Improved upper and lower bounds on the feedback vertex numbers of grids and butterflies

Florent R. Madelaine and Iain A. Stewart
Department of Computer Science, Durham University,
Science Labs, South Road, Durham DH1 3LE, U.K.

Abstract

We improve upon the best known upper and lower bounds on the sizes of minimal feedback vertex sets in butterflies. Also, we construct new feedback vertex sets in grids so that for a large number of pairs (n, m), the size of our feedback vertex set in the grid $M_{n,m}$ matches the best known lower bound, and for all other pairs it differs from this lower bound by at most 2.

Keywords: feedback vertex sets; grids; butterflies.

1 Introduction

A *feedback vertex set* in an undirected graph is a subset of vertices the removal of which (along with their incident edges) results in an acyclic graph. The *feedback vertex set problem* is to find a feedback vertex set of minimum cardinality in a graph G, with the size of such a set known as the *feedback vertex number* $\tau(G)$. Whilst the feedback vertex set problem is NP-hard in general, it has been extensively studied in a wide variety of restricted classes of graphs and shown to be polynomial-time solvable in many of these classes. Furthermore, a number of lower and upper bounds on the feedback vertex number of graphs from these classes have been established. The reader is referred to [4] for an extensive survey of feedback vertex set problems which ranges over polynomially solvable cases, approximation algorithms, exact algorithms, practical heuristics and applications.

In this paper, we are concerned with the classes of graphs known as grids and butterflies. Such graphs are common in the study of interconnection networks for parallel processing as they have particularly attractive properties in this regard (see, for example, [3]). The study of feedback vertex sets in grids and butterflies has traditionally gone hand-in-hand. In [5], Luccio proved upper and lower bounds on the sizes of minimal feedback vertex sets in both grids and butterflies. It was shown in [5] that the feedback vertex number of the grid $M_{n,m}$ is at most

$$\left\lfloor \frac{mn}{3} + \frac{m+n}{6} + o(m,n) \right\rfloor$$

and at least

$$\left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil,$$

and that the feedback vertex number of the butterfly B_d is at most

$$\left\lfloor \frac{(d+\frac{1}{3})2^d+\frac{1}{3}}{3} \right\rfloor$$

(see the analysis in [2]) and at least

$$2^{d-1}\left\lfloor \frac{d+1}{2} \right\rfloor$$

(definitions of $M_{n,m}$ and B_d follow). Subsequently, in [1], Caragiannis, Kaklamanis and Kanellopoulos improved the state of affairs by establishing a general lower bound technique by which they showed that the feedback vertex number of the butterfly B_d is at least

$$\left\lceil \frac{(d-1)2^d+1}{3} \right\rceil.$$

They also showed that the feedback vertex number of the grid $M_{n,m}$ is at most

$$\left\lfloor \frac{mn}{3} - \frac{m+n-5}{6} \right\rfloor$$

and that the feedback vertex number of the butterfly B_d is at most

$$\left\lfloor \frac{(d+\frac{1}{2})2^d}{3} \right\rfloor.$$

Finally, more recently, Chang, Lin and Lee [2] both improved Luccio's analysis of the sizes of feedback vertex sets in butterflies and exhibited an algorithm which constructed a feedback vertex set in B_d of size

$$\left\lfloor \frac{(d-\frac{2}{3})2^d+\frac{2}{3}}{3} \right\rfloor, \text{ if } d \text{ is even,}$$

and of size

$$\left\lfloor \frac{(d-\frac{2}{3})2^d}{3} \right\rfloor + \frac{2^{\lceil \frac{d}{2} \rceil}+2^{\lfloor \frac{d}{2} \rfloor}}{3}, \text{ if } d \text{ is odd.}$$

In this paper, we improve upon Chang, Lin and Lee's algorithm and obtain a smaller upper bound on the feedback vertex number of a butterfly B_d, when $d \geq 5$. Our algorithm is very similar to that of Chang, Lin and Lee except that our 'starting point' in the recursive algorithm is improved feedback vertex sets for B_5 and B_6. We find that we can use Chang, Lin and Lee's analysis to prove our algorithm correct and also to establish our improved bounds. We also improve upon Luccio's lower bound on the feedback vertex numbers of butterflies. As regards grids, we make dramatic progress. We construct new feedback vertex sets in grids so that for a large number of pairs (n, m), the size of our feedback vertex set in the grid $M_{n,m}$ matches the best known lower bound (from [5]), and for all other pairs the size of our feedback vertex set is at most this lower bound plus 2.

This paper is structured as follows. In Section 2, we provide the basic definitions, before dealing with feedback vertex sets in grids in Section 3 and in butterflies in Section 4. Our conclusions are given in Section 5.

2 Basic definitions

Let $n, m \geq 2$. The *rectangular grid with n rows and m columns* (or *$n \times m$ mesh*), denoted by $M_{n,m}$, is the graph with vertex set $V(M_{n,m})$ defined as $\{v_{i,j} : 0 \leq i < n, 0 \leq j < m\}$ and edge set $E(M_{n,m})$ defined as

$$\{(v_{i,j}, v_{i+1,j}) : 0 \leq i < n-1, 0 \leq j < m\}$$
$$\cup\{(v_{i,j}, v_{i,j+1}) : 0 \leq i < n, 0 \leq j < m-1\}.$$

Let $d \geq 1$. The *d-dimensional butterfly B_d* has vertex set $V(B_d)$ partitioned into $(d+1)$ *rows*, whereupon each row contains 2^d vertices. Every vertex of $V(B_d)$ is indexed by the pair (i, j) where i indicates its row and j its *column* in that row: as such, we refer to the vertices of $V(B_d)$ as $\{v_{i,j} : 0 \leq i \leq d, 0 \leq j \leq 2^d - 1\}$. The edge set $E(B_d)$ of B_d consists of the following edges.

- For every pair of adjacent rows, there is an edge joining corresponding vertices, $v_{i,j}$ and $v_{i+1,j}$, on these two rows; that is, there are edges

$$\{(v_{i,j}, v_{i+1,j}) : 0 \leq i < d, 0 \leq j \leq 2^d - 1\}.$$

- For every pair of adjacent rows, there is an edge joining a vertex $v_{i,j}$ on the lower-indexed row to the vertex v_{i+1,j_i} on the higher-indexed row so that the binary representation of the integer j_i differs from that of the integer j only in the ith position (where the right-most bit is bit 0); that is, there are edges

$$\{(v_{i,j}, v_{i+1,j_i}) : 0 \leq i < d, 0 \leq j \leq 2^d - 1\}.$$

Grids and butterflies can be visualized as in Figs. 1 and 13, respectively.

We adopt the following notation (in line with that of Chang, Lin and Lee in [2]). If G is a graph with vertex set $V(G)$ and V' is a subset of vertices of $V(G)$ then the subgraph of G induced by the vertices of V' is denoted $G[V']$ and the subgraph of G induced by the vertices of $V(G) \setminus V'$ is denoted $G \setminus V'$.

3 New upper bounds for grids

In this section, we derive new upper bounds on the sizes of minimal feedback vertex sets in two-dimensional grids. For a large number of pairs (n, m), our upper bound on the size of a minimal feedback vertex set of $M_{n,m}$ matches the lower bound from [5], and on the other pairs it differs from this lower bound by at most 2.

3.1 The bulk of the cases

Case (i) Let $n \geq 4$ be such that $n \equiv 1 \pmod 3$, and let $m \geq 4$ be even.
Define the set of vertices $X_{n,m}$ of $V(M_{n,m})$ as $A_{n,m} \cup B_{n,m} \cup C_{n,m} \cup D_{n,m} \cup E_{n,m} \cup F_{n,m}$ where:

$$\begin{aligned}
A_{n,m} &= \{v_{i,j} : 0 \leq i \leq n-1, i \equiv 1 \pmod 6, 2 \leq j \leq m-2, j \text{ even}\};\\
B_{n,m} &= \{v_{i,j} : 0 \leq i \leq n-1, i \equiv 2 \pmod 6, 1 \leq j \leq m-3, j \text{ odd}\};\\
C_{n,m} &= \{v_{i,j} : 0 \leq i \leq n-1, i \equiv 4 \pmod 6, 1 \leq j \leq m-3, j \text{ odd}\};\\
D_{n,m} &= \{v_{i,j} : 0 \leq i \leq n-1, i \equiv 5 \pmod 6, 2 \leq j \leq m-2, j \text{ even}\};\\
E_{n,m} &= \{v_{i,1} : 0 \leq i \leq n-1, i \equiv 0 \pmod 6\};\\
F_{n,m} &= \{v_{i,m-2} : 0 \leq i \leq n-1, i \equiv 3 \pmod 6\}.
\end{aligned}$$

The set $X_{10,8}$ is shown in Fig. 1, where the vertices of $X_{10,8}$ have been annotated according to their subsets in the above definition.

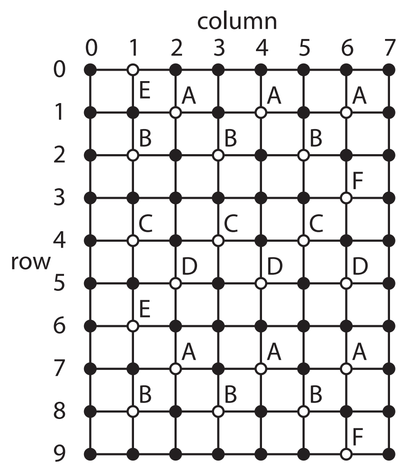


Figure 1. The set of vertices $X_{10,8}$.

We claim that $X_{n,m}$ is a feedback vertex set. Observe that if there is a cycle in $M_{n,m} \setminus X_{n,m}$ then the inclusion of the vertices of $A_{n,m} \cup B_{n,m} \cup C_{n,m} \cup D_{n,m}$ in $X_{n,m}$ means that the cycle must use only the perimeter vertices of $M_{n,m}$ or the vertices on row i, for each $i \equiv 0 \pmod 3$. However, the vertices of $E_{n,m}$ and $F_{n,m}$ preclude any such cycle, and so $X_{n,m}$ is a feedback vertex set of $M_{n,m}$. The size of $X_{n,m}$ is

$$\frac{(n-1)}{3}(m-2) + \frac{(n-1)}{3} + 1 = \frac{(n-1)(m-1)}{3} + 1,$$

while Luccio's lower bound, which we denote $lb_{n,m}$, is

$$lb_{n,m} = \left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil,$$

which, for $n \equiv 1 \pmod 3$ and m even, is identical to the size of $X_{n,m}$. Hence, $X_{n,m}$ is a feedback vertex set of minimal size.

