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Discrete subgroups of PU(2, 1) with screw parabolic elements

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Abstract

We give a version of Shimizu's lemma for groups of complex hyperbolic isometries one of whose generators is a parabolic screw motion. Suppose that G is a discrete group containing a parabolic screw motion A and let B be any element of G not fixing the fixed point of A. Our result gives a bound on the radius of the isometric spheres of B and B^{-1} in terms of the translation lengths of A at their centres. We use this result to give a sub-horospherical region precisely invariant under the stabiliser of the fixed point of A in G.

1. *Introduction*

Let G be a discrete subgroup of PSL(2, $\mathbb{R}$) containing the parabolic map $A(z) = z + t$ for some $t > 0$. Then Shimizu's lemma [**13**] says that for any $B(z) = (az + b)/(cz + d) \in G$ with $c \neq 0$ then $|c| \geqslant 1/t$. Geometrically this result says that the radius r_B of the isometric sphere of B satisfies $r_B \leqslant t$. Equivalently, the horoball U_t of height t is precisely invariant under G_∞ in G. Here G_∞ denotes those elements of G stabilising the point ∞. A set U is said to be *precisely invariant* under a subgroup H of G if $B(U) = U$ for all $B \in H$ and $B(U) \bigcap U = \varnothing$ for all $B \in G - H$.

Shimizu's lemma may be generalised to isometries of higher dimensional real hyperbolic space. If $A \in \mathrm{Isom}(\mathbf{H}^n_{\mathbb{R}})$ is a pure translation then the result generalises directly. On the other hand, when $n \geqslant 4$, parabolic maps do not have to be pure translations but may be screw motions. Ohtake [**9**] showed that when G contains a screw motion A, there is no uniform bound on the radii of isometric spheres of elements of G, nor is there a precisely invariant horoball. However, Waterman [**14**] showed that one may bound the radii of the isometric spheres of B and B^{-1} by a function of the product of the Euclidean translation length of A at their centres. In consequence, one can find a sub-horospherical region that is precisely invariant.

Viewing PSL(2, $\mathbb{R}$) as the isometry group of complex hyperbolic 1-space, $\mathbf{H}^1_{\mathbb{C}}$, we can seek to generalise Shimizu's lemma to higher dimensional complex hyperbolic isometries. For the case when A is a vertical translation this was done in [**5**], and a precisely invariant horoball was found in [**6**]. (See the next section for background material including the definitions of terms used.)

THEOREM 1·1 ([**5**, theorem 3·2]). *Let A be vertical translation by* $(0, t)$ *in* $\mathrm{PU}(2,1)$ *fixing* ∞. *Let B be any element of* $\mathrm{PU}(2,1)$ *not projectively fixing* ∞ *and let* r_B *denote the radius of the isometric sphere of B. If*

$$\frac{t}{{r_B}^2} < 1,$$

then the group $\langle A,\ B\rangle$ *is not discrete.*

COROLLARY 1·2 ([**6**, theorem 2·2]). *Let A be the vertical translation of Theorem* 1·1 *and let G be any discrete subgroup of* $\mathrm{PU}(2,1)$ *containing A. Then the horoball* U_t *of height t is precisely invariant under* G_∞ *in G.*

For the case where the stabiliser of ∞ is a cyclic group of non-vertical translations it was shown in [**10**] that there is no uniform bound on the radii of isometric spheres, nor is there a precisely invariant horoball. However, there is a bound on the radii of isometric spheres in terms of the translation lengths at their centres. This leads to a precisely invariant sub-horospherical region.

THEOREM 1·3 ([**11**, theorem 2·1]). *Let A be a Heisenberg translation by* (τ, t) *in* $\mathrm{PU}(2,1)$ *fixing* ∞. *Let B be any element of* $\mathrm{PU}(2,1)$ *not projectively fixing* ∞ *and let* r_B *denote the radius of the isometric sphere of B. If*

$$\frac{\rho_0(B(\infty), AB(\infty))\rho_0(B^{-1}(\infty), AB^{-1}(\infty)) + 4|\tau|^2}{{r_B}^2} < 1$$

then $\langle A,\ B\rangle$ *is not discrete.*

COROLLARY 1·4 ([**11**, theorem 3·2]). *Let A be the non-vertical translation of Theorem* 1·3 *and let G be any discrete subgroup of* $\mathrm{PU}(2,1)$ *and suppose that the stabiliser of* ∞ *in G is* $G_\infty = \langle A\rangle$. *Then the sub-horospherical region U is precisely invariant under* G_∞ *in G where*

$$U = \left\{z = (\zeta, v, u) \in \mathbf{H}^2_{\mathbb{C}} : u > \rho_0(z, Az)^2 + 8|\tau|^2\right\}.$$

In this paper we consider parabolic screw motions A in $\mathbf{H}^2_{\mathbb{C}}$. If the rotational part of A has finite order then some power is a vertical translation and we can use Theorem 1·1. Thus we concentrate on screw parabolic maps A with infinite order rotational part. If such an A is in a discrete group then the only elements of this group sharing a fixed point with A are screw parabolic and boundary elliptic maps with the same axis as A. Our results concern screw motions where the angle of rotation is small and is positively oriented relative to the direction of translation. It is clear that any parabolic screw motion with infinite order rotational part has a power satisfying these conditions.

