Gert de Cooman and Matthias C. M. Troffaes. Coherent lower previsions in systems modelling: products and aggregation rules. *Reliability Engineering and System Safety*, 85(1-3):113-134, 2004.

COHERENT LOWER PREVISIONS IN SYSTEMS MODELLING: PRODUCTS AND AGGREGATION RULES

GERT DE COOMAN AND MATTHIAS C. M. TROFFAES

ABSTRACT. We discuss why coherent lower previsions provide a good uncertainty model for solving generic uncertainty problems involving possibly conflicting expert information. We study various ways of combining expert assessments on different domains, such as natural extension, independent natural extension and the type-I product, as well as on common domains, such as conjunction and disjunction. We provide each of these with a clear interpretation, and we study how they are related. Observing that in combining expert assessments no information is available about the order in which they should be combined, we suggest that the final result should be independent of the order of combination. The rules of combination we study here satisfy this requirement.

1. INTRODUCTION

In [11], two problem sets are presented, whose challenge consists in modelling diverse kinds of uncertainty about a system's parameters, sometimes from different and possibly conflicting sources, and inferring from that an uncertainty model for the system's output. In the present paper, we concentrate on the first of these problem sets, because we believe it is simpler and at the same time presents the same modelling challenges as the second one. It can be briefly sketched as follows.

The system response y is a function f of two continuous parameters a and b, given by

$$y = f(a, b) = (a + b)^a.$$

The parameters a and b are non-negative real numbers, and consequently, so is the system output y. The task is to give a model for the uncertainty about y, given additional information about the values that a and b assume. This additional information is different in each of the following six problems for this set. In modelling the available information about a and b, it should be kept in mind that they are assumed to be *epistemically independent*, i.e., information about the value that one parameter assumes does not influence our knowledge and beliefs about the value of the other one.

Problem 1. a and b assume a value in the respective closed intervals:

$$A = [0.1, 1.0] \text{ and } B = [0.0, 1.0].$$

Problem 2. a assumes a value in the closed interval A, and for b there are four independent and equally credible sources of information, each of them stating that b belongs to a closed interval B_j ($i = 1, \dots, 4$). There are three different cases.

2a) The intervals B_j are consonant, or nested:

$$A = [0.1, 1.0] \text{ and } B_1 = [0.6, 0.8], B_2 = [0.4, 0.85], B_3 = [0.2, 0.9], B_4 = [0.0, 1.0].$$

2b) The intervals B_j are consistent, i.e., they have a non-empty intersection:

$$A = [0.1, 1.0] \text{ and } B_1 = [0.6, 0.9], B_2 = [0.4, 0.8], B_3 = [0.1, 0.7], B_4 = [0.0, 1.0].$$

2c) the intervals B_j are inconsistent or conflicting, i.e., they have an empty intersection:

$$A = [0.1, 1.0] \text{ and } B_1 = [0.6, 0.8], B_2 = [0.5, 0.7], B_3 = [0.1, 0.4], B_4 = [0.0, 1.0].$$

Problem 3. For a there are three independent and equally credible sources of information, and for b there are four. There are three different cases, where, with obvious notations,

3a) the intervals A_i and B_j are consonant:

$$A_1 = [0.5, 0.7], A_2 = [0.3, 0.8], A_3 = [0.1, 1.0]$$
$$B_1 = [0.6, 0.6], B_2 = [0.4, 0.85], B_3 = [0.2, 0.9], B_4 = [0.0, 1.0];$$

3b) the intervals A_i and B_j are consistent:

$$A_1 = [0.5, 1.0], A_2 = [0.2, 0.7], A_3 = [0.1, 0.6]$$
$$B_1 = [0.6, 0.6], B_2 = [0.4, 0.8], B_3 = [0.1, 0.7], B_4 = [0.0, 1.0];$$

3c) the intervals A_i and B_j are conflicting:

$$A_1 = [0.8, 1.0], A_2 = [0.5, 0.7], A_3 = [0.1, 0.4]$$
$$B_1 = [0.8, 1.0], B_2 = [0.5, 0.7], B_3 = [0.1, 0.4], B_4 = [0.0, 0.2].$$

Key words and phrases. expert information, coherent lower previsions, natural extension, independence, type-I product, marginal extension, conjunction, disjunction.

Email: gert.decooman@rug.ac.be, matthias.troffaes@rug.ac.be.
Address: Onderzoeksgroep SYSTeMS, Technologiepark Zwijnaarde 914, 9052 Zwijnaarde, Belgium.

Problem 4. a belongs to the closed interval A, and b is log-normally distributed, i.e., $\ln b \sim N(\mu, \sigma)$, where μ and σ are known to belong to the respective closed intervals M and S, where

$$A = [0.1, 1.0], M = [0.0, 1.0] \text{ and } S = [0.1, 0.5].$$

Problem 5. There are three independent and equally credible sources of information for a, each specifying a closed interval that a belongs to. There are also three independent and equally credible sources of information for b, each stating that b is log-normally distributed, i.e., $\ln b \sim N(\mu, \sigma)$, and each specifying closed intervals that μ and σ belong to. There are three different cases, where, with obvious notations,

5a) the intervals A_i, M_j and S_k are consonant:

$$A_1 = [0.5, 0.7], A_2 = [0.3, 0.8], A_3 = [0.1, 1.0]$$
$$M_1 = [0.6, 0.8], M_2 = [0.2, 0.9], M_3 = [0.0, 1.0]$$
$$S_1 = [0.3, 0.4], S_2 = [0.2, 0.45], S_3 = [0.1, 0.5];$$

5b) the intervals A_i, M_j and S_k are consistent:

$$A_1 = [0.5, 1.0], A_2 = [0.2, 0.7], A_3 = [0.1, 0.6]$$
$$M_1 = [0.6, 0.9], M_2 = [0.1, 0.7], M_3 = [0.0, 1.0]$$
$$S_1 = [0.3, 0.45], S_2 = [0.15, 0.35], S_3 = [0.1, 0.5];$$

5c) the intervals A_i, M_j and S_k are conflicting:

$$A_1 = [0.8, 1.0], A_2 = [0.5, 0.7], A_3 = [0.1, 0.4]$$
$$M_1 = [0.6, 0.8], M_2 = [0.1, 0.4], M_3 = [0.0, 1.0]$$
$$S_1 = [0.4, 0.5], S_2 = [0.25, 0.35], S_3 = [0.1, 0.2].$$

Problem 6. a belongs to the closed interval A, and b is log-normally distributed, i.e., $\ln b \sim N(\mu, \sigma)$, with known parameters μ and σ:

$$A = [0.1, 1.0], \mu = 0.5 \text{ and } \sigma = 0.5.$$

Below, we propose to use Walley's imprecise probability models, or coherent lower previsions [14], in order to represent the available information about the parameters a and b, and to infer a model for the uncertainty about the output y. We have good reasons for preferring these models to a number of their very popular alternatives, such as Bayesian probabilities and belief functions. First of all, unlike belief functions, imprecise probability models have an operationalisable definition and a definite interpretation in terms of a subject's behaviour. In this respect, they are very much like Bayesian models, and they are also required to satisfy a number of rationality requirements, such as avoiding sure loss and coherence. But they allow for more generality: roughly speaking, it is not claimed that all uncertainty should be represented by probability measures, but rather by *sets of* probability measures. In our view, this makes imprecise probability models more widely applicable and more realistic than their Bayesian counterparts. For an extensive discussion of these issues, we refer to [14, Chapter 5].

Moreover, we are convinced that a model cannot be considered separately from how it is to be used. In many cases, models are used to (help somebody) make decisions, such as deciding which action to take, but also deciding which estimate to prefer, or which inference to make. Like their Bayesian counterparts, but unlike belief functions, imprecise probability models come with a full-fledged decision theory that is closely linked with their behavioural interpretation: roughly speaking, these models reflect a subject's behaviour in certain situations, which, through requirements of consistency or rationality, has implications for how his behaviour should be in other situations (for more information, see [14, Section 3.9]). The imprecise probability models that we shall derive for the value of the system output y, can be used to choose between actions whose outcome depends on the actual value of y. Although we concentrate on modelling itself, we nevertheless feel that this additional aspect of our models should be mentioned.

And finally, imprecise probability models include Bayesian probabilities and belief functions as special cases, as they do a number of other models in the literature, such as 2-monotone capacities [1], possibility measures [2, 7, 16], convex sets of probability measures [10], comparative and modal probabilities (see for instance [15]).

In deriving a model for the uncertainty about the system parameter y, there are three steps to be taken: 1) finding an imprecise probability model for the given information about the parameter a and about the parameter b, taken separately; 2) combining these separate models into a joint model for the values of a and b; and 3) deriving from this joint model an imprecise probability model for the uncertainty about y.

Step 1 is discussed in Section 2. Section 3 deals with Step 2, which can also be described more technically as forming products from independent marginals; and Step 3 is discussed in Section 4. We want to point out here that Problems 1, 4 and 6 are conceptually simpler than Problems 2, 3 and 5, because the latter also involve aggregating information from different sources or experts. The results in Sections 2–4 do not deal with this aggregation problem, and therefore only allow us to solve Problems 1, 4 and 6, which is done in Section 5. The problem of aggregation in Step 1 is dealt with in Section 6, and the solutions to Problems 2, 3 and 5 are presented in Section 7.

In solving these problems, we have had to derive a number of new results about imprecise probability models, for which we have provided detailed proofs. As these are fairly technical and not always essential for understanding the main argument, and as they sometimes use other results proven elsewhere, we advise the reader with a limited knowledge of the theory of imprecise probabilities to simply skip them.

2. Imprecise probability models for the value of a parameter

2.1. **Importance of a common mathematical model.** Each problem in the set provides expert assessments for the values of two parameters, a and b. It is important to note that each of the expert assessments deals with a single parameter. Moreover, it is either of the

- *vacuous type*, such as the interval information: 'a belongs to $A = [0.1, 0.9]$',
- *Bayesian type*, such as 'b has the log-normal distribution with parameters $\mu = 0.5$ and $\sigma = 0.5$', or
- *Bayesian type with vacuous parameters*, such as 'b has a log-normal distribution with parameters μ belonging to the interval $M = [0.0, 1.0]$, and σ belonging to the interval $S = [0.1, 0.5]$'.

Assessments of the last type are *hierarchical*: an assessment of the variable b is made through an assessment about variables μ and σ and an assessment about the variable b conditional on the variables μ and σ.

A first step toward combining these assessments is to express them using mathematical models of the same type. This should allow us to deal with all sources of information in a uniform way. We shall argue in this section that *all* the given expert assessments can be modelled by specific imprecise probability models, called *coherent lower previsions*. But before we do that, let us first explain briefly what those models are. For a more detailed discussion, and many of the technical results used in the proofs further on, we refer to Walley's important book on the subject [14].

2.2. **Coherent lower previsions: a behavioural uncertainty model.**

2.2.1. *The behavioural definition.* Let us consider an agent who is uncertain about something, say, the value of the variable a that takes values in a set $\mathscr{A}$. A *gamble* is a bounded mapping from $\mathscr{A}$ to $\mathbb{R}$, and it is interpreted as an uncertain reward: if some α in $\mathscr{A}$ would turn out to be the true value of the variable a then the agent would receive the amount $X(\alpha)$, expressed in units of some (predetermined) linear utility. Gambles play a similar part in the theory of imprecise probabilities as events do in the classical, or Bayesian, theory of probability. In fact, any event can be interpreted as a very simple gamble that only allows the modeller to distinguish between two situations: the event either occurs, or it doesn't, and the reward depends only on whether or not it does. So, an *event*, modelled as a subset A of the space $\mathscr{A}$ of possible parameter values, corresponds to a gamble I_A (its indicator) that yields one unit of utility if it occurs, i.e., if $a \in A$, and zero units if it doesn't, i.e., if $a \in \complement A$, where $\complement A$ denotes the set-theoretic complement of A. In other words, there is a natural correspondence between events and zero-one-valued gambles. The concept of a gamble can therefore be seen as a generalisation of the concept of an event. The set of all gambles associated with the variable a is denoted by $\mathscr{L}(\mathscr{A})$. It is a real linear space under the point-wise addition of gambles and the scalar point-wise multiplication of gambles with real numbers.

The information the agent has about the value of the parameter a will lead him to accept or reject transactions whose reward depends on this value, and we can formulate a model for his uncertainty by looking at a specific type of transaction: buying gambles. The agent's *lower prevision* (or supremum acceptable buying price, or lower expectation) $\underline{P}(X)$ for a gamble X is the greatest real number s such that he is disposed to buy the gamble X for any price strictly lower than s. If the agent assesses a supremum acceptable buying price for every gamble X in some subset $\mathscr{K}$ of $\mathscr{L}(\Omega)$, the resulting mapping $\underline{P}\colon \mathscr{K} \to \mathbb{R}$ is called the agent's *lower prevision*. $\overline{P}$ will denote the conjugate *upper prevision* of $\underline{P}$. It is defined by $\overline{P}(X) = -\underline{P}(-X)$ for every $X \in -\mathscr{K}$. $\overline{P}(X)$ represents the agent's infimum acceptable price for selling the gamble X.

Lower and upper previsions for gambles are a natural generalisation of probabilities for events. Indeed, any assessment of a probability of an event can be translated into an assessment of a supremum buying price and an infimum selling price for a zero-one-valued gamble. Suppose that the probability of the event A is known to be p. The reward we expect from I_A is then equal to $0 \cdot (1-p) + 1 \cdot p = p$. Therefore, we are willing to buy I_A for any price less than p, and we are willing to sell I_A for any price greater than p. We infer that $\underline{P}(I_A) = \overline{P}(I_A) = p$. The power of lower and upper previsions, compared to classical probability theory, is that lower and upper previsions allow for far more generality. In particular, the theory does not require that your supremum buying price should be equal to your infimum selling price.

As we have already suggested in the Introduction, a particular benefit from this way of modelling available information (or uncertainty) is that it leads naturally to a theory of decision making under uncertainty. For example, making a particular decision d from a set $\mathscr{D}$ of alternatives is behaviourally equivalent to accepting a gamble X_d, which represents the (possibly negative) utility received as a function of the value of the (unknown) parameter a of the decision problem. The agent should strictly prefer one action d_1 over an alternative d_2 if $\underline{P}(X_{d_1} - X_{d_2}) > 0$: this means that he is willing to pay some strictly positive amount of utility for exchanging the rewards of making decision d_2 with those of making decision d_1. More details on decision making with imprecise probability models can be found in [14, Section 3.9]. For a discussion of optimal control and dynamic programming in connection with imprecise probability models, we refer to [4].

2.2.2. *Coherence.* Since a lower prevision $\underline{P}$ represents an agent's commitments to act in certain ways—to buy gambles X in its domain $\mathscr{K}$ up to certain prices $\underline{P}(X)$—it should satisfy a number of requirements that ensure that his behaviour is rational. The strongest such rationality criterion is that of *coherence*. It is easiest to understand and define if the domain $\mathscr{K}$ is the set of all gambles $\mathscr{L}(\mathscr{A})$. A lower prevision $\underline{P}$ on $\mathscr{L}(\mathscr{A})$ is called *coherent* if it satisfies the following three requirements.

Accepting sure gains: The agent should always be willing to buy a gamble X for a price equal to the lowest possible reward he may expect from X, that is, $\inf[X]$. Hence, it should hold that

$$\underline{Q}(X) \geq \inf[X] \text{ for all gambles } X.$$

Positive homogeneity: Next, since we are working with a linear utility, buying prices should be independent of the choice of the scale of the utility. Mathematically, this means that

$$\underline{Q}(\lambda X) = \lambda \underline{Q}(X) \text{ for each gamble } X \text{ and } \lambda > 0.$$

Superadditivity: Finally, since we are working with a linear utility, if the agent is willing to buy X for price $\underline{Q}(X)$ and Y for price $\underline{Q}(Y)$, he should be willing to buy $X+Y$ for at least $\underline{Q}(X)+\underline{Q}(Y)$, whence:

$$\underline{Q}(X+Y) \geq \underline{Q}(X) + \underline{Q}(Y) \text{ for all gambles } X \text{ and } Y.$$

A lower prevision $\underline{P}$ on an arbitrary domain $\mathscr{K} \subseteq \mathscr{L}(\mathscr{A})$ is called coherent if it is the restriction of—can be extended to—some coherent lower prevision on $\mathscr{L}(\mathscr{A})$.

2.2.3. *Avoiding sure loss.* There is a weaker rationality criterion, called *avoiding sure loss*, that is of interest for the development in this paper. A lower prevision $\underline{P}$ defined on a set of gambles $\mathscr{K} \subseteq \mathscr{L}(\mathscr{A})$ avoids sure loss if it is point-wise dominated by some coherent lower prevision, i.e. if there is a coherent lower prevision $\underline{Q}$ on $\mathscr{L}(\mathscr{A})$ such that for all gambles X in $\mathscr{K}$, $\underline{P}(X) \leq \underline{Q}(X)$.

It can be shown that, if this criterion is not satisfied, there are gambles $X_1, \dots, X_n$ in $\mathscr{K}$ such that

$$\sup_{\alpha\in\mathscr{A}} \left[\sum_{k=1}^{n} [X_k(\alpha) - \underline{P}(X_k)]\right] < 0, \tag{1}$$

i.e., the combination of the transactions in which the gambles X_k are bought for a price $\underline{P}(X_k)$ leads to a loss, whatever the actual value of the parameter a. This means that the assessments in $\underline{P}$ are clearly unacceptable. For this reason, avoiding sure loss is a minimal but stringent requirement that an agent's lower prevision should satisfy!

2.2.4. *Natural extension.* If a lower prevision $\underline{P}$ on a set of gambles $\mathscr{K}$ avoids sure loss (which we have argued should always at least be the case), then it has a dominating coherent lower prevision on $\mathscr{L}(\mathscr{A})$, and it is not difficult to see that it has a point-wise *smallest* dominating lower prevision.[1] This lower prevision $\underline{E}$ is called the *natural extension* of $\underline{P}$: it is the most conservative correction of $\underline{P}$ to a coherent lower prevision on $\mathscr{L}(\mathscr{A})$.

