_2 C_{ny,m}\,(m^2 - mn/y + n^2/y^2) & n^2/y^2 \end{pmatrix} \\ &\equiv d + g_2\, t_{\text{scalar}} . \end{aligned} \tag{3.39}$$

Multiplying it on the left and on the right by $S^{-1/2} = 1\!\!1 - g_2(s/2) + \mathcal{O}(g_2^2)$ we get the expression for the matrix Γ introduced in (2.16) at $\mathcal{O}(g_2)$:

$$\Gamma_{\text{scalar}} = d + g_2\left[t_{\text{scalar}} - (1/2)\{s\,,\,d\}\right] \tag{3.40}$$

[7]We use a somewhat simplified, but correct, notation for the indices of the matrices S and T.

$$= \lambda' \begin{pmatrix} m^2 & (g_2 C_{m,ny}/2)\,(m-n/y)^2 \\ (g_2 C_{ny,m}/2)\,(m-n/y)^2 & n^2/y^2 \end{pmatrix} ,$$

from which it follows, using the definition (3.19) of $C_{m,ny}$,

$$\langle \mathcal{O}^J_{ij,m} \, |\Gamma_{\rm scalar}| \, \mathcal{T}^{J,y}_{ij,n} \rangle = \lambda' \, \frac{g_2}{\sqrt{J}} \frac{\sqrt{(1-y)/y}\, \sin^2(\pi m y)}{2\,\pi^2} \; . \qquad (3.41)$$

This result was first found in [17]. (3.41) agrees with (3.9) after choosing the normalisation (3.12) for the string result.

3.2 Matrix elements with mixed BMN states

In this case, using (3.27) and (3.34) we can determine the matrix $T_{\rm mixed}$ in (3.23) for the case of mixed impurities. It is given, at $\mathcal{O}(g_2)$, by the following expression:

$$\begin{aligned} T_{\rm mixed} \;&=\; \lambda' \begin{pmatrix} m^2 & g_2 C_{m,ny}\,(m^2+n^2/y^2)/2 \\ g_2 C_{ny,m}\,(m^2+n^2/y^2)/2 & n^2/y^2 \end{pmatrix} \qquad (3.42) \\ &\equiv\; d \,+\, g_2\, t_{\rm mixed} \; , \end{aligned}$$

where we used $a_{m,ny} + b^{\rm mixed}_{m,ny} = (m^2+n^2/y^2)/2$. It then follows that

$$\Gamma_{\rm mixed} \;=\; d + g_2 \left[t_{\rm mixed} - (1/2) \{ s \, , \, d \} \right] \;=\; \lambda' \begin{pmatrix} m^2 & 0 \\ 0 & n^2/y^2 \end{pmatrix} , \qquad (3.43)$$

and hence

$$\langle \mathcal{O}^J_{i\mu,m} \, |\Gamma_{\rm mixed}| \, \mathcal{T}^{J,y}_{i\mu,n} \rangle \;=\; 0 \; . \qquad (3.44)$$

This verifies in gauge theory the vanishing of the three-string amplitude (3.10) between states with one scalar and one vector impurity, which was predicted in [8].

3.3 Matrix elements with vector BMN states

Finally, we study the case of vector BMN impurities. Using (3.27) and (3.35) we obtain the matrix $T_{\rm vector}$ in (3.23) for the case of vector impurities. At $\mathcal{O}(g_2)$, it is given by:

$$T_{\rm vector} \;=\; \lambda' \begin{pmatrix} m^2 & g_2 C_{m,ny}\,(mn/y) \\ g_2 C_{ny,m}\,(mn/y) & n^2/y^2 \end{pmatrix} \qquad (3.45)$$

$$\equiv \quad d \ + \ g_2 \, t_{\rm vector} \ ,$$

where we used $a_{m,ny} + b^{\rm vector}_{m,ny} = mn/y$. It then follows that

$$\Gamma_{\rm vector} \;=\; d + g_2 \left[t_{\rm vector} - (1/2) \{ s \, , \, d \} \right] \tag{3.46}$$

$$= \lambda' \begin{pmatrix} m^2 & -(g_2 C_{m,ny}/2)\,(m-n/y)^2 \\ -(g_2 C_{m,ny}/2)\,(m-n/y)^2 & n^2/y^2 \end{pmatrix} ,$$

from which we get

$$\langle \mathcal{O}^J_{\mu\nu,m} \, | \Gamma_{\rm vector} | \, \mathcal{T}^{J,y}_{\mu\nu,n} \rangle = -\lambda' \, \frac{g_2}{\sqrt{J}} \frac{\sqrt{(1-y)/y} \, \sin^2(\pi m y)}{2\,\pi^2} \tag{3.47}$$

$$= \; -\langle \mathcal{O}^J_{ij,m} \, | \Gamma_{\rm scalar} | \, \mathcal{T}^{J,y}_{ij,n} \rangle \ .$$

As advertised earlier, the off-diagonal elements $\langle \mathcal{O}^J_{\mu\nu,m} \, | \Gamma_{\rm vector} | \, \mathcal{T}^{J,y}_{\mu\nu,n} \rangle$ of $\Gamma_{\rm vector}$ are precisely the opposite of the corresponding elements $\langle \mathcal{O}^J_{ij,m} \, | \Gamma_{\rm scalar} | \, \mathcal{T}^{J,y}_{ij,n} \rangle$ of $\Gamma_{\rm scalar}$. This again had been predicted in string field theory in [8]. As explained in the introduction, this signals the spontaneous breaking of $\mathbb{Z}_2$ symmetry in pp-wave string theory.

3.4 Generalisation to all representations for two-impurity BMN states

Finally, we extend our previous computations to include all representations of scalar and vector BMN operators with two impurities.

We recall here the results from the previous sections:

$$\langle \mathcal{O}^J_{ij,m} \, | \Gamma_{\rm scalar} | \, \mathcal{T}^{J,y}_{ij,n} \rangle \;=\; -\langle \mathcal{O}^J_{\mu\nu,m} \, | \Gamma_{\rm vector} | \, \mathcal{T}^{J,y}_{\mu\nu,n} \rangle \;=\; \lambda' \, \frac{g_2}{\sqrt{J}} \frac{\sqrt{(1-y)/y} \, \sin^2(\pi m y)}{2\,\pi^2} \ ,$$

$$\langle \mathcal{O}^J_{k\mu,m} \, | \Gamma_{\rm mixed} | \, \mathcal{T}^{J,y}_{l\nu,n} \rangle \;=\; 0 \ , \tag{3.48}$$

($i \neq j$, $\mu \neq \nu$) which correspond to the string field theory amplitude (3.9), (3.10) and (3.11). From these results it is immediate to obtain

$$\langle \mathcal{O}^J_{\nu\mu,m} \, | \, \Gamma_{\rm vector} \, | \, \mathcal{T}^{J,y}_{\mu\nu,n} \rangle = \langle \mathcal{O}^J_{\mu\nu,m} \, | \, \Gamma_{\rm vector} \, | \, \mathcal{T}^{J,y}_{\mu\nu,n} \rangle \ , \tag{3.49}$$

since this amounts to complex conjugate the BMN phase factor contained in $\mathcal{O}^J_{\mu\nu,m}$, i.e. to exchange $m \to -m$ (same considerations apply for the scalar amplitude). Equation (3.49) follows since the first expression in (3.48) is even in m. Therefore we can at once obtain the result for the symmetric-traceless and antisymmetric representations for vectors:

$$\langle \mathcal{O}^J_{(\mu\,\nu),m} \, | \, \Gamma_{\rm vector} \, | \, \mathcal{T}^{J,y}_{(\rho\sigma),n} \rangle \;=\; -\lambda' \frac{g_2}{\sqrt{J}} \, \frac{\sqrt{(1-y)/y} \, \sin^2(\pi m y)}{\pi^2} \delta_{\mu(\rho} \delta_{\sigma)\nu} \ , \tag{3.50}$$

$$\langle \mathcal{O}^J_{[\mu\nu],m} \,|\, \Gamma_{\rm vector} \,|\, \mathcal{T}^{J,y}_{[\rho\sigma],n} \rangle \;=\; 0 \;, \tag{3.51}$$

whereas for scalars,

$$\langle \mathcal{O}^J_{(i\,j),m} \,|\, \Gamma_{\rm scalar} \,|\, \mathcal{T}^{J,y}_{(kl),n} \rangle \;=\; \lambda' \frac{g_2}{\sqrt{J}} \frac{\sqrt{(1-y)/y}\, \sin^2(\pi m y)}{\pi^2} \delta_{i(k}\delta_{l)j} \;, \tag{3.52}$$

$$\langle \mathcal{O}^J_{[ij],m} \,|\, \Gamma_{\rm scalar} \,|\, \mathcal{T}^{J,y}_{[kl],n} \rangle \;=\; 0 \;. \tag{3.53}$$

Here we have defined

$$\mathcal{O}_{(\mu\nu)} = \frac{1}{2}(\mathcal{O}_{\mu\nu} + \mathcal{O}_{\nu\mu}) - \frac{\delta_{\mu\nu}}{4} \sum_\rho \mathcal{O}_{\rho\rho} \;, \qquad \mathcal{O}_{[\mu\nu]} = \frac{1}{2}(\mathcal{O}_{\mu\nu} - \mathcal{O}_{\nu\mu}) \;. \tag{3.54}$$

The vector singlet case can be treated instantly by noticing that the three-point function coefficient for vector singlets is actually the same as the three-point function coefficient for the symmetric-traceless scalars, as it can be seen by comparing (3.29) to (3.31). This, together with (3.52), immediately implies that[8]

$$\begin{aligned} \langle \mathcal{O}^J_{\mathrm{vector}\,\mathbf{1},m} \,|\, \Gamma_{\rm vector} \,|\, \mathcal{T}^{J,y}_{\mathrm{vector}\,\mathbf{1},n} \rangle \;&=\; \lambda' \frac{g_2}{\sqrt{J}} \frac{\sqrt{(1-y)/y}\, \sin^2(\pi m y)}{\pi^2} \\ &=\; -\langle \mathcal{O}^J_{\mathrm{scalar}\,\mathbf{1},m} \,|\, \Gamma_{\rm scalar} \,|\, \mathcal{T}^{J,y}_{\mathrm{scalar}\,\mathbf{1},n} \rangle \;, \end{aligned} \tag{3.55}$$

where $\mathcal{O}_{\mathbf{1}} = (1/2) \sum_\mu \mathcal{O}_{\mu\mu}$. We notice that the result for the scalar singlet amplitude $\langle \mathcal{O}^J_{\mathrm{scalar}\,\mathbf{1},m} \,|\, \Gamma_{\rm scalar} \,|\, \mathcal{T}^{J,y}_{\mathrm{scalar}\,\mathbf{1},n} \rangle$ agrees with the result found in [18]. The opposite sign in (3.55) for the vector singlet compared to the scalar singlet case is again a manifestation of the (spontaneously broken) $\mathbb{Z}_2$ symmetry in pp-wave string theory.