THEOREM 1·5. *Let A be a positively oriented screw parabolic element of* $\mathrm{PU}(2,1)$ *fixing* ∞. *Let* $e^{i\theta} \in \mathrm{U}(1)$ *denote the rotational part of A and suppose that* $|e^{i\theta} - 1| < 1/4$. *Let B be any element of* $\mathrm{PU}(2,1)$ *not projectively fixing* ∞ *and let* r_B *denote the radius of the isometric sphere of B. If*

$$\frac{\rho_0(B(\infty), AB(\infty))\rho_0(B^{-1}(\infty), AB^{-1}(\infty))}{{r_B}^2} < \left(\frac{1 + \sqrt{1 - 4|e^{i\theta} - 1|}}{2}\right)^2,$$

then $\langle A,\ B\rangle$ *is not discrete.*

COROLLARY 1·6. *Let A be the screw parabolic map* $A\colon (\zeta, v, u) \mapsto (e^{i\theta}\zeta, v+t, u)$ *where*

$|e^{i\theta}-1| < 2/9$ *and* $t\sin(\theta) > 0$. *Let* G *be a discrete subgroup of* PU(2, 1) *for which any element of* G_∞ *has the same axis as* A. *Then the sub-horospherical region* U *defined by*

$$U = \left\{ (\zeta, v, u) \colon u > \frac{2|2|\zeta|^2(e^{i\theta}-1)+it|}{1-6|e^{i\theta}-1|+\sqrt{1-4|e^{i\theta}-1|}} \right\}$$

is precisely invariant under G_∞ *in* G.

We remark that if θ is a rational multiple of π then some power of A is a vertical translation and so Corollary 1·2 applies. On the other hand, if θ is an irrational multiple of π then, since G is discrete, G_∞ must be a group of screw parabolic maps (and possibly boundary elliptic maps) with the same axis as A, and so some power of A satisfies the conditions of Corollary 1·6. Observe that in the limit as θ tends to zero, Theorem 1·5 and Corollary 1·6 become Theorem 1·1 and Corollary 1·2 respectively.

In [**4**] a bound on the radii of the isometric spheres was found but this result does not immediately give a precisely invariant sub-horospherical region. In Section 5 we explore the relation between this bound and Theorem 1·5. We also show that, for small θ, Theorem 1·5 implies the relevant case of a result of Basmajian and Miner [**1**] (compare [**7**, **8**]).

2. *Background*

We begin with some background material on complex hyperbolic geometry. Much of this can be found in Goldman's book [**2**] or in the introduction to papers in the bibliography. The Siegel domain model of complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$ with horospherical coordinates is

$$\mathbf{H}^2_{\mathbb{C}} = \{z = (\zeta, v, u) \colon \zeta \in \mathbb{C},\ v \in \mathbb{R},\ u \in \mathbb{R}_+\}.$$

The level sets where u is constant are called *horospheres* and each of these bounds a *horoball*: for each $t > 0$ the horoball U_t of height t is defined to be

$$U_t = \left\{ (\zeta, v, u) \in \mathbf{H}^2_{\mathbb{C}} \colon u > t \right\}.$$

The boundary of the Siegel domain comprises the one point compactification of the horosphere of height $t = 0$:

$$\partial\mathbf{H}^2_{\mathbb{C}} = \{z = (\zeta, v, 0) \colon \zeta \in \mathbb{C},\ v \in \mathbb{R}\} \bigcup \{\infty\}.$$

A subset U of $\mathbf{H}^2_{\mathbb{C}}$ is called a *sub-horospherical region* based at ∞ if

(i) there exists $t > 0$ so that U is contained in U_t;
(ii) for each $\zeta \in \mathbb{C}$ and $v \in \mathbb{R}$ there exists $t = t(\zeta, v) > 0$ so that $(\zeta, v, u) \in U$ for all $u > t$;
(iii) U does not contain any horoball.

Let $\mathbb{C}^{2,1}$ be the complex vector space with the indefinite Hermitian inner product given by the matrix

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Points in $\mathbf{H}^2_{\mathbb{C}}$ may be identified with negative vectors in $\mathbb{C}^{2,1}$ and points of $\partial\mathbf{H}^2_{\mathbb{C}}$ may be identified with null vectors in $\mathbb{C}^{2,1}$ by the map $\psi\colon \overline{\mathbf{H}}^2_{\mathbb{C}} \to \mathbb{C}^{2,1}$ given by

$$\psi\colon (\zeta, v, u) \longmapsto \begin{bmatrix} -|\zeta|^2 - u + iv \\ \sqrt{2}\zeta \\ 1 \end{bmatrix}, \qquad \psi\colon \infty \longmapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Using this identification, we obtain a projective action of U(2, 1), the unitary group of J, on $\overline{\mathbf{H}}^2_{\mathbb{C}}$. The kernel of this action is the set of diagonal matrices, and so we may take the quotient of U(2, 1) by this kernel to obtain PU(2, 1), which we identify with the (holomorphic) isometries of $\mathbf{H}^2_{\mathbb{C}}$. We can characterise elements B of U(2, 1) by saying that $B^{-1} = JB^*J$. That is

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} \overline{j} & \overline{f} & \overline{c} \\ \overline{h} & \overline{e} & \overline{b} \\ \overline{g} & \overline{d} & \overline{a} \end{bmatrix}. \tag{2·1}$$

An immediate consequence of this is

LEMMA 2·1. *If B has the form* (2·1) *then*

$$|g| = |dh - eg|, \quad |d| = |bg - ah|.$$

Proof. From elementary linear algebra, we can express B^{-1} in terms of its determinant and adjoint as

$$B^{-1} = \frac{1}{\det(B)} \begin{bmatrix} ej - fh & ch - bj & bf - ce \\ fg - dj & aj - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{bmatrix}.$$

Comparing this expression with (2·1) we see that

$$\overline{g}\,\det(B) = dh - eg, \quad \overline{d}\,\det(B) = bg - ah.$$

Since B is unitary we have $|\det(B)| = 1$ and the result follows.