The natural extension $\underline{E}$ of $\underline{P}$ can also be calculated as follows: for any gamble X,

$$\begin{aligned} \underline{E}(X) &= \sup_{\substack{n\in\mathbb{N}\\ \lambda_i\geq 0\\ Y_i\in\mathscr{K}}} \left\{\gamma\colon X-\gamma \geq \sum_{i=1}^{n} \lambda_i \left[Y_i(\alpha) - \underline{P}(Y_i)\right]\right\} \\ &= \sup_{\substack{n\in\mathbb{N}\\ \lambda_i\geq 0\\ Y_i\in\mathscr{K}}} \inf_{\alpha\in\mathscr{A}} \left[X(\alpha) - \sum_{i=1}^{n} \lambda_i [Y_i(\alpha) - \underline{P}(Y_i)]\right]. \end{aligned} \tag{2}$$

If the domain $\mathscr{K}$ of $\underline{P}$ is a linear subspace of $\mathscr{L}(\mathscr{A})$, then is is easy to show that

$$\underline{E}(X) = \sup_{Y\in\mathscr{K}} \inf_{\alpha\in\mathscr{A}} \left[X(\alpha) - [Y(\alpha) - \underline{P}(Y)]\right]. \tag{3}$$

If $\underline{P}$ is actually coherent, then it obviously coincides with its natural extension $\underline{E}$ on its domain $\mathscr{K}$: $\underline{E}$ is the most conservative extension of $\underline{P}$ to a coherent lower prevision on $\mathscr{L}(\mathscr{A})$.

Natural extension is a very important tool in imprecise probability theory, as it allows any lower prevision that avoids sure loss to be corrected into a coherent lower prevision, and any coherent lower prevision to be extended to the set of all gambles, with minimal behavioural implications! The Bayesian counterpart of natural extension is de Finetti's fundamental theorem of probability [5]. To give an example of how de Finetti's theorem relates to natural extension, observe for instance that from any probability measure μ on a σ-field $\mathscr{F}$ on Ω, which gives probabilities $\mu(A)$ for all events A in $\mathscr{F}$, we obtain a coherent lower prevision $\underline{P}$ defined by $\underline{P}(I_A) = \overline{P}(I_A) = \mu(A)$ for all $A \in \mathscr{F}$. The natural extension of this lower prevision $\underline{P}$ coincides exactly with the Lebesgue integral with respect to μ, on all μ-measurable gambles.

2.2.5. *Linear previsions.* This section focuses on the important issue of how classical theory of probability is embedded in the theory of coherent lower previsions, and how the theory of coherent lower previsions can interpreted in terms of classical probability theory. We establish that, without loss of generality, experts may represent their information using *sets* of classical probability models, rather than assessing supremum buying prices directly (which one is simpler depends on the application).

If an agent has little or no relevant information about the outcome of the gamble X, his infimum acceptable price $\overline{P}(X)$ for selling the X will typically be substantially higher than his supremum acceptable price $\underline{P}(X)$ for buying it. The bid-ask spread $\overline{P}(X) - \underline{P}(X)$ is a measure for the amount of imprecision in the agent's behavioural dispositions toward the gamble X. The more relevant information the agent has about the outcome of X, the closer $\overline{P}(X)$ and $\underline{P}(X)$ will move to each other. If it should happen that $\overline{P}(X) = \underline{P}(X)$, then this common value is denoted by $P(X)$ and it is called the agent's *fair price*, or *prevision*, for the gamble X.

[1] To prove this, use the definition of a coherent lower prevision to verify that the point-wise infimum of any number of coherent lower previsions that dominate $\underline{P}$ is still a coherent lower prevision that dominates $\underline{P}$.

As we hinted at before, in the Bayesian theory of uncertainty, championed by de Finetti [5, 6], it is assumed that an agent can always give a fair price for a gamble, whatever information he may have about its value. This questionable assumption is not made in imprecise probability theory, but it should be obvious that a precise, or Bayesian, model is nothing but a special case of an imprecise probability model, where the agent's lower and upper previsions happen to coincide. In particular, de Finetti's rationality requirements for fair prices are nothing but particular cases of the coherence requirements for lower (and upper) previsions.

A coherent lower prevision $\underline{P}$ on $\mathscr{L}(\mathscr{A})$ that is *self-conjugate* in the sense that $\underline{P}(X) = \overline{P}(X) = P(X)$ for all X in $\mathscr{L}(\mathscr{A})$, is called a *linear prevision*. It can be characterised alternatively as a real linear functional on the linear space $\mathscr{L}(\mathscr{A})$, that is moreover positive (if $X \geq 0$ then $P(X) \geq 0$) and has unit norm ($P(1) = 1$). A linear prevision on an arbitrary set of gambles $\mathscr{K}$ is the restriction to $\mathscr{K}$ of some linear prevision on $\mathscr{L}(\mathscr{A})$. We shall denote by $\mathcal{P}(\mathscr{A})$ the set of all linear previsions on $\mathscr{L}(\mathscr{A})$. As stated before, these linear previsions are exactly the same thing as de Finetti's coherent previsions, i.e., the Bayesian, or precise, probability models. In general, a linear prevision P on $\mathscr{K}$ has no unique linear extension to all of $\mathscr{L}(\mathscr{A})$: its natural extension is generally imprecise. This fact is emphasised by de Finetti's so-called *fundamental theorem of probability* which gives bounds for linear extensions of linear previsions. It is interesting to note that these bounds correspond exactly to the values obtained by natural extension.

There is another interesting connection between lower and linear previsions. Let $\mathcal{M}(\underline{P})$ be the set of all linear previsions P on $\mathscr{L}(\mathscr{A})$ that dominate the lower prevision $\underline{P}$ on its domain $\mathscr{K}$: $P(X) \geq \underline{P}(X)$ for all $X \in \mathscr{K}$. Then $\underline{P}$ avoids sure loss if and only if $\mathcal{M}(\underline{P})$ is non-empty, i.e., if $\underline{P}$ has a dominating linear prevision. $\underline{P}$ is coherent if and only if

$$\underline{P}(X) = \min\{P(X) \colon P \in \mathcal{M}(\underline{P})\}$$

for all X in $\mathscr{K}$, i.e., if $\underline{P}$ is the *lower envelope* of $\mathcal{M}(\underline{P})$. And, finally, if $\underline{P}$ avoids sure loss then the natural extension $\underline{E}$ is given by

$$\underline{E}(X) = \min\{P(X) \colon P \in \mathcal{M}(\underline{P})\}$$

for all X in $\mathscr{L}(\mathscr{A})$. Moreover, any lower envelope of a set of linear previsions is a coherent lower prevision: it is easily checked using the definition of coherence that the point-wise infimum of a set of coherent lower previsions (and in particular linear previsions) is a coherent lower prevision.

2.2.6. *Lower probabilities.* A *gamble* X on $\mathscr{A}$ can be seen as a very general risky investment that yields a possibly different return $X(\alpha)$ for each possible value α of the parameter a. In more traditional approaches to uncertainty modelling, it is common to work with a restricted class of quite simple gambles, that only allow the modeller to distinguish between two situations: zero-one-valued gambles, which correspond to events in $\mathscr{A}$ as we argued before. Let A be an event in $\mathscr{A}$ and let I_A be its corresponding gamble. An agent's lower prevision $\underline{P}(I_A)$ for this gamble is also denoted by $\underline{P}(A)$ and is called his *lower probability* for the event A: lower probabilities are simply lower previsions for zero-one-valued gambles. $\underline{P}(A)$ can also be interpreted as an agent's supremum acceptable rate for betting on the occurrence of the event A. Similar considerations holds for upper probabilities. It should be mentioned here that most of the lower and upper probabilities in the literature, such as precise probability measures, 2-monotone capacities, belief functions, and possibility measures, are in this way special cases of coherent lower and upper previsions [3, 14].

We prefer to work with the more general notion of gambles, rather than events. In fact, whereas the languages of events and of gambles are equally powerful when dealing with precise, or Bayesian, probabilities or previsions [5], it has been shown [14] that this is no longer the case for imprecise models: we need the more powerful language of gambles in the more general theory of imprecise probability.

This concludes our brief survey of the theory of coherent lower previsions. We are now ready to apply the tools this theory provides to modelling the expert assessments of the values of the variables a and b in the problems described in the Introduction.

2.3. **Vacuous information.** Let us first consider assessments of the type 'a assumes a value in a subset A of $\mathscr{A}$'. This type of assessment includes the various instances of interval information present in the problem set—one may for example think of $\mathscr{A}$ as the positive real line and of A as a closed interval.

This type of information can be represented by the so-called *vacuous lower prevision relative to* A, which will be denoted by $\underline{P}_A$, and is given by

$$\underline{P}_A(X) = \inf_{\alpha \in A} X(\alpha),$$

for all gambles X in $\mathscr{L}(\mathscr{A})$. This is a coherent lower prevision on $\mathscr{L}(\mathscr{A})$ and its conjugate upper prevision is given by

$$\overline{P}_A(X) = \sup_{\alpha \in A} X(\alpha).$$

There are several lines of reasoning to motivate that this lower prevision indeed is the appropriate model for the given information. First of all, if the agent knows that a belongs to A, *and nothing more*, he should be willing to buy a gamble X for any price s strictly lower than $\inf_{\alpha \in A} X(\alpha)$ because doing so results in a sure gain; but he should not be willing to pay a price t strictly higher than that, because then there is some $\alpha \in A$ such that $t > X(\alpha)$, and *for all the agent knows*, α might be the actual value of the parameter a!

A second justification for $\underline{P}_A$ is that it is the natural extension of the single precise probability assessment $P(A) = 1$. Using Eq. (2), we find indeed that:

$$\sup_{\lambda\geq 0}\inf_{\alpha\in\mathscr{A}}[X(\alpha)-\lambda[I_A(\alpha)-P(A)]] = \sup_{\lambda\geq 0}\min\left\{\inf_{\alpha\in A}X(a),\ \inf_{\alpha\in\complement A}[X(\alpha)+\lambda]\right\} = \inf_{\alpha\in A}X(\alpha) = \underline{P}_A(X).$$

This shows that the vacuous lower prevision relative to A follows *uniquely* from the single assessment that the agent's probability of event A is equal to 1, or equivalently, that the agent is practically certain that a belongs to A (since he is prepared to bet at all odds on the occurrence of A).

Yet another way of justifying $\underline{P}_A$ is the following. Consider a gamble X. Take any linear prevision P such that $P(A) = 1$. De Finetti's fundamental theorem of probability imposes bounds on $P(X)$, i.e., states that $P(X)$ belongs to some interval. The union of all these intervals over all linear previsions P such that $P(A) = 1$ is exactly given by $[\underline{P}_A(X), \overline{P}_A(X)]$.

2.4. **Bayesian information.** Let us now look at the expert assessment 'the parameter b is log-normally distributed with parameters μ and σ'. This is a special case of a very common type of assessment, stating that a continuous real random variable b taking values in a subset $\mathscr{B}$ of the set of real numbers $\mathbb{R}$ has (cumulative) distribution function F, or density function ϕ, with respect to some measure μ on the reals, such as the Lebesgue measure λ.

It is well-known (see for instance [5]) that specifying such a model is equivalent to specifying a *linear prevision* P on a set $\mathscr{F}$ of gambles that are measurable with respect to some σ-algebra on $\mathscr{B}$, where for each such gamble X,[2]

$$P(X) = \int_{\mathscr{B}} X\mathrm{d}F = \int_{\mathscr{B}} X\phi\,\mathrm{d}\mu$$

is the expectation of X. For the problem set under study here, $\mathscr{B}$ is the positive real line, μ is the Lebesgue measure λ, $\mathscr{F}$ is the linear space of Lebesgue-measurable gambles, and ϕ is the log-normal density function.

Since linear previsions are special types of coherent lower previsions, Bayesian models fit perfectly into imprecise probability theory. This is also the case for the so-called *robust Bayesian models* that we deal with next.

2.5. **Bayesian information with vacuous parameters.**

2.5.1. *Robust Bayesian models.* The Bayesian approach has the disadvantage that the probability density ϕ must be known exactly. In some situations this is not realistic: there may be a class Φ of probability densities ϕ, each compatible with the given information, and different choices within this class may lead to completely different results: inferences based on particular choices of ϕ will then not be robust, and will not with any confidence reflect the information that is actually available.

The theory of imprecise probabilities deals with this situation in a straightforward manner: it associates with the set Φ of density functions a coherent[3] lower prevision $\underline{P}$ that is the *lower envelope* of the linear previsions associated with the density functions $\phi \in \Phi$: for each measurable gamble X,

$$\underline{P}(X) = \inf_{\phi\in\Phi}\int_{\mathscr{B}} X\phi\,\mathrm{d}\mu. \tag{4}$$

This ensures that our agent will accept to buy a gamble X only for prices that are lower than any of the prices $\int_{\mathscr{B}} X\phi\,\mathrm{d}\mu$ corresponding to specific choices of ϕ in Φ; any conclusions we may draw from the model will be automatically robust.

We have shown in Section 2.2.5 that, in a very specific sense, coherent lower previsions are mathematically equivalent with sets of linear previsions. This way of looking at coherent lower previsions is sometimes referred to as the *Bayesian sensitivity analysis interpretation*, or *robust Bayesian interpretation* of such lower previsions.

2.5.2. *An alternative justification.* There is another way of deriving Eq. (4), which, in our opinion, provides a better justification for associating a lower prevision with a set of density functions.

Assume that the available information about a variable b allows us to specify for it a Bayesian model $P(\cdot|\theta)$ that depends on some additional variable $\theta \in \Theta$. However, the true value of the variable θ is only known to belong to some non-empty subset Θ_0 of Θ. We thus have the following information about the value of the variable b.

(i) Conditionally on θ, a probability density ϕ_θ for b. The inferred *conditional linear prevision* $P(\cdot|\vartheta)$, given by

$$P(X|\vartheta) = \int_{\mathscr{B}} X\phi_\vartheta\,\mathrm{d}\mu$$

for each measurable gamble X, would describe the agent's behavioural dispositions toward gambles on $\mathscr{B}$ if ϑ were the true value of the parameter θ. This conditional linear prevision $P(\cdot|\vartheta)$ is also called a *sampling model*.

(ii) Vacuous information about the variable θ, described by a vacuous lower prevision relative to Θ_0, $\underline{P}_{\Theta_0}$, on $\mathscr{L}(\Theta)$, where $\Theta_0 \subseteq \Theta$ and $\Theta_0 \neq \emptyset$. The coherent lower prevision $\underline{P}_{\Theta_0}$ is also called the *prior*.

[2] To see that P really is a linear prevision, observe that by the Hahn-Banach theorem P can be extended to a linear prevision on the whole space $\mathscr{L}(\mathscr{B})$. Generally speaking, such an extension will not be unique, or equivalently, the natural extension of P to $\mathscr{L}(\mathscr{B})$ will be imprecise (a coherent lower prevision that is not a linear prevision).

[3] A lower envelope of linear previsions is always a coherent lower prevision, see Section 2.2.5.

For a given measurable gamble X on $\mathscr{B}$, we shall denote by $P(X|\Theta)$ the gamble on Θ that assumes the value $P(X|\vartheta)$ in the element ϑ of Θ.

Walley's *marginal extension theorem* [14, Theorem 6.7.2] then tells us that the natural extension $\underline{P}$ of the lower prevision $\underline{P}_{\Theta_0}$ and the conditional linear prevision $P(\cdot|\Theta)$ is given by

$$\underline{P}(X) = \underline{P}_{\Theta_0}(P(X|\Theta)) = \inf_{\vartheta \in \Theta_0} P(X|\vartheta) = \inf_{\vartheta \in \Theta_0} \int_{\mathscr{B}} X \phi_\vartheta \, \mathrm{d}\mu,$$

for all measurable gambles X. $\underline{P}(X)$ is the supremum acceptable buying price for X that can be inferred from the agent's assessments $\underline{P}_{\Theta_0}$ and $P(\cdot|\Theta)$, through arguments of coherence alone! $\underline{P}$ is the smallest (most conservative) coherent extension to measurable gambles on $\mathscr{B}$ of both $P(\cdot|\Theta)$ and $\underline{P}_{\Theta_0}$.

For a full motivation of marginal extension we refer to [14, Chapter 6]. But it may be interesting to observe that it generalises Kolmogorov's definition $\mathsf{P}(B) = \mathsf{E}(\mathsf{P}_u(B))$ of conditional probability [9, Chapter V, Equations (1–3), pp. 47–9], where $\mathsf{P}_u(B)$ is the probability of event B conditional on the outcome of a random variable u, E is the expectation of a gamble depending on the outcome of the random variable u and $\mathsf{P}(B)$ is the probability of the event B.

For the problem set under study, the sampling model is a log-normal distribution with parameters μ and σ; we shall denote the set of possible values for parameter μ by $\mathscr{M}$ and for parameter σ by $\mathscr{S}$. Thus, we have that $\Theta = \mathscr{M} \times \mathscr{S}$ and $\theta = (\mu, \sigma)$. Each expert expresses his information about μ and σ through a vacuous lower prevision on $\mathscr{L}(\mathscr{M})$ or $\mathscr{L}(\mathscr{S})$, relative to some subset (closed interval) of $\mathscr{M}$ or $\mathscr{S}$, respectively.

What we still need, however, in order to be able to apply the marginal extension theorem, is a way to combine the *separate* lower previsions on $\mathscr{M}$ and $\mathscr{S}$ into a *joint* lower prevision on $\Theta = \mathscr{M} \times \mathscr{S}$. If we can do that, we shall be able to arrive at a lower prevision modelling the available information about the parameter b. We have seen above that we are also able to find a lower prevision modelling the available information about parameter a. But here again, we still have to combine these separate lower previsions into a joint lower prevision modelling the available information about a and b taken together. We conclude that in order to proceed, we need a way to form so-called *products* of lower previsions. This is the subject of the next section.

3. Products

Assume that the information about the value that the variable a assumes in $\mathscr{A}$ is modelled by a coherent lower prevision $\underline{P}$. To make things as easy as possible and as complicated as necessary, its domain is assumed to be a linear space $\mathscr{F}_{\mathscr{A}} \subseteq \mathscr{L}(\mathscr{A})$ of gambles on $\mathscr{A}$ — for instance, the set $\mathscr{L}(\mathscr{A})$, or the linear space of all gambles that are measurable with respect to some σ-field on $\mathscr{A}$. Similarly, the information about the variable b is represented by a coherent lower prevision $\underline{Q}$ defined on a linear space of gambles $\mathscr{F}_{\mathscr{B}} \subseteq \mathscr{L}(\mathscr{B})$.