Before continuing, let us look at the feedback vertex set $X_{n,m}$ and why it is of minimal size from a different perspective (this perspective underpins Luccio's lower bound construction in [5] but will be of use to us later in alternative contexts). Consider the perimeter-cycle of $M_{n,m}$; this cycle must contain at least one vertex from any feedback vertex set, so choose such a vertex to be $v_{0,1}$. After removing $v_{0,1}$ and any incident edges from $M_{n,m}$, to get $M^1_{n,m}$, there is a natural perimeter-cycle which is as before except that the sub-path navigating around $v_{0,1}$ is

$$\ldots, v_{2,0}, v_{1,0}, v_{1,1}, v_{1,2}, v_{0,2}, v_{0,3}, \ldots$$

Also note that the sub-graph of the original $M_{n,m}$ induced by the vertices strictly outside the perimeter-cycle but not vertices of our partial feedback vertex set (at present, just the vertex $v_{0,0}$) is acyclic.

Similarly, the perimeter-cycle of $M^1_{n,m}$ contains at least one vertex of any resulting feedback vertex set; so choose the vertex $v_{1,2}$ to be such a vertex. In $M^2_{n,m}$, obtained from $M^1_{n,m}$ by removing $v_{1,2}$ and any incident edges, there is a natural perimeter cycle and the sub-graph of the original $M_{n,m}$ induced by the vertices strictly outside the perimeter-cycle but not vertices of our partial feedback vertex set (at present, just the vertices $v_{0,0}$ and $v_{0,2}$) is acyclic.

Similarly, the perimeter-cycle of $M^2_{n,m}$ contains at least one vertex of any resulting feedback vertex set; so choose the vertex $v_{2,3}$ to be such a vertex. In $M^3_{n,m}$, obtained from $M^2_{n,m}$ by removing $v_{2,3}$ and any incident edges, there is a natural perimeter cycle and the sub-graph of the original $M_{n,m}$ induced by the vertices strictly outside the perimeter-cycle but not vertices of our partial feedback vertex set (at present, just the vertices $v_{0,0}$ and $v_{0,2}$) is acyclic.

Continuing in this fashion ultimately results in a perimeter-cycle the 'breaking' of which results in an empty perimeter-cycle, so that the sub-graph of $M_{n,m}$ induced by those vertices not in the resulting feedback vertex set is indeed acyclic. This constructive approach to the formation of $X_{n,m}$ can be visualized as in Fig. 2, where the order in which perimeter-cycle vertices are chosen is given and where at each stage, the edges incident with the chosen vertex and outside the resulting perimeter-cycle are omitted. Yet another alternative way of viewing the construction of $X_{n,m}$ is via a tessellation of the grid, in a natural way. Note that given any feedback vertex set X, a feedback vertex set $Y \subseteq X$ can be constructed by adopting the above procedure and making appropriate choices.

Apart from the first and last choices of the vertices of $X_{n,m}$ in the above procedure, the choices are optimal in the sense that at any stage, no other choice could decrease the number of cells inside the perimeter-cycle more than the number resulting from the vertex chosen (the most the number of cells can decrease by is 3, as is the case with our choices). Indeed, the first choice of vertex is optimal in this sense too (with a decrease of 2 cells). Initially, there are $(n-1)(m-1)$ cells, and any first choice decreases this number to at least $(n-1)(m-1)-2$. Subsequent choices decrease this number by at most 3 cells per choice, and so as $n \equiv 1 \pmod 3$, after

$$\frac{(n-1)(m-1)}{3}$$

choices, there is at least 1 cell inside the perimeter-cycle. Thus, the size of any feedback vertex set is at least

$$\frac{(n-1)(m-1)}{3} + 1.$$

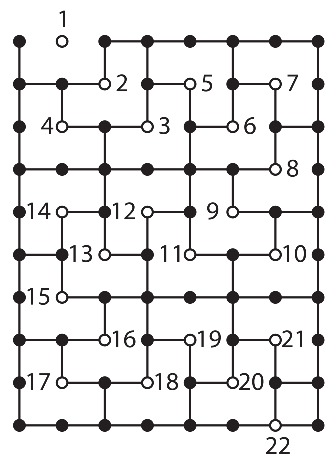


Figure 2. The set of vertices $X_{10,8}$ formed by perimeter-breaking.

<u>Case (ii)</u> Let $n \geq 4$ be such that $n \equiv 1 \pmod 3$, and let $m \geq 7$ be odd.

Partition $M_{n,m}$ into two sub-grids, one, call it $M'_{n,m}$, induced by the vertices in columns 0, 1 and 2, and one, call it $M''_{n,m}$, induced by the vertices in the remaining columns. Note that $M''_{n,m}$ is such that it has an even number of columns. There are two cases, depending upon whether $n \equiv 1 \pmod 6$ or not.

<u>Sub-case $(ii.a)$</u> $n \equiv 4 \pmod 6$.

We can build a set of vertices $X''_{n,m}$ in $M''_{n,m}$, as above, except starting from the right-hand side as opposed to the left. In particular, define $X''_{n,m}$ as $A''_{n,m} \cup B''_{n,m} \cup C''_{n,m} \cup D''_{n,m} \cup E''_{n,m} \cup F''_{n,m}$ where:

$$\begin{aligned}
A''_{n,m} &= \{v_{i,j} : 0 \leq i \leq n-1, i \equiv 1 \pmod 6, 4 \leq j \leq m-3, j \text{ even}\};\\
B''_{n,m} &= \{v_{i,j} : 0 \leq i \leq n-1, i \equiv 2 \pmod 6, 5 \leq j \leq m-2, j \text{ odd}\};\\
C''_{n,m} &= \{v_{i,j} : 0 \leq i \leq n-1, i \equiv 4 \pmod 6, 5 \leq j \leq m-2, j \text{ odd}\};\\
D''_{n,m} &= \{v_{i,j} : 0 \leq i \leq n-1, i \equiv 5 \pmod 6, 4 \leq j \leq m-3, j \text{ even}\};\\
E''_{n,m} &= \{v_{i,m-2} : 0 \leq i \leq n-1, i \equiv 0 \pmod 6\};\\
F''_{n,m} &= \{v_{i,4} : 0 \leq i < n-1, i \equiv 3 \pmod 6\} \cup \{v_{n-2,3}\}.
\end{aligned}$$

In $M'_{n,m}$, define $X'_{n,m}$ as $A'_{n,m} \cup B'_{n,m}$ where:

$$\begin{aligned}
A'_{n,m} &= \{v_{i,1} : 0 \leq i \leq n-2, i \text{ even}\};\\
B'_{n,m} &= \{v_{i,2} : 1 \leq i \leq n-3, i \text{ odd}\}.
\end{aligned}$$

Define $X_{n,m} = X'_{n,m} \cup X''_{n,m}$. The construction of $X_{10,11}$ by perimeter-breaking can be visualized as in Fig. 3. The perimeter-breaking argument applied above yields that $X_{n,m}$ is a feedback vertex set of $M_{n,m}$.

Case $(ii.b)$ $n \equiv 1 \pmod 6$.
We can build a set of vertices $X''_{n,m}$ in $M''_{n,m}$, as above (starting from the left-hand side). In particular, define $X''_{n,m}$ as $A''_{n,m} \cup B''_{n,m} \cup C''_{n,m} \cup D''_{n,m} \cup E''_{n,m} \cup F''_{n,m}$ where:

$$\begin{aligned}
A''_{n,m} &= \{v_{i,j} : 0 \le i \le n-1, i \equiv 1 \pmod 6), 5 \le j \le m-2, j \text{ odd}\}; \\
B''_{n,m} &= \{v_{i,j} : 0 \le i \le n-1, i \equiv 2 \pmod 6), 4 \le j \le m-3, j \text{ even}\}; \\
C''_{n,m} &= \{v_{i,j} : 0 \le i \le n-1, i \equiv 4 \pmod 6), 4 \le j \le m-3, j \text{ even}\}; \\
D''_{n,m} &= \{v_{i,j} : 0 \le i \le n-1, i \equiv 5 \pmod 6), 5 \le j \le m-2, j \text{ odd}\}; \\
E''_{n,m} &= \{v_{i,4} : 0 \le i < n-1, i \equiv 0 \pmod 6)\}; \\
F''_{n,m} &= \{v_{i,m-2} : 0 \le i \le n-1, i \equiv 3 \pmod 6)\} \cup \{v_{n-2,3}\}.
\end{aligned}$$

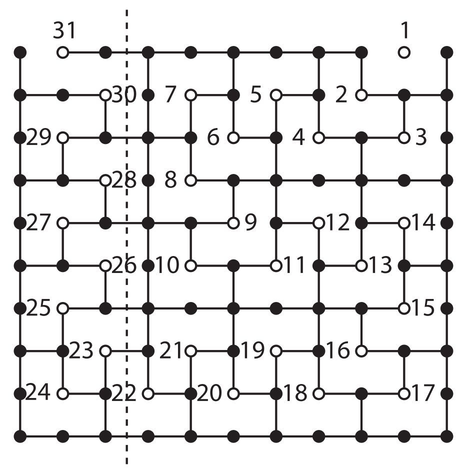


Figure 3. The set of vertices $X_{10,11}$ formed by perimeter-breaking.

In $M'_{n,m}$, define $X'_{n,m}$ as $A'_{n,m} \cup B'_{n,m}$ where:

$$\begin{aligned}
A'_{n,m} &= \{v_{i,1} : 1 \le i \le n-2, i \text{ odd}\}; \\
B'_{n,m} &= \{v_{i,2} : 0 \le i \le n-3, i \text{ even}\}.
\end{aligned}$$

Define $X_{n,m} = X'_{n,m} \cup X''_{n,m}$. The construction of $X_{13,11}$ by perimeter-breaking can be visualized as in Fig. 4. Again, the perimeter-breaking argument applied above yields that $X_{n,m}$ is a feedback vertex set of $M_{n,m}$.

In both of the above cases, the size of $X_{n,m}$ is

$$\frac{(n-1)(m-4)}{3} + 1 + (n-1) = \frac{(n-1)(m-1)}{3} + 1,$$

and so $X_{n,m}$ is a minimal feedback vertex set as in this case $|X_{n,m}| = lb_{n,m}$.

Case (iii) Let $n \ge 9$ be such that $n \equiv 0 \pmod 3$ and let $m \ge 6$ be even such that $m \not\equiv 1 \pmod 3$ (for if $m \equiv 1 \pmod 3$ then we can apply either Case (i) or Case (ii)). Let $M'_{n,m}$ be the sub-grid induced by the vertices in rows $0, 1, \ldots, n-6$. Using the construction from Case (i) in $M'_{n,m}$, starting (in the sense of the perimeter-breaking exposition) with either $v_{0,1}$ or $v_{0,m-2}$, as appropriate, we can build a feedback vertex set $X'_{n,m}$ of $M'_{n,m}$ of size

$$\frac{(n-6)(m-1)}{3}+1$$

so that $v_{n-6,m-2}$ lies in this feedback vertex set.