As well as the Bergman metric, which is its intrinsic hyperbolic metric, the Siegel domain has another metric, the *Cygan metric* ρ_0. The Siegel domain is not complete with respect to the Cygan metric, which should be thought of as an analogue of the Euclidean metric on the upper half plane model of the hyperbolic plane. The Cygan metric is defined by the following distance function

$$\rho_0((\zeta_1, v_1, u_1),\ (\zeta_2, v_2, u_2)) = \left||\zeta_1 - \zeta_2|^2 + |u_1 - u_2| + iv_1 - iv_2 + 2i\Im(\zeta_1\overline{\zeta_2})\right|^{1/2}.$$

Let $B \in \mathrm{PU}(2,1)$, that is B is a holomorphic (Bergman) isometry of $\mathbf{H}^2_{\mathbb{C}}$. Suppose that B does not fix ∞, equivalently $g \neq 0$ when B has the form (2·1). Then the *isometric sphere* of B is the sphere in the Cygan metric with centre $B^{-1}(\infty)$ and radius $r_B = 1/\sqrt{|g|}$. In horospherical coordinates

$$B^{-1}(\infty) = \left(\frac{\overline{h}}{\sqrt{2}\,\overline{g}},\ -\Im\left(\frac{j}{g}\right),\ 0\right).$$

Similarly the isometric sphere of B^{-1} is the Cygan sphere of radius $1/\sqrt{|g|}$ with centre

$$B(\infty) = \left(\frac{d}{\sqrt{2}\,g},\ \Im\left(\frac{a}{g}\right),\ 0\right).$$

We will need the following proposition.

LEMMA 2·2 ([**7**, proposition 2·4]). *Let B be any element of* PU(2, 1) *that does not fix* ∞ *and let r_B be the radius of its isometric sphere. Then for all $z \in \partial\mathbf{H}^2_{\mathbb{C}} - \{\infty,\ B^{-1}(\infty)\}$ we have*

$$\rho_0(B(z),\ B(\infty)) = \frac{{r_B}^2}{\rho_0(z,\ B^{-1}(\infty))}.$$

An isometry of complex hyperbolic space is called *parabolic* if it has a unique fixed point on $\partial\mathbf{H}^2_{\mathbb{C}}$. Conjugating if necessary, we assume that this fixed point is ∞. Any parabolic element of PU(2, 1) is a Cygan isometry. A parabolic isometry is a *screw motion* if and only if it is conjugate to the map A: $\mathbf{H}^2_{\mathbb{C}} \to \mathbf{H}^2_{\mathbb{C}}$ given by

$$A\colon (\zeta,\ v,\ u) \longmapsto (e^{i\theta}\zeta,\ v+t,\ u).$$

The *axis* of A is the complex line $L_A = \{(0, v, u) \in \mathbf{H}^2_{\mathbb{C}}\}$, and A rotates about the axis with *rotational part* $e^{i\theta} \in \mathrm{U}(1)$ and translates along the axis by a Cygan distance $\sqrt{|t|} \in \mathbb{R}_+$, its *translation length*. (If $t = 0$ then such a map is *boundary elliptic*.) A screw motion is *positively oriented* if $t\sin(\theta) > 0$ and *negatively oriented* if $t\sin(\theta) < 0$. As an element of PU(2, 1), the screw motion A is given by

$$A = \begin{bmatrix} 1 & 0 & it \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (2·2)$$

The Cygan translation length of A is

$$\begin{aligned} \rho_0((\zeta, v, u),\ A(\zeta, v, u)) &= \rho_0((\zeta, v, u),\ (e^{i\theta}\zeta, v+t, u)) \\ &= |2|\zeta|^2(e^{i\theta}-1) + it|^{1/2}. \end{aligned}$$

The translation length of positively oriented screw motions is an increasing function $|\zeta|$ but this is not true of those that are negatively oriented. We first estimate the translation length from below:

LEMMA 2·3. *Let A be a positively oriented screw motion. Then*

$$2|\zeta|^2|e^{i\theta}-1| \leqslant |2|\zeta|^2(e^{i\theta}-1) + it|.$$

Proof.

$$\begin{aligned} |2|\zeta|^2(e^{i\theta}-1) + it|^2 &= 4|\zeta|^4|e^{i\theta}-1|^2 + 4|\zeta|^2\sin(\theta)t + t^2 \\ &\geqslant 4|\zeta|^4|e^{i\theta}-1|^2. \end{aligned}$$

We now compare the Cygan translation lengths of A at different points of $\overline{\mathbf{H}}^2_{\mathbb{C}}$.