4 A technical aside: three-point function with mixed impurities

In this section we derive the expression (3.30). The reader not interested in the actual calculation can turn a few pages and proceed to the next section.

Here we would like to compute the coefficient of the three-point function of one vacuum operator (3.1) and two conformal primary Δ-BMN operators with one scalar and one vector impurity,

$$\tilde{\mathcal{O}}^J_{i\mu,m} = \mathcal{O}^J_{i\mu,m} + \cdots \;, \tag{4.1}$$

[8] The reader willing to derive explicitly the result (3.55) for vector singlets should be aware that, for singlets, (3.20) should be modified to $\langle \mathcal{T}^{J,y}_{\mathbf{1},n}(0)\, \bar{\mathcal{O}}^J_{\mathbf{1},m}(x) \rangle = g_2 C_{m,ny} \left[1 + \lambda' \,(a_{m,ny} + b_{m,ny}) \log(x\Lambda)^{-2}\right] + g_2 C_{-m,ny} \left[1 + \lambda' \,(a_{-m,ny} + b_{-m,ny}) \log(x\Lambda)^{-2}\right]$, and that, from the result (3.31) derived in [21], it follows that $[b_{m,ny}]_{\mathrm{vector},\,\mathbf{1}} = m^2 - mn/y$.

where $\mathcal{O}^J_{i\mu,m}$ is defined in (3.4). The operator $\tilde{\mathcal{O}}^J_{i\mu,m}$ has a definite scaling dimension, $\Delta_n = \Delta_{\rm cl} + \delta_n$, which implies that the single-trace expression $\mathcal{O}^J_{i\mu,m}$ on the right hand side of (4.1) must be accompanied by multi-trace corrections (and other mixing effects) at higher orders in g_2 [33, 20]. The dots on the right hand side of (4.1) stand for these corrections.

Nevertheless, our strategy is to study the three-point correlator of the *original* BMN operators $\mathcal{O}^J_{i\mu,m}$,

$$\langle \mathcal{O}^{y\cdot J}_{j\nu,n}(x_1)\mathcal{O}^{(1-y)\cdot J}_{\rm vac}(x_2)\bar{\mathcal{O}}^J_{i\mu,m}(x_3)\rangle = \tag{4.2}$$

$$g_2 C_{m,ny}\left[1-\lambda'\left(a_{m,ny}\log(x_{31}\Lambda)^2 + b^{\rm mixed}_{m,ny}\log\left|\frac{x_{32}x_{31}\Lambda}{x_{12}}\right|\right)\right]\ \delta_{\mu\nu}\delta_{ij}\ ,$$

and to focus on the computation of the coefficient $b^{\rm mixed}_{m,ny}$ of the $\log x_{12}$. This is because, for reasons explained in the paragraph below (3.27), this coefficient can be computed without taking into account mixing altogether, and is directly related to the coefficient of the three-point function of conformal primary BMN operators with mixed impurities through (3.28). Therefore, from now on we will work with the original BMN operator (3.4).

The mixed BMN operator in (3.4) contains two terms: a "pure" BMN part and a compensating term, first and second term on the right hand side of (3.4), respectively. The Feynman diagrams contributing to the Green's function in (4.2) can be divided into two classes: those obtained by taking only the pure BMN part of $\mathcal{O}^{y\cdot J}_{j\nu,n}(x_1)$ and $\bar{\mathcal{O}}^J{}_{i\mu,m}(x_3)$ and those where the compensating part is taken (in one or both operators). As already explained, we will focus only on diagrams which can produce a $\log x_{12}$ dependence, and both classes of diagrams contribute to the coefficient $b^{\rm mixed}_{m,ny}$ in (4.2). For the sake of clarity, we quote here the results from these two classes of diagrams:

$$\left[b^{\rm mixed}_{m,ny}\right]_{\rm BMN} = m\left(m-\frac{n}{y}\right)\ , \tag{4.3}$$

$$\left[b^{\rm mixed}_{m,ny}\right]_{\rm comp} = -\frac{1}{2}\left(m-\frac{n}{y}\right)^2\ . \tag{4.4}$$

The total result

$$[b_{m,ny}]_{\rm scalar-vector} = \left[b^{\rm mixed}_{m,ny}\right]_{\rm BMN} + \left[b^{\rm mixed}_{m,ny}\right]_{\rm comp} = \frac{1}{2}\left(m^2-\frac{n^2}{y^2}\right)\ , \tag{4.5}$$

was anticipated in (3.34). We now compute separately these two classes of diagrams. Our notation and conventions are summarised in Appendix B.

4.1 Diagrams originating from the "pure" BMN parts

We consider first the diagrams where the scalar impurity interacts. These are represented in Figure 1. The result for these diagrams is:

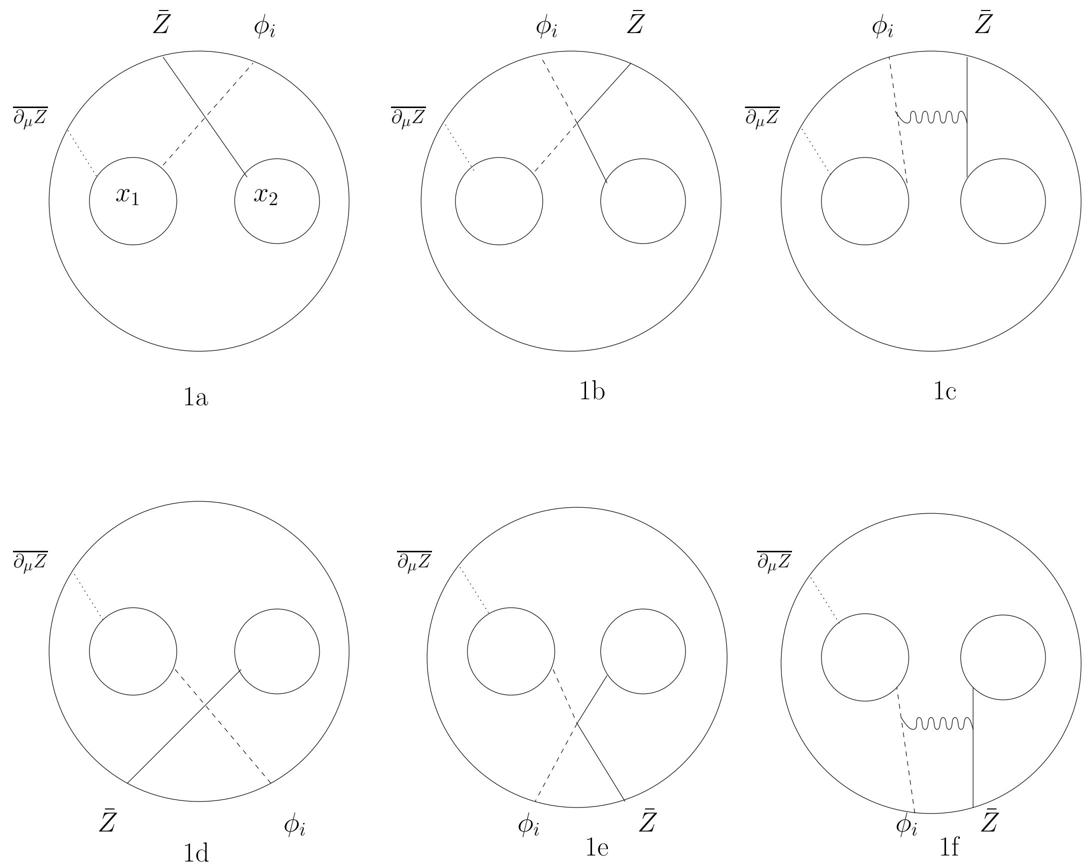

Figure 1: Diagrams with scalar impurity interacting. Diagrams 1a and 1d have positive signs, all the others have negative signs.

$$\left(\frac{2}{g^2}\right)\left(\frac{g^2}{2}\right)^4 \cdot 2(P_I - P_{II} + \bar{P}_I - \bar{P}_{II}) \cdot (2\delta_{\mu\nu})\delta_{ij} \cdot X \ . \tag{4.6}$$

The first term on the right hand side of (4.6) comes from the diagram 1a (the coefficient of 2 is easily seen from $-V_F$ in (B.4)), the second term is the sum of diagrams 1b and 1c. The relative sign is also immediately seen from the commutators in $-V_F$. We have taken into account the fact that diagrams 1b and 1c give the same contribution.[9]

[9]This is a simple corollary of the cancellation of D-terms against gluon interactions and self-energies in three-point functions of BMN operators at $\mathcal{O}(\lambda')$ (in the complex basis) [34, 6, 31, 20]. In our case, self-interactions diagrams do not participate since they cannot generate $\log x_{12}^2$ terms at order $\mathcal{O}(\lambda')$.

The third and fourth term in (4.6) come from the mirror diagrams 1d, 1e and 1f, where the ϕ interaction is now at the bottom. The factor $2\delta_{\mu\nu}$ comes from the free contraction[10] of the $\overline{D_\mu Z}$ impurity with the $D_\nu Z$ impurity. The coefficients P_I and P_{II} come from summing over the BMN phase factors, and their expressions are summarised in Appendix C. Mirror diagrams are associated with the complex conjugate coefficients $\bar{P}_I$ and $\bar{P}_{II}$. Finally, the function X is defined in (D.2) of Appendix D.

There are additional diagrams, drawn in Figures 2 and 3, where the interaction involves now the vector impurity.