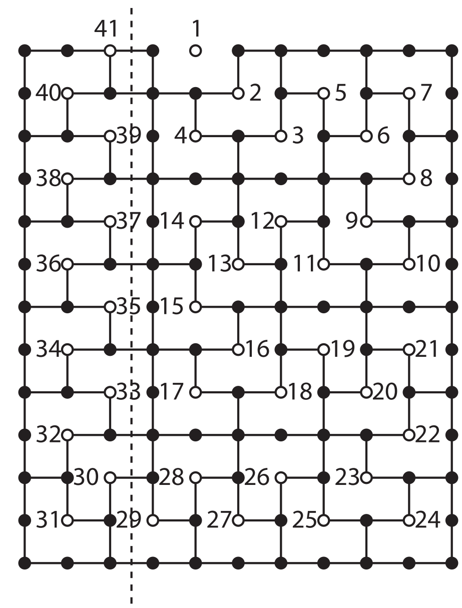


Figure 4. The set of vertices $X_{13,11}$ formed by perimeter-breaking.

Let m' be such that $m' \equiv 1 \pmod 3$ and either $m = m'+1$ or $m = m'+2$. Let $M''_{n,m}$ be the sub-grid induced by the vertices in rows $n-6, n-5, \ldots, n-1$ and in columns $0, 1, \ldots, m'-1$. Using the construction from Case (i) in $M''_{n,m}$, starting with either $v_{n-5,0}$ or $v_{n-2,0}$, as appropriate, we can build a feedback vertex set $X''_{n,m}$ of $M''_{n,m}$ of size

$$\frac{5(m'-1)}{3}+1$$

so that $v_{n-2,m'-1}$ lies in this feedback vertex set (note that even though $M'_{n,m}$ and $M''_{n,m}$ share a row, we do not duplicate vertices in our feedback vertex set).

Consider the partial feedback set $X'_{n,m} \cup X''_{n,m}$ (note that $X'_{n,m} \cup X''_{n,m}$ is a feedback vertex set of the sub-grid of $M_{n,m}$ induced by the vertices of $M'_{n,m}$ and $M''_{n,m}$). If $m' = m-2$ (with $m \equiv 0 \pmod 3$) then the additional 3 vertices $v_{n-4,m-2}$, $v_{n-3,m-2}$ and $v_{n-2,m-1}$ extend this set to a feedback vertex set of $M_{n,m}$. If $m' = m-1$ (with $m \equiv 2 \pmod 3$) then the additional vertex $v_{n-4,m-2}$ extends this set to a feedback vertex set of $M_{n,m}$. The constructions are illustrated for $M_{12,6}$ and $M_{12,8}$ in Fig. 5.

Consequently, we have constructed a feedback vertex set of size

$$\frac{(nm-n-m+6)}{3} = lb_{n,m}+1 \text{ if } m \equiv 0 \pmod 3$$

and of size

$$\frac{(nm - n - m + 5)}{3} = lb_{n,m} + 1 \text{ if } m \equiv 2 \pmod 3.$$

Case (iv) Suppose now that $n \geq 9$ is such that $n \equiv 0 \pmod 3$ and $m \geq 9$ is odd such that $m \not\equiv 1 \pmod 3$.

Let $M'_{n,m}$ be the sub-grid induced by the vertices in rows $0, 1, \ldots, n-6$. Using the constructions in Case (ii), we can build a feedback vertex set $X'_{n,m}$ of $M'_{n,m}$ of size

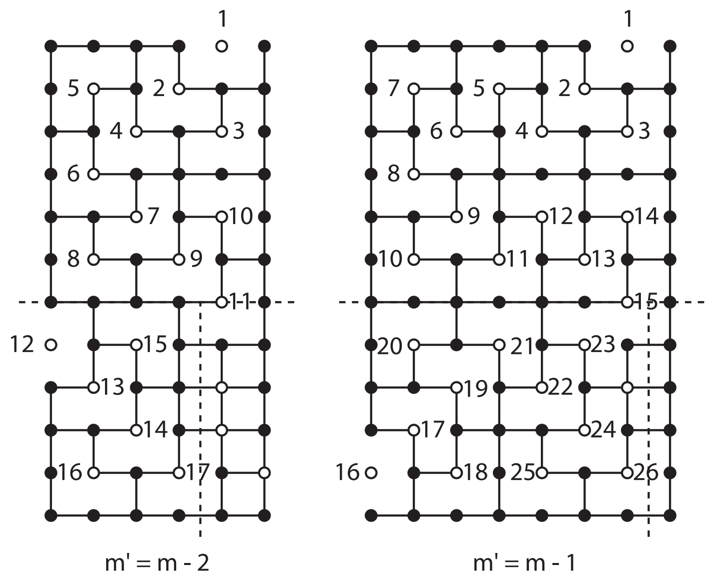


Figure 5. The sets of vertices $X_{12,6}$ and $X_{12,8}$.

$$\frac{(n-6)(m-1)}{3} + 1$$

so that either $v_{n-6,m-2}$ or $v_{n-6,m-3}$ lies in this feedback vertex set. Defining $M''_{n,m}$ as we did in Case (iii), we can build a feedback vertex set $X''_{n,m}$ of $M''_{n,m}$ of size

$$\frac{5(m'-1)}{3} + 1$$

so that $v_{n-2,m'-1}$ lies in this feedback vertex set.

Consider the partial feedback set $X'_{n,m} \cup X''_{n,m}$ (note that $X'_{n,m} \cup X''_{n,m}$ is a feedback vertex set of the sub-grid of $M_{n,m}$ induced by the vertices of $M'_{n,m}$ and $M''_{n,m}$). If $m' = m - 2$ then the additional 3 vertices $v_{n-5,m-2}$, $v_{n-3,m-2}$ and $v_{n-2,m-2}$ extend this set to a feedback vertex set of $M_{n,m}$. If $m' = m - 1$ then the additional 2 vertices $v_{n-5,m-2}$ and $v_{n-4,m-2}$ extend this set to a feedback vertex set of $M_{n,m}$. The constructions are illustrated for $M_{12,9}$ and $M_{12,11}$ in Fig. 6.

Consequently, we have constructed a feedback vertex set of size

$$\frac{(nm - n - m + 6)}{3} = lb_{n,m} + 1 \text{ if } m \equiv 0 \pmod 3)$$

and of size

$$\frac{(nm-n-m+8)}{3} = lb_{n,m} + 2 \text{ if } m \equiv 2 \pmod 3.$$

Case (v) Let $n \geq 11$ be such that $n \equiv 2 \pmod 3$ and let $m \geq 6$ be even such that $m \not\equiv 1 \pmod 3$.

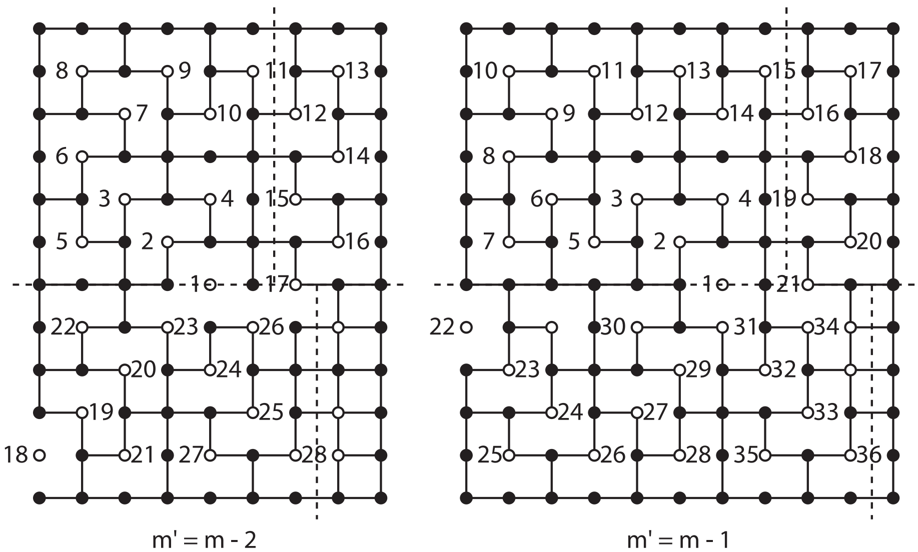


Figure 6. The sets of vertices $X_{12,9}$ and $X_{12,11}$.

Let $M'_{n,m}$ be the sub-grid induced by the vertices in rows $0, 1, \ldots, n-8$. Using the construction in Case (i), we can build a feedback vertex set $X'_{n,m}$ of $M'_{n,m}$ of size

$$\frac{(n-8)(m-1)}{3} + 1$$

so that $v_{n-8,m-2}$ lies in this feedback vertex set.

Let m' be such that $m' \equiv 1 \pmod 3$ and either $m = m'+1$ or $m = m'+2$. Let $M''_{n,m}$ be the sub-grid induced by the vertices in rows $n-8, n-7, \ldots, n-1$ and in columns $0, 1, \ldots, m'-1$. Using the construction from Case (i) in $M''_{n,m}$, we can build a feedback vertex set $X''_{n,m}$ of $M''_{n,m}$ of size

$$\frac{7(m'-1)}{3} + 1$$

so that $v_{n-2,m'-1}$ lies in this feedback vertex set.

Consider the partial feedback set $X'_{n,m} \cup X''_{n,m}$ (note that $X'_{n,m} \cup X''_{n,m}$ is a feedback vertex set of the sub-grid of $M_{n,m}$ induced by the vertices of $M'_{n,m}$ and $M''_{n,m}$). If $m' = m-2$ then the additional 4 vertices $v_{n-6,m-2}$, $v_{n-5,m-3}$, $v_{n-4,m-2}$ and $v_{n-2,m-1}$ extend this set to a feedback vertex set of $M_{n,m}$. If $m' = m-1$ then the additional 2 vertices $v_{n-6,m-2}$ and $v_{n-4,m-2}$ extend this set to a feedback vertex set of $M_{n,m}$

(the situation can be visualized using similar figures to those already detailed, hence we omit them). Consequently, we have constructed a feedback vertex set of size

$$\frac{(nm-n-m+5)}{3} = lb_{n,m} + 1 \text{ if } m \equiv 0 \pmod 3$$

and of size

$$\frac{(nm-n-m+6)}{3} = lb_{n,m} + 1 \text{ if } m \equiv 2 \pmod 3.$$

Case (vi) Let $n \geq 11$ be such that $n \equiv 2 \pmod 3$ and let $m \geq 7$ be odd such that $m \not\equiv 1 \pmod 3$.

Let $M'_{n,m}$ be the sub-grid induced by the vertices in rows $0, 1, \ldots, n-8$. Using the constructions in Case (ii), we can build a feedback vertex set $X'_{n,m}$ of $M'_{n,m}$ of size

$$\frac{(n-8)(m-1)}{3} + 1$$

so that either $v_{n-8,m-2}$ or $v_{n-8,m-3}$ lies in this feedback vertex set.

Let m' be such that $m' \equiv 1 \pmod 3$ and either $m = m'+1$ or $m = m'+2$. Let $M''_{n,m}$ be the sub-grid induced by the vertices in rows $n-8, n-7, \ldots, n-1$ and in columns $0, 1, \ldots, m'-1$. Using the construction from Case (i) in $M''_{n,m}$, we can build a feedback vertex set $X''_{n,m}$ of $M''_{n,m}$ of size

$$\frac{7(m'-1)}{3} + 1$$

so that $v_{n-2,m'-1}$ lies in this feedback vertex set.