LEMMA 2·4. *Let A be a positively oriented screw motion. Then*

$$|2|\xi|^2(e^{i\theta}-1) + it|^{1/2} \leqslant |2|\zeta|^2(e^{i\theta}-1) + it|^{1/2} + |\zeta-\xi|\,|2(e^{i\theta}-1)|^{1/2}.$$

In particular, for any points z and w

$$\rho_0(A(w), w) \leqslant \rho_0(A(z), z) + \rho_0(z, w)|2(e^{i\theta}-1)|^{1/2}.$$

Proof. Using Lemma 2·3 and the triangle inequality we have

$$\begin{aligned}
t_A(w)^2 &= |2|\xi|^2(e^{i\theta}-1)+it| \\
&\leqslant |2|\zeta|^2(e^{i\theta}-1)+it| + |2(|\zeta|^2-|\xi|^2)(e^{i\theta}-1)| \\
&= |2|\zeta|^2(e^{i\theta}-1)+it| + 2||\zeta|-|\xi||(|\zeta|+|\xi|)|e^{i\theta}-1| \\
&\leqslant |2|\zeta|^2(e^{i\theta}-1)+it| + 2|\xi-\zeta|(2|\zeta|+|\xi-\zeta|)|e^{i\theta}-1| \\
&= t_A(z)^2 + 2|2|\zeta|^2(e^{i\theta}-1)|^{1/2}|2|\xi-\zeta|^2(e^{i\theta}-1)|^{1/2} + |2|\xi-\zeta|^2(e^{i\theta}-1)| \\
&\leqslant t_A(z)^2 + 2t_A(z)|2|\xi-\zeta|^2(e^{i\theta}-1)|^{1/2} + |2|\xi-\zeta|^2(e^{i\theta}-1)| \\
&= (t_A(z) + |2|\xi-\zeta|^2(e^{i\theta}-1)|^{1/2})^2.
\end{aligned}$$

3. *Proof of the main theorem*

In this section we prove Theorem 1·5. The basic structure of this proof resembles Shimizu's original proof [**13**] and all its generalisations. In particular, it should be compared to the proof of [**11**, theorem 2·1] or [**14**, theorem 8].

Consider the sequence B_n defined by $B_0 = B$ and $B_{n+1} = B_n A B_n{}^{-1}$. We write

$$B_n = \begin{bmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & h_n & j_n \end{bmatrix}.$$

If A has the form (2·2) then

$$\begin{aligned}
B_{n+1} &= \begin{bmatrix} a_{n+1} & b_{n+1} & c_{n+1} \\ d_{n+1} & e_{n+1} & f_{n+1} \\ g_{n+1} & h_{n+1} & j_{n+1} \end{bmatrix} \\
&= \begin{bmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & h_n & j_n \end{bmatrix} \begin{bmatrix} 1 & 0 & it \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{j}_n & \overline{f}_n & \overline{c}_n \\ \overline{h}_n & \overline{e}_n & \overline{b}_n \\ \overline{g}_n & \overline{d}_n & \overline{a}_n \end{bmatrix}.
\end{aligned}$$

Thus the entries of B_{n+1} are (see [**4**, page 660])

$$a_{n+1} = 1 + b_n\overline{h}_n(e^{i\theta}-1) + a_n\overline{g}_n it, \tag{3·1 a}$$
$$b_{n+1} = b_n\overline{e}_n(e^{i\theta}-1) + a_n\overline{d}_n it, \tag{3·1 b}$$
$$c_{n+1} = |b_n|^2(e^{i\theta}-1) + |a_n|^2 it, \tag{3·1 c}$$
$$d_{n+1} = e_n\overline{h}_n(e^{i\theta}-1) + d_n\overline{g}_n it, \tag{3·1 d}$$
$$e_{n+1} = 1 + |e_n|^2(e^{i\theta}-1) + |d_n|^2 it, \tag{3·1 e}$$
$$f_{n+1} = e_n\overline{b}_n(e^{i\theta}-1) + d_n\overline{a}_n it, \tag{3·1 f}$$
$$g_{n+1} = |h_n|^2(e^{i\theta}-1) + |g_n|^2 it, \tag{3·1 g}$$
$$h_{n+1} = h_n\overline{e}_n(e^{i\theta}-1) + g_n\overline{d}_n it, \tag{3·1 h}$$
$$j_{n+1} = 1 + h_n\overline{b}_n(e^{i\theta}-1) + g_n\overline{a}_n it. \tag{3·1 i}$$

For all $n \geqslant 0$ define the quantities r_n, t_n and t_n' to be the radius of the isometric sphere of B_n and the Cygan translation lengths of A at $B_n^{-1}(\infty)$ and $B_n(\infty)$. That is, $r_n = |g_n|^{-1/2}$ and

$$\begin{aligned}
t_n &= \rho_0\big(AB_n^{-1}(\infty), B_n^{-1}(\infty)\big) = ||h_n/g_n|^2(e^{i\theta}-1)+it|^{1/2}, \\
t_n' &= \rho_0(B_n(\infty), AB_n(\infty)) = ||d_n/g_n|^2(e^{i\theta}-1)+it|^{1/2}.
\end{aligned}$$

Use of (3·1 g) gives:

$$\frac{1}{r_{n+1}} = |g_{n+1}|^{1/2} = |g_n|\,\big||h_n/g_n|^2(e^{i\theta}-1)+it\big|^{1/2} = \frac{t_n}{{r_n}^2}. \tag{3·2}$$

From Lemma 2·3 we have

$$\left|\frac{h_n}{g_n}\right| |e^{i\theta}-1|^{1/2} \leqslant t_n, \quad \left|\frac{d_n}{g_n}\right| |e^{i\theta}-1|^{1/2} \leqslant t_n'. \tag{3·3}$$

Using Lemma 2·1 and (3·3) we have:

$$\begin{aligned}\left|\frac{d_{n+1}}{g_{n+1}} - \frac{d_n}{g_n}\right| &= \left|\frac{(e_n g_n - d_n h_n)\overline{h}_n(e^{i\theta}-1)}{g_{n+1}g_n}\right| \\ &= \left|\frac{h_n}{g_n}\right| \frac{|g_n|}{|g_{n+1}|}|e^{i\theta}-1| \\ &\leqslant t_n\frac{{r_{n+1}}^2}{{r_n}^2}|e^{i\theta}-1|^{1/2} \\ &= r_{n+1}|e^{i\theta}-1|^{1/2}.\end{aligned}$$

