



Integrated Density of States for Ergodic Random Schrödinger Operators on Manifolds

NORBERT PEYERIMHOFF and IVAN VESELIĆ*

Fakultär für Mathematik, Ruhr-Universität Bochum, Universitätsstr. 150, D-44780 Bochum, Germany. e-mail: peyerim@math.ruhr-uni-bochum.de; ivan.veselic@ruhr-uni-bochum.de

(Received: 30 January 2001)

Abstract. We consider the Riemannian universal covering of a compact manifold $M = X/\Gamma$ and assume that Γ is amenable. We show the existence of a (nonrandom) integrated density of states for an ergodic random family of Schrödinger operators on X .

Mathematics Subject Classifications (2000). 82B44, 58J35, 47B80.

Key words. integrated density of states, random Schrödinger operators, Riemannian manifolds with compact quotient, amenable groups, ergodic theorem.

1. Introduction and Statement of Results

The integrated density of states (IDS) is an important notion in the quantum theory of solids and describes the number of electron states below a certain energy level per unit volume. Let us shortly explain this notion in the case of a disordered solid, e.g., an alloy of two metals with a crystal structure where the nuclei of the two metals are randomly distributed at the lattice points. The situation can be described quantum mechanically by a corresponding family $H^\omega = \Delta + V^\omega$ of random Schrödinger operators. Due to the macroscopic dimensions of the solid one can consider operators on the whole \mathbb{R}^3 . Let $\Lambda_n \subset \mathbb{R}^3$ denote a cube of sidelength n , centered at the origin, and let $H_{\Lambda_n}^\omega$ denote the restriction of H^ω to Λ_n with a suitable boundary condition (e.g. Dirichlet or Neumann or periodic). Then the IDS is the limit of the corresponding eigenvalue counting functions of $H_{\Lambda_n}^\omega$, normalized by the volume of the cubes Λ_n . An ergodicity assumption yields the fact that one can associate to the whole family $\{H^\omega\}$ a nonrandom spectrum, i.e., that almost all operators have the same spectrum. Moreover, the points of increase of the IDS coincide with the almost sure spectrum of $\{H^\omega\}$. The nonrandomness of spectral data implies that the alloy exhibits almost surely a particular behavior of conductivity. For the importance of the IDS from the viewpoint of solid state physics see [BBEE+84, ES-84, Lif-85, LGP-88]. An overview over the mathematical results on random Schrödinger operators is given in the books [CL-90] and [PF-92]. Early rigorous

*Partially supported by the SFB 237 of the DFG and the MaPhySto-Centre of The Danish National Research Foundation.

results on the IDS can be found, e.g., in articles by Pastur [Pas-71a, Pas-71b, Pas-80] or Shubin [Shu-79, Shu-82]. A good introductory course on random Schrödinger operators is [Kir-89].

Our aim is to generalize the classical existence result of a nonrandom IDS for random Schrödinger operators to more general spaces. The main results are Theorems 1.3 and 1.4 below. They are generalizations of [PV-00]. We consider the universal Riemannian covering X of a compact Riemannian manifold $M = X/\Gamma$ with an infinite group Γ of deck transformations. In this context, Adachi, Brüning and Sunada [BS-92, AS-93] proved the existence of an IDS for a Γ -periodic elliptic operator H in the case that Γ is amenable. They also proved that this IDS agrees with the Γ -trace of the spectral projections of H . Note also that Dodziuk and Mathai in their paper [DM-97] on L^2 -Betti numbers derived a result for the IDS of the pure Laplace operator on k -forms. All mentioned results on the IDS use Shubin's [Shu-82] convergence criterium based on the Laplace transform.

In this article we consider a family of Schrödinger operators $H^\omega = \Delta + V^\omega$ (on a Riemannian manifold X), which are parametrized by the elements of a probability space. More precisely, we consider the following objects:

DEFINITION 1.1. Let X be the Riemannian universal covering of a compact Riemannian manifold $M = X/\Gamma$ and $\{H^\omega = \Delta + V^\omega\}_{\omega \in \Omega}$ be a family of Schrödinger operators, parametrized by elements of the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The family $\{H^\omega\}$ is called an *ergodic random family of Schrödinger operators*, if the potential $V: \Omega \times X \rightarrow \mathbb{R}$ is jointly measurable and if there exists an ergodic family of measure preserving transformations $\{T_\gamma: \Omega \rightarrow \Omega\}_{\gamma \in \Gamma}$ with $T_{\gamma_1 \gamma_2} = T_{\gamma_1} T_{\gamma_2}$ such that the potential satisfies the following compatibility condition

$$V^{T_\gamma \omega}(x) = V^\omega(\gamma^{-1}x) \quad (1)$$

for all $\omega \in \Omega$, $\gamma \in \Gamma$ and $x \in X$.

According to our convention Δ is a nonnegative operator.

For the notion of measurability of random unbounded selfadjoint operators we refer to [KM-82a, Section 2] and [CL-90, Chapter V]. The ergodicity of such operators is thoroughly investigated in [PF-92]. If $X \times \Omega \ni (x, \omega) \mapsto V^\omega(x)$ is jointly measurable, the multiplication operator $\omega \mapsto V^\omega$ is measurable in the sense of Kirsch and Martinelli [KM-82a]. Furthermore, by their Proposition 2.4 we know that $H^\omega = \Delta + V^\omega$ is measurable, too.

In the Euclidean case $X = \mathbb{R}^n$, mild integrability assumptions on $(x, \omega) \mapsto V^\omega(x)$ ensure the independence of the IDS of the boundary conditions (b.c.) used for its construction (see [KM-82b]). In relation to cubes Λ_n , one can consider Dirichlet and Neumann b.c. as well as periodic ones. In more general geometries, b.c. (in)dependence is a more subtle question. See [Sz-89, Sz-90], where Dirichlet and Neumann IDS on hyperbolic spaces are compared.

Note that ergodicity of the family $\{T_\gamma\}$ means that the only invariant measurable sets $A \subset \Omega$ are measure-theoretically trivial, i.e. $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$. For technical reasons (in order to apply a Sobolev lemma) we require that there exists a constant $C_0 > 0$ for an ergodic random family of Schrödinger operators such that

$$\|\nabla^k V^\omega\|_\infty \leq C_0, \quad \text{for all } \omega \in \Omega \text{ and } k \leq \frac{1}{2} \dim(X) + 2. \quad (2)$$

This implies in particular that V^ω is infinitesimally Δ -bounded, uniformly in ω . It seems that this regularity condition on the potential may be relaxed considerably by using stochastic methods instead of analytical methods for the required heat kernel estimates. Another approach to circumvent strong regularity assumptions could be the use of quadratic forms [Sim-71].

EXAMPLE. Let X and Γ be as in Definition 1.1. Then we can consider the following potential, which is an analogue of an alloy-type potential in the Euclidean setting:

Let $u: X \rightarrow \mathbb{R}$ be a smooth function with compact support. We choose $\Omega = \times_{\gamma \in \Gamma} \mathbb{R}$, equipped with the product measure $\mathbb{P} = \otimes_{\gamma \in \Gamma} \mu$, where μ is a probability measure on \mathbb{R} . Then the random variables $\pi_\gamma: \Omega \rightarrow \mathbb{R}$, $\pi_\gamma(\omega) = \omega_\gamma$ are independent and identically distributed. The transformations $(T_{\gamma_1}(\omega))_{\gamma_2} = \omega_{\gamma_1^{-1}\gamma_2}$ are measure preserving and ergodic. Then $H^\omega = \Delta + V^\omega$ with $V^\omega(x) = \sum_{\gamma \in \Gamma} \pi_\gamma(\omega) u(\gamma^{-1}x)$ defines an ergodic random family of Schrödinger operators. Note that V^ω is a superposition of Γ -translates of the single site potential u with coupling constants given by the random variables.