Consider the partial feedback set $X'_{n,m} \cup X''_{n,m}$ (note that $X'_{n,m} \cup X''_{n,m}$ is a feedback vertex set of the sub-grid of $M_{n,m}$ induced by the vertices of $M'_{n,m}$ and $M''_{n,m}$). If $m' = m-2$ then the additional 5 vertices $v_{n-7,m-2}$, $v_{n-5,m-2}$, $v_{n-4,m-3}$, $v_{n-3,m-2}$ and $v_{n-2,m-1}$ extend this set to a feedback vertex set of $M_{n,m}$. If $m' = m-1$ then the additional 3 vertices $v_{n-7,m-2}$, $v_{n-5,m-2}$ and $v_{n-4,m-2}$ extend this set to a feedback vertex set of $M_{n,m}$. Consequently, we have constructed a feedback vertex set of size

$$\frac{(nm-n-m+8)}{3} = lb_{n,m} + 2 \text{ if } m \equiv 0 \pmod 3$$

and of size

$$\frac{(nm-n-m+9)}{3} = lb_{n,m} + 2 \text{ if } m \equiv 2 \pmod 3.$$

Drawing together the results of this section, we obtain the following theorem.

Theorem 1 *If the pair (n, m) does not lie in the set*

$$\begin{aligned}&\{(2,m),(n,2) : n,m \geq 2\} \cup \{(3,m),(n,3) : n,m \geq 3\} \cup \{(4,5),(5,4)\}\\ &\quad \cup \{(5,m),(n,5) : n,m \geq 5\} \cup \{(6,6),(6,8),(8,6),(8,8)\}\end{aligned}$$

then the size of a minimal feedback vertex set in the grid $M_{n,m}$ is $lb_{n,m}$, $lb_{n,m}+1$ or $lb_{n,m}+2$.

Given any specific pair (n, m) for which Theorem 1 is relevant, an upper bound on the size of the minimal feedback vertex set can be read from the appropriate case considered earlier.

Ignoring the finite number of 'isolated' grids for which Theorem 1 does not apply (for in each of these cases the dimensions are sufficiently small for a simple computer program to find the size of a minimal feedback vertex set), we are left with three (infinite) classes of grids lying outside our analysis.

3.2 Grids with 2 or 3 rows

For the class of grids with 2 rows, we can resolve the situation exactly: when $(n, 2) \in \{(n, 2) : n \geq 2\}$, the size of a minimal feedback vertex set is, trivially,

$$\left\lceil \frac{n-1}{2} \right\rceil.$$

We shall turn to the situation when our grids have 3 rows after we have examined an alternative feedback vertex set construction.

From the constructions above, we have yet to exhibit minimal feedback vertex sets for certain grid dimensions, *i.e.*, when neither n nor m is equivalent to 1 modulo 3. However, we have another construction which enables us to construct a minimal feedback vertex set in some of these cases. Moreover, our construction also allows us to use feedback vertex sets in smaller grids to build feedback vertex sets in larger grids where the size of the constructed feedback vertex set is 'controlled' in terms of the size of the original feedback vertex set.

We can *expand* the grid $M_{n,m}$ by: 'placing' a new *edge-vertex* in the 'middle' of each edge of $M_{n,m}$; 'placing' a new *cell-vertex* in the 'middle' of each cell of $M_{n,m}$; and joining each new cell-vertex to the new edge-vertices on the 'perimeter' of its cell. Note that the expanded grid, which we denote $\mathcal{E}(M_{n,m})$, is actually a copy of $M_{2n-1,2m-1}$.

Let X be a feedback vertex set of $M_{n,m}$. We expand $M_{n,m}$ into $\mathcal{E}(M_{n,m})$ and define the set of vertices $\mathcal{E}(X)$ to consist of the vertices corresponding to the vertices of X in union with the set of cell-vertices of $\mathcal{E}(M_{n,m})$. It is immediate that the set $\mathcal{E}(X)$ is a feedback vertex set of $\mathcal{E}(M_{n,m})$ (essentially, if we remove the cell-vertices from $\mathcal{E}(M_{n,m})$ then cycles correspond to cycles in $M_{n,m}$, and *vice versa*). The construction can be visualized as in Fig. 7, where the white vertices in $M_{9,9}$, on the right, are vertices of its feedback vertex set corresponding to the vertices of the feedback vertex set in $M_{5,5}$, on the left, and the grey, square vertices in $M_{9,9}$ are the added cell-vertices.

The size of the feedback vertex set $\mathcal{E}(X)$ of $\mathcal{E}(M_{n,m})$ is equal to the size of the feedback vertex set X of $M_{n,m}$ plus $(n-1)(m-1)$. That is,

$$|\mathcal{E}(X)| = |X| + (n-1)(m-1).$$

Luccio's lower bounds $lb_{n,m}$ and $lb_{2n-1,2m-1}$ on the sizes of minimal feedback vertex sets of $M_{n,m}$ and $M_{2n-1,2m-1}$ are

$$\left\lceil \frac{nm-n-m+2}{3} \right\rceil \text{ and } \left\lceil \frac{4nm-4n-4m+5}{3} \right\rceil,$$

respectively. Hence,

$$|\mathcal{E}(X)| - lb_{2n-1,2m-1} = |X| - lb_{n,m}.$$

Thus, the 'distance' a feedback vertex set is away from the lower bound $lb_{n,m}$ in $M_{n,m}$ is preserved by the construction in $M_{2n-1,2m-1}$. In particular, if X is a minimal feedback vertex set of $M_{n,m}$ of size $lb_{n,m}$ then $\mathcal{E}(X)$ is a minimal feedback vertex set of $M_{2n-1,2m-1}$ of size $lb_{2n-1,2m-1}$. The feedback vertex set of $M_{5,5}$ shown in Fig. 7 is minimal (and has size $lb_{5,5}$), thus we have effectively constructed minimal feedback vertex sets in all grids $M_{2^r+1,2^r+1}$, for $r \geq 2$. Our construction generalizes, yet simplifies, the construction of Luccio in [5].

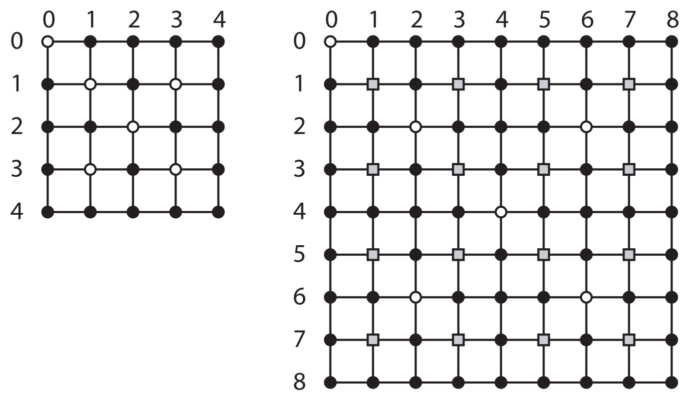


Figure 7. Expanding a grid with a feedback vertex set.

Let us now return to $M_{3,2r-1}$, where $r \geq 2$. As $M_{3,2r-1} = \mathcal{E}(M_{2,r})$, we immediately obtain an upper bound of

$$\left\lceil \frac{3(r-1)}{2} \right\rceil$$

for $\tau(M_{3,2r-1})$.

Consider the grid $M_{3,5}$ and how many of the vertices in columns 0, 1, 2 and 3 must necessarily lie in a minimal feedback vertex set of $M_{3,5}$: a simple case-by-case analysis yields that at least 3 such vertices must do so. Divide $M_{3,2r-1}$, where $r \geq 3$, into copies of $M_{3,5}$, the first copy consisting of the vertices in columns 0, 1, 2, 3 and 4, the second copy of vertices in columns 4, 5, 6, 7 and 8, the third copy of vertices in columns 8, 9, 10, 11, and 12, and so on. By above, at least 3 of the vertices of any feedback vertex set of $M_{3,2r-1}$ must lie in columns 0, 1, 2 and 3, at least 3 must lie in columns 4, 5, 6 and 7, at least 3 must lie in columns 8, 9, 10 and 11, and so on. Hence, if $r \geq 3$ is odd then

$$\tau(M_{3,2r-1}) \geq \frac{3(2r-2)}{4} = \frac{3(r-1)}{2},$$

and if $r \geq 4$ is even then

$$\tau(M_{3,2r-1}) \geq \frac{3(2r-4)}{4} + 2 = \left\lceil \frac{3(r-1)}{2} \right\rceil$$

(in the latter case, we divide $M_{3,2r-1}$ into copies of $M_{3,5}$ and we need at least 2 vertices to break cycles involving vertices in the 'left-over' columns indexed by $2r-4$, $2r-3$ and $2r-2$). Thus, when $r \geq 3$,

$$\tau(M_{3,2r-1}) = \left\lceil \frac{3(r-1)}{2} \right\rceil.$$

Trivially, when $r \geq 3$,

$$\left\lceil \frac{3(r-1)}{2} \right\rceil \leq \tau(M_{3,2r}) \leq \left\lceil \frac{3(r-1)}{2} \right\rceil + 1$$

(simply consider the copy of $M_{3,2r-1}$ induced by the vertices of $M_{3,2r}$ in columns $0, 1, \ldots, 2r-2$).

3.3 Grids with 5 rows

Finally, we are left with the grids $\{M_{5,n} : n \geq 5\}$. The methods above do not suffice to deal with this case and we need to examine the situation in more detail.

We can decompose a grid $M_{5,n}$ in two ways. First, we consider $M_{5,n}$ to be the *concatenation* of the grid $M_{5,p}$ and the grid $M_{5,q}$, where $n = p + q$ (the two smaller grids have no vertices in common and vertices in the rightmost column of $M_{5,p}$ are joined to their corresponding vertices in the leftmost column of $M_{5,q}$). In this case, we write $M_{5,n} = M_{5,p} + M_{5,q}$, and clearly we have that $\tau(M_{5,n}) \geq \tau(M_{5,p}) + \tau(M_{5,q})$. Second, we shall consider $M_{5,n}$ to be the *fusion* of $M_{5,p}$ and $M_{5,q}$, where $n = p+q-1$, by identifying the vertices in the rightmost column of $M_{5,p}$ with their corresponding vertices in the leftmost column of $M_{5,q}$. In this case, we write $M_{5,n} = M_{5,p} \oplus M_{5,q}$.