Similarly

$$\begin{aligned}\left|\frac{\overline{h}_{n+1}}{\overline{g}_{n+1}} - \frac{d_n}{g_n}\right| &= \left|\frac{(e_n g_n - d_n h_n)\overline{h}_n(e^{-i\theta}-1)}{\overline{g}_{n+1}g_n}\right| \\ &\leqslant r_{n+1}|e^{i\theta}-1|^{1/2}.\end{aligned}$$

Thus using Lemma 2·4 we have

$$\begin{aligned}t_{n+1}' &\leqslant t_n' + \left|\frac{d_{n+1}}{g_{n+1}} - \frac{d_n}{g_n}\right| |e^{i\theta}-1|^{1/2} \\ &\leqslant t_n' + r_{n+1}|e^{i\theta}-1|.\end{aligned}$$

Dividing by r_{n+1} and using (3·2) we obtain

$$\frac{t_{n+1}'}{r_{n+1}} \leqslant \frac{t_n t_n'}{{r_n}^2} + |e^{i\theta}-1|. \tag{3·4}$$

Similarly we have

$$\frac{t_{n+1}}{r_{n+1}} \leqslant \frac{t_n t_n'}{{r_n}^2} + |e^{i\theta}-1|. \tag{3·5}$$

Let δ denote the larger root of the equation $\delta^2 + |e^{i\theta}-1| = \delta$, that is

$$\delta = \frac{1+\sqrt{1-4|e^{i\theta}-1|}}{2} < 1.$$

We claim that $t_n/r_n < \delta$ and $t_n'/r_n < \delta$ for all $n \geqslant 1$. This follows inductively from our main hypothesis, namely $t_0t_0'/{r_0}^2 < \delta^2$, and the inequalities (3·4) and (3·5).

We now use $t_n/r_n < \delta$ and $t_n'/r_n < \delta$ for all $n \geqslant 1$ to show that B_n is a converging sequence of distinct elements of PU(2, 1). From (3·2), we see that

$$|g_{n+1}| = \frac{1}{{r_{n+1}}^2} = \frac{{t_n}^2}{{r_n}^2}\cdot\frac{1}{{r_n}^2} < \delta^2|g_n|.$$

Thus $|g_n| < \delta^{2n-2}|g_1|$ and g_n tends to zero as n tends to infinity.

From (3·3) we have

$$|h_n|^2 \leqslant \frac{{t_n}^2|g_n|^2}{|e^{i\theta}-1|} = \frac{{t_n}^2}{{r_n}^2}\frac{|g_n|}{|e^{i\theta}-1|} \leqslant \frac{\delta^{2n}|g_1|}{|e^{i\theta}-1|}.$$

Likewise

$$|d_n|^2 \leqslant \frac{{t'_n}^2}{{r_n}^2}\frac{|g_n|}{|e^{i\theta}-1|} \leqslant \frac{\delta^{2n}|g_1|}{|e^{i\theta}-1|}.$$

Hence d_n and h_n tend to zero as n tends to infinity.

Using Lemma 2·1 we have $|e_n g_n - d_n h_n| = |g_n|$ and so

$$|e_n| \leqslant 1 + \left|\frac{d_n h_n}{g_n}\right| \leqslant 1 + \frac{t_n t'_n}{{r_n}^2}\cdot\frac{1}{|e^{i\theta}-1|} \leqslant 1 + \frac{\delta^2}{|e^{i\theta}-1|} = \frac{\delta}{|e^{i\theta}-1|}.$$

Thus

$$\begin{aligned} ||e_{n+1}|^2 - 1| &\leqslant ||e_n|^2 - 1|\,|e_n|^2\,|e^{i\theta}-1|^2 + 2|d_n|^2|e_n|^2|e^{i\theta}-1|t + |d_n|^4t^2 \\ &\leqslant ||e_n|^2-1|\delta^2 + \frac{2\delta^{2n+2}t|g_1|}{|e^{i\theta}-1|^2} + \frac{\delta^{4n}t^2|g_1|^2}{|e^{i\theta}-1|^2}. \end{aligned}$$

Therefore $|e_n|^2 - 1$ tends to zero and hence e_{n+1} tends to $e^{i\theta}$ as n tends to infinity.

Also using Lemma 2·1 we have $|b_n g_n - a_n h_n| = |d_n|$ and so

$$\left|\frac{a_{n+1}-1}{g_{n+1}} - \frac{a_n}{g_n}\right| = \left|\frac{(b_n g_n - a_n h_n)\overline{h_n}(e^{i\theta}-1)}{g_{n+1}g_n}\right| = \left|\frac{d_n h_n}{g_n}\right|\frac{|e^{i\theta}-1|}{|g_{n+1}|}$$

Therefore

$$|a_{n+1}-1| \leqslant |a_n|\frac{|g_{n+1}|}{|g_n|} + \left|\frac{d_n h_n}{g_n}\right||e^{i\theta}-1| \leqslant |a_n - 1|\delta^2 + 2\delta^2.$$

Hence

$$|a_n - 1| \leqslant |a_0 - 1|\delta^{2n} + \frac{2\delta^2(1-\delta^{2n})}{1-\delta^2}.$$

Thus $|a_n - 1|$ is bounded. Therefore

$$|b_{n+1}| \leqslant |b_n|\,|e_n|\,|e^{i\theta}-1| + |a_n|\,|d_n|t \leqslant |b_n|\delta + |a_n|\,|d_n|t$$

tends to zero as n tends to infinity. So

$$|a_{n+1}-1| \leqslant |b_n|\,|h_n|\,|e^{i\theta}-1| + \delta^{2n-2}|g_1|t + |a_n-1|\delta^{2n-2}|g_1|t$$

also tends to zero as n tends to infinity.

