Sharp upper bounds on the minimum number of components of 2-factors in claw-free graphs

Hajo Broersma[1], Daniël Paulusma[1], Kiyoshi Yoshimoto[2]

[1] Department of Computer Science, Durham University
Science Laboratories, South Road, Durham DH1 3LE, England,
{*hajo.broersma,daniel.paulusma*}*@durham.ac.uk*

[2] Department of Mathematics, College of Science and Technology
Nihon University, Tokyo 101-8308, Japan
yosimoto@math.cst.nihon-u.ac.jp

Abstract. Let G be a claw-free graph with order n and minimum degree δ. We improve results of Faudree et al. and Gould & Jacobson, and solve two open problems by proving the following two results. If $\delta = 4$, then G has a 2-factor with at most $(5n-14)/18$ components, unless G belongs to a finite class of exceptional graphs. If $\delta \geq 5$, then G has a 2-factor with at most $(n-3)/(\delta-1)$ components, unless G is a complete graph. These bounds are best possible in the sense that we cannot replace 5/18 by a smaller quotient and we cannot replace $\delta - 1$ by δ, respectively.

Key words. claw-free graph, 2-factor, minimum degree, edge-degree

1. Introduction

Let $G = (V(G), E(G))$ be a finite and simple graph of order $n(G) = |V(G)|$ and of size $e(G) = |E(G)|$. For notation and terminology not defined below we refer to [3]. We denote the minimum (vertex) degree of G by $\delta(G)$. The neighbor set of a vertex x in G is denoted by $N_G(x)$, and its cardinality by $d_G(x)$. If no confusion can arise we use $n, e, V, E, \delta, N(x)$, etc. without specifying the graph G. A *2-factor* of a graph G is a spanning 2-regular subgraph of G.

In this paper we study *claw-free* graphs, i.e., graphs that do not contain an induced four-vertex star $K_{1,3}$. Our aim is to obtain sharp upper bounds on the minimum number of components of a 2-factor in a claw-free graph. Our research is motivated by the following reasons. Firstly, any hamiltonian cycle is a connected 2-factor, i.e., a 2-factor with only one component. Hence the smallest number of components in a 2-factor can be seen as a measure for how close a graph is to being hamiltonian. This relates to the well-known conjecture of Matthews and Sumner [15] stating that every 4-connected claw-free graph is hamiltonian. Little progress has been made on settling this conjecture, but it is easy to construct nonhamiltonian 3-connected claw-free graphs. Secondly, deciding whether a (claw-free) graph is hamiltonian is a well-known NP-complete decision problem, and consequently deciding whether a (claw-free) graph has a 2-factor with at most k components for some fixed k is also NP-complete. The latter decision problem does not assume any connectivity and hence is a different problem that turns out to be interesting in its own right. Previous upper bounds on the number of components of a 2-factor in a claw-free graph have been presented in [6,13]. However, as we shall show below, these bounds are not sharp, and our third reason is that we want to improve these bounds. Fourthly, in [19] two infinite families of claw-free graphs are given, and it was stated as an open problem whether these claw-free graphs are worst-case with respect to the minimum number of components in a 2-factor. The two main results in this paper show that this is indeed the case. Finally, claw-free graphs form a rich class containing all line graphs and the class of complements of triangle-free graphs. Research on claw-free graphs and graph factors are both considerably popular areas within graph theory, as witnessed by the survey papers [7] and [16], respectively.

1.1. Known results

Results of both Choudum & Paulraj [2] and Egawa & Ota [4] imply that a moderate minimum degree condition already guarantees that a claw-free graph contains a 2-factor.

Theorem 1. ([2, 4]) *A claw-free graph with $\delta \geq 4$ has a 2-factor.*

Note that in the above theorem no connectivity condition is imposed on the graph. It is easy to verify that an analogous result does not hold for general graphs, not even with an arbitrarily high constant lower bound on the minimum degree or connectivity. We observe that the above theorem gives a solution to a weaker form of the conjecture of Matthews and Sumner [15] that every 4-connected claw-free graph is hamiltonian: every 4-connected claw-free graph has minimum degree at least four, and hence has a 2-factor. The connectivity condition can be relaxed, however. It is known that a 2-connected claw-free graph already has a 2-factor if $\delta = 3$ [19], but that without any connectivity restriction a claw-free graph with $\delta \leq 3$ does not necessarily contain a 2-factor.

Regarding upper bounds on the number of components of a 2-factor, Faudree et al. [6] showed that every claw-free graph with $\delta \geq 4$ has a 2-factor with at most $6n/(\delta + 2) - 1$ components. Gould & Jacobson [13] proved that, for every integer $k \geq 2$, every claw-free graph of order $n \geq 16k^3$ with $\delta \geq n/k$ has a 2-factor with at most k components. Fronček, Ryjáček & Skupień [9] showed that, for every integer $k \geq 4$, every claw-free graph G of order $n \geq 3k^2 - 3$ with $\delta \geq 3k - 4$ and $\sigma_k > n + k^2 - 4k + 7$ has a 2-factor with at most $k - 1$ components.

More recent results involving moderate connectivity restrictions were obtained in [10]. If a graph G is claw-free, 2-connected and has $\delta \geq 4$, then G has a 2-factor with at most $(n + 1)/4$ components [10]. If a graph G is claw-free, 3-connected and has $\delta \geq 4$, then G has a 2-factor with at most $2n/15$ components [10]. For more on graph factors we refer the reader to the survey [16].

1.2. Our results

We first note that the number of components of a 2-factor in any graph is at most $n/3$. For claw-free graphs with $\delta = 2$ that have a 2-factor we cannot do better than this trivial upper bound. This is obvious from considering a vertex-disjoint set of triangles (cycles on three vertices). For claw-free graphs with $\delta = 3$ that have a 2-factor, the upper bound $n/3$ on its number of components is also tight. In order to see this we construct a family of graphs. We start with an even path $x_1x_2\dots x_{2k}$. We add k new vertices $y_1,\dots,y_k$ and edges $x_{2i-1}y_i, x_{2i}y_i$ for $i=1,\dots,k$. We connect each y_i with a triangle on vertices a_i,b_i,c_i by adding edges a_iy_i,b_iy_i,c_iy_i. We connect x_1 with a triangle on vertices u_1,v_1,w_1 by adding edges u_1x_1,v_1x_1,w_1x_1, and we connect x_{2k} with a triangle on vertices u_2,v_2,w_2 by adding edges $u_2x_{2k},v_2x_{2k}w_2x_{2k}$. The resulting graph G_k has $\delta=3$ and is claw-free, since the neighborhood of every vertex is either one clique or two vertex-disjoint cliques. Clearly, for $i=1,\dots,k$, x_{2i} forms a cycle with x_{2i-1} and y_i in any 2-factor of G_k. Thus G_k has a unique 2-factor consisting of triangles only. So, indeed the upper bound $n/3$ is tight for the class of claw-free graphs with $\delta=3$. Hence, in order to get a nontrivial result it is natural to consider claw-free graphs with $\delta\geq 4$.

Our two main results provide answers to two open questions posed in [19]. Let K_n denote the complete graph on n vertices.

Theorem 2. *A claw-free graph G on n vertices with $\delta\geq 5$ has a 2-factor with at most $(n-3)/(\delta-1)$ components, unless G is isomorphic to K_n.*

Note that K_n has to be excluded because the bound in the theorem is smaller than one if G is a complete graph. The result is tight in the following sense. Let $f_2(G)$ denote the minimum number of components in a 2-factor of G. In [19], for every integer $d\geq 4$ an infinite family $\{F_i^d\}$ of claw-free graphs with $\delta(F_i^d)\geq d$ is given such that $f_2(F_i^d)>|F_i^d|/d\geq |F_i^d|/\delta(F_i^d)$. Hence we cannot replace $\delta-1$ by δ in Theorem 2.

For $\delta=4$ we are able to give a more precise bound which is better than the analogue of the bound in Theorem 2 for $\delta=4$.

Theorem 3. *A claw-free graph G on n vertices with $\delta=4$ has a 2-factor with at most $(5n-14)/18$ components, unless G belongs to a finite class of exceptional graphs.*

The exceptional graphs of Theorem 3 have at most seventeen vertices. We will specify them in Section 4. The bound in Theorem 3 is tight in the following sense. There exists an infinite family $\{H_i\}$ of claw-free graphs with $\delta(H_i)=4$ such that

$$\lim_{|H_i|\to\infty}\frac{f_2(H_i)}{|H_i|}=\frac{5}{18}.$$

This family can be found in [19] as well.

Theorems 2 and 3 together clearly improve the previously mentioned result of Faudree et al. [6]. Theorem 2 also improves the previously mentioned result of Gould & Jacobson [13]. This can be seen as follows. Let G be a claw-free graph with $n\geq 16k^3$ and $\delta\geq n/k$ for some integer $k\geq 2$. If we apply Theorem 2 we find that G has a 2-factor with at most the following number of components:

$$\frac{n-3}{\delta-1}\leq\frac{n-3}{\frac{n}{k}-1}=\frac{nk-3k}{n-k}=k+\frac{k^2-3k}{n-k}\leq k+\frac{k^2-3k}{16k^3-k}=k+\frac{k-3}{16k^2-1}.$$

This shows that G has a 2-factor with at most k components if $k\geq 3$ and with at most $k-1$ components if $k=2$. Hence Theorem 2 indeed improves the result of [13]. Moreover, for fixed $\delta=c$ the result in [13] only admits a finite number of graphs (possibly zero) since $\delta=c$ requires $k\geq n/c$ and hence $n\geq 16k^3\geq 16n^3/c^3$.

1.3. Open problems

Before we present the proofs of our two main results, we finish this introduction by mentioning two of the main intriguing open problems in this area.

The first open problem deals with 2-connected claw-free graphs. Egawa & Saito [5] constructed 2-connected claw-free graphs in which every 2-factor has at least $n/(3\delta+3)$ components. We have reasons to believe that the following question has an affirmative answer but we cannot prove this. Does any 2-connected claw-free graph have a 2-factor with at most $n/3\delta$ components?

The second open problem is posed in [19] and deals with bridgeless claw-free graphs. The graphs in the family $\{H_i\}$ mentioned in Section 1.2 contain bridges. Does every bridgeless claw-free graph with $\delta \geq 4$ have a 2-factor with at most $(n-1)/\delta$ components? A partial answer was obtained in [19] by showing that this bound holds for claw-free graphs with $\delta = 4$ that do not have a maximal clique of two vertices (i.e., graphs with the additional property that every edge is contained in a triangle).

2. Notation and preliminary results

Before we present the proofs of Theorems 2 and 3, we first introduce some additional terminology and notation, and we show how to relate the statements of the two theorems to statements on certain dominating systems in triangle-free graphs, using known results.

2.1. Restriction to line graphs of triangle-free graphs

The *line graph* of a graph H with edges $e_1, \ldots, e_p$ is the graph $L(H)$ with vertices $u_1, \ldots, u_p$ such that there is an edge between any two vertices u_i and u_j if and only if e_i and e_j share one end vertex in H. It is well-known and easy to verify that every line graph is claw-free. We show that, in order to prove our main results we can restrict ourselves to a subclass of claw-free graphs, namely the class of line graphs of triangle-free graphs. For this purpose we use the *closure* concept as defined in [17].

The closure of a graph is defined as follows. Let G be a claw-free graph. Then, for each vertex x of G, the set $N_G(x)$ induces a subgraph with at most two components. If this subgraph has two components, both of them must be *cliques*, i.e., complete subgraphs. If the subgraph induced by $N_G(x)$ is connected, we add edges joining all pairs of nonadjacent vertices in $N_G(x)$. This operation is called *local completion of* G *at* x. The *closure* $cl(G)$ of G is a graph obtained by recursively repeating the local completion operation, as long as this is possible. Ryjáček [17] showed that the closure of G is uniquely determined, and that G is hamiltonian if and only if $cl(G)$ is hamiltonian. The latter result was extended to 2-factors by Ryjáček, Saito & Schelp [18].

Theorem 4. ([18]) *Let G be a claw-free graph. Then G has a 2-factor with at most k components if and only if $cl(G)$ has a 2-factor with at most k components.*

Ryjáček [17] also established the following relationship between claw-free graphs and triangle-free graphs.

Theorem 5. ([17]) *If G is a claw-free graph, then there is a triangle-free graph H such that $L(H) = cl(G)$.*

It is common knowledge that apart from K_3 which is $L(K_3)$ and $L(K_{1,3})$, every connected line graph G has a unique H with $G = L(H)$. We call H the *preimage graph* of G. For K_3 we let $K_{1,3}$ be its preimage graph. For disconnected graphs we define the preimage graphs according to their components.

By Theorems 4 and 5, we deduce that for a claw-free graph G, $f_2(G) = f_2(cl(G)) = f_2(L(H))$, where H is the preimage graph of $cl(G)$. This implies that we can restrict ourselves to line graphs of triangle-free graphs.

2.2. Translating the problem into finding a dominating system

An *even* graph is a graph in which every vertex has even degree at least two. A connected even graph is called a *circuit.* For $q \geq 2$, a *star* $K_{1,q}$ is a complete bipartite graph with independent sets $A = \{c\}$ and B with $|B| = q$; the vertex c is called the *center* and the vertices in B are called the *leaves* of $K_{1,q}$.