Suppose that we have a decomposition of $M_{5,n}$ (as a concatenation or a fusion) into $M_{5,p}$ and $M_{5,q}$, and partial feedback vertex sets in both $M_{5,p}$ and $M_{5,q}$ (that is, designated sets of vertices). We call a grid together with a partial feedback vertex set a *tile*. If $M_{5,n}$ is the concatenation of $M_{5,p}$ and $M_{5,q}$, and these two grids have partial feedback vertex sets X and Y, respectively, so that we denote these tiles as $M_{5,p}^X$ and $M_{5,q}^Y$, then we obtain, in the natural way, a partial feedback vertex set of $M_{5,n}$. If $M_{5,n}$ is the fusion of $M_{5,p}^X$ and $M_{5,q}^Y$ then we obtain a partial feedback vertex set of $M_{5,n}$, again in the natural way, except that we include any vertex of the fused column of $M_{5,n}$ in our partial feedback vertex set if its image in $M_{5,p}$ is in X or its image in $M_{5,q}$ is in Y (and ignore duplications). We call our fusion *compatible* if the vertices of X in the fused column of $M_{5,p}$ correspond exactly to the vertices of Y in the fused column of $M_{5,q}$.

We shall prove the following result by induction on n.

Proposition 2 *When $n \geq 2$, $\tau(M_{5,n}) = 11 \left\lfloor \frac{n}{8} \right\rfloor + \left\lfloor \frac{3}{2}(n \bmod 8) \right\rfloor - 1$.*

3.3.1 The base cases of the induction

Case (i) The grids $M_{5,2}$ and $M_{5,3}$.

It is not difficult to see that $\tau(M_{5,2}) = 2$ and $\tau(M_{5,3}) = 3$. All minimal feedback vertex sets are depicted in Fig. 8, up to isomorphism.

Figure 8. The minimal feedback vertex sets of $M_{5,2}$ and $M_{5,3}$.

Case (*ii*) The grid $M_{5,4}$.

We have that $lb_{5,4} = 5$, and it is not difficult to show that $\tau(M_{5,4}) = 5$; all minimal feedback vertex sets are depicted in Fig. 9, up to isomorphism (these feedback vertex sets have been generated by hand and checked by computer, and will be required later).

Case (*iii*) The grid $M_{5,5}$.

We have that $lb_{5,5} = 6$, and it is not difficult to show that $\tau(M_{5,5}) = 6$ (see Fig. 12 where we show some minimal feedback vertex sets of $M_{5,5}$, which we shall need later).

Case (*iv*) The grid $M_{5,6}$.

We have that $lb_{5,6} = 7$. Let Z be a minimal feedback vertex set of $M_{5,6}$ and suppose that $|Z| = 7$. Decompose $M_{5,6}$ as $M_{5,2} + M_{5,4}$. From above, there must be 2 vertices of Z in the first two columns and 5 vertices of Z in the final 4 columns. In particular, the minimal feedback vertex set induced on $M_{5,2}$ must be isomorphic to one in Fig. 8, and the minimal feedback vertex set induced on $M_{5,4}$ must be isomorphic to one in Fig. 9. By going through the possibilities of juxtaposing minimal feedback vertex sets from Figs. 8 and 9, it is easy to see that we obtain a contradiction. Hence, $\tau(M_{5,6}) = 8$ with a typical minimal feedback vertex set given in Fig. 10.

Case (*v*) The grid $M_{5,7}$.

We have that $lb_{5,7} = 8$. Let Z be a minimal feedback vertex set of $M_{5,7}$ and suppose that $|Z| = 8$. Decompose $M_{5,7}$ as $M_{5,3} + M_{5,4}$. Proceeding as in the previous case but juxtaposing the minimal feedback vertex sets of $M_{5,3}$ instead of $M_{5,2}$ yields a contradiction. Hence, $\tau(M_{5,7}) = 9$ with a typical minimal feedback vertex set given in Fig. 10.

Case (*vi*) The grid $M_{5,8}$.

We have $lb_{5,8} = 10$ and a feedback vertex set realizing this bound is shown in Fig. 11. Let us go further. Decompose $M_{5,8}$ as $M_{5,4} + M_{5,4}$ and let Z be any minimal feedback vertex set of $M_{5,8}$. By above, there must be 5 vertices of Z in the first 4 columns of $M_{5,8}$ and 5 in the last 4 columns. Also, decompose $M_{5,8}$ as $M_{5,2} + M_{5,6}$. By above, there must be 2 vertices of Z in the first 2 columns of $M_{5,8}$ and (symmetrically) 2 in the last 2 columns. Taking this into consideration and trying all possible pairs of minimal feedback vertex sets of $M_{5,4}$ from Fig. 9 yields that there are no vertices of Z in the leftmost or rightmost column of $M_{5,8}$. We shall require this fact later.

Having dealt with the base cases, we will require later one more result regarding $M_{5,9}$.

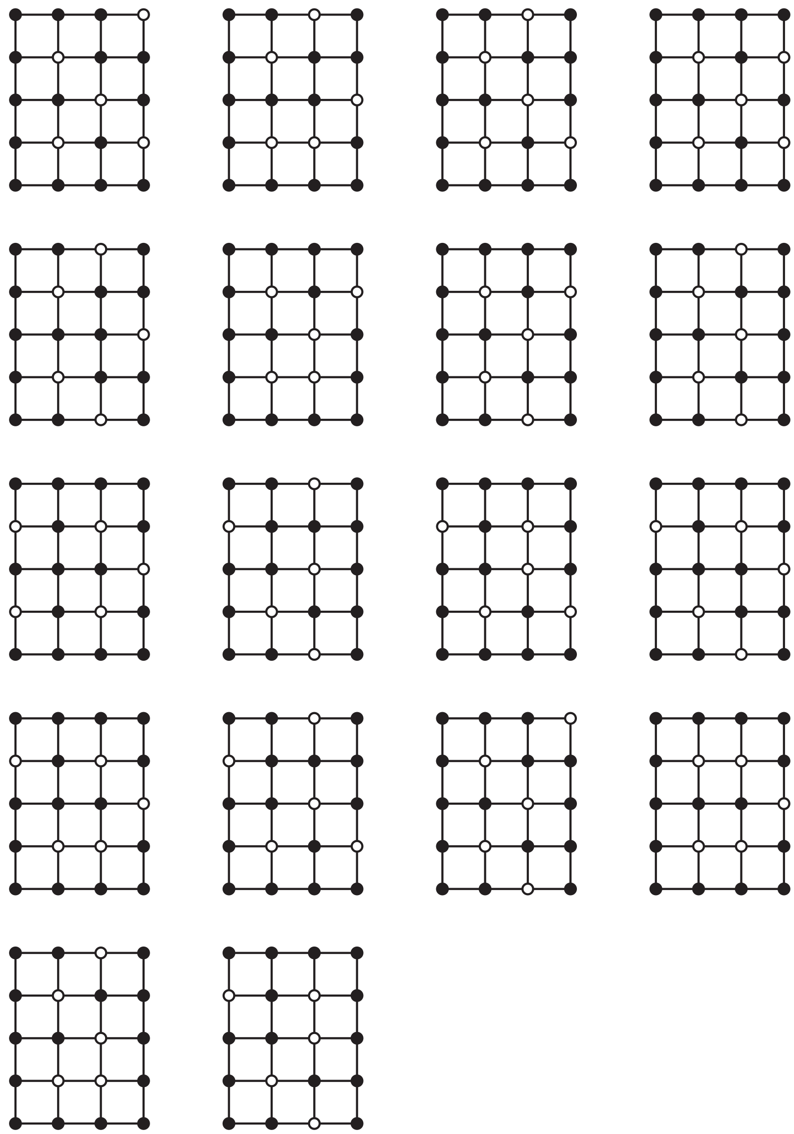


Figure 9. The minimal feedback vertex sets of $M_{5,4}$.

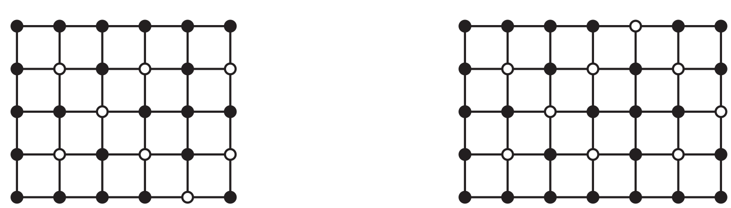


Figure 10. Minimal feedback vertex sets of $M_{5,6}$ and $M_{5,7}$.

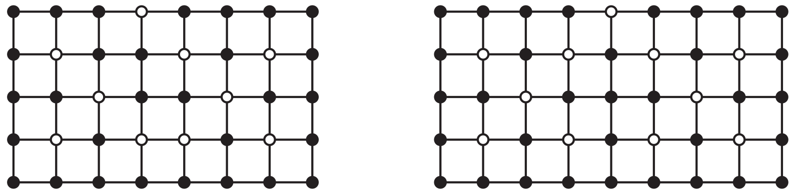


Figure 11. Minimal feedback vertex sets of $M_{5,8}$ and $M_{5,9}$.

Lemma 3 *Up to isomorphism, there is exactly one minimal feedback vertex set of $M_{5,9}$, namely that shown in Fig.* 11.

Proof Let Z be a minimal feedback vertex set of $M_{5,9}$. By above, the number of vertices, a, of Z in the first 4 columns is at least 5, the number of vertices, b, of Z in the last 4 columns is at least 5 and the number of vertices of Z in the first 5 columns is at least 6 (with $|Z| = 11$). The only solution is that $a = b = 5$ and that there is 1 vertex of Z in the 5th column. Given that $\tau(M_{5,5}) = 6$ and $\tau(M_{5,9}) = 11$, we must have that $M^Z_{5,9} = M^X_{5,5} \oplus M^Y_{5,5}$, where this fusion is compatible. The different situations where we have a minimal feedback vertex set of $M_{5,5}$ with exactly 1 vertex of the feedback set in the rightmost column are given in Fig. 12 (to see that this is the case, use the classification given in Fig. 9). It is immediate that the only possible minimal feedback vertex set of $M_{5,9}$ is that shown in Fig. 11 (up to isomorphism). □

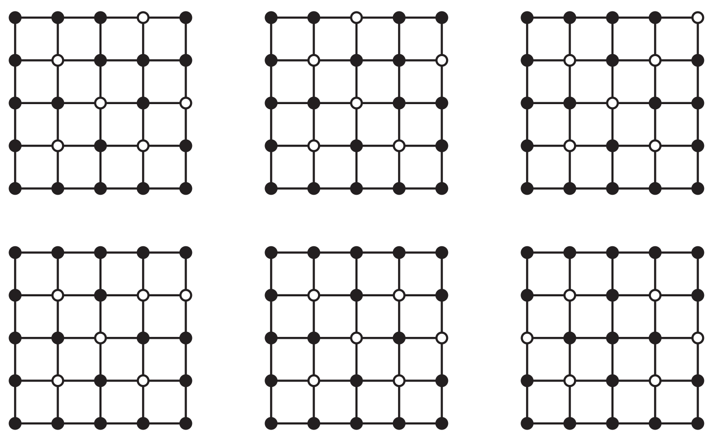


Figure 12. The minimal feedback vertex sets of $M_{5,5}$ with 1 vertex in the rightmost column.

3.3.2 The inductive step

Having dealt with the base cases, we now prove the following result by induction. Proposition 2 is an immediate corollary of the bounds just established and Proposition 4.