Let us introduce some more notation. For a given $h > 0$, the h -boundary of $D \subset X$ is defined as

$$\partial_h D = \{x \in X \mid d(x, \partial D) \leq h\}.$$

A subset of X is called a *regular domain* if it is the nonempty interior of a connected compact set with smooth boundary. A *regular set* D is a finite union $D = \bigcup_{j=1}^k D_j$ of regular domains with disjoint closures $\overline{D_j}$. In the sequel we will often deal with h -approximations and h -regularizations:

DEFINITION 1.2. Let $U, V \subset X$ be open subsets and $h > 0$. V is called an h -approximation of U , if the symmetric difference satisfies the following property:

$$U \Delta V \subset \partial_h U. \quad (3)$$

If, additionally, V is a regular set, we call V an h -regularization of U . Similarly, a sequence $\{V_n\}$ is called an h -approximation (h -regularization) of $\{U_n\}$ if there is a fixed $h > 0$ with $U_n \Delta V_n \subset \partial_h U_n$ for all n (and the sets V_n are regular). If only the existence and not the actual value of $h > 0$ is of importance, we also refer to $\{V_n\}$ as an approximation (regularization) of $\{U_n\}$.

Remarks. (1) For $p \in X$ let $B_r(p)$ denote the open metric r -ball around p . Then $B_{r+h}(p)$ is an h -approximation of $B_r(p)$, but generally not vice versa (think, e.g., of the ball of radius $\pi - h/2$ around any point of the unit 2-sphere).

(2) Relation (3) is equivalent to $U \setminus \partial_h U \subset V \subset (U \cup \partial_h U)$ and implies $\partial V \subset \partial_h U$.

(3) A natural procedure to construct an h -regularization V of a set U goes as follows: Choose a smooth function $g: X \rightarrow [0, 1]$ with

$$g(x) = \begin{cases} 0, & \text{for } x \in X \setminus U, \\ 1, & \text{for } x \in U \setminus \partial_h U, \end{cases}$$

a regular value $t \in (0, 1)$ of g , and $V = g^{-1}((t, 1])$. g can be obtained by smoothing the characteristic function $\chi_{U \setminus \partial_h U}$ via a suitable convolution process.

The restriction of a Schrödinger operator H^ω to a regular set D with Dirichlet boundary conditions is denoted by H_D^ω . It is well known that such an operator has discrete spectrum and, thus, we can define the normalized eigenvalue counting function (including multiplicities) as

$$N_D^\omega(\lambda) = \frac{\#\{i \mid \lambda_i(H_D^\omega) < \lambda\}}{|D|}, \quad (4)$$

where $|D|$ denotes the volume of D . A nonnegative, monotone increasing and left-continuous function on \mathbb{R} is called a *distribution function*. Thus, N_D^ω is a distribution function. Note that a distribution function has at most countably many discontinuity points.

For a better understanding of our general result we first state the simpler case where we assume Γ to be of polynomial growth. We denote the metric open r -ball around $p \in X$ by $B_r(p)$. Then we have, for every $p \in X$, a sequence of increasing radii $r_1 < r_2 < \dots$ satisfying

$$\lim_{n \rightarrow \infty} \frac{|\partial_d B_{r_n}(p)|}{|B_{r_n}(p)|} = 0 \quad \text{for all } d > 0. \quad (5)$$

This follows readily from Lemma 3.2. in [AS-93]. Since metric balls may not be regular sets (due to the existence of conjugate points), we need a regularization of those balls in the following theorem.

THEOREM 1.3. *Let X be the Riemannian universal covering of a compact Riemannian manifold $M = X/\Gamma$ and assume that Γ is of polynomial growth. Let $H^\omega = \Delta + V^\omega$ be a family of ergodic random Schrödinger operators satisfying (2). Then there exists a (nonrandom) distribution function N with the following property: For every $p \in X$ and any regularization D_n of an increasing sequence of balls $B_{r_n}(p)$ satisfying (5) we have, for almost all $\omega \in \Omega$,*

$$N(\lambda) = \lim_{n \rightarrow \infty} N_{D_n}^\omega(\lambda)$$

at all continuity points of N . Note that N_{D_n} denotes the normalized eigenvalue counting function of the restricted operator $H_{D_n}^\omega$ with Dirichlet boundary condition, as defined in (4). N is called the integrated density of states (IDS) of the family $\{H^\omega\}$.

In fact, existence of a nonrandom IDS can be proved in the much more general setting of amenable covering groups Γ . In the following general result, we use the notion of ‘admissible sequences’. This is our generalization of the cubes Λ_n in the Euclidean case. However, to avoid too many technical details, we postpone the precise definition of this notion to the next section.

THEOREM 1.4. *Let X be the Riemannian universal covering of a compact Riemannian manifold $M = X/\Gamma$ and assume that Γ is amenable. Let $H^\omega = \Delta + V^\omega$ be a family of ergodic random Schrödinger operators satisfying (2). Then there exists a (nonrandom) distribution function N such that we have, for every admissible sequence $D_n \subset X$ and almost every $\omega \in \Omega$, $N(\lambda) = \lim_{n \rightarrow \infty} N_{D_n}^\omega(\lambda)$ at all continuity points of N . N is called the integrated density of states of the family $\{H^\omega\}$.*

As mentioned before, we do not present the definition of admissible sequences at this point. We think it is more useful to give some information about the existence of those sequences to give some feeling for the applicability of Theorem 1.4.

PROPOSITION 1.5. *Let X be the Riemannian universal covering of a compact Riemannian manifold $M = X/\Gamma$. For every monotone increasing sequence $D_n \subset X$ of regular sets satisfying the following property*

$$\lim_{n \rightarrow \infty} \frac{|\partial_d D_n|}{|D_n|} = 0 \quad \text{for all } d > 0, \quad (\mathbf{P})$$

there exists a subsequence D_{n_j} which is an admissible sequence. Henceforth, we refer to this isoperimetric property of (not necessarily regular) subsets of X as property (\mathbf{P}) . The existence of a sequence $\{D_n\}$ satisfying property (\mathbf{P}) is equivalent to the fact that Γ is amenable.

The first part of this proposition will be proved after the definition of admissible sequences in the next section. The equivalence statement coincides essentially with [AS-93, Prop. 1.1.].

In the particular case that Γ is of polynomial growth (and, thus, automatically amenable) there are two natural choices for admissible sequences: either via *combinatorial balls* in Γ or via *metric balls* in X . This is the content of Proposition 1.6 below. However, if one drops the assumption on the polynomial growth, metric balls do not seem to be always an appropriate choice for admissible sequences. For example, choose X to be the three-dimensional diagonal horosphere of the Riemannian product of two real hyperbolic planes. X is a solvable Lie group with

a left invariant metric admitting a cocompact lattice Γ .^{*} Thus, Γ is amenable and Proposition 1.5 guarantees the existence of admissible sequences.^{**} On the other hand, metric balls $B_r(p) \subset X$ have exponential volume growth (see [KP-99, p. 669]). This yields strong evidence that a sequence of those balls cannot be used as an admissible sequence.