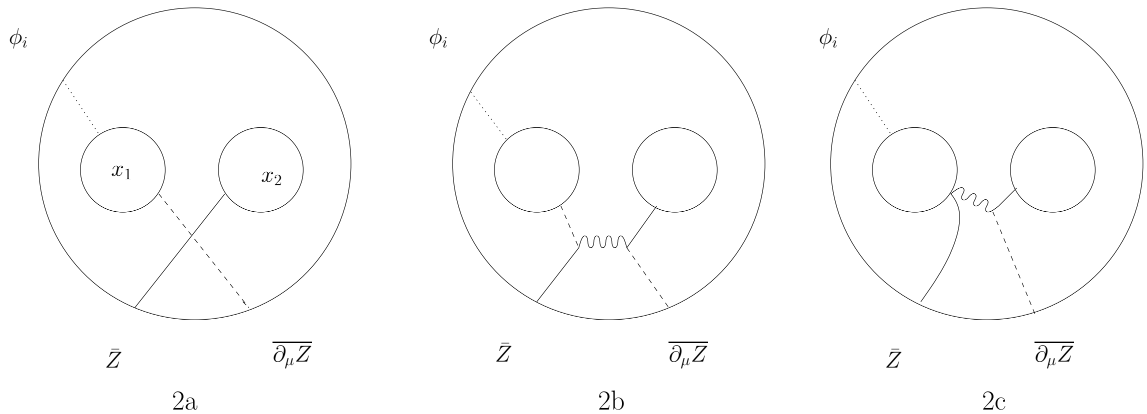

Figure 2: Diagrams with vector impurity interacting associated to P_I.

These diagrams are identical to those in Figure 5 and 6 of [21], with the only modification that the non-interacting impurity is now a scalar impurity (whereas in Figure 5 and 6 of [21] it was a vector impurity). We will not compute again these diagrams, and instead borrow the result from [21]. Their contribution turns out to be precisely the same of the contribution (4.6) from the diagrams where the scalar impurity interacts.

The final result for the pure BMN diagrams is therefore:

$$\left(\frac{2}{g^2}\right)\left(\frac{g^2}{2}\right)^4 \cdot 8(P_I - P_{II} + \bar{P}_I - \bar{P}_{II}) \cdot \delta_{\mu\nu}\delta_{ij} \cdot X \ . \tag{4.7}$$

This quantity has still to be multiplied by the normalisations of the operators, in which we include an extra factor of $J_2 = (1-y) \cdot J$ coming from inequivalent Wick contractions of $\mathcal{O}_{\rm vac}$ with the rest,

$$\frac{1}{\sqrt{J}}\frac{\sqrt{1-y}}{\sqrt{y}}\left(\frac{1}{\sqrt{2}}\right)^2 \ . \tag{4.8}$$

[10]For an extensive discussion of the treatment of BMN operators with vector impurities, the reader is referred to [21]. Free contractions of vector impurities are discussed in Eq. (34) of that paper, and the main results can be summarised as $\langle \overline{D_\mu Z}\ D_\nu Z\rangle_{\rm free} = 2\,\delta_{\mu\nu}$, and $\langle \overline{Z}\ D_\nu Z\rangle_{\rm free} = 0$.

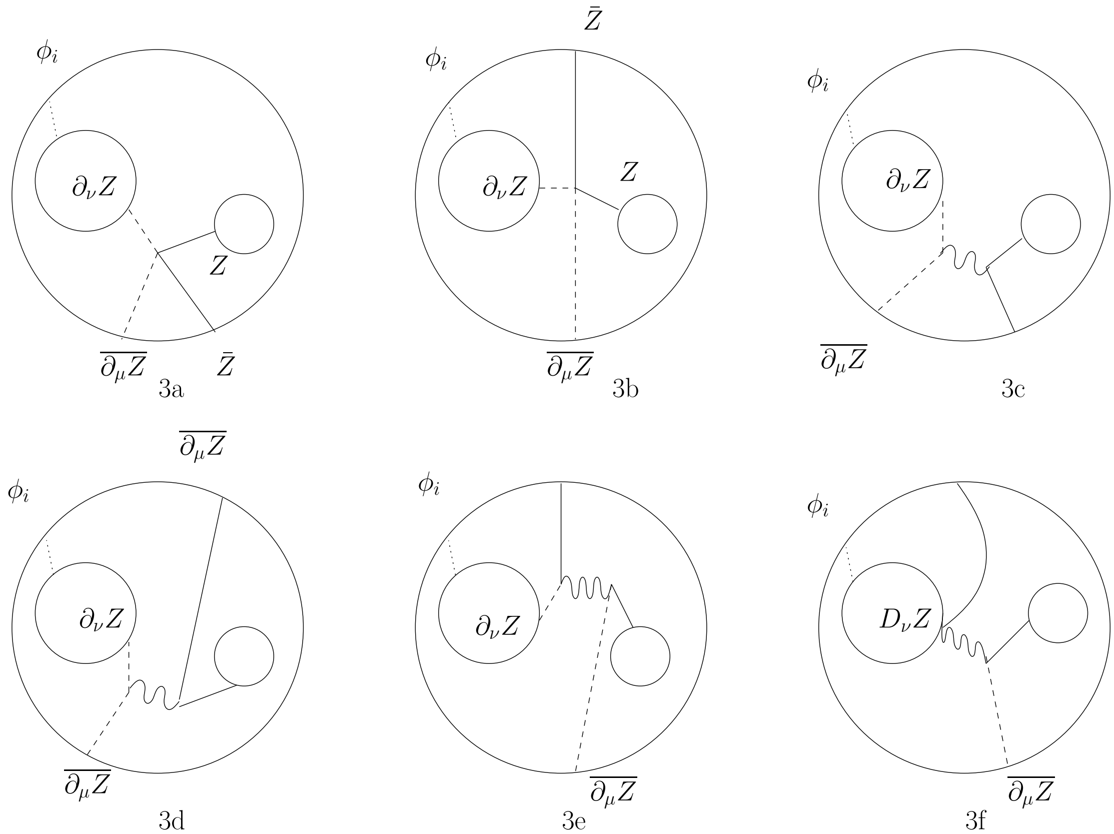

Figure 3: Diagrams with vector impurity interacting associated to P_{II}.

4.2 Diagrams from compensating terms

The compensating term is present in both operators at x_3 and x_1, therefore there are three subclasses of diagrams: (i) diagrams with compensating term in the "external" operator at x_3 and pure BMN part in the "internal" (small) operator at x_1; (ii) diagrams where the compensating term of both operators at x_1 and x_3 is considered; and (iii) diagrams where the compensating term of the small operator at x_1 and the pure BMN part of the operator at x_3 are taken.

Each diagram in class (i) vanishes separately, since the only way to contract the impurity $D_\nu Z$ in $\mathcal{O}^{y\cdot J}_{j\nu,n}(x_1)$ is with a $\bar{Z}$ in $\bar{\mathcal{O}}^{J}_{i\mu,m}(x_3)$, and this contraction vanishes (see footnote 10). Moreover , it is not difficult to see that the total contribution of the diagrams in class (ii) vanishes. Hence we are left with diagrams in class (iii), which we now discuss.

We start by considering diagrams without gluons. The first three diagram are represented in Figure 4, and their contribution is

$$\left(\frac{2}{g^2}\right)\left(\frac{g^2}{2}\right)^4 \cdot \left(2X - X - X\cdot \bar{q}^{J_2}\right)\, \delta_{\mu\nu}\delta_{ij}\ , \tag{4.9}$$

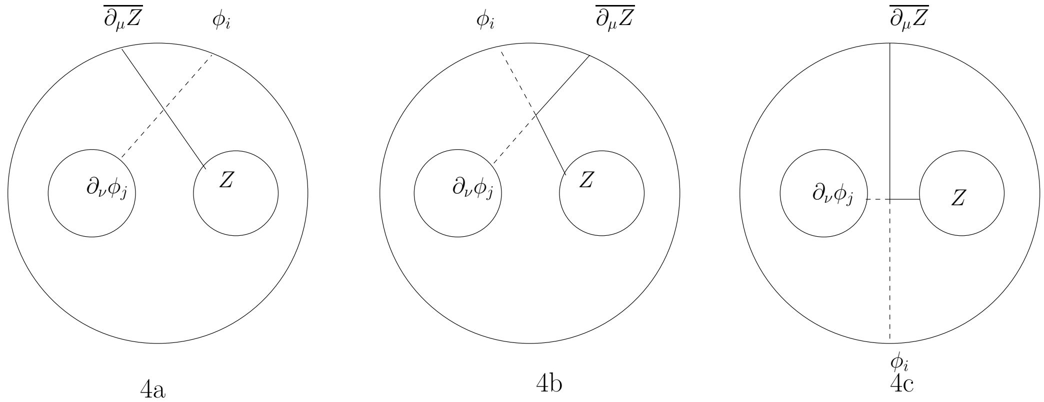

Figure 4: Diagrams originating from the compensating term in the 'internal' operator (at x_1). In these diagrams no gluons are emitted or exchanged.

where $q = \exp(2\pi m/J)$ is the phase factor of the BMN operator at x_3. The first term in the right hand side of (4.9) comes from the first diagram in Figure 4. This diagram is equal to the first diagram in Figure 5 of [21], from which we borrowed the result. The factor of 2 is easily seen from the term $\mathrm{Tr}(2\,Z\phi_i\bar{Z}\phi_i)$ in $-V_F$, see (B.4). The opposite sign of the second term in (4.9) is also seen from the term $-\mathrm{Tr}(\phi_i\phi_i Z\bar{Z})$ in $-V_F$. The third term comes from the term $-\mathrm{Tr}(\phi_i\phi_i\bar{Z}Z)$ in $-V_F$, and carries a BMN phase factor $\bar{q}^{J_2}$. There are also mirror diagrams, where the interaction occurs at the bottom of the diagram (similarly to the fourth, fifth and sixth diagram in Figure 1). As usual, their effect is to add the complex conjugate of the previous result, so that the final result for diagrams without gluons is:

$$\left(\frac{2}{g^2}\right)\left(\frac{g^2}{2}\right)^4 \cdot 2X \cdot \left(1 - \cos\left(2\pi m y\right)\right)\, \delta_{\mu\nu}\delta_{ij}\ . \tag{4.10}$$

We now consider diagrams where a gluon is emitted from the covariant derivative $D_\nu\phi_j$ at x_1. These gluon emission diagrams are represented in Figure 5. The total result for them is:

$$\left(\frac{2}{g^2}\right)\left(\frac{g^2}{2}\right)^4 \cdot \left(3X - 3X \cdot \bar{q}^{J_1}\right)\, \delta_{\mu\nu}\delta_{ij}\ . \tag{4.11}$$