Proposition 4 *For all $p \geq 0$ and so long as the grid has at least 2 columns, we have that*:

$\tau(M_{5,8p}) = 11p - 1$; $\tau(M_{5,8p+1}) = 11p$; $\tau(M_{5,8p+2}) = 11p + 2$
$\tau(M_{5,8p+3}) = 11p + 3$; $\tau(M_{5,8p+4}) = 11p + 5$; $\tau(M_{5,8p+5}) = 11p + 6$
$\tau(M_{5,8p+6}) = 11p + 8$; $\tau(M_{5,8p+7}) = 11p + 9$.

Moreover, any minimal feedback vertex set Z of $M_{5,8p}$ or $M_{5,8p+1}$, for $p \geq 1$, is such that neither the rightmost nor leftmost column contains a vertex of Z.

Proof The base cases tell us that the result is true when $p = 0$, and also that $M_{5,8}$ and $M_{5,9}$ are such that neither the rightmost nor leftmost column of these grids contains a vertex of any minimal feedback vertex set. Suppose, as our induction hypothesis, that the result holds for some $p \geq 0$. Denote the bound given for $\tau(M_{5,j})$ in the statement of the proposition by the function $f(j)$, for all $j \geq 2$.

Fix $i \in \{0, 1, \ldots, 7\}$. Let X be a minimal feedback vertex set of $M_{5,8p+i}$. The fusion of the tile $M^X_{5,8p+i}$ and the tile $M^Y_{5,9}$ depicted in Fig. 11 results in a feedback vertex set Z of $M_{5,8(p+1)+i}$ of size $|X| + 11$ (note that the fusion results in no cycles as there are no paths in $M^Y_{5,9}$ from a vertex in the leftmost column to another vertex in the leftmost column which include vertices not in the leftmost column). By the induction hypothesis, $\tau(M_{5,8(p+1)+i}) \leq f(8p + i) + 11 = f(8(p + 1) + i)$.

Consider $M_{5,8(p+1)}$. As $M_{5,8(p+1)} = M_{5,8p+2} + M_{5,6}$, we have that $\tau(M_{5,8(p+1)}) \geq \tau(M_{5,8p+2}) + \tau(M_{5,6})$; consequently, by the induction hypothesis, $\tau(M_{5,8(p+1)}) \geq (11p + 2) + 8 = 11(p + 1) - 1$, as required.

Let Z be some minimal feedback vertex set of $M_{5,8(p+1)}$ (and so Z has size $11p + 10$). Suppose that column $8p$ has a vertex of Z in it. By the induction hypothesis applied to the first $8p + 1$ columns, these columns contain at least $11p$ vertices of Z. However, if they contain exactly $11p$ vertices of Z then, by the induction hypothesis, we obtain a contradiction (as at least one vertex of Z lies in the rightmost of these $8p + 1$ columns). Hence, the first $8p + 1$ columns contain at least $11p + 1$ vertices of Z, with the last 7 columns of $M_{5,8(p+1)}$ containing at most 9 vertices of Z. By the induction hypothesis, the last 7 columns of $M_{5,8(p+1)}$ must contain exactly 9 vertices of Z, with the first $8p + 1$ columns of $M_{5,8(p+1)}$ containing exactly $11p + 1$ vertices of Z.

If columns $8p, 8p+1, \ldots, 8p+7$ of $M_{5,8(p+1)}$ contain 10 vertices of Z then we have a minimal feedback vertex set of $M_{5,8}$ with a vertex of the feedback vertex set in the leftmost column, which yields a contradiction. Hence, columns $8p, 8p + 1, \ldots, 8p + 7$ contain at least 11 vertices of Z, with column $8p$ containing at least 2 vertices of Z and with the first $8p$ columns of $M_{5,8(p+1)}$ containing at most $11p - 1$ vertices of Z. By the induction hypothesis, the first $8p$ columns of $M_{5,8(p+1)}$ contain exactly $11p-1$ vertices of Z and there is no vertex of Z in column $8p - 1$. By Lemma 3 applied to the last 9 columns of $M_{5,8(p+1)}$, there is no vertex of Z in the rightmost column of $M_{5,8(p+1)}$.

Alternatively, suppose that column $8p$ contains no vertex of Z. By the induction hypothesis, the first $8p + 1$ columns of $M_{5,8(p+1)}$ contain at least $11p$ vertices of Z and the last 8 columns of $M_{5,8(p+1)}$ contain at least 10 vertices of Z. Thus, the first $8p$ columns of $M_{5,8(p+1)}$ contain exactly $11p$ vertices of Z and the last 8 columns contain exactly 10 vertices of Z. By the induction hypothesis (applied to the last 8 columns of $M_{5,8(p+1)}$), there is no vertex of Z in the rightmost column of $M_{5,8(p+1)}$. A symmetric argument holds for the leftmost column of $M_{5,8(p+1)}$.

Consider $M_{5,8(p+1)+1}$. As $M_{5,8(p+1)+1} = M_{5,8p+2} + M_{5,7}$, just as above we have $\tau(M_{5,8(p+1)+1}) \geq \tau(M_{5,8p+2}) + \tau(M_{5,7})$; consequently, by the induction hypothesis, $\tau(M_{5,8(p+1)+1}) \geq (11p+2)+9 = 11(p+1)$, as required.

Let Z be a minimal feedback vertex set of $M_{5,8(p+1)+1}$. Suppose that there is at least 1 vertex of Z in column $8p$. An identical argument to that above shows that: column $8p-1$ contains no vertices of Z; column $8p$ contains 2 vertices of Z; and columns $8p+1, 8p+2, \ldots, 8p+8$ contain 10 vertices of Z. If the rightmost column of $M_{5,8(p+1)+1}$ contains no vertex of Z then we are done; so, assume that the rightmost column of $M_{5,8(p+1)+1}$ contains at least 1 vertex of Z. By the induction hypothesis, columns $8p-1, 8p, \ldots, 8p+7$ contain 11 vertices of Z and column $8p+7$ contains no vertices of Z. This yields a contradiction as column $8p+8$ of $M_{5,8(p+1)+1}$ must contain exactly 1 vertex of Z and Z is supposed to be a feedback vertex set of $M_{5,8(p+1)+1}$. Alternatively, suppose that there are no vertices of Z in column $8p$. An identical argument to that above shows that there is no vertex of Z in the rightmost column of $M_{5,8(p+1)+1}$. A symmetric argument holds for the leftmost column of $M_{5,8(p+1)+1}$.

Consider $M_{5,8(p+1)+2}$. As $M_{5,8(p+1)+2} = M_{5,8p+6} + M_{5,4}$, just as above we have $\tau(M_{5,8(p+1)+2}) \geq \tau(M_{5,8p+6}) + \tau(M_{5,4})$; consequently, by the induction hypothesis, $\tau(M_{5,8(p+1)+2}) \geq (11p+8)+5 = 11(p+1)+2$, as required.

Consider $M_{5,8(p+1)+3}$. As $M_{5,8(p+1)+3} = M_{5,8p+6} + M_{5,5}$, just as above we have $\tau(M_{5,8(p+1)+3}) \geq \tau(M_{5,8p+6}) + \tau(M_{5,5})$; consequently, by the induction hypothesis, $\tau(M_{5,8(p+1)+3}) \geq (11p+8)+6 = 11(p+1)+3$, as required.

Consider $M_{5,8(p+1)+4}$. As $M_{5,8(p+1)+4} = M_{5,8p+6} + M_{5,6}$, just as above we have $\tau(M_{5,8(p+1)+4}) \geq \tau(M_{5,8p+6}) + \tau(M_{5,6})$; consequently, by the induction hypothesis, $\tau(M_{5,8(p+1)+4}) \geq (11p+8)+8 = 11(p+1)+5$, as required.

Consider $M_{5,8(p+1)+5}$. As $M_{5,8(p+1)+5} = M_{5,8p+6} + M_{5,7}$, just as above we have $\tau(M_{5,8(p+1)+5}) \geq \tau(M_{5,8p+6}) + \tau(M_{5,7})$; consequently, by the induction hypothesis, $\tau(M_{5,8(p+1)+5}) \geq (11p+8)+9 = 11(p+1)+6$, as required.

Consider $M_{5,8(p+1)+6}$. As $M_{5,8(p+1)+6} = M_{5,8(p+1)} + M_{5,6}$, just as above we have $\tau(M_{5,8(p+1)+6}) \geq \tau(M_{5,8(p+1)}) + \tau(M_{5,6})$; consequently, by above, $\tau(M_{5,8(p+1)+6}) \geq (11(p+1)-1)+8 = 11(p+1)+7$. Suppose that $\tau(M_{5,8(p+1)+6}) = 11(p+1)+7$ and let Z be a minimal feedback vertex set of $M_{5,8(p+1)+6}$. The first $8(p+1)$ columns of the tile $M^Z_{5,8(p+1)+6}$ yield a minimal feedback vertex set X of $M_{5,8(p+1)}$ and the last 6 columns a minimal feedback vertex set Y of $M_{5,6}$. By above, column $8(p+1)$ contains no vertices of X; hence, column $8(p+1)+1$ contains at least 2 vertices of Y. As $\tau(M_{5,5}) = 6$, column $8(p+1)+1$ must contain exactly 2 vertices of Y. Column $8(p+1)+2$ contains at least 1 vertex of Y, as otherwise there would be a cycle involving vertices on columns $8(p+1)$, $8(p+1)+1$ and $8(p+1)+2$. Also, as $\tau(M_{5,4}) = 5$, column $8(p+1)+2$ must contain exactly 1 vertex of Y. Hence, the rightmost 5 columns of $M^Z_{5,8(p+1)+6}$ induce a minimal feedback vertex set of $M_{5,5}$ with 1 vertex in column $8(p+1)+2$. Such minimal feedback vertex sets are classified in Fig. 12 and it is easy to see that no matter which of the minimal feedback vertex sets we try, we obtain a contradiction (this is even more apparent given that there is only one configuration for the vertices in columns $8(p+1)+1$ and $8(p+1)+2$). Hence, $\tau(M_{5,8(p+1)+6}) = 11(p+1)+8$ as required.

Consider $M_{5,8(p+1)+7}$. As $M_{5,8(p+1)+7} = M_{5,8(p+1)+1} + M_{5,6}$, just as above we have $\tau(M_{5,8(p+1)+7}) \geq \tau(M_{5,8(p+1)+1}) + \tau(M_{5,6})$; consequently, by the induction hypothesis, $\tau(M_{5,8(p+1)+7}) \geq 11(p+1)+8$. Suppose that $\tau(M_{5,8(p+1)+7}) = 11(p+1)+8$.

Reasoning almost identically to as in the previous case yields a contradiction. Hence, $\tau(M_{5,8(p+1)+7}) = 11(p+1)+9$, as required. The result follows by induction. □

Consequently, we can draw all our results together in the following theorem.