We now want to find a way to combine the information about a and that about b into information about the values that the variable (a, b) takes in the product space $\mathscr{A} \times \mathscr{B}$. In other words, we want to find a way to combine $\underline{P}$ and $\underline{Q}$ into a coherent *product* lower prevision $\underline{P}$ whose *marginals* are $\underline{P}$ and $\underline{Q}$. This is made more clear in the following definition. We denote by $\mathscr{F}_{\mathscr{A}} \times \mathscr{F}_{\mathscr{B}}$ the linear space of those gambles Z on $\mathscr{A} \times \mathscr{B}$ whose partial maps belong to $\mathscr{F}_{\mathscr{A}}$ and $\mathscr{F}_{\mathscr{B}}$, i.e., such that for all $\alpha \in \mathscr{A}$ and $\beta \in \mathscr{B}$, $Z(\alpha, \cdot) \in \mathscr{F}_{\mathscr{B}}$ and $Z(\cdot, \beta) \in \mathscr{F}_{\mathscr{A}}$. If X belongs to $\mathscr{F}_{\mathscr{A}}$ then it can also be considered as a gamble on $\mathscr{A} \times \mathscr{B}$ that is constant on $\mathscr{B}$, and which therefore belongs to $\mathscr{F}_{\mathscr{A}} \times \mathscr{F}_{\mathscr{B}}$. A similar remark can be made about gambles Y in $\mathscr{F}_{\mathscr{B}}$.

Definition 1. *A coherent lower prevision $\underline{R}$ whose domain $\mathscr{F}$ includes $\mathscr{F}_{\mathscr{A}} \times \mathscr{F}_{\mathscr{B}}$ is called a* product *of the coherent lower previsions $\underline{P}$ and $\underline{Q}$ if it has these lower previsions as its* marginals*, i.e., if for all X in $\mathscr{F}_{\mathscr{A}}$ and Y in $\mathscr{F}_{\mathscr{B}}$, $\underline{R}(X) = \underline{P}(X)$ and $\underline{R}(Y) = \underline{Q}(Y)$.*

We shall consider three different ways to define a product of $\underline{P}$ and $\underline{Q}$.

3.1. Natural extension.

The *natural extension* $\underline{P} \times_{\mathrm{NE}} \underline{Q}$ of $\underline{P}$ and $\underline{Q}$ is defined as the smallest coherent lower prevision on $\mathscr{F}_{\mathscr{A}} \times \mathscr{F}_{\mathscr{B}}$ that has marginals $\underline{P}$ and $\underline{Q}$. It is, in other words, their least-committal or most conservative product. Since we assume the domains $\mathscr{F}_{\mathscr{A}}$ and $\mathscr{F}_{\mathscr{B}}$ to be linear spaces, we find, using Eq. (3), that for any gamble Z in $\mathscr{F}_{\mathscr{A}} \times \mathscr{F}_{\mathscr{B}}$:

$$\left(\underline{P} \times_{\mathrm{NE}} \underline{Q}\right)(Z) = \sup_{X \in \mathscr{F}_{\mathscr{A}}, Y \in \mathscr{F}_{\mathscr{B}}} \inf_{(\alpha,\beta) \in \mathscr{A} \times \mathscr{B}} \left[Z(\alpha, \beta) - [X(\alpha) - \underline{P}(X)] - [Y(\beta) - \underline{Q}(Y)]\right]. \tag{5}$$

In forming the natural extension, no assumption is made about the possible independence of the variables a and b. Making such an additional assumption generally leads to products that dominate the natural extension, and are therefore less conservative. We look at two types of independent products, each with a different interpretation.

3.2. Independent natural extension.

The *independent natural extension* of $\underline{P}$ and $\underline{Q}$ is denoted by $\underline{P} \times_{\mathrm{INE}} \underline{Q}$ and defined as the smallest coherent lower prevision on $\mathscr{F}_{\mathscr{A}} \times \mathscr{F}_{\mathscr{B}}$ that has marginals $\underline{P}$ and $\underline{Q}$, also taking into account the extra assessment that the variables a and b are *epistemically independent*, i.e., that additional knowledge about the value that a assumes in $\mathscr{A}$ does not affect our knowledge about the value that b assumes in $\mathscr{B}$, and vice versa. It can be shown using the results and ideas in [14, Chapters 6–9] that for any gamble Z in $\mathscr{F}_{\mathscr{A}} \times \mathscr{F}_{\mathscr{B}}$:

$$\left(\underline{P} \times_{\mathrm{INE}} \underline{Q}\right)(Z) = \sup_{X \in \overline{\mathscr{F}}_{\mathscr{A}}, Y \in \overline{\mathscr{F}}_{\mathscr{B}}} \inf_{(\alpha,\beta) \in \mathscr{A} \times \mathscr{B}} \left[Z(\alpha, \beta) - [X(\alpha, \beta) - \underline{P}(X(\cdot, \beta))] - [Y(\alpha, \beta) - \underline{Q}(Y(\alpha, \cdot))]\right], \tag{6}$$

where we have introduced the notations

$$\overline{\mathscr{F}}_{\mathscr{A}} = \{X \in \mathscr{L}(\mathscr{A}\times\mathscr{B})\colon (\forall\beta\in\mathscr{B})(X(\cdot,\beta)\in\mathscr{F}_{\mathscr{A}})\}$$
$$\overline{\mathscr{F}}_{\mathscr{B}} = \{Y \in \mathscr{L}(\mathscr{A}\times\mathscr{B})\colon (\forall\alpha\in\mathscr{A})(Y(\alpha,\cdot)\in\mathscr{F}_{\mathscr{B}})\}\,.$$

Observe that $\mathscr{F}_{\mathscr{A}}\times\mathscr{F}_{\mathscr{B}} = \overline{\mathscr{F}}_{\mathscr{A}}\cap\overline{\mathscr{F}}_{\mathscr{B}}$, $\mathscr{L}(\mathscr{A})\times\mathscr{F}_{\mathscr{B}} = \overline{\mathscr{F}}_{\mathscr{B}}$ and $\mathscr{F}_{\mathscr{A}}\times\mathscr{L}(\mathscr{B}) = \overline{\mathscr{F}}_{\mathscr{A}}$.

3.3. **Type-I product.** There is another way to define an independent product of $\underline{P}$ and $\underline{Q}$ that is compatible with the Bayesian sensitivity analysis interpretation of a lower prevision. On this view, the uncertainty about the variable a is ideally described by a linear prevision P_T; the only problem is that for some reason, we are not able to uniquely identify it. Specifying a coherent lower prevision $\underline{P}$ is then tantamount to stating that the unknown P_T should belong to the set

$$\mathcal{M}(\underline{P}) = \{P\in\mathcal{P}(\mathscr{A})\colon (\forall X\in\mathscr{F}_{\mathscr{A}})(P(X)\geq\underline{P}(X))\}$$

of those linear previsions on $\mathscr{A}$ that dominate the lower prevision $\underline{P}$ on its domain $\mathscr{F}_{\mathscr{A}}$. Similar considerations hold for the uncertainty about the variable b, its ideal precise model Q_T, and the coherent lower prevision $\underline{Q}$. If a and b are independent, then the ideal model describing the uncertainty about the value of the joint variable (a,b) is the *independent product* $P_T\times Q_T$, i.e., the linear prevision defined on gambles Z on $\mathscr{A}\times\mathscr{B}$ by

$$(P_T\times Q_T)(Z) = P_T\left(Q_T(Z)\right) = Q_T\left(P_T(Z)\right),$$

where by $Q_T(Z)$ we denote the gamble on $\mathscr{A}$ taking the value $Q_T(Z(\alpha,\cdot))$ in $\alpha\in\mathscr{A}$ and similarly for $P_T(Z)$.

It then seems appropriate to define the *type-I* product $\underline{P}\times_{\mathrm{TI}}\underline{Q}$ of the lower previsions $\underline{P}$ and $\underline{Q}$ as the lower envelope of all compatible independent linear products (see also Corollary 1 further on):

$$\left(\underline{P}\times_{\mathrm{TI}}\underline{Q}\right)(Z) = \inf\left\{(P\times Q)(Z)\colon P\in\mathcal{M}(\underline{P}) \text{ and } Q\in\mathcal{M}(\underline{Q})\right\} \tag{7}$$

for all Z in $\mathscr{F}_{\mathscr{A}}\times\mathscr{F}_{\mathscr{B}}$.

3.4. **Products with a vacuous lower prevision.** In general, we have that

$$\left(\underline{P}\times_{\mathrm{NE}}\underline{Q}\right)(Z) \leq \left(\underline{P}\times_{\mathrm{INE}}\underline{Q}\right)(Z) \leq \left(\underline{P}\times_{\mathrm{TI}}\underline{Q}\right)(Z)$$

for all gambles Z in $\mathscr{F}_{\mathscr{A}}\times\mathscr{F}_{\mathscr{B}}$, but it turns out that, if at least one of the lower previsions $\underline{P}$ or $\underline{Q}$ is vacuous, then their independent natural extension and their type-I product coincide! This is the case for all of the problems in the problem set described in the Introduction. If both $\underline{P}$ and $\underline{Q}$ are vacuous, then all three products coincide. This is made evident by the following theorem and proposition.

Theorem 1. *Consider a non-empty subset A of $\mathscr{A}$, let $\underline{Q}$ be a coherent lower prevision defined on a linear space of gambles $\mathscr{F}_{\mathscr{B}}$, and let $\underline{P}_A$ be the vacuous lower prevision relative to A, defined on $\mathscr{L}(\mathscr{A})$ by*

$$\underline{P}_A(X) = \inf_{\alpha\in A} X(\alpha),$$

for all gambles X on $\mathscr{A}$. Then the natural extension of $\underline{P}_A$ and $\underline{Q}$ is given by

$$\left(\underline{P}_A\times_{\mathrm{NE}}\underline{Q}\right)(Z) = \underline{E}\left(\inf_{\alpha\in A} Z(\alpha,\cdot)\right)$$

for all gambles Z in $\overline{\mathscr{F}}_{\mathscr{B}} = \mathscr{L}(\mathscr{A})\times\mathscr{F}_{\mathscr{B}}$, where $\underline{E}$ is the natural extension of $\underline{Q}$ to the set $\mathscr{L}(\mathscr{B})$. Moreover, the independent natural extension and the type-I product of $\underline{P}_A$ and $\underline{Q}$ coincide and are given for all Z in $\overline{\mathscr{F}}_{\mathscr{B}}$ by

$$\left(\underline{P}_A\times_{\mathrm{INE}}\underline{Q}\right)(Z) = \left(\underline{P}_A\times_{\mathrm{TI}}\underline{Q}\right)(Z) = \inf_{\alpha\in A}\underline{Q}(Z(\alpha,\cdot)).$$

Proof. Let us first look at the natural extension of $\underline{P}_A$ and $\underline{Q}$. For any Z in $\overline{\mathscr{F}}_{\mathscr{B}}$ we find that

$$\begin{aligned}\left(\underline{P}\times_{\mathrm{NE}}\underline{Q}\right)(Z) &= \sup_{Y\in\mathscr{F}_{\mathscr{B}}}\sup_{X\in\mathscr{F}_{\mathscr{A}}}\inf_{\beta\in\mathscr{B}}\inf_{\alpha\in\mathscr{A}}\left[Z(\alpha,\beta)-[X(\alpha)-\underline{P}(X)]-[Y(\beta)-\underline{Q}(Y)]\right]\\ &\leq \sup_{Y\in\mathscr{F}_{\mathscr{B}}}\inf_{\beta\in\mathscr{B}}\sup_{X\in\mathscr{F}_{\mathscr{A}}}\inf_{\alpha\in\mathscr{A}}\left[Z(\alpha,\beta)-[X(\alpha)-\underline{P}(X)]-[Y(\beta)-\underline{Q}(Y)]\right]\end{aligned}$$

and using the fact that the coherent $\underline{P}_A$ coincides on $\mathscr{L}(\mathscr{A})$ with its natural extension, given by Eq. (3),

$$\begin{aligned}&= \sup_{Y\in\mathscr{F}_{\mathscr{B}}}\inf_{\beta\in\mathscr{B}}\underline{P}_A\left(Z(\cdot,\beta)-[Y(\beta)-\underline{Q}(Y)]\right)\\ &= \sup_{Y\in\mathscr{F}_{\mathscr{B}}}\inf_{\beta\in\mathscr{B}}\left[\inf_{\alpha\in A} Z(\alpha,\beta)-[Y(\beta)-\underline{Q}(Y)]\right]\end{aligned}$$

and invoking Eq. (3) for the natural extension $\underline{E}$ of $\underline{Q}$,

$$= \underline{E}\left(\inf_{\alpha\in A} Z(\alpha,\cdot)\right).$$

To prove the converse inequality, let $X = -\lambda[1-I_A]$ in the definition of the natural extension, where λ is some real number satisfying

$$\lambda \geq \sup_{(\alpha,\beta)\in\mathscr{A}\times\mathscr{B}} Z(\alpha,\beta) - \inf_{(\alpha,\beta)\in\mathscr{A}\times\mathscr{B}} Z(\alpha,\beta).$$

For this choice of λ it is easy to prove that for all $\beta \in \mathscr{B}$:

$$\lambda + \inf_{\alpha \notin A} Z(\alpha, \beta) \geq \inf_{\alpha \in A} Z(\alpha, \beta) \tag{8}$$

It then follows, since $\underline{P}_A(X) = 0$, that

$$\begin{aligned}
\left(\underline{P} \times_{\mathrm{NE}} \underline{Q}\right)(Z) &\geq \sup_{Y \in \mathscr{F}_{\mathscr{B}}} \inf_{\beta \in \mathscr{B}} \inf_{\alpha \in \mathscr{A}} \left[Z(\alpha, \beta) + \lambda[1 - I_A(\alpha)] - [Y(\beta) - \underline{Q}(Y)]\right] \\
&= \sup_{Y \in \mathscr{F}_{\mathscr{B}}} \inf_{\beta \in \mathscr{B}} \left[\min\left\{\inf_{\alpha \in A} Z(\alpha, \beta), \inf_{\alpha \notin A} Z(\alpha, \beta) + \lambda\right\} - [Y(\beta) - \underline{Q}(Y)]\right]
\end{aligned}$$

and taking into account the inequality (8),

$$= \sup_{Y \in \mathscr{F}_{\mathscr{B}}} \inf_{\beta \in \mathscr{B}} \left[\inf_{\alpha \in A} Z(\alpha, \beta) - [Y(\beta) - \underline{Q}(Y)]\right]$$

and using Eq. (3) for the natural extension $\underline{E}$ of $\underline{Q}$,

$$= \underline{E}\left(\inf_{\alpha \in A} Z(\alpha, \cdot)\right).$$

We now turn to the type-I product of $\underline{P}_A$ and $\underline{Q}$:

$$\begin{aligned}
\left(\underline{P} \times_{\mathrm{TI}} \underline{Q}\right)(Z) &= \inf_{Q \in \mathcal{M}(\underline{Q})} \inf_{P \in \mathcal{M}(\underline{P}_A)} P(Q(Z)) \\
&= \inf_{Q \in \mathcal{M}(\underline{Q})} \underline{P}_A(Q(Z)) \\
&= \inf_{Q \in \mathcal{M}(\underline{Q})} \inf_{\alpha \in A} Q(Z(\alpha, \cdot)) \\
&= \inf_{\alpha \in A} \inf_{Q \in \mathcal{M}(\underline{Q})} Q(Z(\alpha, \cdot)) \\
&= \inf_{\alpha \in A} \underline{Q}(Z(\alpha, \cdot)).
\end{aligned}$$

To conclude the proof, we consider the independent natural extension of $\underline{P}_A$ and $\underline{Q}$. For ease of notation, denote by $\underline{R}$ the lower prevision defined on the set of gambles $\mathscr{L}(\mathscr{B})$ by $\underline{R}(Z) = \inf_{\alpha \in A} \underline{E}(Z(\alpha, \cdot))$, where, as before, $\underline{E}$ is the natural extension of $\underline{Q}$. It is not difficult to prove that $\underline{R}$ is coherent. Consequently, $\underline{R}(\cdot) \geq \inf[\cdot]$, and therefore

$$\left(\underline{P}_A \times_{\mathrm{INE}} \underline{Q}\right)(Z) \leq \sup_{X \in \overline{\mathscr{F}}_{\mathscr{A}}, Y \in \overline{\mathscr{F}}_{\mathscr{B}}} \underline{R}\left(Z - [X - \underline{P}_A(X)] - [Y - \underline{Q}(Y)]\right)$$

and from the coherence (superadditivity) of $\underline{R}$

$$\leq \sup_{X \in \overline{\mathscr{F}}_{\mathscr{A}}, Y \in \overline{\mathscr{F}}_{\mathscr{B}}} \left[\underline{R}(Z) - \underline{R}(X - \underline{P}_A(X)) - \underline{R}(Y - \underline{Q}(Y))\right].$$

At the same time, we deduce from the coherence of $\underline{E}$ that

$$\underline{R}\left(X - \underline{P}_A(X)\right) = \inf_{\alpha \in A} \underline{E}\left(X(\alpha, \cdot) - \inf_{\gamma \in A} X(\gamma, \cdot)\right) \geq 0$$

and similarly, since $\underline{Q}$ and its natural extension $\underline{E}$ coincide on $\mathscr{F}_{\mathscr{B}}$,

$$\underline{R}\left(Y - \underline{Q}(Y)\right) = \inf_{\alpha \in A} \underline{Q}\left(Y(\alpha, \cdot) - \underline{Q}(Y(\alpha, \cdot))\right) = 0,$$

whence immediately $\left(\underline{P}_A \times_{\mathrm{INE}} \underline{Q}\right)(Z) \leq \underline{R}(Z)$. To prove the converse inequality, let in the definition for the independent natural extension $Y = Z \in \overline{\mathscr{F}}_{\mathscr{B}}$ and $X = \underline{Q}(Z) \in \mathscr{L}(\mathscr{A})$. □

Proposition 1. *Let A be a non-empty subset of $\mathscr{A}$ and B a non-empty subset of $\mathscr{B}$. Then for all Z in $\mathscr{L}(\mathscr{A} \times \mathscr{B})$,*

$$\left(\underline{P}_A \times_{\mathrm{NE}} \underline{P}_B\right)(Z) = \left(\underline{P}_A \times_{\mathrm{INE}} \underline{P}_B\right)(Z) = \left(\underline{P}_A \times_{\mathrm{TI}} \underline{P}_B\right)(Z) = \underline{P}_{A \times B}(Z) = \inf_{(\alpha,\beta) \in A \times B} Z(\alpha, \beta).$$

Proof. For any Z in $\mathscr{L}(\mathscr{A} \times \mathscr{B})$ we find using Theorem 1 that, since $\underline{P}_A$ is a vacuous lower prevision,

$$\left(\underline{P}_A \times_{\mathrm{NE}} \underline{P}_B\right)(Z) = \underline{P}_B\left(\inf_{\alpha \in A} Z(\alpha, \cdot)\right) = \inf_{\beta \in B} \inf_{\alpha \in A} Z(\alpha, \beta) = \inf_{(\alpha,\beta) \in A \times B} Z(a, b) = \underline{P}_{A \times B}(Z),$$

and similarly,

$$\left(\underline{P}_A \times_{\mathrm{INE}} \underline{P}_B\right)(Z) = \left(\underline{P}_A \times_{\mathrm{TI}} \underline{P}_B\right)(Z) = \inf_{\alpha \in A} \underline{P}_B(Z(\alpha, \cdot)) = \inf_{\alpha \in A} \inf_{\beta \in B} Z(\alpha, \beta) = \inf_{(\alpha,\beta) \in A \times B} Z(a, b) = \underline{P}_{A \times B}(Z). \quad \square$$

3.5. **Products with a linear prevision.** If at least one of the lower previsions $\underline{P}$ or $\underline{Q}$ is a linear prevision, which is the case for Problem 6 described in the Introduction, then their independent natural extension and their type-I product coincide as well.