The first term on the right hand side of (4.11) corresponds to the first diagram in Figure 3. This diagram was computed in [21] (it is the third diagram in Figure 5), from which we took the result. The only difference is that in the present case it is accompanied by phase factor equal to 1. The second term come from the second diagram in Figure 3, and carries a BMN phase factor equal to $\bar{q}^{J_1}$. Again, there are also mirror diagrams, where the interaction occurs at the bottom of the diagram. Their effect is to add the complex conjugate of the previous result, so that the final result for diagrams with gluon emission is:

$$\left(\frac{2}{g^2}\right)\left(\frac{g^2}{2}\right)^4 \cdot 6X \cdot \left(1 - \cos\left(2\pi m y\right)\right)\, \delta_{\mu\nu}\delta_{ij}\ . \tag{4.12}$$

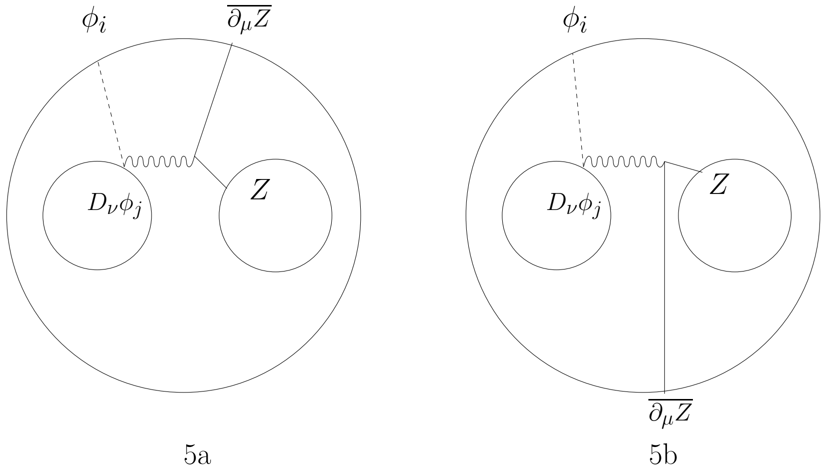

Figure 5: Gluon emission diagrams originating from the compensating term in the internal operator.

Finally, we have to consider gluon interaction diagrams. These are depicted in Figure 6 and, as before, there are also mirror diagrams, where the interaction occurs at the bottom of each diagram. However, each of the diagrams vanishes separately (this is to

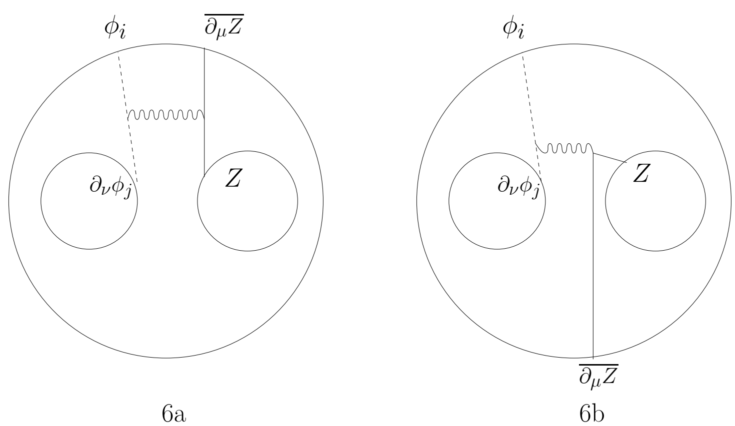

Figure 6: Gluon interaction diagrams, from the compensating term in the internal operator.

be contrasted with the case of scalar interactions, where gluon interactions double up the result for the interaction from the scalar potential, as discussed in the previous subsection). This was shown again in [21] (second diagram in Figure 5 of that paper).

Adding (4.10) and (4.12) we get the total contribution of the compensating term diagrams,

$$\left(\frac{2}{g^2}\right)\left(\frac{g^2}{2}\right)^4 \cdot 16X \cdot \sin^2(\pi m y)\,\delta_{\mu\nu}\delta_{ij}\ . \tag{4.13}$$

The result (4.13) has still to be multiplied by the normalisations of the operators (4.8).

4.3 Summary: the result for mixed impurities

We add the results (4.7) and (4.13), and use (C.3) of Appendix C, to get the total result

$$\left(\frac{g^2}{2}\right)^3 \cdot (-16\,X) \cdot (\delta_{\mu\nu}\delta_{ij}) \cdot \frac{m+n/y}{m-n/y} \sin^2 \pi m y \ . \tag{4.14}$$

Multiplying this result by the normalisation (4.8), and amputating a factor of $[(g^2/2)\,\Delta(x_{13})]^2$ we obtain

$$\begin{aligned} &\langle \mathcal{O}^{y\cdot J}_{j\nu,n}(x_1)\mathcal{O}^{(1-y)\cdot J}_{\text{vac}}(x_2)\bar{\mathcal{O}}^{J}_{i\mu,m}(x_3)\rangle\Big|_{\log x_{12}\ \text{term}} = \\ &\frac{1}{\sqrt{J}}\frac{\sqrt{1-y}}{\sqrt{y}}\left(\frac{1}{\sqrt{2}}\right)^2\left(\frac{g^2}{2}\right)\cdot\left(-16\,\frac{m+n/y}{m-n/y}\,\sin^2\pi m y\right)\frac{\log(x_{12}\Lambda)^{-1}}{8\pi^2}\delta_{\mu\nu}\delta_{ij} \ = \\ &-\lambda' g_2\, C_{m,ny}\cdot\frac{1}{2}\left(m^2-\frac{n^2}{y^2}\right)\cdot\log(x_{12}\Lambda)^{-1}\delta_{\mu\nu}\delta_{ij} \ , \end{aligned} \tag{4.15}$$

where $C_{m,ny}$ is defined in (3.19). Equation (4.15) is the principal result of this section.

The coefficient $[b_{m,ny}]_{\text{scalar}-\text{vector}}$ in (3.34) immediately follows by comparing (4.15) to (4.2). Finally, the three-point function coefficient (3.30) for mixed impurities is obtained from (3.34) and (3.28).

5 The correspondence for an arbitrary number of scalar impurities

In this section we shall evaluate the coefficients of three-point functions of Δ-BMN operators with arbitrary number of scalar impurities, and use this information to derive the single- double-trace two-point function of operators with an arbitrary number of scalar impurities.

5.1 The results in field theory

Every BMN operator with an arbitrary number of impurities can be decomposed into two pieces. The pure BMN part, which contains no $\bar{Z}$, and the compensating part, which contains $\bar{Z}$. In order to make the structure of the general BMN operator clear, let us consider the example of an operator with three impurities. The pure BMN part consists of two terms:

$$\mathcal{O}_{\rm pure} = \sum_{0\le l_2\le l_3} q_2^{l_2} q_3^{l_3}\, {\rm Tr}(\phi_1 Z^{l_2}\phi_2 Z^{l_3-l_2}\phi_3 Z^{J-l_3}) + \sum_{0\le l_3\le l_2} q_2^{l_2} q_3^{l_3}\, {\rm Tr}(\phi_1 Z^{l_3}\phi_3 Z^{l_2-l_3}\phi_2 Z^{J-l_2}) \ . \tag{5.1}$$

In all cases ϕ_1 is positioned first in the trace, while we have to sum over all the different orderings of the remaining impurities. Let us denote by $\phi_{p(i)}$ the impurity which sits in the $i^{\rm th}$ position of a specific ordering of the impurities, and $l_{p(i)}$ the number of Z fields between ϕ_1 and $\phi_{p(i)}$ (in the example given above $p(2) = 2$, $p(3) = 3$ for the first trace while $p(2) = 3$, $p(3) = 2$ for the second trace). Next we consider the compensating terms. In our example, these should be written as

$$\begin{aligned}\mathcal{O}_{\rm comp} = -\delta_{\phi_2\equiv\phi_3}\sum_{0\le l_2}(q_2q_3)^{l_2}\,{\rm Tr}(\phi_1 Z^{l_2}\bar{Z}Z^{J-l_2}) - \delta_{\phi_1\equiv\phi_2}\sum_{0\le l_3} q_3^{l_3}\,{\rm Tr}(\bar{Z}Z^{l_3}\phi_3 Z^{J-l_3})\\ -\delta_{\phi_1\equiv\phi_3}\sum_{0\le l_2} q_2^{l_2}\,{\rm Tr}(\bar{Z}Z^{l_2}\phi_2 Z^{J-l_2}) \ .\end{aligned} \tag{5.2}$$

In other words, whenever two impurities in $\mathcal{O}_{\rm pure}$ are of the same flavour, we add a compensating term where these two impurities are replaced by $\bar{Z}$.

With this example in mind, it is not difficult to write down the most general form of an operator with n impurities,

$$\mathcal{O}_{\{n_i\}} \equiv \frac{1}{\sqrt{J^{n-1}N^{J+n}}}\sum_{p={\rm perm}\{2,\dots,n\}}\mathcal{O}^{1p}_{\{n_i\}} \ , \tag{5.3}$$

where $i = 1,\dots,n$ and

$$\mathcal{O}^{1p(2)\dots p(n)}_{\{n_i\}{\rm pure}} = \sum_{0\le l_{p(2)}\le l_{p(3)}\cdots\le l_{p(n)}}\prod_{i=2}^{n} q_{p(i)}^{l_{p(i)}}\,{\rm Tr}(\phi_1 Z^{l_{p(2)}}\phi_{p(2)}Z^{l_{p(3)}-l_{p(2)}}\phi_{p(3)}\dots\phi_{p(n)}Z^{J-l_{p(n)}}) \ , \tag{5.4}$$

$$\begin{aligned}\mathcal{O}^{1p(2)\dots p(n)}_{\{n_i\}{\rm comp}} = -\frac{1}{2}\sum_{k=2}^{n-1}\delta_{\phi_{p(k)}\equiv\phi_{p(k+1)}}\sum_{0\le l_{p(2)}\cdots\le l_{p(k)}\le l_{p(k+2)}\cdots\le l_{p(n)}}\ \prod_{i=2, i\ne p(k+1)}^{n} q_{p(i)}^{l_{p(i)}}\, q_{p(k+1)}^{l_{p(k)}}\\ {\rm Tr}(\phi_1 Z^{l_{p(2)}}\phi_{p(2)}\dots Z^{l_{p(k)}-l_{p(k-1)}}\bar{Z}Z^{l_{p(k+2)}-l_{p(k)}}\phi_{p(k+2)}\dots\phi_{p(n)}Z^{J-l_{p(n)}})\\ -\delta_{\phi_1\equiv\phi_{p(2)}}\sum_{0\le l_{p(3)}\cdots\le l_{p(n)}}\prod_{i=3}^{n} q_{p(i)}^{l_{p(i)}}\,{\rm Tr}(\bar{Z}Z^{l_{p(3)}}\phi_{p(3)}\cdots\phi_{p(n)}Z^{J-l_{p(n)}}) \ .\end{aligned} \tag{5.5}$$

The origin of the $\frac{1}{2}$ in front of the first term of $\mathcal{O}^{1p(2)\dots p(n)}_{\{n_i\}{\rm comp}}$ is quite clear. It comes from the fact that we have counted twice the same term, since we have two orderings where

$\phi_{p(i)}$ and $\phi_{p(i+1)}$ coincide. The one with $\phi_{p(i)}$ coming first and $\phi_{p(i+1)}$ following and vice versa. Finally, we note that in principle more compensating terms should be added to the right hand side of (5.5) when two or more pairs of impurities coincide. These terms are irrelevant for our purposes in the BMN limit.