Finally, using (3·1 *c*), (3·1*f*) and (3·1 *i*) we see that B_n tends to A as n tends to infinity. Since none of the B_n fix ∞ we see that they are distinct. Hence $\langle A, B\rangle$ is not discrete. This proves the theorem.

4. *A precisely invariant sub-horospherical region*

In this section we prove Corollary 1·6, giving a precisely invariant sub-horospherical region for groups containing a screw parabolic map.

We first suppose that $B \in G - G_\infty$. We must show that $B(U) \cap U = \varnothing$. If $\langle A, B\rangle$ is discrete we have

$${r_B}^2 \leqslant \frac{\rho_0(B(\infty), AB(\infty))\rho_0(B^{-1}(\infty), AB^{-1}(\infty))}{\delta^2}$$

where

$$\delta = \frac{1+\sqrt{1-4|e^{i\theta}-1|}}{2}.$$

In what follows we will need to assume that $\delta^2 > 2|e^{i\theta}-1|$. A brief calculation shows that this is equivalent to $|e^{i\theta}-1| < 2/9$.

Using Lemma 2·2, a Cygan sphere of radius r centred at $B^{-1}(\infty)$ is mapped by B to a Cygan sphere or radius $r' = {r_B}^2/r$ centred at $B(\infty)$. Suppose that $z = (\zeta, v, u)$ is on the sphere S_0 of radius $r = \rho_0(AB^{-1}(\infty), B^{-1}(\infty))/\delta$ centred at $B^{-1}(\infty) = (\zeta_0, v_0, 0)$. Therefore

$$r^2 = |-|\zeta-\zeta_0|^2 - u + iv - iv_0 + 2i\Im(\zeta\overline{\zeta}_0)| \geqslant |\zeta-\zeta_0|^2 + u.$$

Thus we have

$$u \leqslant r^2 - |\zeta-\zeta_0|^2 = \frac{\rho_0(AB^{-1}(\infty), B^{-1}(\infty))^2}{\delta^2} - |\zeta-\zeta_0|^2.$$

Thus

$$\begin{aligned}
\delta^2 u &\leqslant \rho_0(AB^{-1}(\infty), B^{-1}(\infty))^2 - \delta^2|\zeta-\zeta_0|^2\\
&\leqslant (\rho_0(A(z),z) + (2|e^{i\theta}-1|)^{1/2}|\zeta-\zeta_0|)^2 - \delta^2|\zeta-\zeta_0|^2\\
&= \rho_0(A(z),z)^2 + 2\rho_0(A(z),z)(2|e^{i\theta}-1|)^{1/2}|\zeta-\zeta_0| - (\delta^2 - 2|e^{i\theta}-1|)|\zeta-\zeta_0|^2\\
&= \frac{\delta^2\,\rho_0(A(z),z)^2}{\delta^2-2|e^{i\theta}-1|} - (\delta^2-2|e^{i\theta}-1|)\left(\frac{(2|e^{i\theta}-1|)^{1/2}\rho_0(A(z),z)}{\delta^2-2|e^{i\theta}-1|} - |\zeta-\zeta_0|\right)^2\\
&\leqslant \frac{\delta^2\,\rho_0(A(z),z)^2}{\delta^2-2|e^{i\theta}-1|}.
\end{aligned}$$

Therefore

$$u \leqslant \frac{\rho_0(A(z),z)^2}{\delta^2-2|e^{i\theta}-1|} = \frac{\big|2|\zeta|^2(e^{i\theta}-1)+it\big|}{\delta^2-2|e^{i\theta}-1|}.$$

Furthermore, $B(z) = (\zeta', v', u') = z'$ is on the sphere S_1 with radius $r' = {r_B}^2/r$ centred at $B(\infty) = (\zeta_1, v_1, 0)$ where r is defined as above. Hence

$$r' = \frac{{r_B}^2}{r} \leqslant \frac{\rho_0(AB(\infty), B(\infty))\rho_0(AB^{-1}(\infty), B^{-1}(\infty))}{\delta^2 r} = \frac{\rho_0(AB(\infty), B(\infty))}{\delta}.$$

Then, arguing as above,

$$u' \leqslant r'^2 - |\zeta'-\zeta_1|^2 \leqslant \frac{\rho_0(AB(\infty), B(\infty))^2}{\delta^2} - |\zeta'-\zeta_1|^2.$$

A similar argument to that given above shows that

$$u' \leqslant \frac{\rho_0(A(z'),z')^2}{\delta^2-2|e^{i\theta}-1|} = \frac{\big|2|\zeta'|^2(e^{i\theta}-1)+it\big|}{\delta^2-2|e^{i\theta}-1|}.$$

Let U be the sub-horospherical region defined by

$$U = \left\{(\zeta, v, u) : u > \frac{|2|\zeta|^2(e^{i\theta}-1)+it|}{\delta^2-2|e^{i\theta}-1|}\right\}.$$

Then we see that U lies outside the spheres S_0 and S_1. Thus $B(U)$ lies inside S_1 and $B^{-1}(U)$ lies inside S_0. In particular, U is disjoint from its images under B and B^{-1}. Repeating this, we see that $B(U) \cap U = \emptyset$ for all $B \in G - G_\infty$. It is clear that U is mapped to itself under any screw parabolic or boundary elliptic map with the same axis as A, that is by the whole of G_∞. Hence U is precisely invariant under G_∞ in G. This completes the proof of Corollary 1·6.