In order to state Proposition 1.6 below we again need some notation.

Let X and Γ be as before. It was explained in [AS-93, Section 3] how to obtain a connected polyhedral Γ -fundamental domain $\mathcal{F} \subset X$ by lifting simplices of a triangularization of M in a suitable manner. \mathcal{F} consists of finitely many smooth images of simplices which can overlap only at their boundaries. Using a polyhedral fundamental domain \mathcal{F} , any finite subset $I \subset \Gamma$ induces naturally a corresponding subset $\phi(I) \subset X$ defined as

$$\phi(I) = \text{int}(\overline{I\mathcal{F}}) = \text{int}\left(\bigcup_{\gamma \in I} \overline{\gamma\mathcal{F}}\right). \quad (6)$$

PROPOSITION 1.6. *Let X be the Riemannian universal covering of a compact Riemannian manifold $M = X/\Gamma$ and Γ be of polynomial growth.*

(a) *Let \mathcal{F} be a connected polyhedral fundamental domain and ϕ be the corresponding map (see (6)). Let e be the identity element of Γ , E a finite set of generators of Γ with $e \in E = E^{-1}$ and $E^n \subset \Gamma$ the combinatorial ball of radius $n \in \mathbb{N}$ about e . Then there exists an increasing sequence $r_1 < r_2 < \dots$ of integer radii such that*

$$\frac{|E^{r_n+d} \setminus E^{r_n-d}|}{|E^{r_n}|} \rightarrow 0 \quad \text{for all } d \in \mathbb{N}. \quad (7)$$

Any regularization of $\{\phi(E^{r_n})\}$ is an admissible sequence.

(b) *Let $p \in X$ be an arbitrary point. Then there exists an increasing sequence $r_1 < r_2 < \dots$ of radii such that the corresponding metric balls $\{B_{r_n}(p)\}$ satisfy (5). Moreover, any regularization of $\{B_{r_n}\}$ is an admissible sequence.*

We obtain as an immediate consequence of Proposition 1.6 (b) that Theorem 1.3 is a particular case of Theorem 1.4. Thus it suffices to prove Theorem 1.4 which is done in Section 4.

2. Admissible Sequences and Ergodic Theorem

An important tool in the existence proof of a *nonrandom* IDS is an ergodic theorem for the group Γ of deck transformations on X . We will use Lindenstrauss's pointwise ergodic theorem which is related to a maximal ergodic theorem of Shulman (see [Lin-99]; further informations about ergodic theorems can be found in [Kre-85])

^{*} X/Γ coincides with the solvmanifold described in [Thu-97, Example 3.8.9].

^{**} An admissible sequence for the diagonal horosphere is explicitly given in [KP-99, p. 668].

or [Tem-92]). Lindenstrauss's theorem applies to discrete amenable groups. This section contains some useful geometric facts and their interaction with this ergodic theorem.

As in the previous section, let $\mathcal{F} \subset X$ denote a connected polyhedral fundamental domain of Γ and ϕ the associated map from finite subsets of Γ to open subsets of X (see (6)), which we assume to be fixed once and for all.

DEFINITION 2.1. A sequence $\{D_n\}$ of regular subsets of X is called an *admissible sequence of X* if the following properties are satisfied:

- There exists a sequence $\{I_n\}$ of monotone increasing, nonempty, finite subsets of Γ with

$$\lim_{n \rightarrow \infty} \frac{|I_n \Delta I_n \gamma|}{|I_n|} = 0, \quad \text{for all } \gamma \in \Gamma, \quad (8)$$

$$\sup_{n \in \mathbb{N}} \frac{|I_{n+1} I_n^{-1}|}{|I_{n+1}|} < \infty. \quad (9)$$

Let $A_n = \phi(I_n)$. (Lemma 2.4 below implies that $\{A_n\}$ satisfies property **(P)**.)

- Either $\{A_n\}$ is an approximation of $\{D_n\}$ and $\{D_n\}$ satisfies the isoperimetric property **(P)**, or $\{D_n\}$ is an approximation of $\{A_n\}$. (In the second case, $\{D_n\}$ satisfies property **(P)** automatically, see the second statement of Lemma 2.5 below.)

Sequences satisfying only (8) are called *Følner sequences*. Monotone increasing sequences $\{I_n\}$ satisfying (8) and (9) are called *tempered Følner sequences*.

Equation (8) describes geometrically that the group Γ is amenable. Condition (9) and the notion ‘tempered Følner sequence’ are due to A. Shulman who proved a maximal ergodic theorem for those sequences. The following proposition states that (9) is not a serious restriction for Følner sequences:

PROPOSITION 2.2 (see [Lin-99, Prop. 1.5]). *Every Følner sequence has a tempered subsequence. In particular, every amenable group admits a monotone increasing sequence $\{I_n\}$ satisfying (8) and (9).*

Next we state Lindenstrauss' pointwise ergodic theorem [Lin-99, Thm. 1.3].

THEOREM 2.3. *Let Γ be an amenable discrete group and $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Assume that Γ acts ergodically on Ω by measure preserving transformations $\{T_\gamma\}$. Let $\{I_n\}$ be a tempered Følner sequence. Then we have, for every $f \in L^1(\Omega)$ and for almost all $\omega \in \Omega$,*

$$\lim_{n \rightarrow \infty} \frac{1}{|I_n|} \sum_{\gamma \in I_n^{-1}} f(T_\gamma \omega) = \mathbb{E}(f) = \int_{\Omega} f(\omega) d\mathbb{P}(\omega). \quad (10)$$

Furthermore

$$\lim_{n \rightarrow \infty} \int_{\Omega} \left| \mathbb{E}(f) - \frac{1}{|I_n|} \sum_{\gamma \in I_n^{-1}} f(T_{\gamma}\omega) \right| d\mathbb{P}(\omega) = 0.$$

In the statement of the theorem one can replace the space L^1 by L^2 , due to Shulman [Shul-88]. Mean ergodic theorems hold in more general circumstances, see, e.g., [Tem-72, Thm. 6.4] or [Kre-85, §6.4].

The reader might wonder why there is a summation over I_n^{-1} instead of I_n in (10). The reason for this choice is simply that we want it to fit, without modification, for the application later in the paper. Lindenstrauss's theorem contains a summation over I_n . Accordingly, our conditions on I_n agree with those of him only after replacing I_n by I_n^{-1} . Note that condition (8) is equivalent to

$$\lim_{n \rightarrow \infty} \frac{|I_n^{-1} \Delta \gamma I_n^{-1}|}{|I_n^{-1}|} = 0, \quad \text{for all } \gamma \in \Gamma.$$

The following lemma exhibits a useful connection between the isoperimetric property **(P)** and the Følner condition (8).

LEMMA 2.4. *Let $I_n \subset \Gamma$ be a sequence of nonempty, finite sets and let $A_n = \phi(I_n)$. Then the following properties are equivalent:*

- (a) $\{I_n\}$ satisfies the Følner condition (8)
- (b) $\{A_n\}$ satisfies the isoperimetric property **(P)**.