Theorem 2. *Let $\underline{P}$ be a coherent lower prevision defined on a linear space of gambles $\mathscr{F}_{\mathscr{A}} \subseteq \mathscr{L}(\mathscr{A})$, and let Q be a linear prevision defined on $\mathscr{L}(\mathscr{B})$. Then for all gambles Z in $\overline{\mathscr{F}}_{\mathscr{A}} = \mathscr{F}_{\mathscr{A}} \times \mathscr{L}(\mathscr{B})$,*

$$(\underline{P} \times_{\text{INE}} Q)(Z) = (\underline{P} \times_{\text{TI}} Q)(Z) = \underline{E}(Q(Z)),$$

where $\underline{E}$ is the natural extension of $\underline{P}$ to $\mathscr{L}(\mathscr{A})$, and where $Q(Z)$ denotes the gamble on $\mathscr{A}$ whose value in $\alpha \in \mathscr{A}$ is given by $Q(Z(\alpha, \cdot))$.

Proof. Let Z be any gamble in $\overline{\mathscr{F}}_{\mathscr{A}}$. We begin with the last equality. Since $\mathcal{M}(Q) = \{Q\}$, we see that indeed

$$(\underline{P} \times_{\text{TI}} Q)(Z) = \inf_{P \in \mathcal{M}(\underline{P})} P(Q(Z)) = \underline{E}(Q(Z)).$$

To prove the first equality, apply Eq. (6) to see that

$$(\underline{P} \times_{\text{INE}} Q)(Z) = \sup_{X \in \overline{\mathscr{F}}_{\mathscr{A}}, Y \in \mathscr{L}(\mathscr{B})} \inf \left[Z - [X - \underline{P}(X)] - [Y - Q(Y)]\right].$$

If we make the particular choice $Y = Z$ in this supremum, and apply Eq. (3) for the natural extension $\underline{E}$ of $\underline{P}$, we find that

$$(\underline{P} \times_{\text{INE}} Q)(Z) \geq \sup_{X \in \overline{\mathscr{F}}_{\mathscr{A}}} \inf \left[Q(Z) + [X - \underline{P}(X)]\right] = \underline{E}(Q(Z)).$$

To prove the converse inequality, recall that it follows from the coherence of $\underline{P}$ and Q that for any gamble U on $\mathscr{A} \times \mathscr{B}$, $\underline{E}(Q(U)) \geq \inf[U]$, whence

$$\begin{aligned}
(\underline{P} \times_{\text{INE}} Q)(Z) &= \sup_{X \in \overline{\mathscr{F}}_{\mathscr{A}}, Y \in \mathscr{L}(\mathscr{B})} \inf \left[Z - [X - \underline{P}(X)] - [Y - Q(Y)]\right] \\
&\leq \sup_{X \in \overline{\mathscr{F}}_{\mathscr{A}}, Y \in \mathscr{L}(\mathscr{B})} \underline{E}\left(Q\left(Z - [X - \underline{P}(X)] - [Y - Q(Y)]\right)\right) \\
&\leq \sup_{X \in \overline{\mathscr{F}}_{\mathscr{A}}, Y \in \mathscr{L}(\mathscr{B})} \left[\underline{E}(Q(Z)) - \underline{E}(Q(X - \underline{P}(X))) - \underline{E}(Q(Y - Q(Y)))\right]
\end{aligned}$$

It is easy to see that, by Lemma 1,

$$\underline{E}(Q(X - \underline{P}(X))) = \underline{E}(Q(X)) - Q(\underline{P}(X)) \geq 0,$$

and also that

$$\underline{E}(Q(Y - Q(Y))) = 0$$

whence indeed also $(\underline{P} \times_{\text{INE}} Q)(Z) \leq \underline{E}(Q(Z))$. □

Lemma 1. *Let $\underline{P}$ be a coherent lower prevision defined on $\mathscr{L}(\mathscr{A})$, and let Q be a linear prevision defined on $\mathscr{L}(\mathscr{B})$. Then for all gambles Z in $\mathscr{L}(\mathscr{A} \times \mathscr{B})$:*

$$\underline{P}(Q(Z)) \geq Q(\underline{P}(Z)).$$

Proof. For any Z in $\mathscr{L}(\mathscr{A} \times \mathscr{B})$ and any P in $\mathcal{M}(\underline{P})$ we have that $\underline{P}(Z) \leq P(Z)$, whence $Q(\underline{P}(Z)) \leq Q(P(Z)) = P(Q(Z))$, and consequently

$$Q(\underline{P}(Z)) \leq \inf_{P \in \mathcal{M}(\underline{P})} P(Q(Z)) = \underline{P}(Q(Z)). \quad \square$$

Corollary 1. *Let P be a linear prevision defined on $\mathscr{L}(\mathscr{A})$ and let Q be a linear prevision defined on $\mathscr{L}(\mathscr{B})$. Then*

$$P \times_{\text{INE}} Q = P \times_{\text{TI}} Q = P \times Q.$$

It may happen, as in some of the problems sketched in the Introduction, that the variable b is assumed to have a precise probability model Q_T, whose parameters are not well known. This means that the lower prevision $\underline{Q}$ has the Bayesian sensitivity analysis interpretation. If no such additional assumption is made for the variable a, then the lower prevision $\underline{P}$ does not have this interpretation. In this case, if the ideal Q_T were known, the lower prevision describing the joint information would be given by $\underline{P} \times_{\text{INE}} Q_T$, taking into account the epistemic independence of a and b, and the appropriate product of $\underline{P}$ and $\underline{Q}$ is then given by

$$\inf_{Q \in \mathcal{M}(\underline{Q})} (\underline{P} \times_{\text{INE}} Q)(Z) = \inf_{Q \in \mathcal{M}(\underline{Q})} \underline{E}(Q(Z)) = \inf_{Q \in \mathcal{M}(\underline{Q})} \inf_{P \in \mathcal{M}(\underline{P})} P(Q(Z)) = \left(\underline{P} \times_{\text{TI}} \underline{Q}\right)(Z)$$

for any gamble Z, taking into account Theorem 2. In other words, as soon as the lower prevision for at least one of the two epistemically independent variables a and b has the Bayesian sensitivity analysis interpretation, the appropriate product to use is the *type-I product*. If none of the models for these variables has the Bayesian sensitivity analysis interpretation, i.e., if no additional assumption is made that their model is precise but not well known, we should use the *independent natural extension* to form independent products. Fortunately, since it will turn out that at least one of the lower previsions in the problem set is always of the vacuous type, we may deduce from Theorem 1 that both types of independent products coincide, and we need therefore in the rest of this paper not really be concerned with these subtleties of interpretation.

4. INFERENCE

In our line of reasoning so far, we have taken the necessary steps to ensure that we can model the available information about the parameters a and b, or actually their joint value (a, b), by a coherent lower prevision $\underline{P}$ on some linear space $\mathscr{F}$ of gambles on $\mathscr{A} \times \mathscr{B}$. The final step to take is the transformation of $\underline{P}$ into a coherent lower prevision $\underline{P}_y$ on the set of possible values $\mathscr{Y}$ of the parameter y, using the functional relationship $y = f(a, b)$ between (a, b) and y.

This can be achieved quite easily using the following heuristic course of reasoning. Consider a gamble U on $\mathscr{Y}$. If (a, b) assumes the value (α, β) in $\mathscr{A} \times \mathscr{B}$, then y assumes the value $f(\alpha, \beta)$, and consequently U assumes the value $U(f(\alpha, \beta))$. This means that the gamble U on $\mathscr{Y}$ can be interpreted as a gamble $U \circ f$ on $\mathscr{A} \times \mathscr{B}$, whose lower prevision is $\underline{P}(U \circ f)$, provided that $U \circ f$ belongs to the linear space

$$\mathscr{F}_y = \{U \in \mathscr{L}(\mathscr{Y}) \colon U \circ f \in \mathscr{F}\}$$

of gambles on $\mathscr{Y}$. This leads immediately to the definition of the lower prevision $\underline{P}_y(\cdot) = \underline{P}(\cdot \circ f)$ on $\mathscr{F}_y$.

This course of reasoning can be given more weight by arguments of coherence. If we know that the variable (a, b) assumes the value (α, β), then it is absolutely certain that the variable y assumes the value $f(\alpha, \beta)$, and this can be modelled by a degenerate linear prevision all of whose probability mass lies in $f(\alpha, \beta)$: for any gamble U on $\mathscr{Y}$,

$$P(U|\alpha, \beta) = U(f(\alpha, \beta)).$$

The functional relationship f between (a, b) and y can therefore be represented by the *conditional* linear previsions $P(\cdot|\alpha, \beta)$ for all (α, β) in $\mathscr{A} \times \mathscr{B}$, or with the notation established in Section 2.5.2, by the conditional linear prevision $P(\cdot|\mathscr{A} \times \mathscr{B})$ defined on the set $\mathscr{L}(\mathscr{Y})$ of all gambles on $\mathscr{Y}$. Together with the prior $\underline{P}$, it leads, through natural extension (Walley's marginal extension theorem [14, Theorem 6.7.2], see also the similar course of reasoning in Section 2.5.2), to the lower prevision $\underline{P}_y$, defined on the set of gambles $\mathscr{F}_y$ by

$$\underline{P}_y(U) = \underline{P}(P(U|\mathscr{A} \times \mathscr{B})) = \underline{P}(U \circ f), \tag{9}$$

for all U in $\mathscr{F}_y$. $\underline{P}_y(U)$ is the smallest (most conservative) supremum acceptable price for buying the gamble U that can be inferred from the lower prevision $\underline{P}$ and the functional relationship f, using only arguments of coherence!

Observe that, if U is the indicator I_C of some subset C of $\mathscr{Y}$, then it is clear that $(I_C \circ f)(\alpha, \beta)$ equals one if and only if $f(\alpha, \beta) \in C$, or equivalently, $(\alpha, \beta) \in f^{-1}(C)$, and that it equals zero elsewhere:

$$I_C \circ f = I_{f^{-1}(C)},$$

which tells us that $\underline{P}_y(C) = \underline{P}(f^{-1}(C))$. In other words, Eq. (9) is the appropriate generalisation to lower previsions and to gambles of the notion of a probability measure induced by the map f!

5. SOLUTIONS TO PROBLEMS 1, 4 AND 6

We are now ready to apply the results derived so far to the solution of the problems not involving combination of assessments from different experts.

5.1. Solution to Problem 1.

We only know that a and b assume values in the respective closed intervals $A = [0.1, 1.0]$ and $B = [0.0, 1.0]$. Hence, we have the vacuous lower prevision $\underline{P}_A$ on $\mathscr{L}(\mathscr{A})$ and the vacuous lower prevision $\underline{P}_B$ on $\mathscr{L}(\mathscr{B})$. Using Proposition 1, we find that the lower prevision $\underline{P}$ representing the available information about the value of (a, b) is given by:

$$\underline{P}(Z) = (\underline{P}_A \times_{\text{INE}} \underline{P}_B)(Z) = (\underline{P}_A \times_{\text{TI}} \underline{P}_B)(Z) = \inf_{(\alpha,\beta)\in A\times B} Z(\alpha, \beta),$$

for each Z in $\mathscr{L}(\mathscr{A} \times \mathscr{B})$. Eq. (9) tells us that the lower prevision $\underline{P}_y$ representing the available information about the output y is then given by

$$\underline{P}_y(U) = \inf_{(\alpha,\beta)\in A\times B} U(f(\alpha, \beta)),$$

for all gambles U on $\mathscr{Y}$. This lower prevision can be used as a starting point for further inference and decision problems involving the value of the output y. As an illustration, we calculate the lower prevision $\underline{P}_y(1_{\mathscr{Y}}) = \underline{P}(f)$ and upper prevision $\overline{P}_y(1_{\mathscr{Y}}) = \overline{P}(f)$ of the output y, where $1_{\mathscr{Y}}$ is the identity map on $\mathscr{Y}$.[4] We get

$$\underline{P}(f) = 0.692201 \text{ and } \overline{P}(f) = 2.$$

5.2. Solution to Problem 4.

We know that a assumes a value in the closed interval $A = [0.1, 1.0]$, and that b is log-normally distributed, $\ln b \sim N(\mu, \sigma)$, with parameters $\mu \in M = [0.0, 1.0]$ and $\sigma \in S = [0.1, 0.5]$. Hence, we have as appropriate models the vacuous lower prevision $\underline{P}_A$ on $\mathscr{L}(\mathscr{A})$ and, as argued in Section 2.5, a lower envelope over M and S:

$$\underline{Q}(X) = \inf_{(\mu,\sigma)\in M\times S} \int_{\mathscr{B}} X\phi_{\mu,\sigma}\mathrm{d}\lambda = \inf_{\mu\in M}\inf_{\sigma\in S} \int_{\mathscr{B}} X\phi_{\mu,\sigma}\mathrm{d}\lambda,$$

for each integrable gamble X on $\mathscr{B}$, with $\phi_{\mu,\sigma}$ the log-normal distribution and λ the Lebesgue measure on the reals. Since the model $\underline{Q}$ for the parameter b has the Bayesian sensitivity analysis interpretation, we have argued in Section 3.5 that

[4] The reader will perhaps object that $1_{\mathscr{Y}}$ is not bounded, and therefore not a gamble. But we have shown elsewhere [12, 13] that this difficulty can be circumvented, and we shall not go any deeper into this matter here.

the appropriate product to use is the type-I product. But since $\underline{P}_A$ is vacuous, this product coincides with the independent natural extension, by Theorem 1, and it is given by

$$\underline{P}(Z) = \left(\underline{P}_A \times_{\text{TI}} \underline{Q}\right)(Z) = \left(\underline{P}_A \times_{\text{INE}} \underline{Q}\right)(Z) = \inf_{\alpha\in A}\inf_{\mu\in M}\inf_{\sigma\in S}\int_{\mathscr{B}} Z(\alpha,\cdot)\phi_{\mu,\sigma}\mathrm{d}\lambda,$$

for each gamble $Z \in \mathscr{L}(\mathscr{B})$ such that $Z(\alpha,\cdot)$ is integrable for every $\alpha \in \mathscr{A}$. The lower prevision $\underline{P}_y$ representing the available information about the output y is then given by

$$\underline{P}_y(U) = \inf_{\alpha\in A}\inf_{\mu\in M}\inf_{\sigma\in S}\int_{\mathscr{B}} U\left(f(\alpha,\cdot)\right)\phi_{\mu,\sigma}\mathrm{d}\lambda,$$

for all gambles U on $\mathscr{Y}$ such that $U\left(f(\alpha,\cdot)\right)$ is integrable for all $\alpha \in \mathscr{A}$. In particular, using the monotonicity of the integral $\int_{\mathscr{B}} f(\alpha,\cdot)\phi_{\mu,\sigma}\mathrm{d}\lambda$ with respect to α, μ and σ, we easily find the following values for the lower and the upper prevision of the output y:

$$\underline{P}(f) = 1.00966 \text{ and } \overline{P}(f) = 4.08022.$$

5.3. **Solution to Problem 6.** We know that a assumes a value in the closed interval $A = [0.1, 1.0]$, and that b is log-normally distributed, $\ln b \sim N(\mu,\sigma)$, with parameters $\mu = 0.5$ and $\sigma = 0.5$. Hence, we have as appropriate models the vacuous lower prevision $\underline{P}_A$ on $\mathscr{L}(\mathscr{A})$ and the linear prevision

$$Q(X) = \int_{\mathscr{B}} X\phi_{\mu,\sigma}\mathrm{d}\lambda,$$

for each integrable gamble X on $\mathscr{B}$, with $\phi_{\mu,\sigma}$ the log-normal distribution and λ the Lebesgue measure on the reals. By Theorem 1, or alternatively by Theorem 2, the independent natural extension and the type-I product of $\underline{P}_A$ and Q coincide, and the lower prevision $\underline{P}$ representing the available information about the parameters (a, b) is given by:

$$\left(\underline{P}_A \times_{\text{INE}} Q\right)(Z) = \left(\underline{P}_A \times_{\text{TI}} Q\right)(Z) = \inf_{\alpha\in A}\int_{\mathscr{B}} Z(\alpha,\cdot)\phi_{\mu,\sigma}\mathrm{d}\lambda$$

for each gamble $Z \in \mathscr{L}(\mathscr{B})$ such that $Z(\alpha,\cdot)$ is integrable for every $\alpha \in \mathscr{A}$. The available information about the output y is modelled by the lower prevision $\underline{P}_y$, where

$$\underline{P}_y(U) = \inf_{\alpha\in A}\int_{\mathscr{B}} U\left(f(\alpha,\cdot)\right)\phi_{\mu,\sigma}\mathrm{d}\lambda$$

for all gambles U on $\mathscr{Y}$ such that $U\left(f(\alpha,\cdot)\right)$ is integrable for all α in $\mathscr{A}$. In particular, using the monotonicity of $\int_{\mathscr{B}} f(\alpha,\cdot)\phi_{\mu,\sigma}\mathrm{d}\lambda$ with respect to α, we find for the lower and upper prevision of the output y:

$$\underline{P}(f) = 1.05939 \text{ and } \overline{P}(f) = 2.86825.$$

6. Combination of assessments

Whereas Problems 1, 4 and 6 involve only a single assessment for each parameter, problems 2, 3 and 5 also involve different and possibly conflicting assessments about the same parameter. In order to solve these problems, we now aim at finding ways of combining *multiple* lower previsions defined on gambles on the same parameter space into a single lower prevision. To arrive at a joint uncertainty model for all the parameters we can combine lower previsions on each parameter separately and apply one of the product rules described in Section 3. We could alternatively first construct all products between lower previsions on different parameters, and then combine these products. In general, even more scenarios are possible.