Let H be a graph that contains a set $\mathcal{S}$ consisting of stars with at least three edges and circuits, all (stars and circuits) mutually edge-disjoint. We call $\mathcal{S}$ a *system that dominates* H or simply a *dominating system* if for every edge e of H the following holds:

- e is contained in one of the stars of $\mathcal{S}$, or
- e is contained in one of the circuits of $\mathcal{S}$, or
- e shares an end vertex with an edge of at least one of the circuits in $\mathcal{S}$.

For convenience we sometimes use the term *k-D-system* as shorthand for a dominating system with exactly k elements. Gould & Hynds [12] proved the following result.

Theorem 6. ([12]) *The line graph $L(H)$ of a graph H has a 2-factor with k components if and only if H has a k-D-system.*

Combining Theorems 4 and 5 with Theorem 6 yields the following result.

Theorem 7. *Let G be a claw-free graph. Then G has a 2-factor with at most k components if and only if the preimage graph of $cl(G)$ has a dominating system with at most k elements.*

The *edge-degree* of an edge xy in a graph H is defined as $d_H(x) + d_H(y) - 2$, i.e., it is equal to the degree of the vertex corresponding to xy in the line graph $L(H)$. We denote the minimum edge-degree of H by $\delta_e = \delta_e(H)$.

Due to Theorem 7, in order to prove Theorems 2 and 3 it is sufficient to prove the following two theorems, respectively.

Theorem 8. *A triangle-free graph H with $\delta_e(H) \geq 5$ has a dominating system with at most $(e(H) - 3)/(\delta_e(H) - 1)$ elements, unless H is isomorphic to $K_{1,e(H)}$.*

Theorem 9. *A triangle-free graph H with $\delta_e(H) = 4$ has a dominating system with at most $(5e(H) - 14)/18$ elements, unless H belongs to a finite class of exceptional graphs.*

We will specify the six exceptional graphs of Theorem 9 in Section 4.

2.3. Outline of the proofs

We will prove Theorem 8 and Theorem 9 in Sections 3 and 4, respectively. Both proofs are modelled along the following lines. We first prove the statements in case H is a tree or forest. This is done by induction combined with replacement and counting arguments. In case H is not a tree we start with a specific maximum even subgraph X of H. We carefully remove edges from the circuits in X, such that we obtain a new graph F that is a forest. After adding sufficiently many pendant edges to F we can apply the results we have for trees and forests to the components of F. We then translate the dominating system of F into one of H, and counting arguments complete the proofs.

2.4. Additional notation and a useful lemma

We close this section by introducing some additional notation and by proving a technical result that is a common ingredient for the proofs of our two main results.

The set of all vertices with degree k in G is denoted by $V_k(G)$ and we put $V_{\geq k}(G) = \bigcup_{i\geq k} V_i(G)$. A vertex with degree 1 in G is called an *end vertex* or *leaf* of G. An edge which is incident with a leaf is called a *pendant* edge. We say that we *add a pendant edge* to G if we add a new vertex to G and join it to precisely one of the vertices of G. Two edges are called *independent* if they do not share any end vertices.

We say that a graph H is *smaller* than a graph G if $e(H) < e(G)$. If $\mathcal{G}$ is a set of graphs, we write $G \in \mathcal{G}$ if the graph G is isomorphic to a graph in $\mathcal{G}$, and $G \notin \mathcal{G}$ if G is isomorphic to none of the graphs in $\mathcal{G}$. We write $H \subset G$ if H is a proper subgraph of G. For a graph $G = (V, E)$, we denote the subgraph induced by a subset $S \subseteq V$ by $G[S]$. For a subgraph $H \subset G$, we denote $N_G(x) \cap V(H)$ by $N_H(x)$ and its cardinality by $d_H(x)$. The set $(\bigcup_{v\in H} N_G(v)) \setminus V(H)$ is denoted by $N_G(H)$ or $N(H)$. For a subgraph $F \subset G$, we denote $N_G(H) \cap V(F)$ by $N_F(H)$. For simplicity, we sometimes slightly abuse notation and replace $|V(H)|$ by $|H|$, "$u_i \in V(H)$" by "$u_i \in H$", and "$G - V(H)$" by "$G - H$".

Let X be an even subgraph of H and let $\mathcal{C}(X)$ be the set of components of X. Then we say that X is a *maximum* even subgraph of H if $|V(X)|$ is as large as possible, such that, subject to the maximality of $|V(X)|$, the number of components of X is as small as possible, and subject to the minimality of $|\mathcal{C}(X)|$, the number $|E(X)|$ of edges is as large as possible. Obviously, if X is a maximum even subgraph of H, then $H - E(X)$ is a forest: if $H - E(X)$ would contain a cycle D, we could add D to X, and the newly formed even subgraph X' clearly contradicts the choice of X if $|V(X')| > |V(X)|$ or if $|V(X')| = |V(X)|$ and $|\mathcal{C}(X')| < |\mathcal{C}(X)|$. Since $|E(X')| > |E(X)|$, we also obtain a contradiction with the choice of X in case $|V(X')| = |V(X)|$ and $|\mathcal{C}(X')| = |\mathcal{C}(X)|$.

For the purpose of the proofs of Theorems 8 and 9, we would be deleting too many edges if we consider $H - E(X)$, hence we need a stronger statement. Before we present this statement, we first introduce some new terminology and describe a procedure for treating the components of a (maximum) even subgraph.

As before, let $\mathcal{C} = \mathcal{C}(\mathcal{X})$ be the set of components of an even subgraph X of H. For each $C \in \mathcal{C}$ we do as follows. First suppose C is isomorphic to a complete bipartite graph $K_{2,2k}$ for some $k \geq 1$. Let $A(C) = \{s, t\}$ and $B(C) = \{s_1, s_2, \dots, s_{2k}\}$ be the partition classes of C. If $k = 1$, we choose zero or more (possibly both) edges from $\{ss_1, ts_2\}$. If $k \geq 2$, we choose zero or more (possible all) edges from $\{ss_i \mid i = 1, \dots, 2k\}$. If C is not isomorphic to a $K_{2,2k}$ for some $k \geq 1$, we choose at most one edge from C. We call the set of all chosen edges the X-*set* and denote it by M. Let $H^* = (H - E(X)) \cup M$. We call H^* an X-*graph* of H. Note that the following result holds for general graphs, not necessarily triangle-free graphs, although we will only use it for triangle-free graphs in the sequel.

Lemma 1. *Let X be a maximum even subgraph of a graph H. Then any X-graph H^* of H is a forest or there is a $C \in \mathcal{C}$ such that $H[V(C)]$ is isomorphic to K_4.*

Proof. Let X be a maximum even subgraph of a graph H. Let H^* be an X-graph of H and suppose H^* is not a forest.

In order to show that there is a $C \in \mathcal{C}$ such that $H[V(C)]$ is isomorphic to K_4, let D be a cycle in H^*. Let $\mathcal{C}^*$ be the set of circuits of X that share at least one edge with D. Suppose $\mathcal{C}^* = \emptyset$. Then, the even subgraph $X \cup D$ clearly contradicts the choice of X. Hence $\mathcal{C}^* \neq \emptyset$. Note that by construction each $C \in \mathcal{C}^*$ can share at most two edges with D. Clearly, the edges of all components of $X \cap D$ belong to M, and hence these components are paths of length 1 or 2.

Consider the graph $X \cup D$. We are going to construct another even subgraph of $X \cup D$, hence of H. For each $uv \in E(X \cap D)$ we act as follows. Let uv belong to a circuit $C \in \mathcal{C}^*$. Suppose C only shares one edge with D (namely uv). Then we remove uv from $X \cup D$. Note that this way both u and v get even degree in the resulting graph. Suppose C shares two edges with D. Then C is a complete bipartite graph with partition classes $A(C) = \{s, t\}$ and $B(C) = \{s_1, \ldots, s_{2k}\}$ for some $k \geq 1$. If $k = 1$, we may without loss of generality assume $uv = ss_1$. Then the second edge C shares with D is ts_2. We remove ss_2 and ts_1 from $X \cup D$. Note that s, s_1, s_2, t get even degree in the resulting graph. If $k \geq 2$, we may without loss of generality assume $uv = ss_1$ and the second edge C shares with D is ss_2. We remove ss_1 and ss_2 from $X \cup D$. Note that the degree of s stays even in the resulting graph, while the degrees of s_1 and s_2 get even. Hence, after removing all the edges as prescribed as above we obtain an even subgraph Y of $X \cup D$. Since we did not remove any vertices, we must have $|V(Y)| = |V(X)|$ due to our choice of X. Let $\mathcal{C}'$ denote the set of components of $\mathcal{C} \backslash \mathcal{C}^*$ that share one or more vertices (but no edges) with D. Then, also due to our construction, all remaining edges of components in $C \in \mathcal{C}^* \cup \mathcal{C}'$ together with all remaining edges of D form one component D^* in Y. Hence, $\mathcal{C}^* = \{C\}$ and $\mathcal{C}' = \emptyset$, as otherwise Y contains fewer components than X. This would contradict our choice of X.

Suppose $|V(D)| \geq 5$. As $|V(D \cap X)| \leq 3$ we obtain $|V(Y)| > |V(X)|$, which is not possible. Hence $|V(D)| = 4$ and D is a cycle on four vertices. If C is not a cycle on four vertices sharing two edges with D, then again we find $|V(Y)| > |V(X)|$. So we can write $C = ss_1ts_2s$ and $D = ss_1s_2ts$ and we find that $H[V(C)]$ is isomorphic to K_4. This completes the proof of the lemma. □

For trees, a simple counting argument yields the following folklore result, which we will use in both proofs.

Observation 10 *Let T be a tree with $|T| \geq 2$. Then $|V_1(T)| = \sum_{i \geq 3} (i-2)|V_i(T)| + 2$.*

3. Proof of Theorem 8

We first prove the statement of Theorem 8 in case H is a tree or forest.

3.1. Theorem 8 holds for forests

We start with the following useful lemma.

Lemma 2. *A tree H with $\delta_e(H) \geq 5$ has a dominating system $\mathcal{S}$ such that the set of centers of stars in $\mathcal{S}$ is $\bigcup_{i \geq \frac{\delta_e(H)}{2}+1} V_i(H)$.*

Proof. Let $F = H - V_1(H)$. We use induction on $|F|$. For convenience, let $d = \delta_e(H)$. If $|F| = 1$, the statement holds, since then $\mathcal{S} = \{H\}$ is a 1-D-system and the center of the star has degree at least $d+1 > d/2+1$ while all other vertices have degree 1. Similarly, it is easy to check the case that $|F| = 2$: then $F = K_2$, both vertices of F have degree at least $d+1$ in H and are the two centers of stars of a 2-D-system, while the other vertices of H have degree 1. Hence for the rest of the proof we may assume that $|F| \geq 3$. We distinguish two cases.

Case 1. F has a vertex u with $d_H(u) < d/2+1$.
Note that, by the minimum edge-degree condition, $N(u) \cap V_1(H) = \emptyset$. Let $N(u) = \{y_i\}_{i \leq d_H(u)}$ and let H_i be the component of $H - u$ containing y_i. See Figure 1, left hand side.

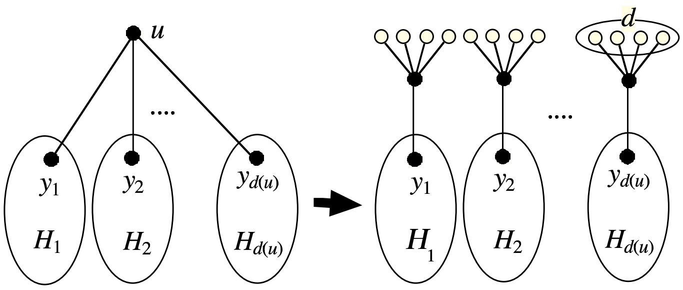

Fig. 1. From the tree H to the trees H_i'.

Let H_i' be the graph obtained from H_i and $K_{1,d+1}$ by identifying y_i and an end vertex of $K_{1,d+1}$. See Figure 1, right hand side. Since the minimum edge-degree of H_i' is at least d and $|H_i' - V_1(H_i')| \leq |F| - 1$, by induction each H_i' has a dominating system $\mathcal{S}_i$ such that the set of centers of stars in $\mathcal{S}_i$ is $\bigcup_{j \geq \frac{d}{2}+1} V_j(H_i')$. Since $d_H(u) < d/2+1$, $d_{H_i'}(y_i) = d_H(y_i) \geq d/2+1$. Therefore, there exists a star $A_i \in \mathcal{S}_i$ whose center is y_i for $1 \leq i \leq d_H(u)$. For $1 \leq i \leq d_H(u)$, let A_i' be the star in H with edge uy_i together with all edges in H that correspond to edges in A_i. For $1 \leq i \leq d_H(u)$, we identify all stars in $\mathcal{S}_i \backslash \{A_i\}$ with stars in H, except the star whose center is in the extra added $K_{1,d+1}$. This way we obtain a desired dominating system of H.

Case 2. $d_H(v) \geq d/2+1$ for every vertex $v \in V(F)$.
Since $|F| \geq 3$, there is a vertex $u \in F$ that is not an end vertex of F. Let $N(u) = \{y_i\}_{i \leq d_H(u)}$ and let H_i be the component of $H - u$ containing y_i. Note that $d_H(u) \geq 4$ as $d \geq 5$. We may assume without loss of generality that $|H_1| \geq |H_2| \geq \cdots \geq |H_{d_H(u)}|$. Let $\ell = \max\{i : |H_i| \geq 2\}$.