In the above expression, $\phi_{p(i)} \in \{\phi_1, \phi_2, \phi_3, \phi_4\}$. This makes the meaning of the Kronecker δ-symbols functions obvious. It should also be noted that the operator given above is normalised so that its two point function is one at the free theory level.[11]

As in section 3, we will need the expressions for double-trace operators,

$$\mathcal{T}^{J,y}_{\{n_i\}\{k_i\}} =: \mathcal{O}^{J\cdot y}_{\{n_i\}} :: \mathcal{O}^{J\cdot(1-y)}_{\{k_i\}} : \ . \tag{5.6}$$

On general grounds, the two-point function of the double- and single-trace BMN operators takes the form

$$\langle \mathcal{T}^{J,y}_{\{n_i\}\{k_i\}}(0)\, \bar{\mathcal{O}}^{J}_{\{m_i\}}(x)\rangle = g_2 C_{\text{free}} \left[1 + \lambda' \left(a+b+c\right) \log(x\Lambda)^{-2}\right] \ . \tag{5.7}$$

In (5.7) we have suppressed the indices of a, b and c. Here $\bar{\mathcal{O}}^{J}_{\{m_i\}}$ contains p_3 impurities, whereas the two single-trace expressions in $\mathcal{T}^{J,y}_{\{n_i\}\{k_i\}}$ contain p_1 and p_2 impurities, respectively.

Compared to (3.20), the above equation contains a new coefficient, c. This is due to the fact that the second operator on the right hand side of (5.6) is no longer just the vacuum, but instead is a generic string state. This results in an additional logarithmic part for the three-point function (3.18), i.e. $c \cdot \log(x_{32}\Lambda)^2)$.

The next step is to calculate the matrix of classical overlaps S. To this end, we will need to compute the correlation functions of single-trace operators with double-trace operators to $\mathcal{O}(g_2)$. We will not need the correlation functions of two different double-trace operators, because these overlaps are of $\mathcal{O}(g_2^2)$. Hence, it is possible to treat each double-trace operator independently and write the expressions (5.9), (5.13) and (5.14) as two by two matrices.

Thus the classical overlap is given by

$$S = \mathbb{1} + g_2 s \ , \tag{5.8}$$

where

$$s = \begin{pmatrix} 0 & C_{\text{free}} \\ C_{\text{free}} & 0 \end{pmatrix} \ , \tag{5.9}$$

[11] Strictly speaking this is true only when all the q's which correspond to a particular ϕ, say ϕ_1, are different. This is the case that we are going to consider. However what follows can be applied with slight modifications to the case where two or more of the q's are the same.

and

$$C_{\text{free}} = \sum_{\text{perm}'} C^p_{\text{free}} \ , \tag{5.10}$$

where

$$C^p_{\text{free}} = \frac{(-1)^{p_2}}{\pi^{p_3}\sqrt{y^{p_1-1}(1-y)^{p_2-1}J}} \prod_{a=1}^{p_1} \frac{\sin(\pi m_{p(a)} y)}{m_{p(a)} - n_a/y} \prod_{b=1}^{p_2} \frac{\sin(\pi m_{p(b+p_1)} y)}{m_{p(b+p_1)} - k_b/(1-y)} \ . \tag{5.11}$$

The sum in (5.10) is over all the admissible permutations of the $\{m_i\}$, which label the barred BMN operator, as on the left hand side of (5.7). A permutation is admissible only when the permuted numbers belong to ϕ's of the same flavour.

Our next goal is to determine the anomalous dimension matrix T,

$$T = d + g_2 t \ , \tag{5.12}$$

where the diagonal part d contains the anomalous dimensions, as in (3.24),

$$d = \lambda' \begin{pmatrix} \sum_{a=1}^{p_3} m_a^2/2 & 0 \\ 0 & \sum_{a=1}^{p_1} n_a^2/2y^2 + \sum_{a=1}^{p_2} k_a^2/2(1-y)^2 \end{pmatrix} \ , \tag{5.13}$$

and

$$t = \lambda' \begin{pmatrix} 0 & t_{12} \\ t_{21} & 0 \end{pmatrix} \ . \tag{5.14}$$

t_{12} can be read from (5.7), and is given by

$$t_{12} = \sum_{\text{perm}'} C^p_{\text{free}} (a + b + c) \ . \tag{5.15}$$

The coefficients a and c are given by the anomalous dimensions of the first and second operators in the definition of $\mathcal{T}$,

$$a = \sum_{a=1}^{p_1} \frac{n_a^2}{2y^2} \ , \qquad c = \sum_{a=1}^{p_2} \frac{k_a^2}{2(1-y)^2} \ . \tag{5.16}$$

The contributions of $\sum_{\text{perm}'} C^p_{\text{free}}\, a$ and $\sum_{\text{perm}'} C^p_{\text{free}}\, c$ to t_{12} in (5.15) factorise, to give

$$C_{\text{free}}\, a \ , \qquad C_{\text{free}}\, c \ , \tag{5.17}$$

respectively.

The remaining contribution to t_{12} is $\sum_{\text{perm}'} C^p_{\text{free}}\, b$. It can be extracted from the coefficient of the corresponding three-point function following the same logic as in section

3. These three-point functions of generic BMN operators with arbitrary numbers of scalar impurities are computed in the following section. Our result is

$$
\begin{aligned}
\sum_{\mathrm{perm}'} C^p_{\mathrm{free}}\, b = \quad & \sum_{\mathrm{perm}'} C^p_{\mathrm{free}}\, \frac{1}{2}\left(\sum_{a=1}^{p_1} m_{p(a)}\left(m_{p(a)} - \frac{n_a}{y}\right) + \sum_{a=1}^{p_2} m_{p(a+p_1)}\left(m_{p(a+p_1)} - \frac{k_a}{1-y}\right)\right. \\
& + \frac{1}{2}\sum_{(a,b)}^{p_1} (m_{p(a)} - n_a)(m_{p(b)} - n_b) + \frac{1}{2}\sum_{(a,b)}^{p_2} (m_{p(p_1+a)} - k_a)(m_{p(p_1+b)} - k_b) \\
& \left. + \frac{1}{2}\sum_{a=1}^{p_1}\sum_{b=1}^{p_2} (m_{p(a)} - n_a)(m_{p(p_1+b)} - k_b)\right) \ . \qquad (5.18)
\end{aligned}
$$

The double sum summation notation (a, b) means that we do not distinguish between the pair a, b and the pair b, a $(a \neq b)$.

As in section 3, the anomalous dimension matrix Γ in the isomorphic to string basis is given by

$$
\Gamma = d + g_2 t' \ , \qquad t' = t - \frac{1}{2}\{s, d\} \ , \qquad (5.19)
$$

where

$$
\{s, d\} = \lambda' \begin{pmatrix} 0 & C_{\mathrm{free}}(\delta_1 + \delta_2 + \delta_3) \\ C_{\mathrm{free}}(\delta_1 + \delta_2 + \delta_3) & 0 \end{pmatrix} \ , \qquad (5.20)
$$

and δ_i is the anomalous dimension of the i^{th} operator $(i = 1, 2, 3)$. After some algebra, we obtain the final result:

$$
\begin{aligned}
\Gamma_{12} = \quad & \frac{\lambda' g_2}{4} \sum_{\mathrm{perm}'} C^p_{\mathrm{free}} \left(\sum_{a=1}^{p_1}\left(m_{p(a)} - \frac{n_a}{y}\right)^2 + \sum_{a=1}^{p_2}\left(m_{p(a+p_1)} - \frac{k_a}{1-y}\right)^2\right. \\
& + \sum_{(a,b)}^{p_1} (m_{p(a)} - n_a)(m_{p(b)} - n_b) \qquad (5.21) \\
& \left. + \sum_{(a,b)}^{p_2} (m_{p(p_1+a)} - k_a)(m_{p(p_1+b)} - k_b) + \sum_{a=1}^{p_1}\sum_{b=1}^{p_2} (m_{p(a)} - n_a)(m_{p(p_1+b)} - k_b)\right) \ .
\end{aligned}
$$

This is our final expression for matrix elements in gauge theory. In the next section we will compute the corresponding three-string amplitude and compare it to (5.21). We will find perfect agreement.

5.2 The results in string field theory

In this subsection we assemble the basic ingredients of the SFT calculation of the string amplitude for states with an arbitrary number of scalar impurities. The amplitude has the form [7,8] (we refer the reader to Appendix A for more details)

$$\langle \Phi | \mathsf{P} | V_B \rangle \ , \tag{5.22}$$

where $\langle \Phi |$ represents the three external string states, $|V_B\rangle$ is the kinematic part of the bosonic vertex (A.6), and the prefactor P is given by

$$\mathsf{P} = C_{\text{norm}} \sum_{r=1}^{3} \sum_{-\infty}^{\infty} \frac{\omega_{rn}}{\mu \alpha_r} \, \alpha_n^{rI\dagger} \alpha_{-n}^{rI} \ . \tag{5.23}$$

C_{norm} is defined in such a way that we get agreement with the field theory calculation. Its value is taken to be

$$C_{\text{norm}} = -\frac{1}{2} g_2 \frac{\sqrt{y(1-y)}}{\sqrt{J}} (-1)^{\frac{1}{2}\sum_{\text{r}=1}^{3}\sum_{m=-\infty}^{+\infty} \alpha_m^{\text{r}\,I^\dagger} \alpha_m^{\text{r}\,I}} \ . \tag{5.24}$$

The prefactor can act on the external bra–state and give a sum of $2p_3$ terms, each of which has an external state identical to the initial $\langle \Phi |$, except one of the α_n's which has changed to α_{-n}. Of course each of these terms is multiplied by the corresponding $\omega_{rn}/\mu\alpha_r$. What we are left with is the action of the exponential in $|V_B\rangle$. In order to keep the comparison to field theory as simple as possible, we choose a certain set of associations between the impurities of the third string and the impurities of the other two strings. The final result will be a sum over all possible such associations, i.e. permutations of this set.