Proof. We first show that (a) implies (b). For an arbitrary fixed $d > 0$ we define the following finite set:

$$B = \{g \in \Gamma \mid d(g\overline{\mathcal{F}}, \overline{\mathcal{F}}) \leq d\}.$$

We first observe that if $A = \overline{\phi(I)}$, then

$$T_d(A) := \{x \in X \mid d(x, A) \leq d\} \subset \overline{\phi(IB)}.$$

In fact, for any $x \in T_d(A)$ there exists an $x_0 \in A$ and a $\gamma \in I$ with $d(x, x_0) \leq d$ and $x_0 \in \overline{\gamma\mathcal{F}}$. Consequently, we have $d(\gamma^{-1}x, \overline{\mathcal{F}}) \leq d$ and, thus, there exists a $g \in B$ with $\gamma^{-1}x \in g\overline{\mathcal{F}}$. This implies $x \in \overline{\gamma g\mathcal{F}} \subset \overline{\phi(IB)}$.

For the proof we apply this observation twice. From $A_n = \phi(I_n)$ we conclude that

$$T_d(\overline{A_n}) \setminus A_n \subset \overline{\phi(I_n B)} \setminus \phi(I_n) = \overline{\phi(I_n B \setminus I_n)}.$$

Let $H_n = I_n B \setminus I_n$. A second application of the above observation yields

$$\begin{aligned} \partial_d A_n &\subset T_d(T_d(\overline{A_n}) \setminus A_n) \subset \overline{\phi(H_n B)} \\ &\subset \overline{\phi\left(\bigcup_{g_1, g_2 \in B} (I_n g_1 \Delta I_n) g_2\right)} = \bigcup_{g_1, g_2 \in B} \overline{\phi((I_n g_1 \Delta I_n) g_2)}. \end{aligned}$$

This implies

$$\frac{|\partial_d A_n|}{|A_n|} \leq \sum_{g_1, g_2 \in B} \frac{|\phi(I_n g_1 g_2 \Delta I_n g_2)|}{|\phi(I_n)|} = |B| \cdot \sum_{g_1 \in B} \frac{|I_n g_1 \Delta I_n|}{|I_n|} \rightarrow 0,$$

finishing the proof of the first implication.

For the proof of ‘(b) \Rightarrow (a)’ it suffices to show that there is a $d > 0$ (not dependent on n) such that $\phi(I_n \Delta I_n \gamma) \subset \partial_d A_n$. We first prove that $\phi(I_n \gamma \setminus I_n) \subset \partial_{d_0+d_1} A_n$, where $d_0 = d(\overline{\gamma \mathcal{F}}, \overline{\mathcal{F}})$ and $d_1 = \text{diam}(\overline{\mathcal{F}})$. Let $g \in I_n \gamma \setminus I_n$. Then $g = g_0 \gamma$ with $g_0 \in I_n$ and $g_0 \mathcal{F} \subset A_n$, $g \mathcal{F} \cap A_n = \emptyset$. Since $d(\overline{g \mathcal{F}}, \overline{g_0 \mathcal{F}}) = d_0$ we conclude that there exists a $z \in \partial A_n$ with $d(z, \overline{g \mathcal{F}}) \leq d_0$. This implies that $g \mathcal{F} \subset \partial_{d_0+d_1} A_n$, finishing this inclusion. We are done if we prove that $\phi(I_n \setminus I_n \gamma) \subset \partial_{2d_0+2d_1} A_n$. Let $g \in I_n \setminus I_n \gamma$. Then we have $g \gamma^{-1} \in I_n \gamma^{-1} \setminus I_n$ and we obtain by the previous considerations that $g \gamma^{-1} \mathcal{F} \subset \partial_{d_0+d_1} A_n$. The required inclusion follows now from $g \mathcal{F} \subset T_{d_0+d_1}(g \gamma^{-1} \mathcal{F})$. \square

LEMMA 2.5. *Let $\{U_n\}$ be a sequence of subsets of X satisfying the isoperimetric property (P). Then we have, for every radius $r > 0$, an index $n_0 = n_0(r)$ such that every set U_n , $n \geq n_0$, contains a metric ball of radius r . Moreover, if $\{V_n\}$ is an approximation of $\{U_n\}$, then $\{V_n\}$ satisfies also property (P).*

Proof. We assume that there exists an $r > 0$ and a sequence $n_1 < n_2 < n_3 < \dots$ such that we have $B_r(p) \not\subset U_{n_j}$ for all $p \in U_{n_j}$ and all $j \geq 1$. This implies $U_{n_j} \subset \partial_r U_{n_j}$, which is a contradiction to property (P). It remains to prove the second statement. We have $U_n \setminus \partial_h U_n \subset V_n \subset (U_n \cup \partial_h U_n)$ and $\partial V_n \subset \partial_h U_n$. This implies that

$$\frac{|\partial_d V_n|}{|V_n|} \leq \frac{|\partial_{h+d} U_n|}{|U_n|} \frac{|U_n|}{|U_n \setminus \partial_h U_n|} = \frac{|\partial_{h+d} U_n|}{|U_n|} \frac{|U_n|}{|U_n| - |U_n \cap \partial_h U_n|} \rightarrow 0. \quad (11)$$

Note that, for n large enough, the denominator $|U_n \setminus \partial_h U_n|$ in (14) is strictly larger than 0. \square

Remark. The second statement in Lemma 2.5 is *not* symmetric w.r.t. V_n and U_n .

Proposition 1.5 is now a consequence of Proposition 2.2 and the previous geometric considerations:

Proof of Proposition 1.5. Let $D_n \subset X$ be as in the proposition. We define $I_n = \{\gamma \in \Gamma \mid \gamma \mathcal{F} \subset D_n\}$. Note that $I_n \subset \Gamma$ is monotone increasing and nonempty, for n sufficiently large, by the first statement of Lemma 2.5. One easily checks that $A_n = \phi(I_n)$ is a d_1 -approximation of D_n with $d_1 = \text{diam}(\overline{\mathcal{F}})$. Thus, by the second

statement of Lemma 2.5, A_n inherits the isoperimetric property of D_n . This implies, by Lemma 2.4, that I_n is an increasing Følner sequence. By Proposition 2.2, there exists a tempered Følner subsequence I_{n_j} . Consequently, D_{n_j} is an admissible sequence. \square

Proof of Proposition 1.6. Note that Γ is of polynomial growth. We first prove (a). The existence of an increasing sequence of radii r_n satisfying (7) was proved in [Ad-93, Prop. 5]. Let $I_n = E^{r_n}$. (8) follows readily from Lemma 2.4. By Gromov's famous result [Gro-81], Γ is almost nilpotent and this implies, together with [Ba-72], that there exist a constant $C \geq 1$ such that

$$\frac{r^k}{C} \leq |E^r| \leq Cr^k, \quad (12)$$

where $k \in \mathbb{N}$ is the degree of Γ . This immediately yields (9):

$$\frac{|I_{n+1}I_n^{-1}|}{|I_{n+1}|} \leq \frac{|E^{2r_{n+1}}|}{|E^{r_{n+1}}|} \leq 2^k C^2.$$

Now we prove (b). W.l.o.g. we can assume that $p \in \overline{\mathcal{F}}$. Let $I_n = \{\gamma \in \Gamma \mid \gamma\mathcal{F} \subset B_{r_n}(p)\}$. The Følner property of I_n follows precisely as in the proof of Proposition 1.5. It remains to prove (9). One easily checks that

$$B_{r_n-d_1}(p) \subset A_n = \phi(I_n) \subset B_{r_n}(p) \quad (13)$$

with $d_1 = \text{diam}(\overline{\mathcal{F}})$. Let $\|\cdot\|$ denote the word norm of Γ with respect to E . Milnor showed in [Mil-68] that there are $a \geq 1$, $b \geq 0$ such that

$$\frac{1}{a}\|\gamma\| - b \leq d(p, \gamma p) \leq a\|\gamma\|.$$

This implies, together with (13), that

$$\{\gamma \in \Gamma \mid \|\gamma\| \leq \frac{1}{a}(r_n - d_1)\} \subset I_n \subset \{\gamma \in \Gamma \mid \|\gamma\| \leq a(r_n + b)\},$$

and the same inclusions hold for I_n^{-1} . Consequently, we have

$$\{\gamma \in \Gamma \mid \|\gamma\| \leq \frac{2}{a}(r_n - d_1)\} \subset I_n I_n^{-1} \subset \{\gamma \in \Gamma \mid \|\gamma\| \leq 2a(r_n + b)\},$$

and the required estimate follows, again, by (12). \square

The final lemma of this section, which we will apply later to heat kernels, is an immediate consequence of Lindenstrauss's ergodic theorem.