Theorem 5 *There exists a computable function $f(n,m)$ such that the size of a minimal feedback vertex set of the grid $M_{n,m}$, where $(n,m) \in \{(n,m) : n \geq 2, m \geq 2\}$, is one of $f(n,m)$, $f(n,m)+1$ or $f(n,m)+2$.*

Of course, the word 'computable', whilst strictly correct, is somewhat inappropriate as the function f can be described very concisely according to the different cases arising in this section. Also, $\tau(M_{n,m})$ is known exactly for a lot of different cases.

4 Feedback vertex sets in butterflies

In this section, we improve known bounds on the size of minimal feedback vertex sets in butterflies. We begin with some basic structural decompositions.

If $0 \leq j \leq 2^d - 1$ then denote by $bit(d,j)$ the bit-string of length d that is the binary representation of j. Also, for any bit-string b of length d, denote by $bin(b)$ the integer whose binary representation as a bit string of length d is b. For any two bit-strings b and b', denote the concatenation of b and b' as bb'.

Fix some $d \geq 2$. Let b be a bit-string whose length, $|b|$, is at least 1 and at most $d-1$. Define the subgraph B_d^b of B_d as the subgraph of B_d induced by the vertices of the set

$$\{v_{i,j} : |b| \leq i \leq d, 0 \leq j \leq 2^d - 1, j \equiv bin(b) \bmod 2^{|b|}\};$$

that is, the vertices on rows $|b|, |b|+1, \ldots, d$ whose column names (when written in binary) end in b. The sub-graphs B_4^1 (with edges in bold) and B_4^{10} (with dashed edges) are illustrated in Fig. 13.

Lemma 6 *Let $d \geq 2$ and let b be a bit-string whose length is at least 1 and at most $d-1$. The subgraph B_d^b of B_d is isomorphic to $B_{d-|b|}$ via the isomorphism $\beta_b : B_{d-|b|} \to B_d^b$ given by $\beta_b(v_{i,j}) = v_{i',j'}$, with $i' = i + |b|$ and $j' = bit(d-|b|, j)b$.*

Proof For B_d^0 and B_d^1 (that is, for the bit-strings $b = 0$ and $b = 1$), the definition of B_d yields the result. For other bit-strings, the result then follows by a simple induction. □

Lemma 7 *Let $d \geq 2$ and let the set of vertices U of B_d be defined as $\{v_{i,j} : 0 \leq i \leq d-1, 0 \leq j \leq 2^d - 1\}$. The subgraph $B_d[U]$ of B_d consists of two disjoint copies B_d^l and B_d^r of B_{d-1} where the isomorphisms $\beta_l : B_{d-1} \to B_d^l$ and $\beta_r : B_{d-1} \to B_d^r$ are given by $\beta_l(v_{i,j}) = v_{i,j}$ and $\beta_r(v_{i,j}) = v_{i,j+2^{d-1}}$.*

Proof Immediate from the definition of B_d. □

We are now in a position to improve the lower bound on the size of a minimal feedback vertex set of B_d as established in [1], namely

$$lb_d^{CKK} = \frac{(d-1)2^d + 1}{3}.$$

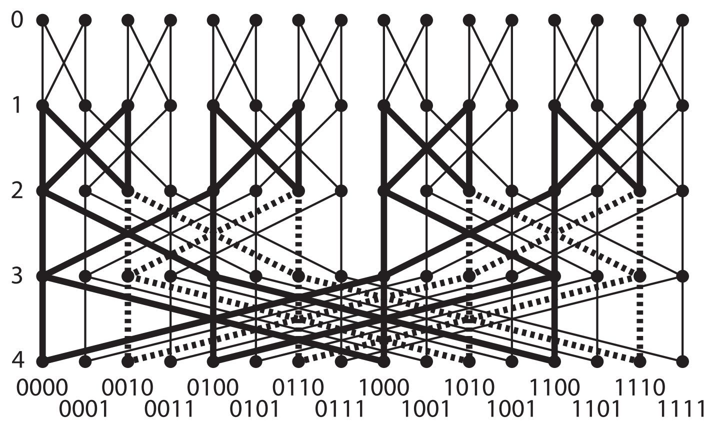


Figure 13. Butterflies contained within butterflies.

Our improvement is obtained by refining the general theorem used in [1] to obtain this lower bound.

Proposition 8 *Let $G = (V, E)$ be a graph of maximal degree δ and let F be a feedback vertex set of G, with H the subgraph $G \setminus F$. Let F_i be the set of vertices in F of degree i in G, let P be the set of edges of G induced by the vertices of F and let c be the number of connected components of H. Then*

$$|E| - (\Sigma_{i=1}^{\delta} i|F_i|) + |P| = |V| - |F| - c.$$

Proof The number of edges of E incident with a vertex of F is

$$(\Sigma_{i=1}^{\delta} i|F_i|) - |P|.$$

As F is a feedback vertex set, H is a forest and so the number of edges of H is equal to the number of vertices of H minus c; that is,

$$|E| - (\Sigma_{i=1}^{\delta} i|F_i|) + |P| = |V| - |F| - c$$

and the result follows. □

By applying Proposition 8 to B_d, we obtain that (with the definitions as in the statement of Proposition 8),

$$3|F| - 2|F_2| = (d-1)2^d + |P| + c.$$

Note that as $|P|$ and $|F_2|$ are at least 0 and c is at least 1, we obtain the lower bound lb_d^{CKK} on the size of a feedback vertex set of B_d, as was done in [1].

However, we can use Proposition 8 to obtain an improved lower bound on the size of a feedback vertex set of B_d.

Proposition 9 *Let $d \geq 4$. Any feedback vertex set of B_d has size at least*

$$\frac{(d-1)2^d + 4}{3}.$$

Proof Consider B_d, for $d \geq 4$, and let F be a minimal feedback vertex set of B_d. Let H be the subgraph $B_d \setminus F$ of B_d. We may assume that there are no vertices of F on row 0 nor on row d (as if, for example, the vertex $v_{0,0}$ is in F then we can replace $v_{0,0}$ in F with the vertex $v_{1,0}$ and still obtain a minimal feedback vertex set).

By Lemma 7, the subgraphs B_d^l and B_d^r of B_d are isomorphic to B_{d-1}. Also, as every cycle (of length 4) in B_d involving only vertices on rows $d-1$ and d must contain at least one vertex of F (with such a vertex of F being on row $d-1$), there is no path in $B_d \setminus F$ from a vertex of $B_d^l \setminus F$ to a vertex of $B_d^r \setminus F$ (also, note that $B_d^l \setminus F$ and $B_d^r \setminus F$ are both non-empty, as no vertex on row 0 is in F).

Consider B_d^l. By Lemma 6, $B_d^l \cap B_d^0$ and $B_d^l \cap B_d^1$ are isomorphic to B_{d-2}. Moreover, as every cycle (of length 4) in B_d involving only vertices on rows 0 and 1 must contain at least one vertex of F (with such a vertex of F being on row 1), there is no path in $B_d \setminus F$ from a vertex of $B_d^l \cap B_d^0$ to a vertex of $B_d^l \cap B_d^1$. Furthermore, as $d \geq 4$, both $(B_d^l \cap B_d^0) \setminus F$ and $(B_d^l \cap B_d^1) \setminus F$ are non-empty. Similar reasoning applies to $(B_d^r \cap B_d^0) \setminus F$ and $(B_d^r \cap B_d^1) \setminus F$. Thus, $B_d \setminus F$ consists of at least 4 connected components. Putting $c \geq 4$ into the equation in Proposition 8 (with G taken as B_d) yields the result. $\square$

So, if we denote our new lower bound from Proposition 9 as lb_d then we have that $lb_d = lb_d^{CKK} + 1$. Whilst our lower bound improvement is somewhat slight, we can make a more significant improvement on the best known upper bound on the size of a minimal feedback vertex set of B_d.

Definition 10 Define

$$V_l = V(B_d^l) = \{v_{i,j} : 0 \leq i \leq d-1, 0 \leq j \leq 2^{d-1} - 1\},$$
$$V_r = V(B_d^r) = \{v_{i,j} : 0 \leq i \leq d-1, 2^{d-1} \leq j \leq 2^d - 1\}$$

and

$$V_d = \{v_{d,j} : 0 \leq j \leq 2^d - 1\},$$

for all $d \geq 1$. $\square$

Note that $V(B_d) = V_d \cup V_l \cup V_r$.

Definition 11 Define

$$\begin{aligned} V_{d-1}^1 &= \{v_{d-1,j} : 0 \leq j \leq 2^{d-2} - 1\}, \\ V_{d-1}^2 &= \{v_{d-1,j} : 2^{d-2} \leq j \leq 2^{d-1} - 1\}, \\ V_{d-1}^3 &= \{v_{d-1,j} : 2^{d-1} \leq j \leq 3 \cdot 2^{d-2} - 1\}, \\ V_{d-1}^4 &= \{v_{d-1,j} : 3 \cdot 2^{d-2} \leq j \leq 2^d - 1\}, \end{aligned}$$

for all $d > 1$. $\square$

Definition 12 Define

$$\begin{aligned} V_{l,0} &= \{v_{i,j} : 0 \leq i \leq d-2, 0 \leq j \leq 2^{d-2} - 1\}, \\ V_{l,1} &= \{v_{i,j} : 0 \leq i \leq d-2, 2^{d-2} \leq j \leq 2^{d-1} - 1\}, \\ V_{r,0} &= \{v_{i,j} : 0 \leq i \leq d-2, 2^{d-1} \leq j \leq 3 \cdot 2^{d-2} - 1\}, \\ V_{r,1} &= \{v_{i,j} : 0 \leq i \leq d-2, 3 \cdot 2^{d-2} \leq j \leq 2^d - 1\}, \end{aligned}$$

for all $d > 1$. $\square$

Note that $B_d[V_{l,0}]$, $B_d[V_{l,1}]$, $B_d[V_{r,0}]$ and $B_d[V_{r,1}]$ are $(d-2)$-dimensional butterflies.

We illustrate the above definitions in Figs. 14 and 15, where as well as showing the decomposition of the butterflies B_5 and B_6 into their constituent parts, we also detail two particular feedback vertex sets. We shall use these feedback vertex sets presently and consequently we name them as $F_B(B_5)$ and $F_B(B_6)$, respectively. (Note that we split B_6 in Fig. 15 into two halves, due to its size.)