When an external oscillator has not been changed by the prefactor, the action of $|V_B\rangle$ gives a factor of $\hat{N}^{3r}_{n_a n'_a}$ where $r = 1, 2, 3$. But if the external oscillator has been changed by the prefactor, the action of $|V_B\rangle$ gives a factor of

$$F^{3r}_{n_a - n'_a} = \hat{N}^{3r}_{-n_a n'_a} \frac{\omega_{3n_a}}{\mu\alpha_3} + \hat{N}^{3r}_{n_a - n'_a} \frac{\omega_{rn'_a}}{\mu\alpha_r} = \hat{N}^{3r}_{n_a - n'_a} \left(\frac{\omega_{3n_a}}{\mu\alpha_3} + \frac{\omega_{rn'_a}}{\mu\alpha_r} \right) \ . \tag{5.25}$$

One can evaluate $F^{3r}_{n_a - n'_a}$ to get

$$\begin{aligned}
F^{31}_{n_a - n'_a} &= (-1)^{n_a + n'_a} \frac{\lambda'}{2\pi\sqrt{y}} \left(n_a - \frac{n'_a}{y} \right)^2 \frac{\sin(\pi n_a y)}{n_a - n'_a/y} \ , \\
F^{32}_{n_a - n'_a} &= (-1)^{n_a + 1} \frac{\lambda'}{2\pi\sqrt{1-y}} (n_a - n'_a/(1-y))^2 \frac{\sin(\pi n_a y)}{n_a - n'_a/(1-y)} \ , \\
F^{33}_{n_a - n'_a} &= (-1)^{n_a + n'_a + 1} \frac{2\sin(\pi n_a y)\sin(\pi n'_a y)}{\pi\mu} \ , \\
F^{11}_{n_a - n'_a} &= \frac{2(-1)^{n_a + n'_a}}{4\pi\mu y} \ , \\
F^{22}_{n_a - n'_a} &= \frac{2}{4\pi\mu(1-y)} \ .
\end{aligned} \tag{5.26}$$

We are now in position to write down the result for a given permutation. This reads

$$
\begin{aligned}
\langle\Phi|\mathsf{P}|V_B\rangle = \quad & C_{\rm norm}\Bigg[\sum_{a}^{p_1} F^{31}_{m_a-n_a}\prod_{b\neq a}^{p_1}\hat{N}^{31}_{m_bn_b}\prod_{b=1}^{p_2}\hat{N}^{32}_{m_{b+p_1}k_b}+\sum_{a}^{p_2}F^{32}_{m_a-k_a}\prod_{b=1}^{p_1}\hat{N}^{31}_{m_bn_b}\prod_{b\neq a}^{p_2}\hat{N}^{32}_{m_{b+p_1}k_b}\\
&+\left(\sum_{(a,b)}^{p_1}F^{33}_{m_a-m_b}\hat{N}^{11}_{n_an_b}+\sum_{(a,b)}^{p_1}F^{11}_{n_a-n_b}\hat{N}^{33}_{m_am_b}\right)\prod_{c\neq a,b}^{p_1}\hat{N}^{31}_{m_cn_c}\prod_{c=1}^{p_2}\hat{N}^{32}_{m_{c+p_1}k_c}\\
&+\left(\sum_{(a,b)}^{p_2}F^{33}_{m_{p_1+a}-m_{p_1+b}}\hat{N}^{22}_{k_ak_b}+\sum_{(a,b)}^{p_2}F^{22}_{k_a-k_b}\hat{N}^{33}_{m_{p_1+a}m_{p_1+b}}\right)\prod_{c=1}^{p_1}\hat{N}^{31}_{m_cn_c}\prod_{c\neq a,b}^{p_2}\hat{N}^{32}_{m_{c+p_1}k_c}\\
&+\left(\sum_{a=1}^{p_1}\sum_{b=1}^{p_2}F^{33}_{m_a-m_{p_1+b}}\hat{N}^{12}_{m_ak_b}+\sum_{a=1}^{p_1}\sum_{b=1}^{p_2}F^{12}_{n_a-k_b}\hat{N}^{33}_{m_am_{p_1+b}}\right)\prod_{c\neq a}^{p_1}\hat{N}^{31}_{m_cn_c}\prod_{c\neq b}^{p_2}\hat{N}^{32}_{m_{c+p_1}k_c}\Bigg]\ .
\end{aligned}
\tag{5.27}
$$

As before, in the double sum over all pairs (a,b) which appears above we do not distinguish between the pair (a,b) and the pair (b,a). Making use of the expressions for the Neumann matrices from [35] it is now easy to obtain the final expression for the matrix element in string theory:

$$
\begin{aligned}
\langle\Phi|\mathsf{P}|V_B\rangle = \frac{C^p_{\rm free}}{4}g_2\lambda'\Bigg(&\sum_{a=1}^{p_1}(m_a-\frac{n_a}{y})^2+\sum_{a=1}^{p_2}(m_{a+p_1}-\frac{k_a}{1-y})^2\\
&+\sum_{(a,b)}^{p_1}(m_a-n_a)(m_b-n_b)+(m_b-n_a)(m_a-n_b)\\
+\sum_{(a,b)}^{p_2}&(m_{p_1+a}-k_a)(m_{p_1+b}-k_b)+(m_{p_1+b}-k_a)(m_{p_1+a}-k_b)\\
+\sum_{a=1}^{p_1}\sum_{b=1}^{p_2}&(m_a-n_a)(m_{p_1+b}-n_b)+(m_{p_1+b}-n_a)(m_a-n_b)\Bigg)\ .
\end{aligned}
\tag{5.28}
$$

One should note that in calculating the three string vertex we did not take into account terms where there were two contractions of oscillators belonging to the same string if the prefactor had not acted on one of these oscillators previously. This is so because these terms are of order $(1/\mu)^4=\lambda'^2$, as can be easily seen.[12]

In order to get the final string theory result, we should not forget to sum (5.28) over all the admissible permutations, as we have done for the field theory result. This means that the first line of (5.28) should be summed over all the possible permutations, while the remaining lines should be summed over all permutations except those which exchange the m's associated with the labels a and b (more precisely, the permutations which exchange

[12]There is a subtlety in writing (5.28), since the C^p for each term in that equation are in fact different: to each term in (5.28) one should associate the C^p corresponding to the permutation of the indices which label m, n and k appearing in the term considered.

m_a with m_b, m_{p_1+a} with m_{p_1+b}, m_a with m_{p_1+b} in the second, third and fourth line of (5.28), respectively). Including these permutations would result in a double-counting. Once this sums are performed, we obtain perfect agreement with the field theory result (5.21).

6 A technical aside: calculation of general scalar BMN three-point functions

This final section is devoted to the derivation of the expression (5.18).

Our goal is to calculate $\langle \mathcal{O}^{J\cdot y}_{\{n_i\}}(x_1)\, \mathcal{O}^{J\cdot(1-y)}_{\{k_i\}}(x_2)\, \bar{\mathcal{O}}^{J}_{\{m_i\}}(0)\rangle$. To simplify the notation, we will rename the operators in this Green's function as $\mathcal{O}_1(x_1)$, $\mathcal{O}_2(x_2)$ and $\bar{\mathcal{O}}_3(0)$. Let us assume that there are $f_1^{(i)}$ ϕ_1's, $f_2^{(i)}$ ϕ_2's, $f_3^{(i)}$ ϕ_3's and $f_4^{(i)}$ ϕ_4's in the i^{th} operator where $i = 1, 2, 3$. We consider the case where $f_1^{(3)} = f_1^{(1)} + f_1^{(2)}$ with similar relations holding for the other three impurities.

There are of course many different diagrams. In order to deal efficiently with them, let us select a particular set of Wick contractions between the impurities of the barred operator and the impurities of the unbarred operators. Obviously, only impurities of the same flavour can be contracted. The full result will then be a sum over all the different permutations of such contractions.

We start by considering the diagrams where the pure BMN part is taken in each of the three operators. These are drawn in Figure 7. It is easy to see that there are $f_1^{(3)}!f_2^{(3)}!f_3^{(3)}!f_4^{(3)}!$ different contributions. Let us select one of them and draw the corresponding diagrams, Figure 7, in which the i^{th} ϕ_1 field of $\bar{\mathcal{O}}_3(0)$ interacts with the n^{th} ϕ_1 field of $\mathcal{O}_1(x_1)$, while all the other fields are freely contracted. The phase factor associated with these free contractions becomes, in the BMN limit:

$$
\begin{aligned}
P_2 &= \prod_{a\neq n}^{p_1} \sum_{l_a=1}^{J_1} (\bar{q}_a r_a)^{l_a} \prod_{b=1}^{p_2} \sum_{l_b=J_1+1}^{J} (\bar{q}_{p_1+b} p_b)^{l_b} \\
&= \prod_{a\neq n}^{p_1} \sum_{l_a}^{J_1} e^{-2\pi i (m_a - \frac{n_a}{y})\frac{l_a}{J}} \prod_{b=1}^{p_2} \sum_{l_b=J_1+1}^{J} e^{-2\pi i (m_{p_1+b} - \frac{k_b}{1-y})\frac{l_b}{J}} \\
&= J^{p_3-1} \prod_{a\neq n}^{p_1} \int_0^y dx e^{-2\pi i (m_a - \frac{n_a}{y})x} \prod_{b=1}^{p_2} \int_y^1 dx e^{-2\pi i (m_{p_1+b} - \frac{k_b}{y})x} \\
&= J^{p_3-1} \prod_{a\neq n}^{p_1} e^{-\pi i m_a y} \frac{\sin \pi m_a y}{(m_a - n_a/y)\pi} \prod_{b=1}^{p_2} e^{-\pi i m_{p_1+b} y} \frac{(-1)^{p_2} \sin \pi m_{p_1+b} y}{(m_{p_1+b} - k_b/(1-y))\pi}\ . \qquad (6.1)
\end{aligned}
$$

Recall that (5.1), (5.2) and (5.4), (5.5) do not contain q_1. In (6.1) and in what follows q_1 is defined as $q_1 = \prod_{i=2}^{J} \bar{q}_i$, and $q_i = e^{2\pi i m_i/J}$ is the phase factor of the i^{th} impurity (r_a and p_a are the phase factors of $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively).