5. *Relation to other results*

Jiang and Parker gave the following theorem.

THEOREM 5·1 ([**4**, theorem 5·1]). *Let A be a screw parabolic element of* PU(2, 1) *fixing* ∞. *Let L_A denote the axis of A and $e^{i\theta} \in \mathrm{U}(1)$ denote the rotational part of A. Suppose that $|e^{i\theta} - 1| < 1$. Let $\sqrt{t}$ denote the Cygan translation length of A on L_A. Suppose that G is a discrete subgroup of* PU(2, 1) *containing A. Let B be any element of G not projectively fixing ∞ and denote the radius of the isometric sphere of B by r_B. Let*

$$R = \max\{\rho_0(L_A, B(\infty)),\ \rho_0(L_A, B^{-1}(\infty))\}.$$

Then

$$r_B{}^2 \leqslant \frac{2R^2|e^{i\theta} - 1|}{(1 - |e^{i\theta} - 1|)} + \frac{t}{(1 - |e^{i\theta} - 1|^{1/2})^2}. \tag{5·1}$$

Observe that

$$(1 - |e^{i\theta} - 1|^{1/2})^2 \leqslant \left(\frac{1 + \sqrt{1 - 4|e^{i\theta} - 1|}}{2}\right)^2 \leqslant 1 - |e^{i\theta} - 1|.$$

This enables us to show:

THEOREM 5·2. *Let A be a positively oriented screw parabolic element of* PU(2, 1) *and let B be any element of* PU(2, 1) *not fixing* ∞. *If $\rho_0(L_A, B(\infty))$ and $\rho_0(L_A, B^{-1}(\infty))$ are both small enough then Theorem* 5·1 *follows from Theorem* 1·5. *On the other hand, if $\rho_0(L_A, B(\infty))$ equals $\rho_0(L_A, B^{-1}(\infty))$ and is sufficiently large then Theorem* 1·5 *follows from Theorem* 5·1.

Proof. Observe that for any point $(\zeta, v, 0) \in \partial\mathbf{H}^2_{\mathbb{C}} - \{\infty\}$ we have

$$\rho_0(L_A, (\zeta, v, 0)) = |\zeta|,$$

hence

$$\rho_0(z, Az)^2 \leqslant 2\rho_0(L_A, z)^2|e^{i\theta} - 1| + t.$$

Therefore

$$\rho_0(AB(\infty), B(\infty))\rho_0(AB^{-1}(\infty), B^{-1}(\infty)) \leqslant 2R^2|e^{i\theta} - 1| + t.$$

So, provided that,

$$2R^2|e^{i\theta} - 1| \leqslant t\, \frac{(1 - |e^{i\theta} - 1|^{1/2})^{-2} - \delta^{-2}}{\delta^{-2} - (1 - |e^{i\theta} - 1|)^{-1}} \tag{5·2}$$

we have

$$\frac{\rho_0(AB(\infty), B(\infty))\rho_0(AB^{-1}(\infty), B^{-1}(\infty))}{\delta^2} \leqslant \frac{2R^2|e^{i\theta} - 1|}{(1 - |e^{i\theta} - 1|)} + \frac{t}{(1 - |e^{i\theta} - 1|^{1/2})^2}.$$

We remark that for $0 < |e^{i\theta} - 1| < 1/4$ the function in (5·2) is bounded above and below by positive multiples of $t\sqrt{(1 - 4|e^{i\theta} - 1|)/|e^{i\theta} - 1|}$.

Conversely, suppose $\rho_0(L_A, B(\infty)) = \rho_0(L_A, B^{-1}(\infty))$. Then using Lemma 2·3 we have

$$2R^2|e^{i\theta} - 1| \leqslant \rho_0(AB(\infty), B(\infty))\rho_0(AB^{-1}(\infty), B^{-1}(\infty)).$$

Thus, provided

$$2R^2|e^{i\theta} - 1| \geqslant t\,\frac{(1 - |e^{i\theta} - 1|^{1/2})^{-2}}{\delta^{-2} - (1 - |e^{i\theta} - 1|)^{-1}}$$

we have

$$\frac{2R^2|e^{i\theta} - 1|}{(1 - |e^{i\theta} - 1|)} + \frac{t}{\big(1 - |e^{i\theta} - 1|^{1/2}\big)^2} \leqslant \frac{\rho_0(AB(\infty), B(\infty))\rho_0(AB^{-1}(\infty), B^{-1}(\infty))}{\delta^2}.$$

We remark that a similar argument can be used to show that if $|e^{i\theta} - 1| < 6/25$ and

$$\frac{\rho_0(L_A, B(\infty))}{\rho_0(L_A, B^{-1}(\infty))} \leqslant \frac{\delta^2}{1 - |e^{i\theta} - 1|}$$

then Theorem 5·1 follows from Theorem 1·5. We leave the details for the reader.