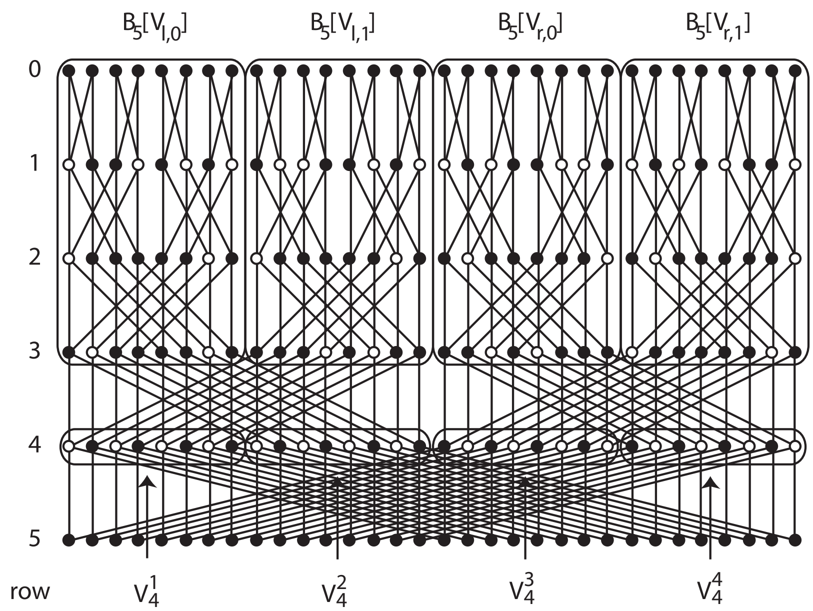


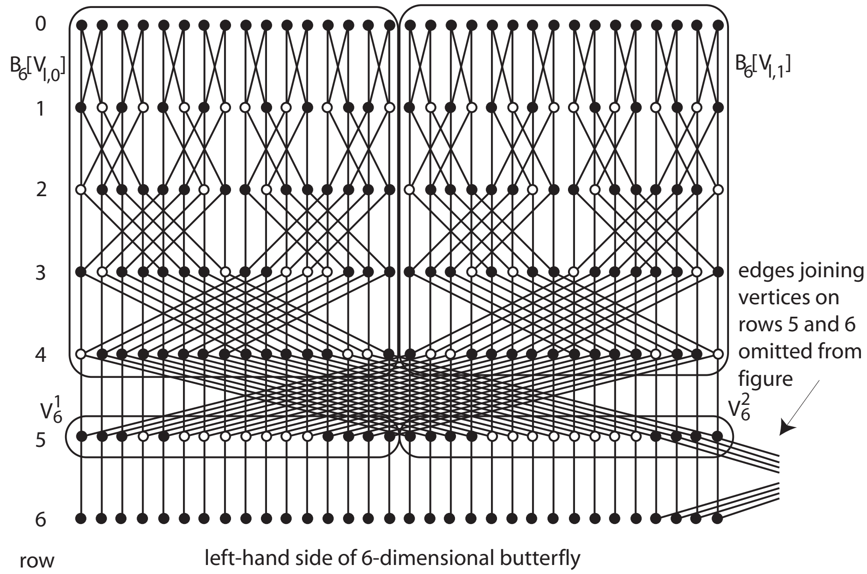
Figure 14. The butterfly B_5 with the feedback vertex set $F_B(B_5)$.

We leave it as an exercise for the reader to check that $F_B(B_5)$ and $F_B(B_6)$ are indeed feedback vertex sets of B_5 and B_6, respectively. (Readers might find it instructive to first of all convince themselves that there are no cycles involving only vertices on two subsequent levels, and then to rule out potential cycles involving vertices on the bottom two levels, then cycles involving vertices on the penultimate and antepenultimate levels, and so on.)

We are now in a position to detail our algorithm. Our algorithm outputs a feedback vertex set for B_d which we denote $F_B(B_d)$, and we denote the feedback vertex sets of B_d resulting from Algorithms A and L, in [2], by $F_A(B_d)$ and $F_L(B_d)$, respectively (recall, Algorithm A is Chang, Lin and Lee's algorithm and Algorithm L is Luccio's algorithm, first derived in [5]).

Algorithm B
Input: The d-dimensional butterfly B_d, where $d \geq 0$.
Output: The feedback vertex set $F_B(B_d)$ of B_d.
  If $d \in \{0,1,2,3,4\}$ then return $F_B(B_d) = F_A(B_d)$
  else if $d = 5$ then return $F_B(B_5)$
  else if $d = 6$ then return $F_B(B_6)$

```
else return F_B(B_d) = (V^2_{d-1} ∪ F_B(B_d[V_{l,0}]) ∪ F_L(B_d[V_{l,1}]))
                       ∪(V^3_{d-1} ∪ F_L(B_d[V_{r,0}]) ∪ F_B(B_d[V_{r,1}])).
```



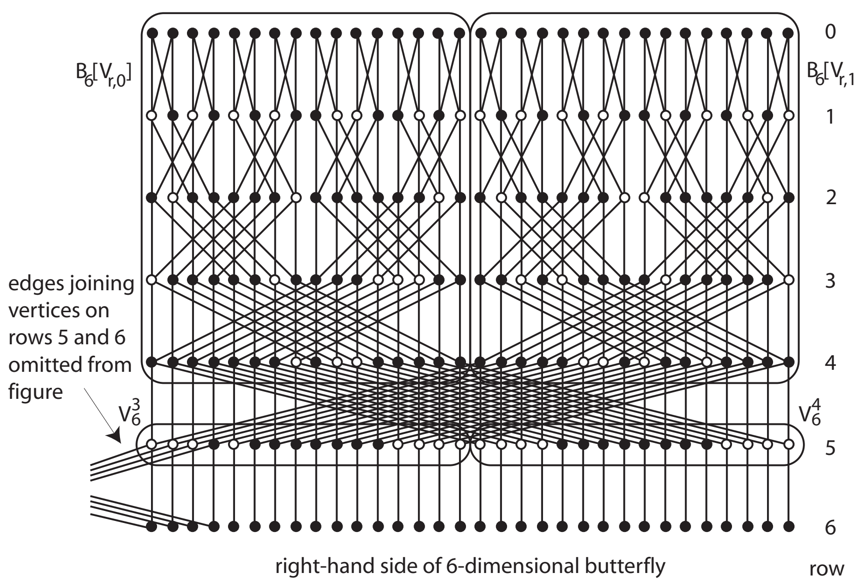


Figure 15. The butterfly B_6 with the feedback vertex set $F_B(B_6)$.

That is, we proceed just as Chang, Lin and Lee did except the base cases of our recursive algorithm are different. The fact that our algorithm produces a feedback vertex set follows from the following lemmas from [2].

Lemma 13 [2] *For $d > 1$, suppose that $F_{l,0}$ is a feedback vertex set of $B_d[V_{l,0}]$ and that $F_{l,1} = F_L(B_d[V_{l,1}])$. Then $F_{l,0} \cup F_{l,1} \cup V^2_{d-1}$ is a feedback vertex set of $B_d[V_l]$.* □

Lemma 14 [2] *For $d > 1$, suppose that $F_{r,0} = F_L(B_d[V_{r,0}])$ and that $F_{r,1}$ is a feedback vertex set of $B_d[V_{r,1}]$. Then $F_{r,0} \cup F_{r,1} \cup V^3_{d-1}$ is a feedback vertex set of $B_d[V_r]$.* □

Lemma 15 [2] *For $d > 1$, suppose that $F_l \supseteq V^2_{d-1}$ and that $F_r \supseteq V^3_{d-1}$ are feedback vertex sets of $B_d[V_l]$ and $B_d[V_r]$, respectively. Then $F_l \cup F_r$ is a feedback vertex set of B_d.* □

Theorem 16 *The set $F_B(B_d)$ is a feedback vertex set of B_d.*

Proof The proof follows from an elementary induction using the above lemmas. □

Not only can we use Chang, Lin and Lee's tools to prove that our algorithm is correct, we can also use their analysis to obtain the size of the feedback vertex set $F_B(B_d)$, for each $d \geq 5$.

From [2], the size $f_A(d)$ of the feedback vertex set $F_A(B_d)$ is

$$\left\lfloor \frac{(3d+1)2^d+1}{9} \right\rfloor - \frac{2^d-1}{3}, \text{ if } d \text{ is even,}$$

and

$$\left\lfloor \frac{(3d+1)2^d+1}{9} \right\rfloor - \frac{2^d - 2^{\lceil \frac{d}{2} \rceil} - 2^{\lfloor \frac{d}{2} \rfloor} + 1}{3}, \text{ if } d \text{ is odd.}$$

Consequently, $f_A(5) = 50$ and $f_A(6) = 114$; whereas, with $f_B(d)$ denoting the size of the feedback vertex set $F_B(B_d)$ produced by Algorithm B, $f_B(5) = 48$ and $f_B(6) = 110$.

A simple observation yields that

$f_A(7) - f_B(7) = 4;$
$f_A(8) - f_B(8) = 8;$
$f_A(9) - f_B(9) = 2(f_A(7) - f_B(7)) = 8;$
$f_A(10) - f_B(10) = 2(f_A(8) - f_B(8)) = 16;$
$f_A(11) - f_B(11) = 2(f_A(9) - f_B(9)) = 16;$
$f_A(12) - f_B(12) = 2(f_A(10) - f_B(10)) = 32;$
...

and a simple induction yields that $f_B(d)$ is equal to

$$\left\lfloor \frac{(3d+1)2^d+1}{9} \right\rfloor - \frac{2^d-1}{3} - 2^{\frac{d-2}{2}}, \text{ if } d \geq 6 \text{ is even,}$$

and

$$\left\lfloor \frac{(3d+1)2^d+1}{9} \right\rfloor - \frac{2^d - 2^{\lceil \frac{d}{2} \rceil} - 2^{\lfloor \frac{d}{2} \rfloor} + 1}{3} - 2^{\frac{d-3}{2}}, \text{ if } d \geq 5 \text{ is odd.}$$

Hence, the above are upper bounds on $\tau(B_d)$.

5 Conclusion

In this paper, we have improved the known upper and lower bounds on the sizes of minimal feedback vertex sets in grids and butterflies. We feel that the closeness of the resulting upper and lower bounds in grids should essentially close the investigation. The situation for butterflies is not so clear cut. Whilst we have managed to improve both upper and lower bonds, there is still some distance between the two bounds. We conjecture that the feedback vertex number for butterflies lies closer to our upper bound than our lower bound. Intuitively, we feel that our lower bound technique, which has only been applied at the 'extremities' of the butterfly, should be applicable 'within' the butterfly. Of course, we have so far been unable to do this.

References

[1] I. Caragiannis, C. Kaklamanis and P. Kanellopoulos, New bounds on the size of the minimum feedback vertex set in meshes and butterflies, *Information Processing Letters* **83** (2002) 275–280.

[2] M-S. Chang, C-H. Lin and C-M. Lee, New upper bounds on feedback vertex numbers in butterflies, *Information Processing Letters* **90** (2004) 279–285.

[3] W.J. Dally and B. Towles, *Principles and Practices of Interconnection Networks*, Morgan Kaufmann (2004).

[4] P. Festa, P.M. Pardalos and M.G.C. Resende, Feedback set problems, in: *Handbook of Combinatorial Optimization* (D.-Z. Du and P.M. Pardalos, eds.), Kluwer Academic Publishers, Supplement Vol. A (2000) 209-259.

[5] F.L. Luccio, Almost exact minimum feedback vertex sets in meshes and butterflies, *Information Processing Letters* **66** (1998) 59–64.