LEMMA 2.6. *Let $I_n \subset \Gamma$ be a tempered Følner sequence and $A_n = \phi(I_n) \subset X$. Assume that Γ acts ergodically on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ by measure preserving transformations $\{T_\gamma\}$. Let $f: \Omega \times X \rightarrow \mathbb{R}$ be a jointly measurable bounded function*

satisfying the compatibility condition

$$f(T_\gamma \omega, x) = f(\omega, \gamma^{-1}x)$$

for all $\omega \in \Omega$, $\gamma \in \Gamma$ and $x \in X$. Then we have, for almost all $\omega \in \Omega$,

$$\lim_{n \rightarrow \infty} \frac{1}{|A_n|} \int_{A_n} f(\omega, x) dx = \frac{1}{|\mathcal{F}|} \mathbb{E} \left(\int_{\mathcal{F}} f(\bullet, x) dx \right), \quad (14)$$

where \mathbb{E} denotes the expectation on Ω . The convergence holds in the $L^1(\Omega)$ -topology, as well.

Proof. Let $F(\omega) = \int_{\mathcal{F}} f(\omega, x) dx$. Then we obviously have $F \in L^1(\Omega)$. We conclude that

$$\begin{aligned} \frac{1}{|A_n|} \int_{A_n} f(\omega, x) dx &= \frac{1}{|I_n|} \sum_{\gamma \in I_n} \frac{1}{|\mathcal{F}|} \int_{\mathcal{F}} f(\omega, x) dx \\ &= \frac{1}{|I_n|} \sum_{\gamma \in I_n} \frac{1}{|\mathcal{F}|} \int_{\mathcal{F}} f(\omega, \gamma x) dx \\ &= \frac{1}{|I_n|} \sum_{\gamma \in I_n} \frac{1}{|\mathcal{F}|} \int_{\mathcal{F}} f(T_{\gamma^{-1}} \omega, x) dx \\ &= \frac{1}{|I_n|} \sum_{\gamma \in I_n^{-1}} \frac{1}{|\mathcal{F}|} F(T_\gamma \omega). \end{aligned}$$

Now Theorem 2.3 implies

$$\lim_{n \rightarrow \infty} \frac{1}{|A_n|} \int_{A_n} f(\omega, x) dx = \frac{1}{|\mathcal{F}|} \mathbb{E}(F),$$

for almost all $\omega \in \Omega$ and in L^1 -sense. \square

3. Heat Kernel Estimates

In this section we derive heat kernel estimates for a family of Schrödinger operators $\{H^\omega = \Delta + V^\omega\}_{\omega \in \Omega}$ satisfying the regularity condition

$$\|\nabla^k V^\omega\|_\infty \leq C_0, \quad \text{for all } \omega \in \Omega \quad \text{and} \quad k \leq \frac{1}{2} \dim(X) + 2. \quad (15)$$

These estimates are, besides an ergodic theorem, the second crucial tool for our existence proof of an IDS.

Due to the Kato-Rellich Theorem [RSII-75] all H^ω are densely defined selfadjoint operators on $L^2(X)$ and their domains coincide. By the spectral theorem we can define the operator $\exp(-tH^\omega)$, which has an integral kernel $k^\omega(t, \cdot, \cdot)$. Let $D \subset X$ be a regular set. We denote the restriction of H^ω to D with Dirichlet boundary conditions by H_D^ω and the corresponding heat kernel of $\exp(-tH_D^\omega)$ by $k_D^\omega(t, \cdot, \cdot)$. We will need the following estimates.

PROPOSITION 3.1. *Let $H^\omega = \Delta + V^\omega$, $\omega \in \Omega$, be a family of operators satisfying (15). Then the following estimates hold:*

(a) **Domain Monotonicity:** *For every regular set $D \subset X$ we have*

$$0 \leq k_D^\omega(t, x, y) \leq k^\omega(t, x, y),$$

for all $x, y \in D$ and $t > 0$.

(b) **Upper Bound:** *There exists a function $C(t)$, $t > 0$, such that*

$$0 \leq k^\omega(t, x, y) \leq C(t),$$

for all $x, y \in X$ and $\omega \in \Omega$.

(c) **Principle of not feeling the boundary:** *For all $t > 0$ there exists an $h = h(t, \varepsilon) > 0$ such that, for all regular sets $D \subset X$ and all $\omega \in \Omega$, we have*

$$|k^\omega(t, x, y) - k_D^\omega(t, x, y)| \leq \varepsilon \quad \text{for all } x, y \in D \setminus \partial_h D.$$

Proof. Inequality (a) is a consequence of the maximum principle for solutions of the heat equation (see, e.g., [Tay-96] or [Cha-84]).

We now consider assertion (b). Let $t > 0$ be fixed. The heat kernel $k(t, x, y)$ of the Laplacian on X (i.e., without potential) is a continuous function (see, e.g., [Dav90]) satisfying $k(t, x, y) = k(t, \gamma x, \gamma y)$ for any $\gamma \in \Gamma$. Since Γ acts cocompactly on X , we conclude the existence of a constant $C_1(t)$ with $0 \leq k(t, x, x) \leq C_1(t)$. A simple application of the semigroup property yields the same off-diagonal estimate

$$0 \leq k(t, x, y) \leq \sqrt{k(t, x, x)k(t, y, y)} \leq C_1(t).$$

The potential can be treated with stochastic arguments which was proposed to us by A. Thalmaier: Since X is stochastically complete, we can apply the Feynman–Kac formula for manifolds (see, e.g., [Elw-82]) and obtain, for every $f \in C_0^\infty(X)$,

$$\int_X k^\omega(t, x, y) f(y) dy = \mathbb{E}_x \left(f(b_t) \exp \left(\int_0^t V^\omega(b_s) ds \right) \right),$$

where b_t is the Brownian motion on X starting in x and \mathbb{E}_x is the corresponding expectation. Using $\|V^\omega\| \leq C_0$ we obtain for every nonnegative $f \in C_0^\infty(X)$:

$$\left| \int_X k^\omega(t, x, y) f(y) dy \right| \leq \mathbb{E}_x(f(b_t)) e^{C_0 t} \leq C_1(t) e^{C_0 t} \|f\|_1.$$

Continuity of k^ω implies

$$|k^\omega(t, x, y)| \leq C_1(t) e^{C_0 t},$$

finishing the proof of (b).