Since u is not an end vertex of F, the graph $H - u$ contains at least two components with edges, so $\ell \geq 2$. Let H_1' be the graph obtained from $H_1 \cup H_2$ by adding the edge y_1y_2. If $\ell \geq 3$, then for $3 \leq i \leq \ell$, we let H_i' be the graph obtained from H_i and a copy H_i^* of H_i by joining y_i and its copy y_i^* in H_i^* by an edge. See Figure 2 for an example with $\ell = 3$ and $d_H(u) = 6$.

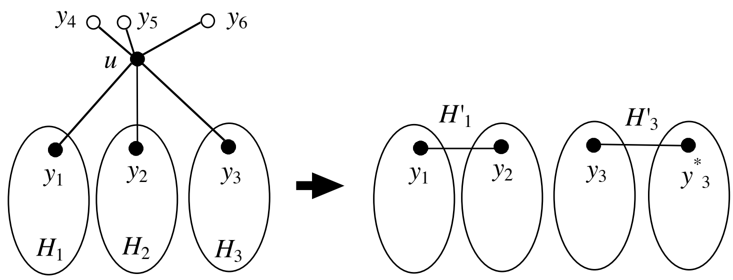

Fig. 2. Case 2 with $\ell = 3$ and $d_H(u) = 6$.

Since $u \notin H_1'$, we find that $|H_1' - V_1(H_1')| \leq |F| - 1$. Obviously, if $\ell \geq 3$, then $|H_i' - V_1(H_i')| \leq |F| - 1$ for $3 \leq i \leq \ell$. Furthermore, every H_i' has minimum edge-degree at least d. Then, by induction, each H_i' has a dominating system $\mathcal{S}_i$ such that the set of centers of stars in $\mathcal{S}_i$ is $\bigcup_{j \geq \frac{d}{2}+1} V_j(H_i')$.

Since y_1 and y_2 have degree at least $d/2+1$ in H_1', there exist stars $A_1, A_2 \in \mathcal{S}_1$ whose centers are y_1, y_2, respectively. Let $\mathcal{S}_1' = \{A \in \mathcal{S}_1 : A \subseteq H_1 \cup H_2\}$. Note that $\mathcal{S}_1'$ contains all stars of $\mathcal{S}_1$ except the star that contains the edge y_1y_2 (this star is either A_1 or A_2). Similarly, if $\ell \geq 3$ then for every pair y_i, y_i^* with $i \geq 3$, there exist stars A_i, A_i^* in $\mathcal{S}_i$ whose centers are y_i, y_i^*, respectively. By symmetry, we may assume that $y_iy_i^* \in E(A_i^*)$ for all $i \geq 3$. Let $\mathcal{S}_i' = \{A \in \mathcal{S}_i : A \subseteq H_i\}$ for all $i \geq 3$.

Suppose $y_1y_2 \in A_2$. We define a star A_0 with center u and with all vertices in $N(u)\backslash\{y_2\}$ as leaves. As $d_H(u) \geq 4$, A_0 contains at least three edges. We define a star A_2' of H that contains the edge uy_2 (as "replacement" for y_1y_2) together with all edges of H that correspond to edges of A_2. We identify all stars in $\mathcal{S}_1'$ with stars in H. Also, for all $i \geq 3$, we identify all stars in $\mathcal{S}_i'$ with stars in H. By combining all these stars we obtain a desired dominating system of H.

The case that $y_1y_2 \in A_1$ is symmetric. This completes the proof of Lemma 2. □

The previous result, together with the next lemma, implies that Theorem 8 holds for trees. Note that this lemma holds for trees with $\delta_e \geq 2$, so we do not need to impose $\delta_e \geq 5$ here.

Lemma 3. *If H is a tree with $\delta_e(H) \geq 2$ that is not isomorphic to $K_{1,e(H)}$, then*

$$\sum_{i \geq \frac{\delta_e(H)}{2}+1} |V_i(H)| \leq \frac{e(H) - 3}{\delta_e(H) - 1}.$$

Proof. By contradiction. Suppose the lemma is false and choose a smallest counterexample H. For convenience, we let $d = \delta_e(H)$. Let $F = H - V_1(H)$. If $|F| = 1$, then H is isomorphic to $K_{1,d+1} = K_{1,e(H)}$, which is a contradiction. Hence we may assume $|F| \geq 2$. We need the following claim.

Claim 1. $N(V_1(H)) = V_1(F)$.

We prove Claim 1 as follows. By definition, $V_1(F) \subseteq N(V_1(H))$, so it is sufficient to prove that $N(V_1(H)) \subseteq V_1(F)$. Suppose that there exists a vertex $u \in N(V_1(H)) \setminus V_1(F)$. Let $N(u) = \{y_j\}_{j \le d(u)}$ and let H_j denote the component of $H - u$ containing y_j. By symmetry, we may assume that $|H_1| \ge |H_2| \ge \cdots \ge |H_{d(u)}|$. Let $r = \max\{j : |H_j| \ge 2\}$. As $u \notin V_1(F)$, $r \ge 2$. Since the minimum edge-degree is d, we know that $d_H(u) \ge d+1$ and $|V_1(H) \cap N(u)| \ge d+1-r$. Therefore,

$$\sum_{j=1}^{r} |H_j| \le |H| - |V_1(H) \cap N(u)| - 1 \le |H| - d - 2 + r. \tag{1}$$

For $1 \le j \le r$, let H'_j be the graph obtained from H_j and $K_{1,d+1}$ by identifying y_j and a leaf of $K_{1,d+1}$. It is easy to check that $\delta_e(H'_j) \ge d$ for $j = 1, \ldots, r$. Since $r \ge 2$, we also have that $|H'_j| < |H|$ for $j = 1, \ldots, r$. Then H'_j is not a counterexample (and H'_j is not a star either). We use this and $|H'_j| \ge d+3$ for $j = 1, \ldots, r$ to obtain for any fixed $j \le r$ that

$$\sum_{i \ge \frac{d}{2}+1} |V_i(H'_j)| \le \frac{e(H'_j) - 3}{d-1} = \frac{|H'_j| - 4}{d-1} = \frac{|H_j| + d - 3}{d-1}. \tag{2}$$

We use inequalities (1), (2), $d \ge 2$, and $r \ge 2$ to obtain

$$\begin{aligned}
\sum_{i \ge \frac{d}{2}+1} |V_i(H)| &= \sum_{j=1}^{r} \sum_{i \ge \frac{d}{2}+1} |V_i(H'_j)| - r + 1 \\
&\le \sum_{j=1}^{r} \frac{|H_j| + d - 3}{d-1} - r + 1 \\
&= \frac{\sum_{j=1}^{r} |H_j| + dr - 3r}{d-1} - \frac{dr - r + d - 1}{d-1} \\
&= \frac{\sum_{j=1}^{r} |H_j| - 2r - d + 1}{d-1} \\
&\le \frac{|H| - d - 2 + r - 2r - d + 1}{d-1} \\
&= \frac{|H| - 2d - 1 - r}{d-1} \\
&\le \frac{e(H) - 3}{d-1}.
\end{aligned}$$

Hence, H cannot be a counterexample. This completes the proof of Claim 1.

By Claim 1, we find that

$$\bigcup_{i \ge \frac{d}{2}+1} V_i(H) = \bigcup_{i \ge \frac{d}{2}+1} V_i(F) \cup V_1(F).$$

Let $n_i = V_i(F)$ for all $i \geq 1$. Then the above implies that

$$\sum_{i \geq \frac{d}{2}+1} |V_i(H)| = \sum_{i \geq \frac{d}{2}+1} n_i + n_1. \tag{3}$$

Since $N(V_1(H)) = V_1(F)$ and every vertex of $N(V_1(H))$ has degree at least $d+1$, for any $u \in N(V_1(H))$ there are at least d end vertices of H which are adjacent to u. Hence

$$|H| \geq |F| + dn_1. \tag{4}$$

As $|F| \geq 2$ we can use Observation 10, which we can translate into

$$n_1 = \sum_{i \geq 3} (i-2)n_i + 2.$$

We use this equality, together with (in)equalities (3) and (4) to deduce that

$$\begin{aligned}
&e(H) - 3 - (d-1) \sum_{i \geq d/2+1} |V_i(H)| \\
&= |H| - 4 - (d-1)\Big(\sum_{i \geq d/2+1} n_i + n_1\Big) \\
&\geq |F| + dn_1 - 4 - (d-1) \sum_{i \geq d/2+1} n_i - (d-1)n_1 \\
&= \sum_{i \geq 1} n_i + n_1 - 4 - (d-1) \sum_{i \geq d/2+1} n_i \\
&= \sum_{i \geq 2} n_i + 2n_1 - 4 - (d-1) \sum_{i \geq d/2+1} n_i \\
&= \sum_{i \geq 2} n_i + 2\Big(\sum_{i \geq 2} (i-2)n_i + 2\Big) - 4 - (d-1) \sum_{i \geq d/2+1} n_i \\
&= \sum_{i \geq 2} (2i-3)n_i - (d-1) \sum_{i \geq d/2+1} n_i \\
&= \sum_{d/2+1 > i \geq 2} (2i-3)n_i + \sum_{i \geq d/2+1} (2i-d-2)n_i \geq 0.
\end{aligned}$$

Hence, H cannot be a counterexample. This completes the proof of Lemma 3. □

Combining Lemmas 2 and 3, we immediately find that the upper bound in Theorem 8 is valid for trees. We complete this section by showing that the upper bound in Theorem 8 holds for forests as well.

Corollary 1. *Let H be a forest with $\delta_e(H) \geq 5$. Then H has a dominating system with at most $(e(H)-3)/(\delta_e(H)-1)$ elements, unless H is isomorphic to $K_{1,e(H)}$.*

Proof. Let H be a forest with $d = \delta_e(H) \geq 5$. Let $D_1, \ldots, D_p$ be the components of H for some $p \geq 1$. By combining Lemmas 2 and 3 we obtain that each D_i has a dominating system $\mathcal{S}_i$ with at most $(e(D_i) - 3)/(\delta_e(D_i) - 1)$ elements unless D_i is isomorphic to $K_{1,e(D_i)}$. In the latter case D_i has a dominating system with one element. Without loss of generality we may assume that for some $0 \leq r \leq p$ all components D_i for $i = r+1, \ldots, p$ are isomorphic to $K_{1,e(D_i)}$ (if there are any), while the other D_i (if any) are not isomorphic to $K_{1,e(D_i)}$. We combine the dominating systems $\mathcal{S}_i$ to obtain a dominating system $\mathcal{S}$ of H. We determine an upper bound on $|\mathcal{S}|$ by using the following three observations. Firstly, $\delta_e(D_i) \geq d$ for $i = 1, \ldots, r$. Secondly, $e(D_i) = \delta_e(D_i) + 1 \geq d + 1$ for $i = r+1, \ldots, p$ as such D_i are isomorphic to $K_{1,e(D_i)}$ and also have $\delta_e(D_i) \geq d$. Thirdly, we may assume $p \geq 2$ or $p = r = 1$; otherwise, if $p = 1$ and $r = 0$, then H is isomorphic to $K_{1,e(H)}$. Using these observations we get

$$\begin{aligned}
|\mathcal{S}| &\leq \sum_{i=1}^{r} \frac{e(D_i) - 3}{\delta_e(D_i) - 1} + p - r \\
&\leq \sum_{i=1}^{r} \frac{e(D_i) - 3}{d - 1} + p - r \\
&= \frac{\sum_{i=1}^{r} e(D_i) - 3r + (p-r)(d-1)}{d-1} \\
&= \frac{e(H) - 3 + 3 - \sum_{i=r+1}^{p} e(D_i) - 3r + (p-r)(d-1)}{d-1} \\
&\leq \frac{e(H) - 3 + 3 - (p-r)(d+1) - 3r + (p-r)(d-1)}{d-1} \\
&= \frac{e(H) - 3 + 3 - dp - p + dr + r - 3r + dp - p - dr + r}{d-1} \\
&= \frac{e(H) - 3 + 3 - 2p - r}{d-1} \\
&\leq \frac{e(H) - 3}{d-1}.
\end{aligned}$$

This completes the proof of Corollary 1. □

3.2. Theorem 8 holds for general triangle-free graphs

For convenience we repeat the statement of Theorem 8.

Theorem 8. *A triangle-free graph H with $\delta_e(H) \geq 5$ has a dominating system with at most $(e(H) - 3)/(\delta_e(H) - 1)$ elements, unless H is isomorphic to $K_{1,e(H)}$.*

Proof. Let H be a triangle-free graph with $d = \delta_e(H) \geq 5$. If H is a forest, the statement follows from Corollary 1. Suppose H is not a forest. Let X be a maximum even subgraph of H. Let $\mathcal{C}$ be the set of components of X.

The proof idea is to construct an X-graph H^* of H. Then, by Lemma 1, H^* is a forest.

For each $C \in \mathcal{C}$ we partition $V(C)$ into two sets $I(C) \cup J(C)$, where $I(C)$ denotes the set of vertices in C that are only adjacent to vertices in $C \cup V_1(H)$ and $J(C)$ denotes the set $V(C)\backslash I(C)$. Before we continue our analysis we first show that we may assume $|J(C)| \geq 1$ for all $C \in \mathcal{C}$.