However, the order in which we apply combination and product rules is not prescribed, and we therefore should demand that this order has no influence on the final result. We call this the *order of combination invariance principle*: the joint should not depend on the order in which combinations and products are applied. Our approach satisfies this principle (see Propositions 4, 5, 6 and 7).

In order to fix terminology and notation, suppose we have n (male) agents, called the *experts*. Their assessments about the value that a parameter ω assumes in a set of possible values Ω are expressed through coherent lower previsions $\underline{P}_k$ on some subset $\mathscr{K}_k$ of $\mathscr{L}(\Omega)$, for $k = 1,\dots,n$. We show how these lower previsions can be combined into a single coherent lower prevision defined on the set of gambles $\mathscr{L}(\Omega)$.

6.1. **Consistency and conjunction.** Consider a new (female) agent, called the *modeller*. She wishes to aggregate all the assessments $\underline{P}_k$ to a single lower prevision $\underline{P}_M$ defined on $\mathscr{L}(\Omega)$. Say that she *trusts* an expert's assessment $\underline{P}_k$ whenever she is willing to accept every decision he makes, that is, whenever she is willing to accept his price s for buying X as her price for buying X, and this for each gamble $X \in \mathscr{K}_k$ and each buying price $s < \underline{P}_k(X)$. We immediately have the following result.

Theorem 3. *The modeller trusts an expert's assessment $\underline{P}_k$ if and only if her coherent lower prevision $\underline{P}_M$ point-wise dominates $\underline{P}_k$ on its domain $\mathscr{K}_k$.*

This leads to the notion of *consistency* of the expert assessments $\underline{P}_k$.

Definition 2. *If there is at least one coherent lower prevision that point-wise dominates the coherent lower previsions $\underline{P}_k$ on their respective domains $\mathscr{K}_k$ for $k = 1,\dots,n$, then these lower previsions $\underline{P}_k$ are called* consistent.

It is not difficult to show that if the $\underline{P}_k$ are consistent, then there is (point-wise) smallest coherent lower prevision that dominates all the $\underline{P}_k$ on their respective domains $\mathscr{K}_k$. This coherent lower prevision is called the *conjunction* of the $\underline{P}_k$ and is denoted by $\sqcap_{k=1}^n \underline{P}_k$. By Theorem 3, it is the smallest, or most conservative, coherent lower prevision on $\mathscr{L}(\Omega)$ that the modeller can have such that she still trusts each of the experts. The conjunction of two consistent coherent lower previsions $\underline{P}_1$ and $\underline{P}_2$ is also denoted by $\underline{P}_1 \sqcap \underline{P}_2$. By the theorem given below, the conjunction is an associative and commutative operator, and hence, it satisfies the order of combination invariance principle.

Theorem 4. *Consider, for each of the coherent lower previsions $\underline{P}_k$ defined on the set $\mathscr{K}_k$, the set of dominating linear previsions*

$$\mathcal{M}(\underline{P}_k) = \{P \in \mathcal{P}(\Omega) \colon (\forall X \in \mathscr{K}_k)(\underline{P}_k(X) \leq \underline{P}(X))\}.$$

Then the $(\underline{P}_k)_{k=1}^n$ are consistent if and only if

$$\bigcap_{k=1}^n \mathcal{M}(\underline{P}_k) \neq \emptyset.$$

In that case, the conjunction $\sqcap_{k=1}^n \underline{P}_k$ is the lower envelope of this intersection: for all gambles X on Ω,

$$\left(\sqcap_{k=1}^n \underline{P}_k\right)(X) = \inf \left\{ P(X) \colon P \in \bigcap_{k=1}^n \mathcal{M}(\underline{P}_k) \right\}.$$

Proof. The first part of the theorem is immediate if we recall that a coherent lower prevision is always dominated by some linear prevision, and that a linear prevision is in particular a coherent lower prevision.

To prove the second part, assume that the $(\underline{P}_k)_{k=1}^n$ are consistent. Then it suffices to show that

$$\mathcal{M}(\sqcap_{k=1}^n \underline{P}_k) = \bigcap_{k=1}^n \mathcal{M}(\underline{P}_k).$$

Assume that the linear prevision P belongs to $\bigcap_{k=1}^n \mathcal{M}(\underline{P}_k)$. Then it dominates all of the $\underline{P}_k$ on their domains, and therefore, since it is in particular a coherent lower prevision, it also dominates smallest coherent lower prevision $\sqcap_{k=1}^n \underline{P}_k$ that dominates all of the $\underline{P}_k$ on their domains. So P belongs to $\mathcal{M}(\sqcap_{k=1}^n \underline{P}_k)$.

Conversely, assume that P belongs to $\mathcal{M}(\sqcap_{k=1}^n \underline{P}_k)$, i.e., it dominates $\sqcap_{k=1}^n \underline{P}_k$. Since $\sqcap_{k=1}^n \underline{P}_k$ dominates all of the $\underline{P}_k$ on their domains, so does P, and consequently P is an element of $\bigcap_{k=1}^n \mathcal{M}(\underline{P}_k)$. □

Theorem 5. *Assume that the domain $\mathscr{K}_k$ of the coherent lower prevision $\underline{P}_k$ is a linear space for each $k \in \{1, \ldots, n\}$. Consider the map $\underline{E}$ assigning to each gamble Z on Ω the (possibly infinite) real number*

$$\underline{E}(Z) = \sup_{X_k \in \mathscr{L}(\Omega)} \left\{ \alpha \in \mathbb{R} \colon Z - \alpha \geq \sum_{k=1}^n [X_k - \underline{P}_k(X_k)] \right\}, \tag{10}$$

If $\underline{E}(Z) = +\infty$ for some Z (and hence for all Z), then the lower previsions $(\underline{P}_k)_{k=1}^n$ are inconsistent. Otherwise, the conjunction $\sqcap_{k=1}^n \underline{P}_k$ coincides with $\underline{E}$ on all gambles on Ω.

Proof. We first show that $\underline{E}$ dominates $\underline{P}_k$ on $\mathscr{K}_k$, for all $k \in \{1, \ldots, n\}$. To see this, let $Z \in \mathscr{K}_k$ and let $X_k = Z$ and $X_\ell = 0$ for all $\ell \in \{1, \ldots, n\} \setminus \{k\}$ in Eq. (10).

Next, let $\underline{F}$ be another coherent lower prevision that dominates each $\underline{P}_k$ on its domain $\mathscr{K}_k$. Then we show that $\underline{F}$ also dominates $\underline{E}$. Indeed, for any gamble Z on Ω,

$$\begin{aligned}
\underline{E}(Z) &= \sup_{X_k \in \mathscr{K}_k} \left\{ \alpha \in \mathbb{R} \colon Z - \alpha \geq \sum_{k=1}^n [X_k - \underline{P}_k(X_k)] \right\} \\
&\leq \sup_{X_k \in \mathscr{K}_k} \left\{ \alpha \in \mathbb{R} \colon \alpha \leq \underline{F}\left(Z - \sum_{k=1}^n [X_k - \underline{P}_k(X_k)] \right) \right\} \\
&\leq \sup_{X_k \in \mathscr{K}_k} \left[\underline{F}(Z) - \sum_{k=1}^n \underline{F}\left(X_k - \underline{P}_k(X_k)\right) \right] \\
&\leq \underline{F}(Z),
\end{aligned}$$

where we used the coherence of $\underline{F}$ and the fact that $\underline{F}(X_k) \geq \underline{P}(X_k)$ for all $X_k \in \mathscr{K}_k$.

It is now easily checked that provided that $\underline{E}$ is everywhere finite, it is a coherent lower prevision, and therefore coincides with the conjunction.

To complete the proof, assume first that the conjunction exists. Denote this conjunction by $\underline{F}$. Then $\underline{F}$ is coherent and dominates each $\underline{P}_i$. But then $\underline{F}$ must dominate $\underline{E}$ too, as we already showed before. Hence, if the conjunction exists, $\underline{E}(Z) \leq \underline{F}(Z) \neq +\infty$ for all gambles Z on Ω, or equivalently, if $\underline{E}(Z) = +\infty$ for some gamble Z on Ω then the conjunction does not exist.

Finally, if $\underline{E}(Z) = +\infty$ for some gamble Z on Ω, then it must hold that

$$(\forall \alpha \in \mathbb{R})(\exists X_k \in \mathscr{K}_k) \left(Z - \alpha \geq \sum_{k=1}^n [X_k - \underline{P}_k(X_k)] \right).$$

Now, observe that since gambles are is bounded, this condition is equivalent to

$$(\forall\alpha\in\mathbb{R})(\exists X_k\in\mathscr{K}_k)\left(Y-\alpha\geq\sum_{k=1}^{n}\left[X_k-\underline{P}_k(X_k)\right]\right),$$

for any other gamble Y on Ω. Hence, it holds indeed that $\underline{E}(Z)=+\infty$ for some gamble Z if and only if $\underline{E}(Y)=+\infty$ for any gamble Y. □

If the consistent coherent lower previsions $\underline{P}_k$ have a common domain $\mathscr{K}$ then the conjunction $\underline{E}$ can also be derived as the natural extension to $\mathscr{L}(\mathscr{A}\times\mathscr{B})$ of the lower prevision $\underline{P}_C$ defined on $\mathscr{K}$ by

$$\underline{P}_C(X)=\max_{k=1}^{n}\underline{P}_k(X),\ \text{for all } X\in\mathscr{K}.$$

Observe that the $\underline{P}_k$ are consistent if and only if $\underline{P}_C$ avoids sure loss.

Let us now take a closer look at a number of interesting special cases.

6.2. **Conjunction of vacuous lower previsions.** If the modeller trusts all the experts' assessments '$\omega\in T_1$', …, '$\omega\in T_n$' then she should at least conclude that '$\omega\in\bigcap_{i\in I}T_i$'. This is the essence of the following proposition.

Proposition 2. *Consider the non-empty subsets T_k, $k=1,\dots,n$ of Ω, and the associated vacuous lower previsions $\underline{P}_{T_k}$ on $\mathscr{L}(\Omega)$. Let $T=\bigcap_{k=1}^{n}T_k$. Then the $\underline{P}_{T_k}$ are consistent if and only if $T\neq\emptyset$. If the $\underline{P}_{T_k}$ are consistent, then their conjunction is equal to the vacuous lower prevision $\underline{P}_T$ on Ω relative to the intersection T.*

Proof. First, assume that $T=\emptyset$. Then the $\underline{P}_{T_k}$ cannot be consistent. Indeed, assume that there is a coherent lower prevision $\underline{R}$ that point-wise dominates them. Since $\bigcup_{i\in I}\complement T_i=\Omega$, it follows from the coherence of $\underline{R}$ that

$$-1=\underline{R}(-1)=\underline{R}\left(-\sum_{i\in I}I_{\complement T_i}\right)\geq\sum_{i\in I}\underline{R}\left(-I_{\complement T_i}\right)\geq\sum_{i\in I}\underline{P}_{T_i}\left(-I_{\complement T_i}\right)=0,$$

a contradiction. This means that the $\underline{P}_{T_k}$ are inconsistent.

Conversely, assume that $T\neq\emptyset$. Then $\underline{P}_T$ is a coherent lower prevision, and it point-wise dominates all the $\underline{P}_{T_k}$: for each k and each X in $\mathscr{L}(\Omega)$,

$$\underline{P}_T(X)=\inf_{\omega\in T}X(\omega)\geq\inf_{\omega\in T_k}X(\omega)=\underline{P}_{T_k}(X).$$

This means that the $\underline{P}_{T_k}$ are consistent. Now let $\underline{R}$ be any coherent lower prevision on Ω that dominates all the $\underline{P}_{T_k}$. It now only remains to show that $\underline{R}\geq\underline{P}_T$. Indeed, let X be any gamble. Then it is always possible to find $\lambda_k\geq 0$ such that

$$\inf\left[X+\sum_{k=1}^{n}\lambda_k\,\complement T_k\right]=\min\left\{\inf_{\omega\in T}X(\omega),\min_{k=1}^{n}\left[\lambda_k+\inf_{\omega\in\complement T_k}X(\omega)\right]\right\}=\inf_{\omega\in T}X(\omega)=\underline{P}_T(X).$$

Consequently, taking into account the coherence of $\underline{R}$, and the fact that $\underline{R}\left(-I_{\complement T_k}\right)\geq\underline{P}_{T_k}\left(-I_{\complement T_k}\right)=0$, whence also $\underline{R}\left(-\sum_{k=1}^{n}\lambda_k I_{\complement T_k}\right)\geq 0$, and consequently

$$\begin{aligned}\underline{R}(X)&\geq\underline{R}(X)-\underline{R}\left(-\sum_{k=1}^{n}\lambda_k I_{\complement T_k}\right)=\underline{R}(X)+\overline{R}\left(\sum_{k=1}^{n}\lambda_k I_{\complement T_k}\right)\\&\geq\underline{R}\left(X+\sum_{k=1}^{n}\lambda_k I_{\complement T_k}\right)\geq\inf\left[X+\sum_{k=1}^{n}\lambda_k I_{\complement T_k}\right]=\underline{P}_T(X).\end{aligned}$$

Thus $\underline{P}_T$ is indeed the smallest coherent lower prevision that point-wise dominates all of the $\underline{P}_k$. □

Proposition 3. *Let $\underline{P}$ be any coherent lower prevision with domain $\mathscr{K}$ and let $\underline{P}_\Omega$ the vacuous lower prevision relative to Ω. Then $\underline{P}\sqcap\underline{P}_\Omega=\underline{E}$, where $\underline{E}$ is the natural extension of $\underline{P}$ to $\mathscr{L}(\Omega)$.*

Proof. The proof is immediate if we recall that any coherent lower prevision $\underline{P}$ point-wise dominates $\underline{P}_\Omega$ on its domain $\mathscr{K}$. □

6.3. **Products of conjunctions.** The conjunction distributes over the two independent products we have defined previously. So, using conjunction to combine expert assessments on common domains, and independent natural extension or type-I product to combine expert assessments on different domains, the order of combination invariance principle is satisfied.

Proposition 4. *For each $k=1,\dots,n$ and each $\ell=1,\dots,m$, let $\underline{P}_k$ be a coherent lower prevision defined on the linear subspace $\mathscr{F}_k$ of $\mathscr{L}(\mathscr{A})$ and let $\underline{Q}_\ell$ be a coherent lower prevision defined on the linear subspace $\mathscr{G}_\ell$ of $\mathscr{L}(\mathscr{B})$. Then consistency of $(\underline{P}_k\times_{\mathrm{INE}}\underline{Q}_\ell)_{k,\ell}$ is equivalent to consistency of both $(\underline{P}_k)_{k=1}^{n}$ and $(\underline{Q}_\ell)_{\ell=1}^{m}$, and in such a case the following equality holds:*

$$\sqcap_{k,\ell}\left(\underline{P}_k\times_{\mathrm{INE}}\underline{Q}_\ell\right)=\left(\sqcap_{k=1}^{n}\underline{P}_k\right)\times_{\mathrm{INE}}\left(\sqcap_{\ell=1}^{m}\underline{Q}_\ell\right).\tag{11}$$

Proof. Let us, for convenience, define the *inconsistent lower prevision* on Ω as $\underline{I}(Z) = +\infty$ for all gambles Z on Ω. In this way, the conjunction is always defined and the formula for conjunction, Eq. (10), always holds, whether the assessments are consistent or not (if they are not, it is equal to $\underline{I}$). Also let the independent natural extension be the inconsistent lower prevision whenever at least one of the factors is the inconsistent lower prevision. Then both sides of Eq. (11) are well defined and if we can prove equality, equivalence of consistency follows naturally.

Let $\underline{P}$ denote the conjunction of $(\underline{P}_k)_{k=1}^n$ (i.e., the usual conjunction if they are consistent, and $+\infty$ otherwise). Similarly, let $\underline{Q}$ denote the conjunction of $(\underline{Q}_\ell)_{\ell=1}^n$.

Let Z be any gamble on $\Omega = \mathscr{A} \times \mathscr{B}$. Using the definition of independent natural extension (Eq. (6)), we have that

$$(12)\qquad \left(\underline{P} \times_{\text{INE}} \underline{Q}\right)(Z) = \sup_{X,\,Y \in \mathscr{L}(\Omega)} \left\{\gamma \in \mathbb{R} \colon Z - \gamma \geq X - \underline{P}(X) + Y - \underline{Q}(Y)\right\}$$

Note that this equation holds whether both $(\underline{P}_k)_{k=1}^n$ and $(\underline{Q}_\ell)_{\ell=1}^m$ are consistent or not because the right hand side is $+\infty$ (independently of Z) in case of inconsistency, conforming with our definition of the inconsistent lower prevision.