The proof of (c) is based on finite propagation speed of the wave equation. The roots of this approach can be found in [CGT-82]. We follow the arguments given

in [LS-99, Thm 2.26] which are attributed to U. Bunke. For the reader's convenience, we present the proof in detail (see also [DM-97] for a related method).

To simplify the notation, we omit the index ω . Due to condition (15), all inequalities hold uniformly in $\omega \in \Omega$.

In what follows, $t > 0$ is fixed and $h > 0$ is kept variable. Let $x_0, y_0 \in D^h := D \setminus \partial_h D$ and $B_1 = B_{h/3}(x_0)$ and $B_2 = B_{h/3}(y_0)$ be the corresponding balls. Our first aim is to prove existence of a function $C(h)$ with $C(h) \rightarrow 0$, as $h \rightarrow \infty$ such that, for every $u \in C_0^\infty(B_2)$ and $f = (e^{-tH} - e^{-tH_D})u$, the following pointwise estimate holds:

$$|H^k f(x_0)| \leq C(h) \|u\|_2. \quad (16)$$

Our departure point is the following Fourier transform identity

$$\frac{(-1)^m}{\sqrt{\pi t}} \int_0^\infty \left(\frac{d^{2m}}{ds^{2m}} e^{-s^2/4t} \right) \cos(s\xi) ds = \xi^{2m} e^{-t\xi^2}. \quad (17)$$

Applying the spectral theorem to (17) with $\xi = \sqrt{H}$ and $\xi = \sqrt{H_D}$ we obtain

$$H^{k+l} f = \int_0^\infty P(s) e^{-s^2/4t} (\cos(s\sqrt{H}) - \cos(s\sqrt{H_D})) u ds,$$

where $P(s)$ is a fixed polynomial. Note that the coefficients of P are expressions in t and that t is considered as a fixed positive constant. Unit propagation speed (see, e.g., [Tay-96]) implies, for $g_s = \cos(s\sqrt{H})u$ and $h_s = \cos(s\sqrt{H_D})u$ that

$$\text{supp}(g_s), \text{supp}(h_s) \subset B_{2h/3}(y_0) \subset D$$

for $s < h/3$. Since g_s and h_s both satisfy the wave equation with initial conditions $g(0, \cdot) = u$, $\partial g / \partial s(0, \cdot) = 0$, we conclude that $g_s - h_s \equiv 0$, for $0 < s < h/3$. The Cauchy-Schwarz inequality yields

$$\begin{aligned} \|H^{k+l} f\|_{L^2(B_1)}^2 &\leq \int_{B_1} \left(\int_{h/3}^\infty |P(s)(g_s(x) - h_s(x))| e^{-s^2/4t} ds \right)^2 dx \\ &\leq A_1(h) \int_{h/3}^\infty |P(s)| e^{-s^2/4t} \int_{B_1} (g_s(x) - h_s(x))^2 dx ds, \end{aligned}$$

where $A_1(h) = \int_{h/3}^\infty |P(s)| e^{-s^2/4t} ds \rightarrow 0$, as $h \rightarrow \infty$. Using, again, the spectral theorem, we conclude from $|\cos(s\xi)| \leq 1$ that

$$\|H^{k+l} f\|_{L^2(B_1)} \leq 2A_1(h) \|u\|_{L^2(B_2)}.$$

In order to obtain the pointwise estimate (16), we would like to apply a Sobolev inequality of the type

$$|g(x)| \leq \sum_{l=0}^N a_l \|H^l g\|_{L^2(B_{h/3}(x))} \quad (18)$$

for all $g \in C_0^\infty(B_{h/3}(x))$, where $N = [(\dim X/2) + 2]$, and the coefficients a_l are

independent of $x \in X$ and $\omega \in \Omega$. This is possible since X/Γ is compact. Moreover, the condition $g \in C_0^\infty(B_{h/3}(x))$ can be relaxed to $g \in C^\infty(B_{h/3}(x))$, since, for $h \geq h_0 > 0$, we can choose, for every point $x \in X$, cut-off functions $\rho_x \in C_0^\infty(B_{h/3}(x))$ with universal bounds on the derivatives and, thus, apply an estimate

$$\|H^l \rho_x g\|_{L^2(B_{h/3}(x))} \leq \beta_l \|g\|_{L^2(B_{h/3}(x))} + \gamma_l \|H^l g\|_{L^2(B_{h/3}(x))}$$

with universal constants β_l, γ_l . This proves (16), namely

$$|H^k f(x_0)| \leq \sum_{l=0}^N a_l \|H^{k+l} f\|_{L^2(B_1)} \leq A_2(h) \|u\|_{L^2(B_2)},$$

where $A_2(h) \rightarrow 0$, as $h \rightarrow \infty$.

Next, the heat kernels come into play:

$$\begin{aligned} & |\langle H_y^k(k(t, x_0, \cdot) - k_D(t, x_0, \cdot)), u \rangle_{L^2(B_2)}| \\ &= \left| \int_{B_2} H_y^k(k(t, x_0, y) - k_D(t, x_0, y)) u(y) dy \right| \\ &= \left| \int_{B_2} (k(t, x_0, y) - k_D(t, x_0, y)) (H^k u)(y) dy \right| \\ &= |((e^{-tH} - e^{-tH_D}) H^k u)(x_0)| \\ &= |H^k f(x_0)| \leq A_2(h) \|u\|_{L^2(B_2)}. \end{aligned}$$

Since $u \in C_0^\infty(B_2)$ was arbitrary, we conclude that

$$\|H_y^k(k(t, x_0, \cdot) - k_D(t, x_0, \cdot))\|_{L^2(B_2)} \leq A_2(h).$$

Again, using the Sobolev inequality (18), we end up with

$$|k(t, x_0, y_0) - k_D(t, x_0, y_0)| \leq A_3(h),$$

where $A_3(h) \rightarrow 0$, as $h \rightarrow \infty$. Choosing h large enough, we obtain the required estimate of the lemma. \square

Remarks. (1) Estimate (b) of Proposition 3.1 is very crude, but sufficient for our purposes. For a much better estimate, we refer the reader to [LY-86].

(2) For the Neumann heat kernel, estimate (c) is still valid [LS-99]. However, Domain Monotonicity for Neumann heat kernels is a subtle question. See [Cha-86, BB-93] and [CZ-94].

The following lemma states, in concise form, the crucial fact about heat kernels, which is needed in the next section.

LEMMA 3.2 (Heat Kernel Lemma). *Let $\{A_n\}$ and $\{D_n\}$ be two sequences of subsets of X satisfying both the isoperimetric property (P). Moreover, we assume that the sets*

D_n are regular and that either $\{A_n\}$ is an approximation of $\{D_n\}$, or vice versa. Moreover, let $H^\omega = \Delta + V^\omega$, $\omega \in \Omega$, be a family of operators satisfying (15). Then we have, for $n \rightarrow \infty$,

$$\sup_{\omega \in \Omega} \left| \frac{1}{|A_n|} \int_{A_n} k^\omega(t, x, x) dx - \frac{1}{|D_n|} \int_{D_n} k_{D_n}^\omega(t, x, x) dx \right| \rightarrow 0. \quad (19)$$

Remark. Note that (19) can be also interpreted as the following limit of traces:

$$\sup_{\omega \in \Omega} \left| \frac{1}{|A_n|} \text{Tr}(\chi_{A_n} e^{-tH^\omega}) - \frac{1}{|D_n|} \text{Tr}(e^{-tH_{D_n}^\omega}) \right| \rightarrow 0,$$

where χ_A denotes the characteristic function of $A \subset X$.