Suppose $J(C) = \emptyset$ for some $C \in \mathcal{C}$. Then $I(C) = C$. Hence $V(H) = V(C) \cup V_1(H)$, and H has a dominating system $\mathcal{S} = \{C\}$ consisting of exactly one element. As H contains a circuit (namely C), H contains two independent edges. Since H is triangle-free and $\delta_e(H) = d$, we obtain that $e(H) \geq d + 1 + d + 1 - 2 = 2d$. Since $d \geq 5$, this implies that $e(H) \geq d + 2$, hence $(e(H) - 3)/(d - 1) \geq 1$, and thus the statement of the theorem is true.

We now deal with each circuit separately in the sequel, and we let $\alpha(C)$ denote the total number of edges we remove from C.

We will add sufficiently many pendant edges to each remaining vertex of each C such that the following two conditions are valid for H^*.

A. Each edge has edge-degree at least d.
B. Each remaining vertex of each C has at least one pendant edge.

As we will treat each circuit separately, we say that a circuit C *satisfies* conditions **A** and **B** if after our modifications in H^* the resulting graph has minimum edge-degree at least d and each remaining vertex of C has at least one pendant edge.

Due to condition **A** we may apply Corollary 1 to H^* in order to obtain a dominating system $\mathcal{S}^*$ with

$$|\mathcal{S}|^*| \leq \frac{e(H^*) - 3}{\delta_e(H^*) - 1} \leq \frac{e(H^*) - 3}{d - 1}.$$

Let $\beta(C)$ denote the total number of pendant edges we added to vertices in C. It will turn out that in this procedure we might have to add more edges than we remove, i.e., $\beta(C) > \alpha(C)$ for some circuits $C \in \mathcal{C}$. However, we will have the following advantage. By condition **B**, the pendant edge(s) of each remaining vertex of each circuit C must be dominated by a number of stars, say $\gamma(C)$ stars, in any dominating system of H^*, so also in $\mathcal{S}^*$. If for each $C \in \mathcal{C}$ we replace these $\gamma(C)$ stars by C and keep all other elements of $\mathcal{S}^*$, then we obtain a dominating system $\mathcal{S}$ in the original graph H with

$$\begin{aligned}
|\mathcal{S}| &\leq |\mathcal{S}^*| - \sum_{\mathcal{C}}(\gamma(C) - 1) \\
&\leq \frac{e(H^*) - 3}{d - 1} - \sum_{\mathcal{C}}(\gamma(C) - 1) \\
&\leq \frac{e(H) - 3 - \sum_{\mathcal{C}}(\beta(C) - \alpha(C))}{d - 1} - \sum_{\mathcal{C}}(\gamma(C) - 1) \\
&= \frac{e(H) - 3}{d - 1} + \sum_{\mathcal{C}} \frac{\beta(C) - \alpha(C) - (d - 1)(\gamma(C) - 1)}{d - 1}.
\end{aligned}$$

Since we want to prove that $|\mathcal{S}| \leq \frac{e(H)-3}{d-1}$, it is sufficient to prove for all $C \in \mathcal{C}$ that

$$\beta(C) - \alpha(C) - (d - 1)(\gamma(C) - 1) \leq 0. \tag{5}$$

Before proving (5) for each $C \in \mathcal{C}$, we first need some new terminology. A *cycle decomposition* of a circuit C is a collection of cycles $D_1, \ldots, D_p$ such that $E(D_i) \cap E(D_j) = \emptyset$ for all $1 \leq i < j \leq p$ and $E(D_1) \cup \cdots \cup E(D_p) = E(C)$. For a vertex u, let $w(u)$ denote the number of cycles of a cycle decomposition to which u belongs. Clearly, $w(u) = d_C(u)/2$ and we obtain the following inequality which we will frequently use: $E(C) = \sum_{V(C)} w(u)$. By Veblen's Theorem (cf. [1]), each circuit has a cycle decomposition.

We now deal with each circuit separately in order to prove (5). Let C be a circuit in $\mathcal{C}$. We write $q = |V(C)|$, $q' = |J(C)|$, $I = I(C)$ and $J = J(C)$. From the above discussion we may assume $q' \geq 1$. Let $d^*(u) = d(u) - 2w(u)$ denote the number of edges incident with a vertex $u \in I$ in the subgraph of H obtained from $H[V(C) \cup V_1(H)]$ after removing $E(C)$.

Let J^* be the subset of J that consists of all vertices u with $d(u) \geq d + 2w(u)$. First suppose J^* is nonempty. Then we remove all edges in $E(C)$. We also remove all vertices of I together with their possible neighbors in $V_1(H)$. To each $u \in J^*$ we add one pendant edge and to each $u \in J \backslash J^*$ we add $d - d(u) + 2w(u) + 1 \geq 2$ edges. In the latter number of edges we need the extra "+1" in order to ensure C satisfies condition **A**. Clearly, C also satisfies condition **B**. We write $q^* = |J^*|$ and $\bar{q} := |J \backslash J^*|$. Then we use $\alpha(C) \geq \sum_C w(u) + \frac{1}{2} \sum_I d^*(u)$ (as two vertices in I might be adjacent to each other), $d(u) \geq 2w(u) + 1$ for all $u \in J \backslash J^*$, $w(u) \geq 1$ for all $u \in C$, $q \geq q^* + \bar{q}$, $q^* \geq 1$ and $d \geq 5$, respectively, in order to obtain

$$
\begin{aligned}
& \beta(C) - \alpha(C) - (d-1)(\gamma(C) - 1) \\
&\leq q^* + \sum_{J \backslash J^*} (d - d(u) + 2w(u) + 1) - \sum_C w(u) - \frac{1}{2} \sum_I d^*(u) - (d-1)(q' - 1) \\
&\leq q^* + d\bar{q} - \sum_{J \backslash J^*} (d(u) - 2w(u) - 1) - \sum_{V(C)} w(u) - dq' + q' + d - 1 \\
&\leq q^* + d\bar{q} - q - dq' + q' + d - 1 \\
&\leq q^* + d\bar{q} - q - d\bar{q} - dq^* + \bar{q} + q^* + d - 1 \\
&\leq (1 - d)(q^* - 1) \\
&\leq 0.
\end{aligned}
$$

Hence inequality (5) holds. From now we will assume that $d(u) \leq d + 2w(u) - 1$ for all $u \in J$. Here, we need the extra "−1" as sometimes we add $d - d(u) + 2w(u)$ new pendant edges to such a vertex u, and we want this number to be strictly positive in order to ensure condition **B** is satisfied.

Suppose $q' = 1$. Say $J = \{u\}$. Then we remove all vertices from C except u. We add d new pendant edges to u. Hence C satisfies conditions **A** and **B**. As C contains an edge xy with $u \notin \{x, y\}$, we obtain

$$\beta(C) - \alpha(C) - (d-1)(\gamma(C) - 1) = d - \alpha(C) - (d-1)(1-1) \leq d - e(C) \leq 0,$$

so inequality (5) holds.

Suppose $q' \geq 2$, say $u_1, u_2 \in J$. Recall $e(C) \geq 4$. Then C contains two vertices v_1 and v_2 such that u_1v_1 and u_2v_2 are independent edges in C. First we assume $C = u_1v_1u_2v_2u_1$. Then $H[V(C)] = C$ as H is triangle-free. We remove the edges u_1v_2 and v_1u_2. For each $u \in C$, we add $d - d(u) + 2w(u) = d - d(u) + 2 \geq 1$ new pendant edges. Then conditions **A** and **B** are satisfied for C.

We use $d(u_i) + d(v_i) \geq d + 2$ for $i = 1, 2$ and $d \geq 5$ in order to obtain

$$\begin{aligned}
&\beta(C) - \alpha(C) - (d-1)(\gamma(C) - 1)\\
&= d - d(u_1) + 2 + d - d(v_1) + 2 + d - d(u_2) + 2 + d - d(v_2) + 2 - 2 - (d-1)(4-1)\\
&= 4d - d - 2 - d - 2 + 6 - 4d + 4 + d - 1\\
&= -d + 5\\
&\leq 0.
\end{aligned}$$

Hence inequality (5) holds.

Now suppose C is not a cycle on four vertices, so $q \geq 5$. We distinguish three cases: $v_1, v_2 \in I$, or one of them is in I, while the other one is in J, or $v_1, v_2 \in J$.

First suppose $v_1, v_2 \in I$. We remove all edges of $E(C)$ and all vertices of I together with their possible neighbors in $V_1(H)$. We add $d - d(u) + 2w(u) + 1 \geq 2$ new pendant edges to each $u \in J$. Note that this way conditions **A** and **B** are satisfied for C. Then we use $\alpha(C) \geq \sum_C w(u) + \frac{1}{2}\sum_I d^*(u)$, $d(u) \geq 2w(u) + 1$ for all $u \in J$, $d(u_i) + d(v_i) = d(u_i) + 2w(v_i) + d^*(v_i) \geq d + 2$ for $i = 1, 2$, $w(u) \geq 1$ for all $u \in C$, and $q \geq q' + 2$ (since we have $v_1, v_2 \in I$) respectively, in order to obtain that indeed inequality (5) holds, i.e.,

$$\begin{aligned}
&\beta(C) - \alpha(C) - (d-1)(\gamma(C) - 1)\\
&\leq \sum_J (d - d(u) + 2w(u) + 1) - \sum_C w(u) - \frac{1}{2}\sum_I d^*(u) - (d-1)(q'-1)\\
&\leq dq' - \sum_{J\backslash\{u_1,u_2\}} (d(u) - 2w(u) - 1) + 2 - \left(\frac{1}{2}d(u_1) - w(u_1)\right) - \left(\frac{1}{2}d(u_2) - w(u_2)\right)\\
&\quad -\frac{1}{2}(d(u_1) + 2w(v_1) + d^*(v_1) + d(u_2) + 2w(v_2) + d^*(v_2)) - \sum_{C\backslash\{u_1,u_2,v_1,v_2\}} w(u) - dq' + q' + d - 1\\
&\leq 2 - \tfrac{1}{2} - \tfrac{1}{2} - d - 2 - q + 4 + q' + d - 1\\
&\leq 0.
\end{aligned}$$

Secondly, suppose $v_1 \in I, v_2 \in J$ or $v_1 \in J, v_2 \in I$. By symmetry, we may assume without loss of generality that $v_1 \in I$ and $v_2 \in J$. We remove all edges of $E(C)$ except u_1v_1. We also remove all vertices of I together with their neighbors in $V_1(H)$. We add $d - d(u) + 2w(u) + 1 \geq 2$ new pendant edges to each $u \in J\backslash\{u_1, v_1\}$. We add $d - d(u) + 2w(u) \geq 1$ new pendant edges to $u \in \{u_1, v_1\}$. Note that this way conditions **A** and **B** are satisfied for C. Then we use $\alpha(C) \geq \sum_C w(u) + \frac{1}{2}\sum_I d^*(u)$, $d(u) \geq 2w(u)+1$ for all $u \in J$, $d(u_2)+d(v_2) = d(u_2)+2w(v_2)+d^*(v_2) \geq d+2$, $w(u) \geq 1$ for all $u \in C$, and $q \geq q'+1$ (since we have $v_2 \in I$) respectively, in order to obtain

$$\begin{aligned}
&\beta(C) - \alpha(C) - (d-1)(\gamma(C)-1)\\
&\leq \sum_J (d - d(u) + 2w(u) + 1) - 2 - \sum_C w(u) - \frac{1}{2}\sum_I d^*(u) + 1 - (d-1)(q'-1)\\
&\leq dq' - \sum_{J\backslash\{u_1,u_2,u_3\}} (d(u) - 2w(u) - 1) + 3 - 2 - \left(\frac{1}{2}d(u_1) - w(u_1)\right) - \left(\frac{1}{2}d(u_2) - w(u_2)\right)\\
&\quad - \left(\frac{1}{2}d(v_1) - w(v_1)\right) - \tfrac{1}{2}(d(u_1) + d(v_1) + d(u_2) + 2w(v_2) + d^*(v_2))\\
&\quad - \sum_{C\backslash\{u_1,u_2,v_1,v_2\}} w(u) + 1 - dq' + q' + d - 1\\
&\leq 1 - \tfrac{1}{2} - \tfrac{1}{2} - \tfrac{1}{2} - d - 2 - q + 4 + 1 + q' + d - 1\\
&\leq \tfrac{1}{2},
\end{aligned}$$

so also in this case we find that $\beta(C)-\alpha(C)+(d-1)(\gamma(C)-1) \leq 0$, and hence inequality (5) holds.

Thirdly, suppose $v_1, v_2 \in J$. We now show that we may assume without loss of generality that $V(C) = J$, so $q = q'$. Suppose otherwise. Then there exists a vertex $y \in I$ such that there exists a path P from u_1 to y in C that besides y only uses vertices from J. Let x be the neighbor of y on P. If $x \notin \{u_2, v_2\}$ we find that edges xy and u_2v_2 are independent. Otherwise xy and u_1v_1 are independent. In both cases we return to the previous case. Hence, we may indeed assume $q = q'$.