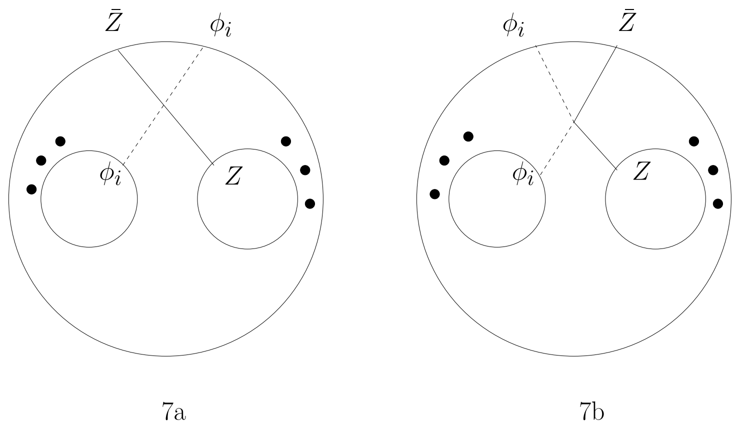

Figure 7: Feynman diagrams where a scalar impurity from $\mathcal{O}_1$ interacts with a Z field. The dots stand for impurities which have free contractions. These diagrams are also accompanied by their mirror images, where the interaction occurs in the bottom part of the diagram.

Note that in obtaining the above formula we have taken into account all the possible orderings of the freely contracting impurities.

We also need the phase factor associated with the fields which are involved in the interaction. This is given by

$$P_1 = (\bar{q}_i^{J_1} - 1)(\bar{q}_i - 1)g^2 = \frac{-4\pi m_i}{J} e^{-\pi i m_i y} \sin \pi m_i y \ . \tag{6.2}$$

The total phase factor will be $P = P_1 P_2$ [13] Taking into account the normalisation of the operators, and evaluating the space-time integral associated with the vertex (see (D.1)), one gets:

$$G_3^{(1)} = \frac{J^{p_3}}{N\sqrt{J^{p_3-1} J_1^{p_1-1} J_2^{p_2-1}}} \prod_{a\neq n}^{p_1} \frac{\sin \pi m_a y}{m_a - n_a/y} \prod_{b=1}^{p_2} \frac{\sin \pi m_{p_1+b} y}{m_{p_1+b} - k_b/(1-y)} (-1)^{p_2}$$
$$(-\lambda') \frac{m_i \sin \pi m_i y}{4\pi^{p_3}} \log \frac{x_1^2 x_2^2}{x_{12}^2} \ . \tag{6.3}$$

In the previous expression we are keeping only the $\log x_{12}$ terms, which are relevant to determine the coefficient b in (5.7), similarly to what we did in section 4. We denoted by

[13] In this section the Lagrangian and Feynman rules of [31] are used.

$G_3^{(1)}$ the part of the three-point function which corresponds to the diagrams of Figure 7, and by $p_3 = p_1 + p_2$ the number of the impurities of $\mathcal{O}_3$.

In order to make easier the comparison with the string theory result, we rewrite (6.3) as

$$g_2 C^p_{\text{free}} b^{(1)} = \frac{J^{p_3}}{N\sqrt{J^{p_3-1} J_1^{p_1-1} J_2^{p_2-1}}} \prod_{a\neq n}^{k} \frac{\sin \pi m_a y}{m_a - n_a/y}$$
$$\prod_{b=1}^{p_2} \frac{\sin \pi m_{p_1+b} y}{m_{p_1+b} - k_b/(1-y)} (-1)^{p_2} \frac{m_i \sin \pi m_i y}{2\pi^{p_3}} \ . \tag{6.4}$$

Equation (6.4) corresponds to the first term in the first line of (5.18). Until now we have considered only diagrams where the interacting ϕ belongs to the operator $\mathcal{O}_1$. Of course, there are diagrams where it is an impurity in $\mathcal{O}_2$ which interacts. These contributions produce the second term in the first line of (5.18).

Now we consider the diagrams in Figure 8. In these diagrams we take for $\mathcal{O}_1$ the

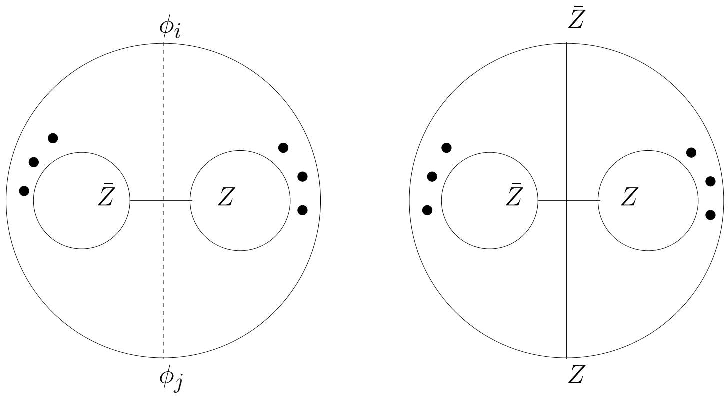

Figure 8: In Figure 8a we take the pure BMN part in the external operator, whereas in 8b we take the compensating term. In both cases we take the compensating term in $\mathcal{O}_1$ and the pure BMN part in $\mathcal{O}_2$. Here i and j label the position of the corresponding ϕ in the barred BMN operator.

compensating term, for $\mathcal{O}_2$ the pure BMN part, and for $\bar{\mathcal{O}}_3$ the pure BMN part (Figure 8a) or the compensating term (Figure 8b). In the case where two ϕ_1's interact, the number of different diagrams is $f_1^{(3)}!f_2^{(3)}!f_3^{(3)}!f_4^{(3)}!\frac{f_1^{(1)}(f_1^{(1)}-1)}{4}$. This number is obtained as follows. There are $f_1^{(1)}(f_1^{(1)}-1)/2$ different ways to single out two ϕ_1's from the operator $\mathcal{O}_1$. This should be multiplied by the $f_1^{(3)}(f_1^{(3)}-1)/2$ different ways in which we can choose two ϕ_1's from the barred operator $\bar{\mathcal{O}}_3$, times the number $(f_1^{(3)}-2)!f_2^{(3)}!f_3^{(3)}!f_4^{(3)}!$ of independent free contractions of all the remaining impurities.

Evaluating the phase factor associated with the interacting fields we get

$$P_1 = (\bar{q}_i^{J_1} - 1)(\bar{q}_j^{J_1} - 1)g^2. \tag{6.5}$$

For the total phase factor one obtains

$$P = J^{p_1+p_2-2} \prod_{a\neq i,j}^{p_1} e^{-\pi i m_a y} \frac{\sin \pi m_a y}{(m_a - n_a/y)\pi} \prod_{b=1}^{p_2} e^{-\pi i m_{p_1+b} y} \frac{\sin \pi m_{p_1+b} y}{(m_{p_1+b} - k_b/(1-y))\pi} (-1)^{p_2}$$
$$g^2(-4) \sin \pi m_i y \sin \pi m_j y e^{-\pi i(m_i+m_j)y} \ . \tag{6.6}$$

The contribution to G_3 which corresponds to the diagrams of Figure 8 is therefore

$$G_3^{(2)} = \frac{J^{p_3}}{N\sqrt{J^{p_3-1} J_1^{p_1-1} J_2^{p_2-1}}} \prod_{a\neq i,j}^{p_1} \frac{\sin \pi m_a y}{m_a - n_a/y} \prod_{b=1}^{p_2} \frac{\sin \pi m_{p_1+b} y}{m_{p_1+b} - k_b/(1-y)} (-1)^{p_2}$$
$$(-\lambda') \frac{\sin \pi m_i y \sin \pi m_j y}{4\pi^{p_3}} \log \frac{x_1^2 x_2^2}{x_{12}^2} \ . \tag{6.7}$$

From the last equation we can extract

$$g_2 C^p_{\text{free}} b^{(2)} = \frac{J^{p_3}}{N\sqrt{J^{p_3-1} J_1^{p_1-1} J_2^{p_2-1}}} \prod_{a\neq i,j}^{k} \frac{\sin \pi m_a y}{m_a - n_a/y}$$
$$\prod_{b=1}^{p_2} \frac{\sin \pi m_{p_1+b} y}{m_{p_1+b} - k_b/(1-y)} (-1)^{p_2} \frac{\sin \pi m_i y \sin \pi m_i y}{2\pi^{p_3}} \ . \tag{6.8}$$

This term corresponds to the first term of the second line in (5.18). There are also diagrams where we consider the pure BMN part in $\mathcal{O}_1$ and the compensating part in $\mathcal{O}_2$ (rather than the opposite). These terms produce the second term in the second line of (5.18).

We now consider the last set of diagrams, which are represented in Figure 9. In order to draw these diagrams we have used (5.4) for the operators $\mathcal{O}_1$, $\mathcal{O}_2$, and (5.4), (5.5) for the barred operator. In these diagrams the x^{th} ϕ_1 belongs to $\mathcal{O}_1$, while the z^{th} ϕ_1 belongs to $\mathcal{O}_2$.

In this case the phase factor associated with the fields which interact becomes

$$P_1 = (\bar{q}_i^{J_1} - 1)(\bar{q}_j^{J_1} - 1)(-g^2) \ . \tag{6.9}$$

For the total phase factor one obtains

$$P = J^{p_3-2} \prod_{a\neq x}^{p_1} e^{-\pi i m_a y} \frac{\sin \pi m_a y}{m_a - n_a/y} \prod_{b\neq z}^{p_2} \exp^{-\pi i m_{p_1+b} y} \frac{\sin \pi m_{p_1+b} y}{m_{p_1+b} - k_b/y} (-1)^{p_2-1}$$
$$\frac{(-g^2)(-4)}{\pi^{p_1-1}\pi^{p_2-1}} \sin \pi m_i y \ \sin \pi m_j y e^{-\pi i(m_i+m_j)y} \ . \tag{6.10}$$

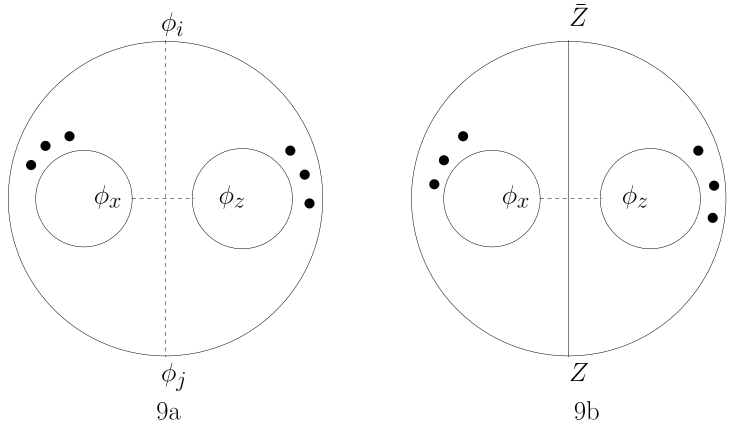

Figure 9: In Figure 9a we take the BMN part in the external operator, whereas in 9b we take the compensating term.