From the formula for conjunction (Eq. (10)), we know that, for any $X \in \mathscr{L}(\Omega)$, and any $\beta \in \mathscr{B}$,

$$(13)\qquad \underline{P}(X(\cdot,\beta)) = \sup_{X_k \in \mathscr{F}_k} \left\{\eta \in \mathbb{R} \colon X(\cdot,\beta) - \eta \geq \sum_{k=1}^n [X_k - \underline{P}_k(X_k)]\right\}$$

Again, this holds whether the $(\underline{P}_k)_{k=1}^n$ are consistent or not. We now consider two cases.

(1) If both $(\underline{P}_k)_{k=1}^n$ and $(\underline{Q}_\ell)_{\ell=1}^m$ are consistent, it is instructive to rewrite Eq. (13) as follows:

(14)

$$(\forall X \in \mathscr{L}(\Omega))(\forall \beta \in \mathscr{B})(\forall \epsilon > 0)(\forall k)(\exists U_{k,X,\beta,\epsilon} \in \mathscr{F}_k)\left(X(\cdot,\beta) - \underline{P}(X(\cdot,\beta)) + \epsilon \geq \sum_{k=1}^n [U_{k,X,\beta,\epsilon} - \underline{P}_k(U_{k,X,\beta,\epsilon})]\right)$$

Using (14), and a similar expression for $\underline{Q}(Y(a,\cdot))$, we find that

$$(15)\qquad (\exists X, Y \in \mathscr{L}(\Omega))(Z - \gamma \geq X - \underline{P}(X) + Y - \underline{Q}(Y))$$

implies that

$$(16)\qquad (\forall \epsilon > 0)(\exists X_k \in \overline{\mathscr{F}}_k)(\exists Y_\ell \in \overline{\mathscr{G}}_\ell)\left(Z - \gamma \geq \sum_{k=1}^n [X_k - \underline{P}_k(X_k)] - \epsilon + \sum_{\ell=1}^m \left[Y_\ell - \underline{Q}_\ell(Y_\ell)\right] - \epsilon\right),$$

for any gamble Z on Ω and any $\gamma \in \mathbb{R}$. From this implication, (15) $\Longrightarrow$ (16), we may infer that the right hand side of (12) is less than or equal to

$$(17)\qquad 2\epsilon + \sup_{\substack{X_k \in \overline{\mathscr{F}}_k \\ Y_\ell \in \overline{\mathscr{G}}_\ell}} \left\{\gamma \in \mathbb{R} \colon Z - \gamma \geq \sum_{k=1}^n [X_k - \underline{P}_k(X_k)] + \sum_{\ell=1}^m \left[Y_\ell - \underline{Q}_\ell(Y_\ell)\right]\right\},$$

for every $\epsilon > 0$. Now also observe that

$$(18)\qquad (\exists X_k \in \overline{\mathscr{F}}_k)(\exists Y_\ell \in \overline{\mathscr{G}}_\ell)\left(Z - \gamma \geq \sum_{k=1}^n [X_k - \underline{P}_k(X_k)] + \sum_{\ell=1}^m \left[Y_\ell - \underline{Q}_\ell(Y_\ell)\right]\right)$$

implies (15). Indeed, take $X = \sum_{k=1}^n X_k$ and $Y = \sum_{k=1}^n Y_\ell$, and use the fact that $\underline{P} \geq \underline{P}_k$ on $\overline{\mathscr{F}}_k$ and $\underline{Q} \geq \underline{Q}_\ell$ on $\overline{\mathscr{G}}_\ell$, e.g.:

$$\begin{aligned}\sum_{k=1}^n [X_k - \underline{P}_k(X_k)] &= X - \sum_{k=1}^n \underline{P}_k(X_k) \geq X - \sum_{k=1}^n \underline{P}(X_k) \\ &\geq X - \underline{P}\left(\sum_{k=1}^n X_k\right) = X - \underline{P}(X).\end{aligned}$$

From this implication, (18) $\Longrightarrow$ (15), we may infer that the right hand side of (12) is greater or equal to

$$(19)\qquad \sup_{\substack{X_k \in \overline{\mathscr{F}}_k \\ Y_\ell \in \overline{\mathscr{G}}_\ell}} \left\{\gamma \in \mathbb{R} \colon Z - \gamma \geq \sum_{k=1}^n [X_k - \underline{P}_k(X_k)] + \sum_{\ell=1}^m \left[Y_\ell - \underline{Q}_\ell(Y_\ell)\right]\right\}.$$

Thus, since that the right hand side of (12) is less or equal to (17) for every $\epsilon > 0$, and greater or equal to (19), we must conclude that it is equal to (19).

(2) In case that the $(\underline{P}_k)_{k=1}^n$ are inconsistent, we may rewrite Eq. (13) as

$$(\forall \beta \in \mathscr{B})(\forall \eta \in \mathbb{R})(\forall k)(\exists U_{k,X,\beta,\eta} \in \mathscr{F}_k)\left(Z(\cdot,\beta) - \eta \geq \sum_{k=1}^n [U_{k,X,\beta,\eta} - \underline{P}_k(U_{k,X,\beta,\eta})]\right).$$

In particular, taking $X_k(\cdot,\beta) = U_{k,X,\beta,\eta}$ and $Y_\ell = 0$, we find that

$$(\forall \gamma \in \mathbb{R})(\forall k)(\exists X_k \in \overline{\mathscr{F}}_k)(\forall \ell)(\exists Y_\ell \in \overline{\mathscr{G}}_\ell)\left(Z - \gamma \geq \sum_{k=1}^n [X_k - \underline{P}_k(X_k)] + \sum_{\ell=1}^m \left[Y_\ell - \underline{Q}_\ell(Y_\ell)\right]\right).$$

Hence, in this case, (19) is equal to $+\infty$, which is also equal to the right hand side of (12).

(3) A similar argument shows that the equality also holds if the $(\underline{Q}_\ell)_{\ell=1}^m$ are inconsistent.

Hence, we showed that, in all cases, (19) is equal to the right hand side of (12)

Now, by the definition of independent natural extension, equation (6), it holds that

$$\left(\underline{P}_k \times_{\text{INE}} \underline{Q}_\ell\right)(Z_{k\ell}) = \sup_{\substack{X \in \overline{\mathscr{F}}_k \\ Y \in \overline{\mathscr{G}}_\ell}} \left\{\gamma_{k\ell} \in \mathbb{R} \colon Z_{k\ell} - \gamma_{k\ell} \geq X - \underline{P}_k(X) + Y - \underline{Q}_\ell(Y)\right\}.$$

for any gamble $Z_{k\ell}$ on Ω. It is instructive to rewrite this equality as follows:

$$(\forall \epsilon > 0)(\exists U_\epsilon \in \overline{\mathscr{F}}_k)(\exists V_\epsilon \in \overline{\mathscr{G}}_\ell)\left(Z_{k\ell} - \left(\underline{P}_k \times_{\text{INE}} \underline{Q}_\ell\right)(Z_{k\ell}) + \epsilon \geq U_\epsilon - \underline{P}_k(U_\epsilon) + V_\epsilon - \underline{Q}_\ell(V_\epsilon)\right). \tag{20}$$

Using (20) we find that

$$(\exists Z_{k\ell} \in \mathscr{L}(\Omega))\left(Z - \gamma \geq \sum_{k,\ell}\left[Z_{k\ell} - \left(\underline{P}_k \times_{\text{INE}} \underline{Q}_\ell\right)(Z_{k\ell})\right]\right) \tag{21}$$

implies that

$$(\forall \epsilon > 0)(\exists X_{k\ell} \in \overline{\mathscr{F}}_k)(\exists Y_{k\ell} \in \overline{\mathscr{G}}_\ell)\left(Z - \gamma \geq \sum_{k,\ell}\left[X_{k\ell} - \underline{P}_k(X_{k\ell}) + Y_{k\ell} - \underline{Q}_\ell(Y_{k\ell}) - \epsilon\right]\right),$$

which implies that

$$(\forall \epsilon > 0)(\exists X_k \in \overline{\mathscr{F}}_k)(\exists Y_k \in \overline{\mathscr{G}}_\ell)\left(Z - \gamma + nm\epsilon \geq \sum_{k=1}^n [X_k - \underline{P}_k(X_k)] + \sum_{\ell=1}^m \left[Y_\ell - \underline{Q}_\ell(Y_\ell)\right]\right). \tag{22}$$

Indeed, take $X_k = \sum_{\ell=1}^m X_{k\ell}$ and $Y_\ell = \sum_{k=1}^n Y_{k\ell}$, and use the fact that, e.g.,

$$\sum_{\ell=1}^m [X_{k\ell} - \underline{P}_k(X_{k\ell})] = X_k - \sum_{\ell=1}^m \underline{P}_k(X_{k\ell}) \geq X_k - \underline{P}_k\left(\sum_{\ell=1}^m X_{k\ell}\right) = X_k - \underline{P}_k(X_k)$$

Now, the implication (21) $\Longrightarrow$ (22) implies that (19) is greater or equal to

$$-nm\epsilon + \sup_{Z_{k\ell} \in \mathscr{L}(\Omega)}\left\{\gamma \in \mathbb{R} \colon Z - \gamma \geq \sum_{k,\ell}\left[Z_{k\ell} - \left(\underline{P}_k \times_{\text{INE}} \underline{Q}_\ell\right)(Z_{k\ell})\right]\right\} \tag{23}$$

for any $\epsilon > 0$. Also

$$(\exists X_k \in \overline{\mathscr{F}}_k)(\exists Y_\ell \in \overline{\mathscr{G}}_\ell)\left(Z - \gamma \geq \sum_{k=1}^n [X_k - \underline{P}_k(X_k)] + \sum_{\ell=1}^m \left[Y_\ell - \underline{Q}_\ell(Y_\ell)\right]\right) \tag{24}$$

implies (21). Indeed, taking $Z_{k\ell} = \frac{1}{m}(X_k - \underline{P}_k(X_k)) + \frac{1}{n}\left(Y_\ell - \underline{Q}_\ell(Y_\ell)\right)$ we find that

$$\left(\underline{P}_k \times_{\text{INE}} \underline{Q}_\ell\right)(Z_{k\ell}) \geq \frac{1}{m}\left(\underline{P}_k \times_{\text{INE}} \underline{Q}_\ell\right)(X_k - \underline{P}_k(X_k)) + \frac{1}{n}\left(\underline{P}_k \times_{\text{INE}} \underline{Q}_\ell\right)(Y_\ell - \underline{Q}_\ell(Y_\ell)) \geq 0.$$

The last inequality follows from the fact that, e.g.,

$$\begin{aligned}\left(\underline{P}_k \times_{\text{INE}} \underline{Q}_\ell\right)(X_k - \underline{P}_k(X_k)) &= \sup_{\substack{X \in \overline{\mathscr{F}}_k \\ Y \in \overline{\mathscr{G}}_\ell}} \left\{\eta \in \mathbb{R} \colon X_k - \underline{P}_k(X_k) - \eta \geq X - \underline{P}_k(X) + Y - \underline{Q}_\ell(Y)\right\} \\ &\geq \sup\left\{\eta \in \mathbb{R} \colon X_k - \underline{P}_k(X_k) - \eta \geq X_k - \underline{P}_k(X_k)\right\} = 0.\end{aligned}$$

Hence, for this choice of $Z_{k\ell}$ we have that

$$\sum_{k=1}^n [X_k - \underline{P}_k(X_k)] + \sum_{\ell=1}^m \left[Y_\ell - \underline{Q}_\ell(Y_\ell)\right] = \sum_{k,\ell} Z_{kl} \geq \sum_{k,\ell}\left[Z_{k\ell} - \left(\underline{P}_k \times_{\text{INE}} \underline{Q}_\ell\right)(Z_{k\ell})\right],$$

which establishes (24) $\Longrightarrow$ (21). This implication implies that (19) is less or equal to

$$\sup_{Z_{k\ell} \in \mathscr{L}(\Omega)}\left\{\gamma \in \mathbb{R} \colon Z - \gamma \geq \sum_{k,\ell}\left[Z_{k\ell} - \left(\underline{P}_k \times_{\text{INE}} \underline{Q}_\ell\right)(Z_{k\ell})\right]\right\} \tag{25}$$

Thus, if (19) is greater or equal to (23) for all $\epsilon > 0$ and less or equal to (25), then it must be equal to (25). But (25) is exactly equal to $\sqcap_{k,\ell}\left(\underline{P}_k \times_{\text{INE}} \underline{Q}_\ell\right)(Z)$, no matter whether the $(\underline{P}_k \times_{\text{INE}} \underline{Q}_\ell)_{k,\ell}$ are consistent or not. □

Proposition 5. *Let $\underline{P}_k$ be a coherent lower prevision defined on a linear subspace $\mathscr{F}_k$ of $\mathscr{L}(\mathscr{A})$, $k = 1, \dots, n$, and let $\underline{Q}_\ell$ be a coherent lower previsions defined on a linear subspace $\mathscr{G}_\ell$ of $\mathscr{L}(\mathscr{B})$, $\ell = 1, \dots, m$. Then the $(\underline{P}_k \times_{\mathrm{TI}} \underline{Q}_\ell)_{k,\ell}$ are consistent if and only if both the $(\underline{P}_k)_{k=1}^n$ and $(\underline{Q}_\ell)_{\ell=1}^m$ are, and in such a case, the following equality holds:*

$$\sqcap_{k,\ell} \left(\underline{P}_k \times_{\mathrm{TI}} \underline{Q}_\ell \right) = \left(\sqcap_{k=1}^n \underline{P}_k \right) \times_{\mathrm{TI}} \left(\sqcap_{\ell=1}^m \underline{Q}_\ell \right). \tag{26}$$

Proof. The proof is immediate if we recall that

$$\left(\bigcap_{k=1}^n \mathcal{M}(\underline{P}_k) \right) \times \left(\bigcap_{\ell=1}^m \mathcal{M}(\underline{Q}_\ell) \right) = \bigcap_{k=1}^n \bigcap_{\ell=1}^m \left[\mathcal{M}(\underline{P}_k) \times \mathcal{M}(\underline{Q}_\ell) \right].$$

The equality holds too whenever either side is equal to the empty set; this establishes equivalence of consistency. □

6.4. **Combination of conflicting assessments.** If the assessments $(\underline{P}_k)_{k=1}^n$ are conflicting—if there exists no conjunction—then there is no coherent way to accept *every* decision of *every* expert, since the modeller incurs a sure loss if she would do so: using Theorem 5 it is easily established that in case of inconsistency there are gambles $X_k \in \mathcal{K}_k$ such that (compare with Eq. (1))

$$\sup_{\omega \in \Omega} \left[\sum_{k=1}^n [X_k(\omega) - \underline{P}_k(X_k)] \right] < 0, \tag{27}$$

i.e., the combination of the transactions in which the gambles X_k are bought for a price $\underline{P}_k(X_k)$ leads to a loss, whatever the actual value of the parameter a. Blindly accepting decisions of all the experts $(\underline{P}_k)_{k=1}^n$ is clearly unacceptable in case of inconsistency.

This issue can be resolved using hierarchical models. In order to avoid sure loss, this procedure must somehow involve the weakening of some of the decisions of at least some of the experts. This can be done for instance by only accepting particular decisions at some prescribed rate r strictly less than one. But we do not know how much the experts' decisions can be weakened.

We propose a simpler and more straightforward method, based on the principle that there should be *at least one* of the experts that provides a reasonable model. Briefly, in case of inconsistency the modeller is certain that some of the experts' assessments $(\underline{P}_k)_{k=1}^n$ cannot be trusted, but she does not necessarily know which ones. Therefore, instead of looking for an aggregate that implies trust to all experts—a conjunction—we now look for an aggregate that is trusted *by* all experts. This is reasonable if we assume that at least one of the experts has a reasonable model.

Say that an expert trusts the modeller whenever he is willing to accept every decision she makes, that is, whenever he is willing to accept her price s for buying X as his price for buying X, and this for each gamble X on Ω and each buying price $s < \underline{P}_M(X)$. We have the following result.

Theorem 6. *The modeller is trusted by an expert if and only if the natural extension $\underline{E}_k$ of his assessment $\underline{P}_k$ point-wise dominates her coherent lower prevision $\underline{P}_M$ on $\mathscr{L}(\Omega)$.*

We need to consider the natural extension $\underline{E}_k$ because $\underline{P}_k$ may not be defined on all gambles, which may complicate comparison with $\underline{P}_M$.

Now define the *disjunction* of $(\underline{P}_k)_{k=1}^n$ as the largest, or least conservative, coherent lower prevision that the modeller can have, such that all of the experts still agree with her decisions. By Theorem 6, this is the (point-wise) largest coherent lower prevision that is dominated by all of the experts' natural extensions $(\underline{E}_k)_{k=1}^n$. This way of combining lower previsions is also called the *unanimity rule* [14, Section 4.3.9].

It is easy to see that the disjunction always exists (see Theorem 7 below). We shall denote the disjunction of n coherent lower previsions $(\underline{P}_k)_{k=1}^n$ by $\sqcup_{k=1}^n \underline{P}_k$, and the disjunction of two coherent lower lower previsions $\underline{P}_1$ and $\underline{P}_2$ by $\underline{P}_1 \sqcup \underline{P}_2$. By the next theorem, the disjunction is an associative and commutative operator, and hence, it satisfies the order of combination invariance principle.