Now suppose there exists a third independent edge u_3v_3 in C. Then we remove all edges of $E(C)$ except u_1v_1. We add $d - d(u) + 2w(u) + 1 \geq 2$ new pendant edges to each $u \in C\backslash\{u_1, v_1\}$. We add $d - d(u) + 2w(u) \geq 1$ new pendant edges to $u \in \{u_1, v_1\}$. Note that this way conditions **A** and **B** are satisfied for C. Let $Z = \{u_1, u_2, u_3, v_1, v_2, v_3\}$ (so $|Z| = 6$). Then we use $d(u) \geq 2w(u)+1$ for all $u \in C$, $d(u_i)+d(v_i) \geq d+2$ for $i = 1, 2, 3$, $w(u) \geq 1$ for all $u \in C$, respectively, in order to obtain

$$\begin{aligned}
&\beta(C)-\alpha(C)-(d-1)(\gamma(C)-1)\\
\leq&\sum_{C}(d-d(u)+2w(u)+1)-2-\sum_{C}w(u)+1-(d-1)(q-1)\\
\leq&\, dq-\sum_{C\backslash Z}(d(u)-2w(u)-1)+6-2-\sum_{i=1}^{3}\left(\frac{1}{2}d(u_i)-w(u_i)\right)-\sum_{i=1}^{3}\left(\frac{1}{2}d(v_i)-w(v_i)\right)\\
&-\frac{1}{3}\sum_{i=1}^{3}(d(u_i)+d(v_i))-\frac{1}{6}\sum_{i=1}^{3}(d(u_i)+d(v_i))-\sum_{C\backslash Z}w(u)+1-dq+q+d-1\\
\leq&\, 4-3\times\tfrac{1}{2}-3\times\tfrac{1}{2}-d-2-\tfrac{1}{2}d-1-q+6+1+q+d-1\\
\leq&\, 4-\tfrac{1}{2}d.
\end{aligned}$$

The final term is at most $\frac{1}{2}$ if $d \geq 7$, so inequality (5) holds if $d \geq 7$. Suppose $d = 5$. We add 3 new pendant edges to each $u \in C$. As $d(u) \geq 3$ for all $u \in C$, we find that besides condition **B** also condition **A** is satisfied for C. Then

$$\beta(C)-\alpha(C)-(d-1)(\gamma(C)-1)=3q-4(q-1)=4-q\leq 0,$$

as $q \geq 5$. Suppose $d = 6$. We add 4 new pendant edges to each $u \in C$. Again, conditions **A** and **B** are satisfied for C. Then

$$\beta(C)-\alpha(C)-(d-1)(\gamma(C)-1)=4q-5(q-1)=5-q\leq 0,$$

as $q \geq 5$. So also in these two cases inequality (5) holds.

Now suppose C does not have three independent edges. Then all cycles in C must contain either four or five vertices. Suppose C has a cycle D on five vertices. Then C must be isomorphic to D as otherwise we can easily find three independent edges, namely two edges of D and one edge with exactly one end vertex in D (note that $G[D] = D$ as H is triangle-free). Let $C = u_1u_2u_3u_4u_5u_1$. Then $H[V(C)] = C$ as H is triangle-free. We remove all edges from C except u_1u_2. We add $d - d(u_i) + 2w(u_i) = d - d(u_i) + 2 \geq 1$ new pendant edges for $i = 1, 2$ and $d-d(u_i)+2w(u_i)+1 = d-d(u_i)+3 \geq 2$ new pendant edges for $i = 3, 4, 5$. Then conditions **A** and **B** are valid for C. We use $2\sum_{i=1}^{5} d(u_i) \geq 5d + 10$

and $d \geq 5$ in order to obtain

$$
\begin{array}{rl}
& \beta(C) - \alpha(C) - (d-1)(\gamma(C)-1) \\
\leq & \sum_{i-1}^{5}(d - d(u_i) + 3) - 2 - 4 - (d-1)(5-1) \\
\leq & 5d - \sum_{i=1}^{5} d(u_i) + 15 - 6 - 4d + 4 \\
\leq & d + 13 - \frac{5}{2}d - 5 \\
= & 8 - \frac{3}{2}d \\
\leq \frac{1}{2}, &
\end{array}
$$

which means $\beta(C) - \alpha(C) - (d-1)(\gamma(C)-1) \leq 0$, so inequality (5) holds.

Now suppose C only contains cycles on four vertices. Then C is a complete bipartite graph (or else we would be in one of the previous cases) with independent sets $A(C)$ of cardinality 2 and $B(C)$ of cardinality $2k$ for some $k \geq 1$. Note that $G[C] = C$ as H is triangle-free, and that we already treated with the case $k = 1$. Hence, we may assume $k \geq 2$. Let $A(C) = \{s, t\}$ and let $B(C) = \{s_1, \ldots, s_k\} \cup \{t_1, \ldots, t_k\}$.

First we assume that $d(s) \leq d$. We remove all edges ts_i of $E(C)$ for $i = 1, \ldots, 2k$. We add $d - d(t) + 2w(t) + 1 = d - d(t) + 2k + 1 \geq 2$ new pendant edges to t and add $d - d(s) + 1$ new pendant edges to s. We add $d - d(s_i) + 2w(s_i) = d - d(s_i) + 2 \geq 1$ new pendant edges to each s_i for $i = 1, \ldots, 2k$. Note that this way conditions **A** and **B** are satisfied for C. We use $d(s_i) \geq 3$ for all $i = 1, \ldots, k$, $d(s) \geq 2k + 1$, $d(t) \geq 2k + 1$ and $d(u) + d(v) \geq d + 2$ for all edges uv, in order to obtain

$$
\begin{aligned}
&\beta(C) - \alpha(C) - (d-1)(\gamma(C)-1) \\
&\leq \sum_{i=1}^{2k} (d - d(s_i) + 2) + d - d(s) + 1 + d - d(t) + 2k + 1 - 2k - (d-1)(2k+2-1) \\
&\leq 2dk - \sum_{i=1}^{2k} \left(\frac{1}{2}d(s_i) - 1\right) - \left(\frac{1}{2}d(s) - k\right) - \left(\frac{1}{2}d(t) - k\right) - \frac{1}{2k}\left(\sum_{i=1}^{2k} d(s_i) + kd(s) + kd(t)\right) \\
&\quad - \frac{k-1}{2k}\left(\sum_{i=1}^{2k} d(s_i)\right) + 2d + 2 - 2k - 2dk - d + 2k + 1 \\
&= -2k \times \tfrac{1}{2} - \tfrac{1}{2} - \tfrac{1}{2} - d - 2 - 6k \times \tfrac{k-1}{2k} + d + 3 \\
&= -k - 3(k-1) \\
&= -4k + 3 \\
&\leq 0,
\end{aligned}
$$

so inequality (5) holds. If $d(s) \geq d+1$ we do exactly the same as above except that we add only one new pendant edge to s. Then again conditions **A** and **B** are satisfied for C and using $d(u) \geq 3$ for all $u \in C$ we obtain

$$
\begin{aligned}
&\beta(C) - \alpha(C) - (d-1)(\gamma(C)-1) \\
&\leq \sum_{i=1}^{2k} (d - d(s_i) + 2) + 1 + d - d(t) + 2k + 1 - 2k - (d-1)(2k+2-1) \\
&\leq 2dk - 2k + 2 + d - d(t) - 2dk - d + 2k + 1 \\
&= 3 - d(t) \\
&\leq 0,
\end{aligned}
$$

so inequality (5) also holds in this case. This completes the proof of Theorem 8. □

4. Proof of Theorem 9

The proof of Theorem 9 is modelled along the same lines as the proof of Theorem 8 but the details are different. We first present the six exceptional graphs of Theorem 9.

Let H^1 be the tree that is obtained from an edge uv by adding four pendant edges to u and four pendant edges to v. Let H^2 be the tree that is obtained from a path uvw by adding four pendant edges to u, three pendant edges to v, and four pendant edges to w. Let H^3 be the tree that is obtained from a path $uvwx$ by adding four pendant edges to u, three pendant edges to v, three pendant edges to w and four pendant edges to x. Let H^4 be the graph that is obtained from a star $K_{1,3}$ with center s and leaves x, y, z by adding two pendant edges to s, four pendant edges to x, four pendant edges to y and four pendant edges to z. See Figure 3.

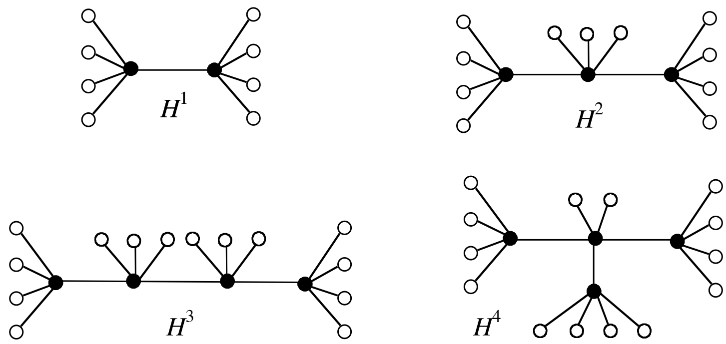

Fig. 3. The graphs H^1, H^2, H^3, H^4.

We note that H^1 has a 2-D-system, H^2 has a 3-D-system, and both H^3 and H^4 have a 4-D-system, while these graphs have 9, 13 and 17 edges, respectively. The other two exceptional graphs are $K_{1,5}$ and $K_{1,6}$. Computing the bound $\frac{5e(H)-14}{18}$ from Theorem 9 for these graphs, we see that these graphs should be excluded from Theorem 9, and therefore the corresponding claw-free graphs should be excluded from Theorem 3. In particular, if a claw-free graph G has a closure $cl(G)$ that is isomorphic to $L(H^i)$ for $i \in \{1, 2, 3, 4\}$ or to K_j for $j \in \{5, 6\}$, then G has to be excluded from Theorem 3.

As in the proof of Theorem 8, we first prove the statement of Theorem 9 in case H is a tree.

4.1. Theorem 9 holds for trees

We will use the following result.

Theorem 11. ([19]) *A tree H with $\delta_e(H) \geq 4$ that does not contain any vertices of degree two has a dominating system with at most $(e(H)-1)/4$ elements.*

Another lemma we need is the following.

Lemma 4. *Let u be a vertex with degree $d_H(u) \geq 3$ in a tree H with $\delta_e(H) \geq 4$ and $V_2(H) = \emptyset$. Then H has a dominating system $\mathcal{S}$ with a star that has center u and $d_H(u)$ leaves.*

Proof. We let u be the root of H. Because $\delta_e(H) \geq 4$ and $V_2(H) = \emptyset$, we can define the following dominating system of H. All vertices of $H \backslash V_1(H)$ on even distance from u become centers of stars with all their neighbors as leaves. All vertices of $H \backslash V_1(H)$ on odd distance from u that have a neighbor in $V_1(H)$ become centers of stars with only leaves in $V_1(H)$. All other vertices do not become star centers. □

Let us denote the set of exceptional preimage graphs by $\mathcal{H}$, so we have that $\mathcal{H} = \{K_{1,5}, K_{1,6}, H^1, H^2, H^3, H^4\}$. Our main result of this section confirms that our second main result holds for trees.

Lemma 5. *Let $H \notin \mathcal{H}$ be a tree with $\delta_e(H) \geq 4$. Then H has a dominating system with at most $(5e(H)-14)/18$ elements.*

Proof. Let H be a smallest counterexample to the claim. Let $F = H - V_1(H)$. We write $n_i = |V_i(F)|$. Since F is a tree we find

$$e(F) = |F| - 1 = \sum_{i \geq 1} n_i - 1. \tag{6}$$

Clearly, H is not isomorphic to a star $K_{1,k}$. This implies that $|F| \geq 2$ and we can use Observation 10 to obtain the following equation.

$$n_1 = \sum_{i \geq 3} (i-2)n_i + 2. \tag{7}$$

We start with the following claim.

Claim 1 $V_2(H) \neq \emptyset$.

We prove this claim by contradiction. Suppose $V_2(H) = \emptyset$. Then the claim immediately follows from Theorem 11, if we can show the following: if $H \notin \mathcal{H}$, then $\lfloor (e(H)-1)/4 \rfloor \leq \lfloor (5e(H)-14)/18 \rfloor$. Below we prove this statement.

First note that $(e(H)-1)/4 \leq (5e(H)-14)/18$ if $e(H) \geq 19$. Since $\delta_e(H) \geq 4$, we find that $e(H) \geq 5$, and that H would be isomorphic to $K_{1,5}$ if $e(H) = 5$, and to $K_{1,6}$ if $e(H) = 6$. Hence we may assume that $7 \leq e(H) \leq 18$. We then observe that $\lfloor (e(H)-1)/4 \rfloor > \lfloor (5e(H)-14)/18 \rfloor$ only if $e(H) \in \{9, 13, 17\}$. We consider each of these three cases, where we will use the following simple observations on F. Since F is a tree and $|F| \geq 2$, $n_1 \geq 2$. Since $\delta_e(H) \geq 4$, each vertex in $V_1(F)$ has at least four pendant edges in H. For the same reason, each vertex in $V_2(F)$ that is adjacent to a leaf of H has at least three pendant edges in H, and each vertex in $V_3(F)$ that is adjacent to a leaf of H has at least two pendant edges in H.

Suppose $e(H) = 9$. Then, using the above, we find that $n_1 = 2$ and $n_i = 0$ for all $i \geq 2$. Hence H is isomorphic to H^1.