The contribution to G_3 which corresponds to the diagrams of Figure 9 is therefore

$$G_3^{(3)} = \frac{J^{p-3}}{N\sqrt{J^{p_3-1}J_1^{p_1-1}J_2^{p_2-1}}} \prod_{a\neq x}^{p_1} \frac{\sin \pi m_a y}{m_a - n_a/y} \prod_{b\neq z}^{p_2} \frac{\sin \pi m_{p_1+b} y}{m_{p_1+b} - k_b/(1-y)} (-1)^{p_2}$$
$$(-\lambda') \frac{\sin \pi m_i y \, \sin \pi m_j y}{4\pi^{p_3}} \log \frac{x_1^2 x_2^2}{x_{12}^2} \ . \tag{6.11}$$

From this equation we extract

$$g_2 C^p_{\text{free}} b^{(3)} = \frac{J^{p-3}}{N\sqrt{J^{p_3-1}J_1^{p_1-1}J_2^{p_2-1}}} \prod_{a\neq x}^{p_1} \frac{\sin \pi m_a y}{m_a - n_a/y}$$
$$\prod_{b\neq z}^{p_2} \frac{\sin \pi m_{p_1+b} y}{m_{p_1+b} - k_b/(1-y)} (-1)^{p_2} \frac{\sin \pi m_i y \sin \pi m_i y}{2\pi^{p_3}} \ . \tag{6.12}$$

Finally, by adding (6.4), (6.8) and (6.12) (and the terms similar to (6.4) and (6.8), as discussed in the text), and summing over all permutations, it is immediate to obtain (5.18).

Acknowledgements

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Appendix A: the three-string vertex

We first specify the notation and conventions used in pp-wave string field theory. The combination $\alpha' p^+$ for the r-th string is denoted $\alpha_{\rm r}$ and $\sum_{\rm r=1}^3 \alpha_{\rm r} = 0$. As is standard in the literature, we will choose a frame in which $\alpha_3 = -1$

$$\alpha_{\rm r} = \alpha' p^+_{(\rm r)} : \qquad \alpha_3 = -1, \qquad \alpha_1 = y, \qquad \alpha_2 = 1-y. \tag{A.1}$$

In terms of the $U(1)$ R-charges of the BMN operators in the gauge theory three-point function, $\langle \mathcal{O}_1^{J_1}\, \mathcal{O}_2^{J_2}\, \bar{\mathcal{O}}_3^{J} \rangle$ we have

$$y = \frac{J_1}{J} \ , \qquad 1-y = \frac{J_2}{J}, \qquad y \in (0,1) \ , \tag{A.2}$$

and $J = J_1 + J_2$.

The effective SYM coupling constant (1.2) in the frame (A.1) takes the simple form

$$\lambda' = \frac{1}{(\mu p^+ \alpha')^2} \ \equiv \frac{1}{(\mu \alpha_3)^2} \ = \frac{1}{\mu^2}. \tag{A.3}$$

Here μ is the mass parameter which appears in the pp-wave metric, in the chosen frame it is dimensionless[14] and the expansion in powers of $1/\mu^2$ is equivalent to the perturbative expansion in λ'. Finally, the frequencies are defined via

$$\omega_{{\rm r}m} = \sqrt{m^2 + (\mu \alpha_{\rm r})^2} \ . \tag{A.4}$$

The three-string vertex $|H_3\rangle$ can be represented as a ket-state in the tensor product of three string Fock spaces. It has the form [7,8]

$$\frac{1}{\mu}|H_3\rangle = \mathsf{P}|V_F\rangle|V_B\rangle\delta\Big(\sum_{\rm r=1}^{3}\alpha_{\rm r}\Big) \ , \tag{A.5}$$

where the kets $|V_B\rangle$ and $|V_F\rangle$ are constructed to satisfy the bosonic and fermionic kinematic symmetries, and $\alpha_{\rm r}$ are defined in (A.1). The bosonic factor $|V_B\rangle$ is given by

$$|V_B\rangle = \exp\left(\frac{1}{2}\sum_{\rm r,s=1}^{3}\sum_{m,n=-\infty}^{\infty}\sum_{I=1}^{8}\alpha_m^{{\rm r}\,I\dagger}\,\hat{N}^{\rm rs}_{mn}\alpha_n^{{\rm s}\,I\dagger}\right)|0\rangle|0\rangle|0\rangle \ , \tag{A.6}$$

where the $\hat{N}^{\rm rs}_{mn}$ are the Neumann matrices in the BMN-basis of string oscillators. The complete perturbative expansion of the Neumann matrices in the pp-wave background in

[14]It is $p^+\mu$ which is invariant under longitudinal boosts and is frame-independent.

the vicinity of $\mu = \infty$, was constructed in [35][15]. The fermionic factor $|V_F\rangle$ is not going to be relevant for the present paper, where only external bosonic string states are considered.

The prefactor P is a polynomial in the bosonic and fermionic oscillators, and is determined from imposing the remaining symmetries of the pp-wave background. The relevant for us bosonic part of the prefactor, as determined by Spradlin and Volovich in [8], reads

$$\mathsf{P} = C_{\rm norm} \sum_{r=1}^{3} \sum_{-\infty}^{\infty} \frac{\omega_{rn}}{\mu \alpha_r} \alpha_n^{rI\dagger} \alpha_{-n}^{rJ} v_{IJ} \ , \tag{A.7}$$

where $v_{IJ} = \mathrm{diag}(\mathbb{1}_4, -\mathbb{1}_4)$, and the overall normalisation $C_{\rm norm}$ is left undetermined. Notice that it is the expression for v_{IJ} which leads to the relative minus sign between the string amplitudes involving states with excitations along the two different $SO(4)$'s as e.g. in (3.9) and (3.11).

Appendix B: notation and conventions in gauge theory

We write the bosonic part of the $\mathcal{N} = 4$ Lagrangian as

$$\mathcal{L} = \frac{2}{g^2} \ \mathrm{Tr} \left(\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2} (D_\mu \phi_i)(D_\mu \phi_i) - \frac{1}{4} [\phi_i, \phi_j][\phi_i, \phi_j] \right) \ , \tag{B.1}$$

where ϕ_i, $i = 1, \ldots, 6$ are the six real scalar fields transforming under an R-symmetry group $SO(6)$. The covariant derivative is $D_\mu \phi_i = \partial_\mu \phi_i - i[A_\mu, \phi_i]$, where $A_\mu = A_\mu^a T^a$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$.

If we define the complex combination

$$Z = \frac{\phi_5 + i\phi_6}{\sqrt{2}} \ , \tag{B.2}$$

the $\mathcal{N} = 4$ Lagrangian can be re-expressed as

$$\mathcal{L} = \frac{2}{g^2} \ \mathrm{Tr} \left(\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \overline{(D^\mu Z)} (D_\mu Z) + \frac{1}{2} (D^\mu \phi^i)(D_\mu \phi^i) \right) + V_F + V_D \ , \tag{B.3}$$

where the F-term and D-term potential are

$$V_F \ = \ -\frac{2}{g^2} \ \mathrm{Tr} \left(2\, Z \phi_i \bar{Z} \phi_i - \phi_i \phi_i (Z \bar{Z} + \bar{Z} Z) + \cdots \right) \ , \tag{B.4}$$

$$V_D \ = \ -\frac{2}{g^2} \ \mathrm{Tr} \left(Z Z \bar{Z} \bar{Z} - Z \bar{Z} Z \bar{Z} + \cdots \right) \ , \tag{B.5}$$

[15]We refer the reader to the Appendix of Ref. [36] for some useful properties of the perturbative Neumann matrices and relations between different string-oscillator bases.

are the F-term and D-term of the scalar potential respectively. In the last equalities the dots stand for impurity flavour changing terms, which mutually cancel between the F- and the D-term.

Our $SU(N)$ generators are normalised as

$$\mathrm{Tr}\left(T^a T^b\right) = \delta^{ab} \ , \tag{B.6}$$

so that, for example,

$$\left\langle Z^i_j(x)\bar{Z}^l_m(0)\right\rangle = \frac{g^2}{2}\,\delta^i_m\delta^l_j\,\Delta(x) \quad , \quad \Delta(x) = \frac{1}{4\pi^2 x^2} \ . \tag{B.7}$$

Finally, we will use the definitions $J := J_1 + J_2$ and $J_1 = y \cdot J$, where $y \in (0,1)$.

Appendix C: summing over BMN phase factors

We report here the expressions for the coefficients P_I and P_{II} which arise after summing over the BMN phase factors in the interacting diagrams of section 4. Defining

$$q = e^{2\pi i m/J} \ , \qquad q_1 = e^{2\pi i n/J_1} \ , \tag{C.1}$$

the expressions for P_I and P_{II} are given by

$$P_I = \sum_{l=0}^{J_1} (\bar{q}q_1)^l\ \bar{q} \ , \qquad P_{II} = \sum_{l=0}^{J_1} (\bar{q}q_1)^l \ . \tag{C.2}$$

We also need to evaluate the quantity $2(P_I + \bar{P}_I) - 2(P_{II} + \bar{P}_{II})$, which in the BMN limit is

$$2(P_I + \bar{P}_I) - 2(P_{II} + \bar{P}_{II}) \ = \ -\frac{8m}{m - n/y}\sin^2 \pi m y \ . \tag{C.3}$$

Appendix D: the function X

The expression for three-point functions of BMN operators with scalar, vector, or mixed impurities involves the ubiquitous integral

$$X_{1234} = \int d^4z\ \Delta(x_1 - z)\Delta(x_2 - z)\Delta(x_3 - z)\Delta(x_4 - z) \ . \tag{D.1}$$

X_{1234} develops a $\log x_{12}^2$ term X as x_1 approaches x_2, which repeatedly appears in section 4. The expression for X is [21]

$$X \ := \ X_{1234}\big|_{x_3 = x_4} \ = \ \frac{\log\,(x_{12}\Lambda)^{-1}}{8\pi^2\,(4\pi^2 x_{31}^2)^2} \ . \tag{D.2}$$

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