Proof. Proposition 3.1 (b) and property (P) easily imply that

$$\left| \frac{1}{|A_n|} \int_{A_n} k^\omega(t, x, x) dx - \frac{1}{|D_n|} \int_{D_n} k^\omega(t, x, x) dx \right| \rightarrow 0.$$

Thus we have to prove (19) only in the case $A_n = D_n$. Using, again, Proposition 3.1 and property (P) we conclude, for $h = h(t, \varepsilon)$, that

$$\begin{aligned} & \left| \frac{1}{|D_n|} \int_{D_n} k^\omega(t, x, x) dx - \frac{1}{|D_n|} \int_{D_n} k_{D_n}^\omega(t, x, x) dx \right| \\ & \leq \frac{1}{|D_n|} \left(\int_{(D_n \setminus \partial_h D_n)} + \int_{(D_n \cap \partial_h D_n)} \right) (k^\omega(t, x, x) - k_{D_n}^\omega(t, x, x)) dx \\ & \leq \frac{|D_n \setminus \partial_h D_n|}{|D_n|} \varepsilon + \frac{|\partial_h D_n|}{|D_n|} C(t) \rightarrow \varepsilon. \end{aligned}$$

This finishes the proof, since $\varepsilon > 0$ was arbitrary. \square

4. Proof of the Main Theorem

In this section we present the proof of Theorem 1.4. We assume that $\{D_n\}$ is an admissible sequence of X , and that I_n and $A_n = \phi(I_n)$ are the associated sequences (see Definition 2.1). Let $\{H^\omega\}_{\omega \in \Omega}$ be an ergodic random family of Schrödinger operators satisfying the regularity condition (15). In order to show almost-sure-convergence of the normalized eigenvalue counting functions $N_{D_n}^\omega$ to a nonrandom distribution function N at all continuity points it suffices to prove pointwise convergence of the corresponding Laplace transformations. This fact is a consequence of the following lemma. Recall that a distribution function is a nonnegative, left-continuous, monotone increasing function.

LEMMA 4.1 (Shubin). *Let N_n be a sequence of distribution functions such that*

- (a) *there exists a $c \in \mathbb{R}$ such that $N_n(\lambda) = 0$ for all $\lambda \leq c$ and $n \in \mathbb{N}$,*

- (b) *there exists a $C_1: \mathbb{R}^+ \rightarrow \mathbb{R}$ such that $\tilde{N}_n(t) := \int e^{-\lambda t} dN_n(\lambda) \leq C_1(t)$ for all $n \in \mathbb{N}$, $t > 0$,*
- (c) *$\lim_{n \rightarrow \infty} \tilde{N}_n(t) =: \psi(t)$ exists for all $t > 0$.*

Then the limit $N(\lambda) := \lim_{n \rightarrow \infty} N_n(\lambda)$ exists at all continuity points. N is, again, a distribution function, and its Laplace transform is ψ .

Proof of Theorem 1.4. Now let $t > 0$ be fixed. The Laplace transforms of the normalized eigenvalue counting functions can be written in terms of heat kernels:

$$\begin{aligned} \tilde{N}_{D_n}^\omega(t) &= \int_{D_n} e^{-t\lambda} dN_{D_n}^\omega(\lambda) = \frac{1}{|D_n|} \text{Tr}(e^{-tH_{D_n}^\omega}) \\ &= \frac{1}{|D_n|} \int_{D_n} k_{D_n}^\omega(t, x, x) dx. \end{aligned} \quad (20)$$

Applying the heat kernel Lemma 3.2, we obtain

$$\lim_{n \rightarrow \infty} \left| \tilde{N}_{D_n}^\omega(t) - \frac{1}{|A_n|} \int_{A_n} k^\omega(t, x, x) dx \right| = 0.$$

Note that the function $(\omega, x) \mapsto k^\omega(t, x, x)$ is jointly measurable. Equation (1) and the spectral theorem imply that

$$e^{-tH_{\gamma^\omega}} = U_\gamma e^{-tH^\omega} U_\gamma^*,$$

where $U_\gamma: L^2(X) \rightarrow L^2(X)$ are unitary operators, defined by $U_\gamma f(x) = f(\gamma^{-1}x)$. This yields $k_{\gamma^\omega}^\omega(t, x, x) = k^\omega(t, \gamma^{-1}x, \gamma^{-1}x)$, and we can apply Lemma 2.6. Consequently, we have, for almost all $\omega \in \Omega$:

$$\lim_{n \rightarrow \infty} \tilde{N}_{D_n}^\omega(t) = \lim_{n \rightarrow \infty} \frac{1}{|A_n|} \int_{A_n} k^\omega(t, x, x) dx = \frac{1}{|\mathcal{F}|} \mathbb{E} \left(\int_{\mathcal{F}} k^\bullet(t, x, x) dx \right).$$

Note that the conditions of Shubin's Lemma are satisfied: The previous considerations imply (c), for almost all $\omega \in \Omega$. (a) holds with $c = -C_0$, where C_0 is the constant in (15), and (b) follows from (20) and Proposition 3.1. Consequently, an application of Shubin's lemma finishes the proof of Theorem 1.4. Moreover, we obtain an explicit formula for the Laplace transform of the nonrandom IDS:

$$\tilde{N}(t) = \int e^{-\lambda t} dN(\lambda) = \frac{1}{|\mathcal{F}|} \mathbb{E} \left(\int_{\mathcal{F}} k^\bullet(t, x, x) dx \right).$$

5. Discussion

There are at least two natural extensions of our results including random higher order terms. From the physical point of view it would be interesting to include a magnetic field term in the Schrödinger operator. The non-Euclidean setting raises the question whether one can consider the pure Laplace operator on a differentiable

manifold equipped with a family of Riemannian metrics depending ergodically on a random parameter. This may model, e.g., a quantum mechanical system of a membrane with random hollows. For these cases, as well as for more singular potentials, we would need to extend the methods of Section 3.

We restricted ourselves to the case where the group action is *discrete* and the configuration space is *continuous*. One could also consider actions of Lie groups on manifolds; or a graph instead of a manifold as the configuration space.

In collaboration with Daniel Lenz we currently investigate whether our IDS coincides with a trace of an appropriate von Neumann algebra. The concept of a von Neumann algebra of random operators may be also useful as a common abstract setting for all the situations described in the previous paragraph. Such an abstract setting for the case of an Abelian group acting on another Abelian group was studied, e.g., in [Le-99].

Furthermore, from the physical point of view, it would be interesting to investigate finer properties of the IDS for particular models: the continuity or differentiability of $\lambda \mapsto N(\lambda)$, and the asymptotic behavior as λ approaches an edge of the spectrum of $\{H^\omega\}$, cf. [Sz-89, Sz-90].

Acknowledgements

We would like to thank Uwe Abresch, Jozef Dodziuk, Werner Kirsch, Gerhard Knieper, Daniel Lenz, Thomas Schick and Anton Thalmaier for many discussions concerning this paper. In particular, we are grateful to J. Dodziuk for helpful remarks about heat kernel estimates and to Th. Schick for the useful reference to Linderstrauss's paper [Lin-99].