Theorem 7. *Let $\underline{E}_k$ be the natural extension of the coherent lower prevision $\underline{P}_k$, and let $\mathcal{M}(\underline{P}_k)$ be its set of dominating linear previsions, for each $k \in \{1, \dots, n\}$. Then for each gamble Z on Ω:*

$$\left(\sqcup_{k=1}^n \underline{P}_k \right)(Z) = \min_{k=1}^n \underline{E}_k(Z) = \inf \left\{ P(Z) \colon P \in \bigcup_{k=1}^n \mathcal{M}(\underline{P}_k) \right\}. \tag{28}$$

Proof. For notational convenience, let $\underline{E} = \min_{k=1}^n \underline{E}_k$. It is obvious that $\underline{E}$ is the largest functional on $\mathscr{L}(\Omega)$ that is point-wise dominated by all the $\underline{E}_k$. Since it is a point-wise minimum of coherent lower previsions, $\underline{E}$ is a coherent lower prevision, which proves the first equality. The proof of the second equality is now immediate: for each gamble Z on Ω,

$$\inf \left\{ P(Z) \colon P \in \bigcup_{k=1}^n \mathcal{M}(\underline{P}_k) \right\} = \inf \bigcup_{k=1}^n \{ P(Z) \colon P \in \mathcal{M}(\underline{P}_k) \} = \min_{k=1}^n \inf \{ P(Z) \colon P \in \mathcal{M}(\underline{P}_k) \} = \min_{k=1}^n \underline{E}_k(Z). \quad \square$$

Obviously, if the modeller only accepts those decisions which are supported by all the experts, her model is going to be at most as precise as the least precise expert only. Disjunction, in contradistinction to conjunction, aims at reconciling all the experts' assessments. Therefore, if the assessments are highly conflicting, any reconciliation of them will be highly imprecise. This is illustrated by Eq. (28). We therefore suggest the following general strategy:

(i) if the experts are consistent, the modeller takes their conjunction $\sqcap_{k=1}^{n}\underline{P}_k$ as her coherent lower prevision,
(ii) if they are not, this points to the fact that the modeller cannot trust all of the experts, so she should try to find out what experts she really can trust, and then discard those experts which she does not trust,
(iii) if the remaining experts, say $(\underline{P}_k)_{k=1}^{n'}$ with $n' \leq n$, are consistent, she takes their conjunction $\sqcap_{k=1}^{n'}\underline{P}_k$ as her coherent lower prevision,
(iv) if there is still inconsistency, she reconciles the remaining experts by taking the disjunction $\sqcup_{k=1}^{n'}\underline{P}_k$ of the remaining assessments as her coherent lower prevision.

Discarding experts will lead to less precise results in case of consistency (smaller conjunction), but may increase precision in case of inconsistency (larger disjunction). There is no unique solution in case of inconsistency: everything depends on how much information is available about the reliability of the given information.

6.5. **Products of disjunctions.** The disjunction also distributes over the two independent products we have introduced. Hence, we obtain order of combination invariance for disjunction and independent natural extension, and disjunction and the type-I product.

Proposition 6. *For all $k = 1, \dots, n$, let $\underline{P}_k$ be a coherent lower prevision defined on the linear subspace $\mathscr{F}_k$ of $\mathscr{L}(\mathscr{A})$ containing all constant gambles, and for all $\ell = 1, \dots, m$, let $\underline{Q}_\ell$ be a coherent lower prevision defined on the linear subspace $\mathscr{G}_\ell$ of $\mathscr{L}(\mathscr{B})$ containing all constant gambles. Then the following equality holds:*

$$\sqcup_{k,\ell}\left(\underline{P}_k \times_{\mathrm{INE}} \underline{Q}_\ell\right) = \left(\sqcup_{k=1}^{n}\underline{P}_k\right) \times_{\mathrm{INE}}\left(\sqcup_{\ell=1}^{m}\underline{Q}_\ell\right). \tag{29}$$

Proof. Consider a gamble Z on $\mathscr{A} \times \mathscr{B}$. Using Eqs. (28) and (6) we find that

$$\left(\sqcup_{k=1}^{n}\underline{P}_k\right) \times_{\mathrm{INE}}\left(\sqcup_{\ell=1}^{m}\underline{Q}_\ell\right)(Z) = \sup_{X,Y\in\mathscr{L}(\mathscr{A}\times\mathscr{B})}\left\{\gamma\colon Z-\gamma \geq X - \left(\min_{k=1}^{n}\underline{E}_k\right)(X) + Y - \left(\min_{\ell=1}^{m}\underline{F}_\ell\right)(Y)\right\},$$

where $\underline{E}_k$ is the natural extension of $\underline{P}_k$ and $\underline{F}_\ell$ is the natural extension of $\underline{Q}_\ell$. Now observe that

$$Z-\gamma \geq X - \left(\min_{k=1}^{n}\underline{E}_k\right)(X) + Y - \left(\min_{\ell=1}^{m}\underline{F}_\ell\right)(Y) \tag{30}$$

is equivalent to

$$(\forall(\alpha,\beta)\in\mathscr{A}\times\mathscr{B})\left(Z(\alpha,\beta)-\gamma \geq X(\alpha,\beta) - \min_{k=1}^{n}\underline{E}_k(X(\cdot,\beta)) + Y(\alpha,\beta) - \min_{\ell=1}^{m}\underline{F}_\ell(Y(\alpha,\cdot))\right),$$

which is also equivalent to

$$(\forall(\alpha,\beta)\in\mathscr{A}\times\mathscr{B})\,(\forall k,\ell)\,(Z(\alpha,\beta)-\gamma \geq X(\alpha,\beta) - \underline{E}_k(X(\cdot,\beta)) + Y(\alpha,\beta) - \underline{F}_\ell(Y(\alpha,\cdot))),$$

which can also be written as

$$(\forall k,\ell)\,(Z-\gamma \geq X - \underline{E}_k(X) + Y - \underline{F}_k(Y)). \tag{31}$$

Using the equivalence of (30) and (31), we find that

$$\left(\sqcup_{k=1}^{n}\underline{P}_k\right) \times_{\mathrm{INE}}\left(\sqcup_{\ell=1}^{m}\underline{Q}_\ell\right)(Z) = \sup_{X,Y\in\mathscr{L}(\Omega)}\{\gamma\colon\ (\forall k,\ell)\,(Z-\gamma \geq X - \underline{E}_k(X) + Y - \underline{F}_\ell(Y))\}.$$

Next, observe that, since $\underline{P}_k(X) = \underline{E}_k(X)$ for $X \in \overline{\mathscr{F}}_k$, and similarly for $\underline{Q}_\ell$,

$$(\forall k,\ell)\,\left(\exists X_k \in \overline{\mathscr{F}}_k, Y_\ell \in \overline{\mathscr{G}}_\ell\right)\left(Z-\gamma \geq X_k - \underline{P}_k(X_k) + Y_\ell - \underline{Q}_\ell(Y_\ell)\right) \tag{32}$$

implies that

$$(\exists X,Y \in \mathscr{L}(\mathscr{A}\times\mathscr{B}))\,(\forall k,\ell)\,(Z-\gamma \geq X - \underline{E}_k(X) + Y - \underline{F}_\ell(Y)). \tag{33}$$

Indeed, let $X = \frac{1}{n+m}\sum_{k=1}^{n}X_k$ and $Y = \frac{1}{n+m}\sum_{\ell=1}^{m}Y_\ell$, and observe that for this choice

$$\underline{E}_k(X) = \underline{P}_k(X) \geq \frac{1}{n+m}\sum_{k=1}^{n}\underline{P}_k(X_k) \text{ and } \underline{F}_\ell(Y) = \underline{Q}_\ell(Y) \geq \frac{1}{n+m}\sum_{\ell=1}^{m}\underline{Q}_\ell(Y_\ell).$$

for all k and ℓ. Thus, if (32) is satisfied then

$$Z-\gamma \geq \sum_{k,\ell}[X_k - \underline{P}_k(X) + Y - \underline{P}_\ell(Y)] \geq X - \underline{E}_k(X) + Y - \underline{F}_\ell(Y),$$

for all k and ℓ, which means that (33) must hold too. Consequently,

$$\left(\sqcup_{k=1}^{n}\underline{P}_k\right) \times_{\mathrm{INE}}\left(\sqcup_{\ell=1}^{m}\underline{Q}_\ell\right)(Z) \geq \sup\left\{\gamma\colon\ (\forall k,\ell)\,\left(\exists X_k \in \overline{\mathscr{F}}_k, Y_\ell \in \overline{\mathscr{G}}_\ell\right)\left(Z-\gamma \geq X_k - \underline{P}_k(X_k) + Y_\ell - \underline{Q}_\ell(Y_\ell)\right)\right\}. \tag{34}$$

To prove the converse inequality, we recall that (see [14, Section 3.1.4])

$$\underline{E}_k(X(\cdot,\beta)) = \sup_{\substack{U\in\mathscr{F}_k\\ U\leq X(\cdot,\beta)}}\underline{P}_k(U(\cdot,\beta)) = \sup_{\substack{U\in\overline{\mathscr{F}}_k\\ U\leq X}}\underline{P}_k(U(\cdot,\beta))$$

and similarly

$$\underline{F}_\ell(Y(\alpha,\cdot)) = \sup_{\substack{V\in\mathscr{G}_\ell\\ V\leq Y(\alpha,\cdot)}} \underline{Q}_\ell(V(\alpha,\cdot)) = \sup_{\substack{V\in\overline{\mathscr{G}}_\ell\\ V\leq Y}} \underline{Q}_\ell(V(\alpha,\cdot)).$$

Consequently,

$$(\forall\epsilon>0)(\forall k,\ell)(\forall X,Y\in\mathscr{L}(\mathscr{A}\times\mathscr{B}))\left(\exists U_{\epsilon,k,X}\in\overline{\mathscr{F}}_k\right)\left(\exists V_{\epsilon,\ell,Y}\in\overline{\mathscr{G}}_\ell\right)$$
$$\Big(U_{\epsilon,k,X}\leq X,\ \underline{P}_k(U_{\epsilon,k,X})+\epsilon\geq\underline{E}_k(X),\ V_{\epsilon,\ell,Y}\leq Y \text{ and } \underline{Q}_\ell(V_{\epsilon,\ell,Y})+\epsilon\geq\underline{F}_\ell(Y)\Big).$$

But this means that (33) implies that

$$(\forall\epsilon>0)(\forall k,\ell)\left(\exists X_k\in\overline{\mathscr{F}}_k\right)\left(\exists Y_\ell\in\overline{\mathscr{G}}_\ell\right)\Big(Z-\gamma\geq X_k-\underline{P}_k(X_k)-\epsilon+Y_\ell-\underline{Q}_\ell(Y_\ell)-\epsilon\Big)$$

[identify X_k with $U_{\epsilon,k,X}$ and Y_ℓ with $V_{\epsilon,\ell,Y}$]. From this implication we infer that

$$\left(\sqcup_{k=1}^n\underline{P}_k\right)\times_{\mathrm{INE}}\left(\sqcup_{\ell=1}^m\underline{Q}_\ell\right)(Z)\leq 2\epsilon+\sup\left\{\gamma\colon(\forall k,\ell)\left(\exists X_k\in\overline{\mathscr{F}}_k\right)\left(\exists Y_\ell\in\overline{\mathscr{G}}_\ell\right)\Big(Z-\gamma\geq X_k-\underline{P}_k(X_k)+Y_\ell-\underline{Q}_\ell(Y_\ell)\Big)\right\}$$

for all $\epsilon>0$, and hence also for $\epsilon=0$. We may thus infer that

$$\left(\sqcup_{k=1}^n\underline{P}_k\right)\times_{\mathrm{INE}}\left(\sqcup_{\ell=1}^m\underline{Q}_\ell\right)(Z)=\sup\left\{\gamma\colon(\forall k,\ell)\left(\exists X_k\in\overline{\mathscr{F}}_k\right)\left(\exists Y_\ell\in\overline{\mathscr{G}}_\ell\right)\Big(Z-\alpha\geq X_k-\underline{P}_k(X_k)+Y_\ell-\underline{Q}_\ell(Y_\ell)\Big)\right\}.$$

If we now define the $A_{k,\ell}$ as the subsets

$$\left\{\alpha\in\mathbb{R}\colon\left(\exists X_k\in\overline{\mathscr{F}}_k\right)\left(\exists Y_\ell\in\overline{\mathscr{G}}_\ell\right)\Big(Z-\alpha\geq X_k-\underline{P}_k(X_k)+Y_\ell-\underline{Q}_\ell(Y_\ell)\Big)\right\}$$

of the reals, then we can rewrite this as

$$\left(\sqcup_{k=1}^n\underline{P}_k\right)\times_{\mathrm{INE}}\left(\sqcup_{\ell=1}^m\underline{Q}_\ell\right)(Z)=\sup\bigcap_{k,\ell}A_{k,\ell}.$$

Since the $A_{k,\ell}$ are down-sets [if $\alpha_1\in A_{k,\ell}$ and $\alpha_2\leq\alpha_1$ then also $\alpha_2\in A_{k,\ell}$], it follows that

$$\sup\bigcap_{k,\ell}A_{k,\ell}=\min_{k,\ell}\sup A_{k,\ell}.$$

Now observe that the right hand side is equal to $\left(\sqcup_{k,\ell}\left(\underline{P}_k\times_{\mathrm{INE}}\underline{Q}_\ell\right)\right)(Z)$. □

Proposition 7. *Let $\underline{P}_k$ be a coherent lower prevision defined on the linear subspace $\mathscr{F}_k$ of $\mathscr{L}(\mathscr{A})$, $k=1,\dots,n$, and let $\underline{Q}_\ell$ be a coherent lower prevision defined on the linear subspace $\mathscr{G}_\ell$ of $\mathscr{L}(\mathscr{B})$, $\ell=1,\dots,m$. Then the following equality holds:*

$$\sqcup_{k,\ell}\left(\underline{P}_k\times_{\mathrm{TI}}\underline{Q}_\ell\right)=\left(\sqcup_{k=1}^n\underline{P}_k\right)\times_{\mathrm{TI}}\left(\sqcup_{\ell=1}^m\underline{Q}_\ell\right).\tag{35}$$

Proof. The proof is an immediate consequence of Theorem 7 if we recall that

$$\left(\bigcup_{k=1}^n\mathcal{M}(\underline{P}_k)\right)\times\left(\bigcup_{\ell=1}^m\mathcal{M}(\underline{Q}_\ell)\right)=\bigcup_{k=1}^n\bigcup_{\ell=1}^m\Big[\mathcal{M}(\underline{P}_k)\times\mathcal{M}(\underline{Q}_\ell)\Big].\quad\square$$

7. Solutions to Problems 2, 3 and 5

The additional difficulty in Problems 2, 3 and 5 is that there are a number of experts, or sources, giving information about the parameters a and b. In the formulation of the problem set [11], nothing is said about how these experts are related: an expert for a and an expert for b might be same person, or they may be different people. We take the independence assumption in [11] stating that "knowledge about the value of one parameter implies nothing about the value of the other" to refer to all the experts: their assessments about the values of one parameter, are in no way influenced by (their own or other experts') assessments about the values of the other parameter. This means that for each a-expert and each b-expert, we can take an independent product of the lower previsions modelling their assessments, and then combine these products using conjunction, or if there is inconsistency, using disjunction. But the results in Propositions 4, 5, 6 and 7 tell us that we get the same result if we first combine, for each parameter, the information about its value coming from the different experts, and then take an independent product: our approach satisfies the order of combination invariance principle.

7.1. Solution to Problem 2.

The argument in Section 2.3 tells us that the expert assessment $A=[a_1,a_2]$ is described by the coherent vacuous lower prevision $\underline{P}_A$ defined on $\mathscr{L}(\mathscr{A})$, and that the expert assessments $B_j=[b_1^j,b_2^j]$ are described by the coherent vacuous lower previsions $\underline{P}_{B_j}$ defined on $\mathscr{L}(\mathscr{B})$ $(j=1,\dots,4)$. Let us first look at the case that the $\underline{P}_{B_j}$ are consistent, or equivalently, by Proposition 2, that $\bigcap_{j=1}^4B_j\neq\emptyset$. We then find for the lower prevision $\underline{P}$ modelling the available information about the pair (a,b) that, again using Proposition 2 and also Propositions 1, 4 and 5:

$$\underline{P}=\sqcap_{j=1}^4\left(\underline{P}_A\times_{\mathrm{INE}}\underline{P}_{B_j}\right)=\sqcap_{j=1}^4\left(\underline{P}_A\times_{\mathrm{TI}}\underline{P}_{B_j}\right)=\underline{P}_A\times_{\mathrm{INE}}\left(\sqcap_{j=1}^4\underline{P}_{B_j}\right)=\underline{P}_A\times_{\mathrm{TI}}\left(\sqcap_{j=1}^4\underline{P}_{B_j}\right)=\underline{P}_{A\times\left(\bigcap_{j=1}^4B_j\right)}.$$

The available information about the output y is then represented by the coherent lower prevision $\underline{P}_y$, where

$$\underline{P}_y(U)=\inf_{\alpha\in A}\inf_{\beta\in\bigcap_{j=1}^4B_j}U(f(\alpha,\beta)),$$

for all gambles U on $\mathscr{Y}$. As an illustration, we can calculate the lower prevision $\underline{P}(f)$ and the upper prevision $\overline{P}(f)$ of the system output y, for the given subproblems 2a and 2b.

Problem 2a. With $A \times \cap_{j=1}^{4} B_j = [0.1, 1.0] \times [0.6, 0.8]$, we find that

$$\underline{P}(f) = 0.956196 \text{ and } \overline{P}(f) = 1.8.$$

Problem 2b. With $A \times \cap_{j=1}^{4} B_j = [0.1, 1.0] \times [0.6, 0.7]$, we find that

$$\underline{P}(f) = 0.956196 \text{ and } \overline{P}(f) = 1.7.$$

In Problem 2c, the lower previsions $\underline{P}_{B_1}, \ldots, \underline{P}_{B_4}$ on $\mathscr{L}(\mathscr{B})$ are inconsistent, as $B_1 \cap B_2 \cap B_3 = \emptyset$. Applying disjunction, this leads to the following lower prevision $\underline{P}$, using Propositions 1, 6 and 7,

$$\underline{P} = \sqcup_{j=1}^{4} \left(\underline{P}_A \times_{\text{INE}} \underline{P}_{B_j}\right) = \sqcup_{j=1}^{4} \left(\underline{P}_A \times_{\text{TI}} \underline{P}_{B_j}\right) = \underline{P}_A \times_{\text{INE}} \left(\sqcup_{j=1}^{4} \underline{P}_{B_j}\right) = \underline{P}_A \times_{\text{TI}} \left(\sqcup_{j=1}^{4} \underline{P}_{B_j}\right) = \underline{P}_{A \times (\bigcup_{j=1}^{4} B_j)}.$$

The information about the output y is now represented by the coherent lower prevision $\underline{P}_y$, where

$$\underline{P}_y(U) = \inf_{\alpha \in A} \inf_{\beta \in \bigcup_{j=1}^{4} B_j} U(f(\alpha, \beta)),$$

for all gambles U on $\mathscr{Y}$. As an illustration, we can calculate the lower prevision $\underline{P}(f)$ and the upper prevision $\overline{P}(f)$ of the system output y.