Suppose $e(H) = 13$. Again, we find that $n_1 = 2$. Then, due to equality (7), $n_i = 0$ for $i \geq 3$. Then $n_2 = 1$. Hence F is a path on three vertices. This implies that H is isomorphic to H^2.

Suppose $e(H) = 17$. We find that $2 \leq n_1 \leq 3$. Suppose $n_1 = 2$. Due to equality (7) we find that $n_i = 0$ for $i \geq 3$. Then $n_2 = 2$. Hence F is a path on four vertices. This implies that H is isomorphic to H^3. Suppose $n_1 = 3$. Due to equality (7) we find that $n_3 = 1$ and $n_i = 0$ for $i \geq 4$. Then $n_2 = 0$. Hence F is a star on four vertices. We find that H is isomorphic to H^4.

This completes the proof of Claim 1.

The following claim immediately follows from our assumption that H is a smallest counterexample and from the definition of the graphs H^i.

Claim 2 *Let H' be a tree with $\delta_e(H') \geq 4$ and $e(H') < e(H)$. Then the minimum number of elements in a dominating system of H' is at most*

– $(5e(H')-14)/18$ *if $H' \notin \mathcal{H}$;*

– $(5e(H') - 14)/18 + 5/18$ *if* H' *is isomorphic to* H^1*;*
– $(5e(H') - 14)/18 + 3/18$ *if* H' *is isomorphic to* H^2*;*
– $(5e(H') - 14)/18 + 1/18$ *if* H' *is isomorphic to* H^3 *or* H^4*.*

We need a few other claims as well.

Claim 3 *Any vertex* $u \in V \backslash V_1(H)$ *with* p *neighbors in* $V \backslash V_1(H)$ *has at most* $\max\{1, 5-p\}$ *neighbors in* $V_1(H)$*.*

We prove Claim 3 by contradiction. Suppose $u \in V \backslash V_1(H)$ is adjacent to more than $\max\{1, 5-p\}$ leaves of H. Then we remove one of these leaves of H to obtain a smaller graph H' with minimum edge-degree at least four. If the new graph H' is not in $\mathcal{H}$, then H' is a smaller counterexample, see Claim 2. This contradicts our assumption on H. Since $V_2(H) \neq \emptyset$ due to Claim 1, H is not a star, and hence H' is neither isomorphic to $K_{1,5}$ nor to $K_{1,6}$. Suppose $H' \in \{H^1, H^2, H^3, H^4\}$. Then it is easy to check that H is not a counterexample. This completes the proof of Claim 3.

Claim 4 $N(V_1(H)) = V_1(F)$*.*

We prove Claim 4 as follows. By definition of F, any vertex in $V_1(F)$ has (at least) four neighbors in $V_1(H)$. Hence, $V_1(F) \subseteq N(V_1(H))$.

We use a proof by contradiction to show that $N(V_1(H)) \subseteq V_1(F)$. Suppose $u \in V(F) \backslash V_1(F)$ is adjacent to $p \geq 2$ vertices of F and $q \geq 1$ leaves of H. By Claim 3, we know that $q \leq \max\{1, 5-p\} \leq 3$. Let $\{v_1, \ldots, v_p\}$ be the neighbors of u in F. We distinguish three cases.

Case 1. $q = 3$ or $d_H(v_i) = d_H(v_j) = 2$ for some $1 \leq i < j \leq p$.
We obtain new trees H_i for $i = 1, \ldots, p$ as follows: for a fixed value of i we remove the edges uv_j for all $j \neq i$ and we add $4 - q \geq 1$ new pendant edges to u. We denote the component of the resulting graph that contains v_i by H_i. See Figure 4 for an example with $q = 1$ and $p = 4$ (so with $d_H(v_i) = d_H(v_j) = 2$ for some $1 \leq i < j \leq p$).

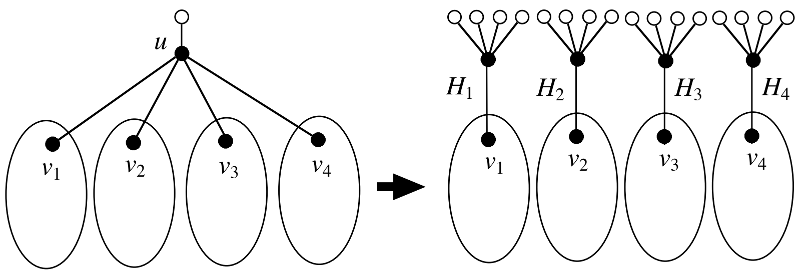

Fig. 4. Case 1 with $q = 1$ and $p = 4$.

We observe that each H_i is smaller than H, has minimum edge-degree at least four, and is neither isomorphic to $K_{1,5}$ nor to $K_{1,6}$. By Claim 2, each H_i has a dominating system $\mathcal{S}_i$ with at most $\frac{5e(H_i)-14}{18} + c_i$ elements with $c_i \in \{0, \frac{1}{18}, \frac{3}{18}, \frac{5}{18}\}$. Due to our assumption that $q = 3$ or $d_H(v_i) = d_H(v_j) = 2$ for some $1 \leq i < j \leq p$, we can unite the stars with center u in each $\mathcal{S}_i$ to obtain a dominating system $\mathcal{S}$ of H (in which indeed u is the center of a star with at least three leaves). We distinguish two subcases.

Case 1a. $p \geq 3$.

Due to $\delta_e(H) \geq 4$, $p+q \geq 5$. Due to Claim 3, $q \leq 2$. Then $d_H(v_i) = d_H(v_j) = 2$ for at least two vertices v_i, v_j. Hence at least two graphs H_i, H_j are not in $\{H^1, H^2, H^3, H^4\}$. Then the number of elements of $\mathcal{S}$ is at most

$$\begin{aligned}\sum_{i=1}^{p} |\mathcal{S}_i| - (p-1) &\leq \frac{5\sum_{i=1}^{p} e(H_i) - 14p}{18} + \frac{5(p-2)}{18} - (p-1)\\ &= \frac{5(e(H)+4p-q) - 9p - 10}{18} - (p-1)\\ &= \frac{5e(H)-14}{18} + \frac{22-2p-5(p+q)}{18}\\ &\leq \frac{5e(H)-14}{18} + \frac{22-2\times 3-5\times 5}{18}\\ &\leq \frac{5e(H)-14}{18},\end{aligned}$$

where we used that $p \geq 3$ and $p+q \geq 5$.

Case 1b. $p = 2$.
Due to $\delta_e(H) \geq 4$ and Claim 3, $q = 3$. If both H_1 and H_2 are isomorphic to H^1, then H is isomorphic to H^2. If one of the graphs, H_1, H_2, is isomorphic to H^1 and the other one is isomorphic to H^2, then H is isomorphic to H^3. So we may assume that these cases do not occur. Then the number of elements of $\mathcal{S}$ is at most

$$\begin{aligned}|\mathcal{S}_1| + |\mathcal{S}_2| - 1 &\leq \frac{5(e(H_1)+e(H_2)) - 28}{18} + \frac{6}{18} - 1\\ &= \frac{5(e(H)+5) - 22}{18} - 1\\ &= \frac{5e(H)-14}{18} - \frac{1}{18}\\ &\leq \frac{5e(H)-14}{18}.\end{aligned}$$

Case 2. Either $q = 2$, or $q = 1$ and $d_H(v_i) = 2$ for some $1 \leq i \leq p$.
Since $\delta_e(H) \geq 2$, we have $p \geq 5 - q \geq 3$, and we may assume that at least two vertices v_i, v_j, say v_1 and v_2, have degree at least three, otherwise we return to Case 1. We remove uv_1 and uv_2, add the edge v_1v_2 and we also add two new pendant edges to u. This way we obtain two graphs H' and H''. See Figure 5 for an example with $p = 3$ and $q = 2$.

We observe that both H' and H'' are smaller than H, have minimum edge-degree at least four, and are neither isomorphic to $K_{1,5}$ nor to $K_{1,6}$. If both H' and H'' are isomorphic to H^1, then H is isomorphic to H^4. So we may assume that this is not the case. Then, by Claim 2, we may without loss of generality assume that H' has a dominating system $\mathcal{S}'$ with at most $\frac{5e(H')-14}{18} + \frac{5}{18}$ elements and H'' has a dominating system $\mathcal{S}''$ with at most $\frac{5e(H'')-14}{18} + \frac{3}{18}$ elements.

Suppose without loss of generality that the edge v_1v_2 is covered by a star with center v_1. Then we can combine $\mathcal{S}'$ and $\mathcal{S}''$ to obtain a dominating system $\mathcal{S}$ of H, in which the edge uv_1 belongs to a star with center v_1 and the edge uv_2 belongs to a star with center u. This, together with our assumption that either $q = 2$, or $q = 1$ and $d_H(v_i) = 2$ for some

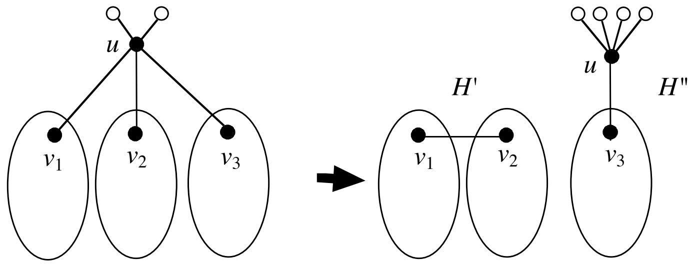

Fig. 5. Case 2 with $p = 3$ and $q = 2$.

$1 \leq i \leq p$, ensures that vertex u is indeed the center of a star in $\mathcal{S}$ with at least three edges. The number of elements of $\mathcal{S}$ is at most

$$\begin{aligned}|\mathcal{S}'| + |\mathcal{S}''| &\leq \frac{5(e(H') + e(H'')) - 28}{18} + \frac{8}{18} \\ &= \frac{5(e(H) + 1) - 20}{18} \\ &\leq \frac{5e(H) - 14}{18}.\end{aligned}$$

Case 3. $q = 1$ and $d_H(v_i) \geq 3$ for all $1 \leq i \leq p$.
Because $\delta_e(H) \geq 4$, we have $p \geq 5 - q = 4$. For $i = 1, \ldots, 4$, we remove the edges uv_i from H. We add an edges v_1v_2 to obtain a graph H_1, an edge v_3v_4 to obtain a graph H_2, and three new pendant edges to u to obtain a graph H_3. See Figure 6 for an example with $p = 4$.

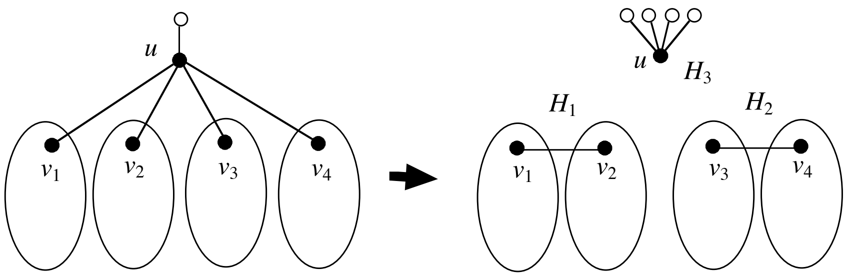

Fig. 6. Case 3 with $p = 4$.

We observe that H_1 and H_2 have minimum edge-degree at least four, are smaller than H, and are neither isomorphic to $K_{1,5}$ nor to $K_{1,6}$. By Claim 2, each H_i has a dominating system $\mathcal{S}_i$ with at most $\frac{5e(H_1)-14}{18} + c_i$ elements with $c_i \in \{0, \frac{1}{18}, \frac{3}{18}, \frac{5}{18}\}$. Suppose without loss of generality that the edge v_1v_2 is covered by a star in $\mathcal{S}_1$ with center v_1, and that the edge v_3v_4 is covered by a star in $\mathcal{S}_2$ with center v_3.

We distinguish two subcases.

Case 3a. H_3 is isomorphic to $K_{1,4}$.
Suppose both H_1 and H_2 are isomorphic to H^1. Then $e(H) = 21$, and H has a 5-D-system. Hence, H would not be a counterexample. Suppose H_1 is isomorphic to H^1 and

H_2 is isomorphic to H^2. Then $e(H) = 25$ and H has a 6-D-system. Again we find that H is not a counterexample. So we may assume these cases do not occur. Then we can combine the dominating systems $\mathcal{S}_i$ for $i = 1, 2$ to obtain a dominating system $\mathcal{S}$ of H, in which the edge uv_1 belongs to a star with center v_1, the edge uv_3 belongs to a star with center v_3, and the edges uv_2, uv_4 belong to a star with center u. The number of elements of $\mathcal{S}$ is at most

$$\begin{aligned} |\mathcal{S}_1| + |\mathcal{S}_2| + 1 &\leq \frac{5(e(H_1) + e(H_2)) - 28}{18} + \frac{6}{18} + 1 \\ &= \frac{5(e(H) - 2) - 4}{18} \\ &= \frac{5e(H) - 14}{18}. \end{aligned}$$

Case 3b. H_3 is not isomorphic to $K_{1,4}$.
Then H_3 has a dominating system $\mathcal{S}_3$ with at most $\frac{5e(H_3)-14}{18} + c_3$ elements with $c_3 \in \{0, \frac{1}{18}, \frac{3}{18}, \frac{5}{18}\}$. We combine $\mathcal{S}_1$ and $\mathcal{S}_2$ in the same way as we did for Case 3a. We also use all stars of $\mathcal{S}_3$ except the star with center u. This way we obtain a dominating system $\mathcal{S}$ of H (in which u is the center of a star with at least three leaves). The number of elements of $\mathcal{S}$ is at most

$$\begin{aligned} |\mathcal{S}_1| + |\mathcal{S}_2| + 1 + |\mathcal{S}_3| - 1 &\leq \frac{5(e(H_1) + e(H_2) + e(H_3)) - 42}{18} + \frac{15}{18} \\ &= \frac{5(e(H) + 1) - 27}{18} \\ &\leq \frac{5e(H) - 14}{18}. \end{aligned}$$

This completes the proof of Claim 4.