References

- [Ad-93] Adachi, T.: A note on the Følner condition for amenability (dedicated to Professor Nobunori Ikebe), *Nagoya Math. J.* **131** (1993) 67–74.
- [AS-93] Adachi, T. and Sunada, T.: Density of states in spectral geometry, *Comment. Math. Helv.* **68** (1993), 480–493.
- [Ba-72] Bass, H.: The degree of polynomial growth of finitely generated nilpotent groups. *Proc. London Math. Soc.* **25** (1972), 603–614.
- [BB-93] Bass, R. F. and Burdzy, K.: On domain monotonicity of the Neumann heat kernel, *J. Funct. Anal.* **116** (1993), 215–224.
- [BBEE+84] Bonch-Bruевич, V. L., Enderlein, R., Esser, B., Keiper, R., Mirnov, A. G. and Zvyagin, I. P.: *Elektronentheorie ungeordneter Halbleiter*, VEB Deutscher Verlag der Wissenschaften, 1984. Russian original: Nauka, Moscow, 1981.
- [BS-92] Brüning J. and Sunada, T.: On the spectrum of periodic elliptic operators, *Nagoya Math. J.* **126** (1992), 159–171.
- [CL-90] Carmona, R. and Lacroix, J.: *Spectral Theory of Random Schrödinger Operators*, Birkhäuser, Boston, 1990.
- [CZ-94] Carmona, R. and Zheng, W.: Reflecting Brownian motions and comparison theorems for Neumann heat kernels, *J. Funct. Anal.* **123**(1) (1994), 109–128.

- [Cha-84] Chavel, I.: *Eigenvalues in Riemannian Geometry*, Academic Press, New York, 1984.
- [Cha-86] Chavel, I.: Heat diffusion in insulated convex domains, *J. London Math. Soc.* **34** (1986), 473–478.
- [CGT-82] Cheeger, J. Gromov, M. and Taylor, M.: Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds, *J. Differential Geom.* **17** (1982), 15–53.
- [Dav90] Davies, E. B.: *Heat Kernels and Spectral Theory*, Cambridge Univ. Press, Cambridge, 1990.
- [DM-97] Dodziuk, J. and Mathai, V.: Approximating L^2 invariants of amenable covering spaces: A heat kernel approach, In: *Lipa's Legacy*, Contemp. Math. 211, Amer. Math. Soc., Providence, 1997, pp. 151–167.
- [ES-84] Efros, A. L. and Shklovski, B. I.: *Electronic Properties of Doped Semi-conductors*, Springer, Berlin, 1984.
- [Elw-82] Elworthy, K. D.: *Stochastic Differential Equations on Manifolds*, London Math. Soc. Lecture Note Ser. 70, Cambridge Univ. Press, 1982.
- [Gro-81] Gromov, M.: Groups of polynomial growth and expanding maps, *IHES Publ. Math.* **53** (1981), 53–73.
- [KP-99] Karp, L. and Peyerimhoff, N.: Horospherical means and uniform distribution of curves, *Math. Z.* **231** (1999), 655–677.
- [Kir-89] Kirsch, W.: Random Schrödinger operators, In: H. Holden and A. Jensen (eds), *Schrödinger Operators*, Lecture Notes in Phys. 345, Springer, Berlin, 1989.
- [KM-82a] Kirsch, W. and Martinelli, F.: On the ergodic properties of the spectrum of general random operators, *J. Reine Angew. Math.* **334** (1982), 141–156.
- [KM-82b] Kirsch, W. and Martinelli, F.: On the density of states of Schrödinger operators with a random potential, *J. Phys. A: Math. Gen.* **15** (1982), 2139–2156.
- [Kre-85] Krengel, U.: *Ergodic Theorems*, with a Supplement by Antoine Brunel, Stud. Math., de Gruyter, Berlin, 1985.
- [Le-99] Lenz, D.: Random operators and crossed products, *Math. Phys. Anal. Geom.* **2** (1999), 197–220.
- [LY-86] Li, P. and Yau, S. T.: On the parabolic kernel of the Schrödinger operator, *Acta Math.* **156** (1986), 153–201.
- [LGP-88] Lifshitz, I. M. Gredeskul, S. A. and Pastur, L. A.: *Introduction to the Theory of Disordered Systems*, Wiley, New York, 1988. Russian original: Nauka, Moscow, 1982.
- [Lif-85] Lifschitz Memorial Issue, *J. Statist. Phys.* **38**(1–2) (1985).
- [Lin-99] Lindenstrauss, E.: Pointwise theorems for amenable groups, *Electr. Res. Ann. Amer. Math. Soc.* **5** (1999), 82–90.
- [LS-99] Lück, W. and Schick, T.: L^2 -Torsion of hyperbolic manifolds of finite volume, *Geom. Funct. Anal.* **9** (1999), 518–567.
- [Mil-68] Milnor, J.: A note on curvature and fundamental group, *J. Differential Geom.* **2** (1968) 1–7.
- [Pas-71a] Pastur, L. A.: The Schrödinger operator with random potential, *Theoret. Math. Phys.* **6** (1971), 415–424.
- [Pas-71b] Pastur, L. A.: Selfaverageability of the number of states of the Schrödinger equation with a random potential, *Mat. Fiz. i Funktsional. Anal.* **238** (1971), 111–116.
- [Pas-80] Pastur, L. A.: Spectral properties of disordered systems in the one-body approximation, *Comm. Math. Phys.* **75** (1980), 179–196.

- [PF-92] Pastur, L. A. and Figotin, A. L.: *Spectra of Random and Almost-Periodic Operators*, Springer, Berlin, 1992.
- [PV-00] Peyerimhoff, N. and Veselić, I.: Integrated density of states for random Schrödinger operators on manifolds, *MaPhySto Res. Rep.*, No. 33, 2000.
- [RSII-75] Reed, M. and Simon, B.: *Methods of Modern Mathematical Physics*. Vol. II: *Fourier Analysis, Self-adjointness*, Academic Press, Boston, 1975.
- [Shu-79] Shubin, M. A.: Spectral theory and the index of elliptic operators with almost-periodic coefficients, *Russian Math. Surveys* **34** (1979), 109–158.
- [Shu-82] Shubin, M. A.: Density of states of self adjoint operators with almost periodic coefficients, *Amer. Math. Soc. Trans* **118** (1982), 307–339.
- [Shul-88] Shulman, A.: Maximal ergodic theorems on groups, *Dep. Lit. NIINTI*, No. 2184, 1988.
- [Sim-71] Simon, B.: *Quantum Mechanics for Hamiltonians Defined as Quadratic Forms*, Princeton Ser. Phys., Princeton Univ. Press, Princeton, NJ, 1971.
- [Sz-89] Sznitman, A.-S.: Lifschitz tail and Wiener sausage on hyperbolic space, *Comm. Pure Appl. Math.* **42** (1989), 1033–1065.
- [Sz-90] Sznitman, A.-S.: Lifschitz tail on hyperbolic space: Neumann conditions, *Comm. Pure Appl. Math.* **43** (1990), 1–30.
- [Tay-96] Taylor, M. E.: *Partial Differential Equations I: Basic Theory*, Springer, New York, 1996.
- [Tem-72] Tempelman, A.: Ergodic theorems for general dynamical systems, *Trans. Moscow Math. Soc.* **26** (1972), 95–132.
- [Tem-92] Tempelman, A.: *Ergodic Theorems for Group Actions. Informational and Thermodynamical Aspects*, *Math. Appl.* 78, Kluwer Acad. Publ., Dordrecht, 1992.
- [Thu-97] Thurston, W. P.: *Three-Dimensional Geometry and Topology*, Princeton Univ. Press, Princeton, 1997.