Problem 2c. With $A \times \bigcup_{j=1}^{4} B_j = [0.1, 1.0] \times [0.0, 1.0]$, we find that

$$\underline{P}(f) = 0.692201 \text{ and } \overline{P}(f) = 2.$$

7.2. **Solution to Problem 3.** Using the results of Section 2.3, the expert assessments $A_i = [a_1^i, a_2^i]$ are described by the coherent vacuous lower previsions $\underline{P}_{A_i}$ defined on $\mathscr{L}(\mathscr{A})$ ($i = 1, \ldots, 3$), and the expert assessments $B_j = [b_1^j, b_2^j]$ by the coherent vacuous lower previsions $\underline{P}_{B_j}$ defined on $\mathscr{L}(\mathscr{B})$ ($j = 1, \ldots, 4$). We first consider the case that the $\underline{P}_{A_i}$ and $\underline{P}_{B_j}$ are consistent, or equivalently, by Proposition 2, that $\bigcap_{i=1}^{3} A_i \neq \emptyset$ and $\bigcap_{j=1}^{4} B_j \neq \emptyset$. We then find for the lower prevision $\underline{P}$ modelling the available information about the pair (a, b) that, using Proposition 2 and Propositions 1, 4 and 5:

$$\begin{aligned}\underline{P} &= \sqcap_{\substack{i=1,\cdots,3\\ j=1,\cdots,4}} \left(\underline{P}_{A_i} \times_{\text{INE}} \underline{P}_{B_j}\right) = \sqcap_{\substack{i=1,\cdots,3\\ j=1,\cdots,4}} \left(\underline{P}_{A_i} \times_{\text{TI}} \underline{P}_{B_j}\right)\\ &= \left(\sqcap_{i=1}^{3} \underline{P}_{A_i}\right) \times_{\text{INE}} \left(\sqcap_{j=1}^{4} \underline{P}_{B_j}\right) = \left(\sqcap_{i=1}^{3} \underline{P}_{A_i}\right) \times_{\text{TI}} \left(\sqcap_{j=1}^{4} \underline{P}_{B_j}\right)\\ &= \underline{P}_{(\bigcap_{i=1}^{3} A_i) \times (\bigcap_{j=1}^{4} B_j)}.\end{aligned}$$

The available information about the output y is represented by the coherent lower prevision $\underline{P}_y$, where

$$\underline{P}_y(U) = \inf_{\alpha \in \bigcap_{i=1}^{3} A_i} \inf_{\beta \in \bigcap_{j=1}^{4} B_j} U(f(\alpha, \beta)),$$

for all gambles U on $\mathscr{Y}$. As an illustration, we give the lower prevision $\underline{P}(f)$ and the upper prevision $\overline{P}(f)$ of the output, for the subproblems 3a and 3b.

Problem 3a. For $(\bigcap_{i=1}^{3} A_i) \times (\bigcap_{j=1}^{4} B_j) = [0.5, 0.7] \times \{0.6\}$, we find

$$\underline{P}(f) = 1.04881 \text{ and } \overline{P}(f) = 1.2016.$$

Problem 3b. For $(\bigcap_{i=1}^{3} A_i) \times (\bigcap_{j=1}^{4} B_j) = [0.5, 0.6] \times \{0.6\}$, we find that

$$\underline{P}(f) = 1.04881 \text{ and } \overline{P}(f) = 1.1156.$$

In Problem 3c, the lower previsions corresponding with the different expert assessments $\underline{P}_{A_i}$ and $\underline{P}_{B_j}$ are conflicting, since $\bigcap_{i=1}^{3} A_i = \bigcap_{j=1}^{4} B_j = \emptyset$. The modeller has no additional information regarding the reliability of any of the sources; she considers all of the assessments equally credible and applies the disjunction rule to all of them:

$$\begin{aligned}\underline{P} &= \sqcup_{\substack{i=1,\cdots,3\\ j=1,\cdots,4}} \left(\underline{P}_{A_i} \times_{\text{INE}} \underline{P}_{B_j}\right) = \sqcup_{\substack{i=1,\cdots,3\\ j=1,\cdots,4}} \left(\underline{P}_{A_i} \times_{\text{TI}} \underline{P}_{B_j}\right)\\ &= \left(\sqcup_{i=1}^{3} \underline{P}_{A_i}\right) \times_{\text{INE}} \left(\sqcup_{j=1}^{4} \underline{P}_{B_j}\right) = \left(\sqcup_{i=1}^{3} \underline{P}_{A_i}\right) \times_{\text{TI}} \left(\sqcup_{j=1}^{4} \underline{P}_{B_j}\right)\\ &= \underline{P}_{(\bigcup_{i=1}^{3} A_i) \times (\bigcup_{j=1}^{4} B_j)}.\end{aligned}$$

The available information about the output y is represented by the coherent lower prevision $\underline{P}_y$, where

$$\underline{P}_y(U) = \inf_{\alpha \in \bigcup_{i=1}^{3} A_i} \inf_{\beta \in \bigcup_{j=1}^{4} B_j} U(f(\alpha, \beta)),$$

for all gambles U on $\mathscr{Y}$. As an illustration, we give the lower prevision $\underline{P}(f)$ and the upper prevision $\overline{P}(f)$ of the output.

Problem 3c. We find that

$$\underline{P}(f) = 0.692201 \text{ and } \overline{P}(f) = 2.$$

7.3. **Solution to Problem 5.** Using the results of Section 2.3, the assessments $A_i = [a_i, a_i]$ are described by the vacuous lower previsions $\underline{P}_{A_i}$ on $\mathscr{L}(\mathscr{A})$ $(i = 1, \ldots, 3)$, the assessments $M_j = [\mu_1^j, \mu_2^j]$ by the vacuous lower previsions $\underline{P}_{M_j}$ on $\mathscr{L}(\mathscr{M})$ $(j = 1, \ldots, 3)$, and the assessments $S_k = [\sigma_1^k, \sigma_2^k]$ by the vacuous lower previsions $\underline{P}_{S_k}$ on $\mathscr{L}(\mathscr{S})$ $(k = 1, \ldots, 3)$. Information about the value of the parameter b in $\mathscr{B}$ is derived from information about μ and σ through the common sampling model, $\ln b \sim N(\mu, \sigma)$, and using the marginal extension theorem (see Section 2.5.2).

Let us first concentrate on the information about the parameter b. When combining the assessments about the values of the parameters μ and σ, there are in principle two possible ways to proceed:

(i) first obtain, for each expert, the marginal extensions based on his assessments of the values of μ and σ, and then combine these marginal extensions;
(ii) first combine the different experts' assessments about the values of μ and σ, and then calculate the marginal extension.

The two strategies are not equivalent; in fact, for the types of combination we have studied, it turns out that the second strategy gives more precise results (i.e., the resulting buying prices are higher). This is not entirely unexpected, as the second strategy leads to more precise results because it involves *additional assumptions*: the experts use the *same* sampling model and make assessments about the *same* sampling model parameters. Since this is precisely the case in the problem under study, we have opted for the second strategy.

We first consider Problems 5a and 5b, where the lower previsions $\underline{P}_{A_i}$, the lower previsions $\underline{P}_{M_j}$, and the lower previsions $\underline{P}_{S_k}$ are consistent, because $\bigcap_{i=1}^3 A_i \neq \emptyset$, $\bigcap_{j=1}^3 M_j \neq \emptyset$ and $\bigcap_{k=1}^3 S_k \neq \emptyset$, see Proposition 2.

We can then model the information about the parameters μ and σ by the following lower prevision on $\mathscr{L}(\mathscr{M} \times \mathscr{S})$:

$$\sqcap_{\substack{j=1,\cdots,3\\k=1,\cdots,3}} \left(\underline{P}_{M_j} \times_{\mathrm{INE}} \underline{P}_{S_k}\right) = \sqcap_{\substack{j=1,\cdots,3\\k=1,\cdots,3}} \left(\underline{P}_{M_j} \times_{\mathrm{TI}} \underline{P}_{S_k}\right)$$
$$= \left(\sqcap_{j=1}^3 \underline{P}_{M_i}\right) \times_{\mathrm{INE}} \left(\sqcap_{k=1}^3 \underline{P}_{S_k}\right) = \left(\sqcap_{j=1}^3 \underline{P}_{M_j}\right) \times_{\mathrm{TI}} \left(\sqcap_{k=1}^3 \underline{P}_{S_k}\right) = \underline{P}_{(\bigcap_{j=1}^3 M_j)\times(\bigcap_{k=1}^3 S_k)}.$$

Using the results of Section 2.5, we find that the information about the value of the parameter b can then be modelled by the lower prevision $\underline{Q}$, where

$$\underline{Q}(Y) = \inf_{\mu \in \bigcap_{j=1}^3 M_j} \inf_{\sigma \in \bigcap_{k=1}^3 S_k} \int_{\mathscr{B}} Y \phi_{\mu,\sigma} \mathrm{d}\lambda,$$

for all integrable gambles Y on $\mathscr{B}$. This lower prevision has the Bayesian sensitivity analysis interpretation.

By Proposition 2, the conjunction of the sources A_i is the vacuous lower prevision $\underline{P}_{\bigcap_{i=1}^3 A_i}$. By Proposition 1, the type-I product of this conjunction with the lower prevision $\underline{Q}$ coincides with their independent natural extension, and it is given by the lower prevision $\underline{P}$, where

$$\underline{P}(Z) = \inf_{\alpha \in \bigcap_{i=1}^3 A_i} \inf_{\mu \in \bigcap_{j=1}^3 M_j} \inf_{\sigma \in \bigcap_{k=1}^3 S_k} \int_{\mathscr{B}} Z(\alpha, \cdot)\phi_{\mu,\sigma}(\cdot) \mathrm{d}\lambda$$

for all gambles Z on $\mathscr{A} \times \mathscr{B}$ such that $Z(\alpha, \cdot)$ is integrable for all α in $\mathscr{A}$.

The available information about the output y is represented by the coherent lower prevision $\underline{P}_y$, where

$$\underline{P}_y(U) = \inf_{\alpha \in \bigcap_{i=1}^3 A_i} \inf_{\mu \in \bigcap_{j=1}^3 M_j} \inf_{\sigma \in \bigcap_{k=1}^3 S_k} \int_{\mathscr{B}} U\left(f(\alpha, \cdot)\right) \phi_{\mu,\sigma}(\cdot) \mathrm{d}\lambda$$

for all gambles U on $\mathscr{Y}$ such that $U\left(f(\alpha, \cdot)\right)$ is integrable for all α in $\mathscr{A}$. As an illustration, we give the lower prevision $\underline{P}(f)$ and the upper prevision $\overline{P}(f)$ of the system output y.

Problem 5a. With $\bigcap_{i=1}^3 A_i = [0.5, 0.7]$, $\bigcap_{j=1}^3 M_j = [0.6, 0.8]$ and $\bigcap_{k=1}^3 S_k = [0.3, 0.4]$ we find that

$$\underline{P}(f) = 1.54027 \text{ and } \overline{P}(f) = 2.19107.$$

Problem 5b. With $\bigcap_{i=1}^3 A_i = [0.5, 0.6]$, $\bigcap_{j=1}^3 M_j = [0.6, 0.7]$ and $\bigcap_{k=1}^3 S_k = [0.3, 0.35]$ we find that

$$\underline{P}(f) = 1.54027 \text{ and } \overline{P}(f) = 1.81496.$$

In Problem 5c, all the experts give conflicting information about the parameters a, μ and σ, as $\bigcap_{i=1}^3 A_i = \bigcap_{j=1}^3 M_j = \bigcap_{k=1}^3 S_k = \emptyset$, see Proposition 2. As discussed above, we first combine the expert assessments $M_1, \ldots, M_3$ and $S_1, \ldots, S_3$. Both μ-sources and σ-sources are conflicting, so we combine them separately using disjunction and take their independent product. We find the following lower prevision on $\mathscr{L}(\mathscr{M} \times \mathscr{S})$, representing the available information about (μ, σ):

$$\underline{P}_{(\bigcup_{i=1}^3 M_i)\times(\bigcup_{i=1}^3 S_i)}.$$

This lower prevision has the Bayesian sensitivity analysis interpretation.

The sources $A_1, \ldots, A_3$ are conflicting too, so we use disjunction. We find the lower prevision $\underline{P}_{\bigcup_{i=1}^3 A_i}$ on $\mathscr{L}(\mathscr{A})$. Applying the marginal extension theorem, and taking the independent product—independent natural extension and type-I product coincide—we eventually find that the available information about (a, b) can be modelled by the lower prevision $\underline{P}$, where

$$\underline{P}(Z) = \inf_{\alpha \in \bigcup_{i=1}^3 A_i} \inf_{\mu \in \bigcup_{j=1}^3 M_j} \inf_{\sigma \in \bigcup_{k=1}^3 S_k} \int_{\mathscr{B}} Z(\alpha, \cdot)\phi_{\mu,\sigma}(\cdot) \mathrm{d}\lambda$$

for all gambles Z on $\mathscr{A} \times \mathscr{B}$ such that $Z(\alpha, \cdot)$ is integrable for all α in $\mathscr{A}$.

The available information about the output y is represented by the coherent lower prevision $\underline{P}_y$, where

$$\underline{P}_y(U) = \inf_{\alpha \in \bigcup_{i=1}^3 A_i} \inf_{\mu \in \bigcup_{j=1}^3 M_j} \inf_{\sigma \in \bigcup_{k=1}^3 S_k} \int_{\mathscr{B}} U\left(f(\alpha, \cdot)\right) \phi_{\mu,\sigma}(\cdot) \mathrm{d}\lambda$$

for all gambles U on $\mathscr{Y}$ such that $U\left(f(\alpha, \cdot)\right)$ is integrable for all α in $\mathscr{A}$. As an illustration, we give the lower prevision $\underline{P}(f)$ and the upper prevision $\overline{P}(f)$ of the system output y.

Problem 5c. For $\bigcup_{i=1}^3 A_i = [0.1, 1.0]$, $\bigcup_{j=1}^3 M_j = [0.0, 1.0]$ and $\bigcup_{k=1}^3 S_k = [0.1, 0.2] \cup [0.25, 0.35] \cup [0.4, 0.5]$, we find that

$$\underline{P}(f) = 1.00966 \text{ and } \overline{P}(f) = 4.08022.$$

8. Conclusion

We have argued extensively that the theory of coherent lower previsions is eminently suited for solving the first set of problems posed in [11]. For easy reference, and in order to allow easy comparison with other solution methods, we have listed the calculated lower and upper previsions $\underline{P}(f)$ and $\overline{P}(f)$ for the output y in Table 1. We want to emphasise that these numbers are by no means the only information that our models provide. In fact the lower previsions $\underline{P}_y$ we have given as solutions in Sections 5 and 7 contain much more information than just these two numbers, and they can also be used to in decision making and estimation problems. But we feel a more extensive discussion of these issues to be beyond the scope of the modelling challenges presented in [11].

problem	$\underline{P}(f)$	$\overline{P}(f)$
1	0.692201	2.0
2a	0.956196	1.8
2b	0.956196	1.7
2c	0.692201	2.0
3a	1.04881	1.2016
3b	1.04881	1.1156
3c	0.692201	2.0
4	1.00966	4.08022
5a	1.54027	2.19107
5b	1.54027	1.81496
5c	1.00966	4.08022
6	1.05939	2.86825

TABLE 1. Lower and upper previsions for the output y calculated using our imprecise probability models for the available information

References

[1] G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, 5:131–295, 1953–1954.
[2] G. de Cooman. Possibility theory I–III. *International Journal of General Systems*, 25:291–371, 1997.
[3] G. de Cooman and D. Aeyels. Supremum preserving upper probabilities. *Information Sciences*, 118:173–212, 1999.
[4] G. de Cooman and M. Troffaes. Dynamic programming under imprecise gain. submitted to the Third International Symposium on Imprecise Probabilities and Their Applications (ISIPTA '03). 2003.
[5] B. de Finetti. *Theory of Probability*. John Wiley & Sons, Chichester, 1974–1975. English Translation of *Teoria delle Probabilità*.
[6] Bruno de Finetti. Sul significato soggettivo della probabilità. *Fundamenta Mathematicae*, 17:298–329, 1931.
[7] D. Dubois and H. Prade. *Possibility Theory*. Plenum Press, New York, 1988.
[8] A. N. Kolmogorov. *Grundbegriffe der Wahrscheinlichkeitsrechnung*. Springer, Berlin, 1933.
[9] A. N. Kolmogorov. *Foundations of Probability*. Chelsea Publishing Co., New York, 1950. English translation of [8].
[10] I. Levi. *The Enterprise of Knowledge*. MIT Press, London, 1980.
[11] W. L. Oberkampf, J. C. Helton, C. A. Joslyn, S. F. Wojtkiewicz, and S. Ferson. Challenge problems: uncertainty in system response given uncertain parameters. *Reliability Engineering and System Safety*. (this issue).
[12] M. C. M. Troffaes and G. de Cooman. Extension of coherent lower previsions to unbounded random variables. In *Proceedings of the Ninth International Conference IPMU 2002 (Information Processing and Management of Uncertainty in Knowledge-Based SystemsJuly 1–5, Annecy, France)*, pages 735–42. ESIA – Université de Savoie, 2002.
[13] M. C. M. Troffaes and G. de Cooman. Lower previsions for unbounded random variables. In P. Grzegorzewski, O. Hryniewicz, and M. Ángeles Gil, editors, *Soft Methods in Probability, Statistics and Data Analysis*, Advances in Soft Computing, pages 146–155. Physica-Verlag, New York, 2002.
[14] P. Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London, 1991.
[15] P. Walley and T. L. Fine. Varieties of modal (classificatory) and comparative probability. *Synthese*, 41:321–374, 1979.
[16] L. A. Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1:3–28, 1978.