By Claims 3 and 4 we find that $V_i(H) = V_i(F)$ for all $i \neq 1, 5$ and $V_5(H) = V_1(F) \cup V_5(F)$. Hence

$$e(H) = e(F) + 4n_1. \tag{8}$$

We need two more claims regarding vertices in $V_2(F)$ before we can complete the proof of Lemma 5.

Claim 5 *Each vertex in $V_{\geq 4}(F)$ has at most one neighbor in $V_2(F) = V_2(H)$.*

We prove Claim 5 by contradiction. Suppose $u \in V_{\geq 4}(F)$ is adjacent to $q \geq 2$ vertices of $V_2(F)$. Assume u is adjacent to $p \geq 0$ other vertices of H, which are all in $V(F) \backslash V_2(F)$ due to Claim 4. Let $\{x_1, \ldots, x_q\}$ be the neighbors of u in $V_2(F)$. For $i = 1, \ldots, q$, let $y_i \neq u$ be the other neighbor of x_i in F. Since $\delta_e(H) \geq 4$, we have $p + q \geq 4$. Then we distinguish three cases.

Case 1. $q \geq 3$.
We remove the vertices x_1 and x_2. We add a new vertex z only adjacent to y_1 and y_2 to obtain a tree H'. We add three new pendant edges to u to obtain a tree H''. Both H' and

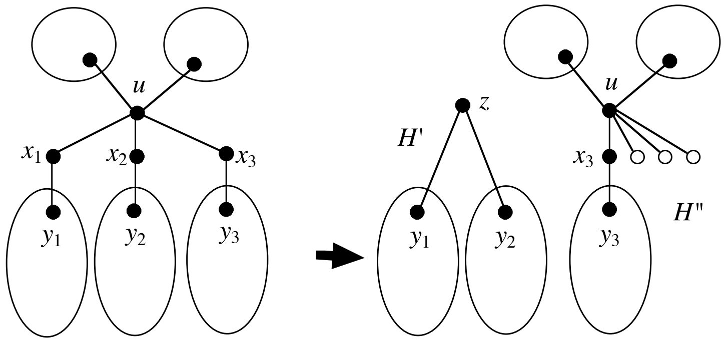

Fig. 7. Case 1 with $p = 2$ and $q = 3$.

H'' are smaller than H, have minimum edge-degree at least four, and are not isomorphic to a graph in $\mathcal{H}$. See Figure 7 for an example with $p = 2$ and $q = 3$.

Due to Claim 2, we then find that H' has a dominating system $\mathcal{S}'$ with at most $\frac{5e(H')-14}{18}$ elements and H'' has a dominating system $\mathcal{S}''$ with at most $\frac{5e(H'')-14}{18}$ elements.

Since $q \geq 3$, we can combine $\mathcal{S}'$ and $\mathcal{S}''$ to obtain a dominating system $\mathcal{S}$ of H (in which indeed vertex u is the center of a star with at least three leaves). The number of elements of $\mathcal{S}$ is at most

$$\begin{aligned} |\mathcal{S}'| + |\mathcal{S}''| &\leq \frac{5(e(H') + e(H'')) - 28}{18} \\ &= \frac{5(e(H) + 1) - 28}{18} \\ &\leq \frac{5e(H) - 14}{18}. \end{aligned}$$

Case 2. $p \geq 3$ and $q = 2$.
Since $p \geq 3$, u has two neighbors v_1 and v_2 in $V(F)\backslash V_2(F)$. Then both v_1 and v_2 have degree at least three in H. We remove uv_1 and uv_2. We add the edge v_1v_2 to obtain a graph H'. We add two new pendant edges to u to obtain a graph H''. See Figure 8 for an example with $p = 3$ and $q = 2$.

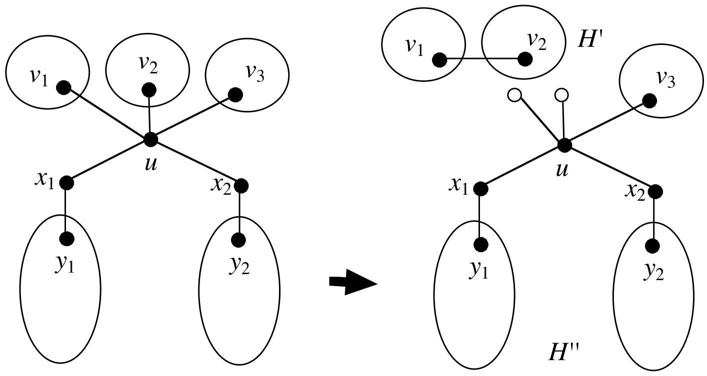

Fig. 8. Case 2 with $p = 3$ and $q = 2$.

Both H' and H'' are smaller than H, and have minimum edge-degree at least four. Furthermore, H' is neither isomorphic to $K_{1,5}$ nor to $K_{1,6}$, and H'' is not in $\mathcal{H}$. Then, by Claim 2, we find that H' has a dominating system $\mathcal{S}'$ with at most $\frac{5e(H')-14}{18}+\frac{5}{18}$ elements and H'' has a dominating system $\mathcal{S}''$ with at most $\frac{5e(H'')-14}{18}$ elements.

Suppose without loss of generality that the edge v_1v_2 is covered by a star with center v_1. Then we can combine $\mathcal{S}'$ and $\mathcal{S}''$ to obtain a dominating system $\mathcal{S}$ of H, in which the edge uv_1 belongs to a star with center v_1 and the edge uv_2 belongs to a star with center u. This, together with $q=2$, ensures that u is indeed the center of a star in $\mathcal{S}$ with at least three edges. The number of elements of $\mathcal{S}$ is at most

$$\begin{aligned}|\mathcal{S}'|+|\mathcal{S}''| &\le \frac{5(e(H')+e(H''))-28}{18}+\frac{5}{18}\\ &= \frac{5(e(H)+1)-23}{18}\\ &\le \frac{5e(H)-14}{18}.\end{aligned}$$

Case 3. $p=q=2$.
Then u has two neighbors v_1 and v_2 in $V(F)\backslash V_2(F)$. Then both v_1 and v_2 have degree at least three. We remove uv_1 and uv_2 from H. We add the edge v_1v_2 to obtain a graph H'. We replace u, x_1, x_2 by a new vertex z only adjacent to y_1 and y_2 to obtain a graph H'''. See Figure 9.

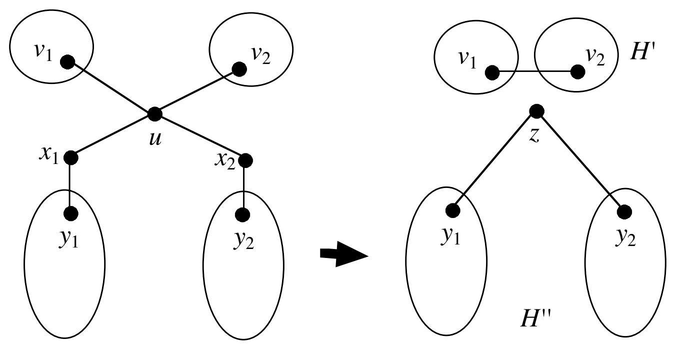

Fig. 9. Case 3 with $p=q=2$.

Both H' and H'' are smaller than H and have minimum edge-degree at least four. Furthermore, H' is neither isomorphic to $K_{1,5}$ nor to $K_{1,6}$, and H'' is not in $\mathcal{H}$. Then, by Claim 2, we find that H' has a dominating system $\mathcal{S}'$ with at most $\frac{5e(H')-14}{18}+\frac{5}{18}$ elements and H'' has a dominating system $\mathcal{S}''$ with at most $\frac{5e(H'')-14}{18}$ elements.

Suppose without loss of generality that the edge v_1v_2 is covered by a star with center v_1. We obtain a dominating system $\mathcal{S}$ of H as follows. We let u be the center of a star with leaves v_2, x_1, x_2. We replace v_2 by u in the star from $\mathcal{S}'$ with center v_1. We use all other stars of $\mathcal{S}'$ as well. We replace z by x_1 in the star from $\mathcal{S}''$ with center y_1. We replace z by x_2 in the star from $\mathcal{S}''$ with center y_2. We use all other stars of $\mathcal{S}''$ as well. Then the number of elements of $\mathcal{S}$ is at most

$$
\begin{aligned}
|\mathcal{S}'| + |\mathcal{S}''| + 1 &\leq \frac{5(e(H') + e(H'')) - 28}{18} + \frac{5}{18} + 1 \\
&= \frac{5(e(H) - 3) - 5}{18} \\
&\leq \frac{5e(H) - 14}{18}.
\end{aligned}
$$

This completes the proof of Claim 5.

Claim 6 *Each vertex in $V_2(F) = V_2(H)$ has one of its neighbors in $V_{\geq 4}(F)$ and the other one in $V_1(F) \cup V_{\geq 4}(F)$.*

We prove Claim 6 as follows. Let $x \in V_2(F)$. Recall that vertices in $V_2(F)$ and $V_3(F)$ are not adjacent to leaves in H due to Claim 4. This implies the following two statements. First, x has exactly two neighbors in H. Secondly, since $\delta_e(H) \geq 4$, x does not have a neighbor in $V_2(F) \cup V_3(F)$. Now suppose x has both its two neighbors in $V_1(F)$. Then F is a path wxy, where both w and y have four pendant edges in H due to Claim 3. This means that $e(H) = 10$ and that H has a 2-D-system. Then H would not be a counterexample. This finishes the proof of Claim 6.

We now have sufficient ingredients to complete the proof of Lemma 5. Recall that $V_2(F) \neq \emptyset$ due to Claim 1. This observation, together with Claim 5 implies that

$$\sum_{i \geq 4} n_i \geq n_2. \tag{9}$$

Let $V_2^*(F)$ denote the set of vertices in $V_2(F)$ that are adjacent to a vertex in $V_1(F)$. We write $n_2^* = |V_2^*(F)|$. For a vertex $u \in V_{\geq 4}(F)$ with (exactly) one neighbor v in $V_2(F)$, we call the component of $H - \{uv\}$ that contains u the *u-tree* of H. We call a u-tree with no vertices in $V_2(F)$ a *proper* u-tree. Let $V_{\geq 4}^*(F)$ consist of all vertices $u \in V_{\geq 4}(F)$ for which H has a proper u-tree. We write $n_{\geq 4}^* = |V_{\geq 4}^*(F)|$. We use Claim 5 and Claim 6 to deduce that

$$n_2^* + n_{\geq 4}^* \geq 2. \tag{10}$$

Recall that a claw-free graph with minimum degree at least 4 contains a 2-factor due to Theorem 1. This implies that our graph H has a dominating system $\mathcal{S}$. Let s be the number of centers of stars in $\mathcal{S}$ that do not contain a leaf of H. By Claim 4 we find that

$$|\mathcal{S}| = n_1 + s. \tag{11}$$

Below we show that we may assume without loss of generality that

$$3s + 4n_1 + n_2^* + n_{\geq 4}^* \leq e(H). \tag{12}$$

We prove inequality (12) as follows. Each star in $\mathcal{S}$ with center in $V(F) \backslash V_1(F)$ has at least three edges. This explains the term $3s$. Each star in $\mathcal{S}$ with center in $V_1(F)$ has at least four edges. This explains the term $4n_1$. Each vertex in $V_2^*(F)$ is a leaf of a star in $\mathcal{S}$ with center in $V_1(F)$, and consequently these stars have five edges. This explains the term n_2^*. Each vertex u in $V_{\geq 4}^*(F)$ is a center vertex of a star S in $\mathcal{S}$, because u is adjacent to a vertex v of $V_2(F)$ that has degree two in H due to Claim 4. By Lemma 4, any proper u-tree H' of H has a dominating system $\mathcal{S}'$ that contains a star S' with center u and $d_{H'}(u) = d_H(u) - 1 \geq 3$ leaves. Then we may without loss of generality assume that S has at least four edges (namely the edges of S' plus the edge uv). This explains the term $n_{\geq 4}^*$. Hence, we have deduced the lower bound on $e(H)$ of inequality (12).

Using (in)equalities (11),(12),(8),(6),(7), (10), and (9) consecutively, we find that

$$\begin{aligned}
&5e(H) - 14 - 18|\mathcal{S}| \\
&= 5e(H) - 14 - 18n_1 - 18s \\
&\geq 5e(H) - 14 - 18n_1 - 6e(H) + 24n_1 + 6n_2^* + 6n_{\geq 4}^* \\
&= -e(H) - 14 + 6n_1 + 6n_2^* + 6n_{\geq 4}^* \\
&= -e(F) - 4n_1 - 14 + 6n_1 + 6n_2^* + 6n_{\geq 4}^* \\
&= -e(F) + 2n_1 - 14 + 6n_2^* + 6n_{\geq 4}^* \\
&= -\sum_{i\geq 1} n_i + 1 + 2n_1 - 14 + 6n_2^* + 6n_{\geq 4}^* \\
&= n_1 - \sum_{i\geq 2} n_i - 13 + 6n_2^* + 6n_{\geq 4}^* \\
&= \sum_{i\geq 3}(i-2)n_i + 2 - \sum_{i\geq 2} n_i - 13 + 6n_2^* + 6n_{\geq 4}^* \\
&= \sum_{i\geq 4}(i-3)n_i - n_2 - 11 + 6n_2^* + 6n_{\geq 4}^* \geq \sum_{i\geq 4}(i-3)n_i - n_2 + 1 \geq 0.
\end{aligned}$$

Hence S has at most $\frac{5e(H)-14}{18}$ elements, and H can not be a counterexample. This completes the proof of Lemma 5. □

So we know that Theorem 9 holds in the case that H is a tree. We will use this result in the next part to show that the theorem holds for general triangle-free graphs.

4.2. Theorem 9 holds for general triangle-free graphs

For convenience we mention the statement of Theorem 9 with the explicit class of exceptional graphs.

Theorem 9. *Let $H \notin \{K_{1,5}, K_{1,6}, H^1, H^2, H^3, H^4\}$ be a triangle-free graph with $\delta_e(H) \geq 4$. Then H contains a dominating system $\mathcal{S}$ with at most $\frac{5e(H)-14}{18}$ elements.*

Proof. Let $H \notin \mathcal{H}$ be a triangle-free graph with $\delta_e(H) \geq 4$. Suppose H is a tree. Then the result follows from Lemma 5. Suppose H is not a tree. Let X be a maximum even subgraph of H. As in the proof of Theorem 8, the proof idea is to construct an X-graph H^* of H. Then, by Lemma 1, H^* is a forest. After some preprocessing we apply Lemma 5 to every component of H^* after adding sufficiently many pendant edges to ensure that each edge has edge-degree at least 4. In this procedure we have to add more edges than we remove. However, we will have the same advantage as in Theorem 8 if we also ensure that each remaining vertex in each circuit of X has at least one pendant edge. The added pendant edges have to be dominated by (extra) stars in any dominating system of H^*, and these stars can be merged together into fewer elements of a dominating system in the original graph H. In other words, the larger number of stars we get by applying the upper bounds to H^* provide the necessary compensation for the larger number of edges that we created. This way we are able to establish our upper bound for H.

We will now describe the procedure. Let C be a circuit in X. Let $I(C)$ be the set of vertices in C that are only adjacent to vertices in $C \cup V_1(H)$. If $I(C) = C$ then $V(H) = V(C) \cup V_1(H)$, and H has a dominating system $\mathcal{S} = \{C\}$ consisting of one element. Since H is a triangle-free graph with $\delta_e(H) \geq 4$ and H is not isomorphic to $K_{1,5}$ or $K_{1,6}$, we know that $|E(H)| \geq 7$. Then the statement of the theorem is true.

From now on, we assume that $I(C) \subsetneq C$ for each circuit C in X. We consider each C in X separately, and distinguish three cases. For each case we determine the net increase in the number of edges in order to restore the minimum edge-degree.

Case 1. $|C| \geq 5$ and $|I(C)| \leq 1$.
Then C contains an edge uv with u and v not in $I(C)$. From H, we remove all vertices in $I(C)$ together with all their neighbors in $V_1(H)$. We also remove all edges in $E(C)\backslash\{uv\}$. We add three new pendant edges to u and three new pendant edges to v. To every other vertex in $C - I(C)$ we add four new pendant edges. This together with $e(C) \geq |C|$ implies that the net increase in the number of edges is at most

$$4(|C| - 2 - |I(C)|) + 6 - (e(C) - 1) \leq 3|C| - 4|I(C)| - 1. \tag{13}$$

Case 2. $|C| \geq 5$ and $|I(C)| \geq 2$.
From H, we remove all vertices in $I(C)$ together with all their neighbors in $V_1(H)$. We also remove all edges in $E(C)$ from H. We add four new pendant edges to every vertex in $C - I(C)$. This, together with $e(C) \geq |C|$, implies that the net increase in the number of edges is at most

$$4(|C| - |I(C)|) - e(C) \leq 3|C| - 4|I(C)|. \tag{14}$$

Case 3. $|C| = 4$.
Then C is a cycle $stuvs$. We remove the edges sv and tu. Recall that st has edge-degree at least four in H. If $d_H(s) = 2$, then $d_H(t) \geq 4$, and we add four pendant edges to s and two pendant edges to t. If $d_H(s) = 3$, then $d_H(t) \geq 3$, and we add three pendant edges to s and three pendant edges to t. If $d_H(s) \geq 4$, then we add two pendant edges to s and four pendant edges to t. Since also uv has edge-degree at least four in H, we can do exactly the same for uv. This way the net increase in the number of edges is $2 \times 6 - 2 = 10$.

After we have performed one of the above operations as considered in Case 1, Case 2 or Case 3 for every circuit C in X, we have obtained a forest H^* by Lemma 1. From the above procedure it is clear that H^* has minimum edge-degree $\delta_e(H^*) \geq 4$. For each circuit C in Case 1 and Case 2 we have removed all vertices in $I(C)$ together with their neighbors in $V_1(H)$. Then the forest H^* does not contain a component isomorphic to $K_{1,5}$ or $K_{1,6}$. Hence, we can apply Lemma 5 to each component D of H^* to obtain a dominating system $\mathcal{S}(D)$ of D that has at most $(5e(D)-14)/18$ elements if D is not isomorphic to a graph in $\{H^1, H^2, H^3, H^4\}$. Otherwise $|\mathcal{S}^*| = (5e(D)-14)/18 + c$ with $c = 5/18$ if D is isomorphic to H^1, $c = 3/18$ if D is isomorphic to H^2, and $c = 1/18$ if D is isomorphic to H^3 or H^4.

For $i = 1, 2, 3$, let $\mathcal{C}^i$ denote the set of circuits that fall under Case i. Then the net increase in the number of edges due to circuits in $\mathcal{C}^3$ is $10|\mathcal{C}^3|$. This, together with inequality (13) and inequality (14) gives

$$e(H^*) \le e(H) + \sum_{C\in\mathcal{C}^1}(3|C| - 4|I(C)| - 1) + \sum_{C\in\mathcal{C}^2}(3|C| - 4|I(C)|) + 10|\mathcal{C}^3|. \qquad (15)$$

From our procedure it is clear that every vertex in any $C - I(C)$ with $C \in \mathcal{C}^1 \cup \mathcal{C}^2$ and every vertex in any $C \in \mathcal{C}^3$ is a center vertex of a star in any dominating system of H^*. By uniting the star centers in each $\mathcal{S}(D)$ that correspond to the same circuit we obtain a dominating system $\mathcal{S}$ of H. Let $D_1, \ldots, D_k$ be the components in H^*. We distinguish two cases.

Case 1. $k \ge 2$.
Then the number of elements in $\mathcal{S}$ is at most

$$\begin{aligned}
&\sum_{i=1}^{k}|\mathcal{S}(D_i)| - \sum_{C\in\mathcal{C}^1\cup\mathcal{C}^2}(|C| - |I(C)| - 1) - 3|\mathcal{C}^3| \\
&\le \sum_{i=1}^{k}\left(\frac{5e(D_i)-14}{18} + \frac{5}{18}\right) - \sum_{C\in\mathcal{C}^1\cup\mathcal{C}^2}(|C| - |I(C)| - 1) - 3|\mathcal{C}^3| \\
&= \frac{5e(H^*)-9k}{18} - \sum_{C\in\mathcal{C}^1\cup\mathcal{C}^2}(|C| - |I(C)| - 1) - 3|\mathcal{C}^3| \\
&\le \frac{5\{e(H) + \sum_{C\in\mathcal{C}^1}(3|C| - 4|I(C)| - 1) + \sum_{C\in\mathcal{C}^2}(3|C| - 4|I(C)|) + 10|\mathcal{C}^3|\} - 14}{18} \\
&\quad - \sum_{C\in\mathcal{C}^1\cup\mathcal{C}^2}(|C| - |I(C)| - 1) - 3|\mathcal{C}^3| \\
&= \frac{5e(H)-14}{18} + \frac{\sum_{C\in\mathcal{C}^1}(13 - 3|C| - 2|I(C)|)}{18} + \frac{\sum_{C\in\mathcal{C}^2}(18 - 3|C| - 2|I(C)|)}{18} - \frac{4|\mathcal{C}^3|}{18} \\
&\le \frac{5e(H)-14}{18} + \frac{\sum_{C\in\mathcal{C}^1}(13 - 3|C|)}{18} + \frac{\sum_{C\in\mathcal{C}^2}(14 - 3|C|)}{18} \\
&\le \frac{5e(H)-14}{18},
\end{aligned}$$

where we used $k \ge 2$, (15), $|I(C)| \ge 2$ for all $C \in \mathcal{C}^2$, and $|C| \ge 5$ for all $C \in \mathcal{C}^1 \cup \mathcal{C}^2$.

Case 2. $k = 1$.
It is easy to check that H^* is not isomorphic to H^1. Suppose H^* is isomorphic to H^2. Then X does not contain a circuit on four vertices, because we would have added pendant

edges to each of those vertices. Consider the three vertices in H^* that are not leaves. Since we assume $I(C) \subsetneq C$ for each circuit C in X, these vertices do not belong to the same circuit in X. By construction they do not belong to three different circuits either. Then H has at least 10 edges and a 2-D-system. Hence, the statement is true.

Suppose H^* is neither isomorphic to H^1 nor to H^2. Then the number of elements in $\mathcal{S}$ is at most

$$\frac{5e(H^*) - 14}{18} + \frac{1}{18} - \sum_{C \in \mathcal{C}^1 \cup \mathcal{C}^2} (|C| - |I(C)| - 1) - 3|\mathcal{C}^3| \leq \frac{5e(H) - 14}{18}.$$

Here we used the same deduction as in the case $k \geq 2$. Since there is at least one circuit C, we can compensate for the constant $1/18$. This completes the proof of the upper bound in Theorem 9 for connected triangle-free graphs with $\delta_e \geq 4$. This completes the proof of Theorem 9. □

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References

1. J.A. Bondy and U.S.R. Murty, *Graph Theory* (Springer, 2007).

2. S.A. Choudum and M.S. Paulraj, *Regular factors in $K_{1,3}$-free graphs*, Journal of Graph Theory **15** (1991) 259–265.

3. R. Diestel, *Graph Theory, Second edition*, Graduate Texts in Mathematics **173** (Springer, 2000).

4. Y. Egawa and K. Ota, *Regular factors in $K_{1,n}$-free graphs*, Journal of Graph Theory **15** (1991) 337–344.

5. Y. Egawa and A. Saito, private communications.

6. R.J. Faudree, O. Favaron, E. Flandrin, H. Li, and Z. Liu, *On 2-factors in claw-free graphs*, Discrete Mathematics **206** (1999) 131–137.

7. R. Faudree, E. Flandrin, and Z. Ryjáček *Claw-free graphs—a survey*, Discrete Mathematics **164** (1997) 87–147.

8. H. Fleischner, *Spanning Eulerian subgraphs, the splitting lemma, and Petersen's theorem*, Discrete Mathematics **101** (1992) 33–37.

9. D. Fronček, Z. Ryjáček, and Z. Skupień, *On traceability and 2-factors in claw-free graphs*, Discussiones Mathematicae Graph Theory **24** (2004) 55–71.

10. B. Jackson and K. Yoshimoto, *Even subgraphs of bridgeless graphs and 2-factors of line graphs*, Discrete Mathematics **307** (2007) 2775–2785.

11. B. Jackson and K. Yoshimoto, *Spanning even subgraphs of 3-edge-connected graphs*, Journal of Graph Theory, to appear.

12. R.J. Gould and E. Hynds, *A note on cycles in 2-factors of line graphs*, Bulletin of the ICA **26** (1999) 46–48.

13. R.J. Gould and M.S. Jacobson, *Two-factors with few cycles in claw-free graphs*, Discrete Mathematics **231** (2001) 191–197.

14. F. Harary and C. St. J.A. Nash-Williams *On eulerian and hamiltonian graphs and line graphs*, Canadian Mathematical Bulletin **8** (1965) 701-710

15. M. M. Matthews and D. P. Sumner, *Hamiltonian results in $K_{1,3}$-free graphs*, Journal of Graph Theory **8** (1984) 139–146.

16. M.D. Plummer, *Graph factors and factorization: 1985-2003: A survey*, Discrete Mathematics **307** (2007) 791–821.

17. Z. Ryjáček, *On a closure concept in claw-free graphs*, Journal of Combinatorial Theory, Series B **70** (1997) 217–224.

18. Z. Ryjáček, A. Saito, and R.H. Schelp, *Closure, 2-factors, and cycle coverings in claw-free graphs*, Journal of Graph Theory **32** (1999) 109–117.

19. K. Yoshimoto, *On the number of components in 2-factors of claw-free graphs*, Discrete Mathematics **307** (2007) 2808–2819.

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