Jørgensen's inequality is a generalisation of Shimizu's lemma and deals with groups with loxodromic and elliptic generators. Complex hyperbolic versions of Jørgensen's inequality were given in [**1**], [**3**] and [**12**]. In particular, in [**1**] Basmajian and Miner give a version of Jørgensen's inequality for groups with a loxodromic generator, [**1**, theorem 9.1]. As a corollary, they give a generalisation of Shimizu's lemma, [**1**, theorem 9·11]. It was shown in [**3**, theorem 6·1] that the hypotheses of Basmajian and Miner's main theorem could be weakened somewhat (see also [**12**] for discussion of this result and Basmajian and Miner's stable basin theorem). With this change, their corollary is:

THEOREM 5·3 ([**1**, theorem 9·11]). *Fix positive numbers r and ϵ so that $r^2 + 2\epsilon < 1$. Let $A \in \mathrm{PU}(2,1)$ be a parabolic map fixing ∞. Let $B \in \mathrm{PU}(2,1)$ be a loxodromic map with attractive fixed point p and repulsive fixed point q. Suppose that neither p nor q is ∞. Suppose that B has complex dilation factor λ with $|\lambda| > 1$ and $|\lambda - 1| < \epsilon$. If*

$$\rho_0(A(p), p)\frac{1 + r^2 + \sqrt{1 + r^2}}{r^2} \leqslant \rho_0(p, q)$$

then the group generated by A and B is not discrete.

We now show that when $|e^{i\theta} - 1| < 3/16$ Theorem 5·3 follows from Theorem 1·5. This should be compared to [**7**] and [**8**] where a similar comparison was made between Theorem 5·3 and Theorem 1·3.

THEOREM 5·4. *Fix positive numbers r and ϵ satisfying $r^2 + 2\epsilon < 1$. Let $A \in \mathrm{PU}(2,1)$ be the screw parabolic map $A\colon (\zeta, v, u) \mapsto (e^{i\theta}\zeta, v + t, u)$ where $|e^{i\theta} - 1| < 3/16$ and $t\sin(\theta) > 0$. Let $B \in \mathrm{PU}(2,1)$ be loxodromic with attractive fixed point p and repelling fixed point q. Suppose that neither p nor q is ∞. Suppose that B has complex dilation factor λ with $|\lambda| > 1$ and $|\lambda - 1| < \epsilon$. Suppose that the isometric spheres of B and B^{-1} have radius r_B. If*

$$\rho_0(A(p), p)\frac{1 + r^2 + \sqrt{1 + r^2}}{r^2} \leqslant \rho_0(p, q)$$

then

$$\frac{\rho_0(AB^{-1}(\infty), B^{-1}(\infty))\rho_0(AB(\infty), B(\infty))}{{r_B}^2} < \left(\frac{1+\sqrt{1-4|e^{i\theta}-1|}}{2}\right)^2.$$

We first prove a lemma.

LEMMA 5·5. *Let B be a loxodromic map with complex dilation factor $\lambda \in \mathbb{C}$, attractive fixed point p and repulsive fixed point q and isometric sphere of radius r_B. Suppose that p, $q \neq \infty$, and let $M = |\lambda - 1| + |\lambda^{-1} - 1|$. Then $\rho_0(p, q) \leqslant M^{1/2} r_B$.*

Proof. Let C be any element of PU(2, 1) with $C(o) = p$ and $C(\infty) = q$ where $o = (0, 0, 0) \in \partial\mathbf{H}^2_{\mathbb{C}}$. Let r_C be the radius of its isometric sphere. Then $A = C^{-1}BC$ fixes o and ∞ and has complex dilation factor λ. Using Lemma 2·2 first for B with $z = q = B(q)$ and then with C and $z = AC^{-1}(\infty)$, $A^{-1}C^{-1}(\infty)$ we have

$$\begin{aligned}{r_B}^2 &= \rho_0(q, B(\infty))\rho_0(q, B^{-1}(\infty)) \\ &= \rho_0(C(\infty), CAC^{-1}(\infty))\rho_0(C(\infty), CA^{-1}C^{-1}(\infty)) \\ &= \frac{{r_C}^4}{\rho_0(C^{-1}(\infty), AC^{-1}(\infty))\rho_0(C^{-1}(\infty), A^{-1}C^{-1}(\infty))}.\end{aligned}$$

Now using [**12**, lemma 2.1] we have

$$\rho_0(C^{-1}(\infty), AC^{-1}(\infty)) \leqslant |\lambda|^{1/2} M^{1/2} \rho_0(o, C^{-1}(\infty))$$
$$\rho_0(C^{-1}(\infty), A^{-1}C^{-1}(\infty)) \leqslant |\lambda|^{-1/2} M^{1/2} \rho_0(o, C^{-1}(\infty)),$$

where $M = |\lambda - 1| + |\lambda^{-1} - 1|$.

Therefore

$${r_B}^2 \geqslant \frac{{r_C}^4}{M\rho_0(o, C^{-1}(\infty))^2} = \frac{\rho_0(C(o), C(\infty))^2}{M},$$

where we have used Lemma 2·2 again, but this time with $z = o$. Substituting $p = C(o)$ and $q = C(\infty)$ gives the result.

We can now prove the theorem:

Proof of Theorem 5·4. Since $|\lambda - 1| < \epsilon < 1/2$ we have $1 < |\lambda| < 3/2$. Hence we have $|\lambda|^{1/2} + |\lambda|^{-1/2} < 5/\sqrt{6}$. Also, from Lemma 5·5 we have

$$\rho_0(p, q)^2 \leqslant M{r_B}^2 < 2|\lambda - 1|{r_B}^2 < 2\epsilon {r_B}^2 < (1 - r^2){r_B}^2.$$

Thus

$$\begin{aligned}\frac{\rho_0(A(p), p)}{r_B} &< \frac{r^2\sqrt{1-r^2}}{1 + r^2 + \sqrt{1+r^2}} \\ &= \frac{(\sqrt{1+r^2}+1)(\sqrt{1+r^2}-1)\sqrt{1-r^2}}{\sqrt{1+r^2}(\sqrt{1+r^2}+1)} \\ &= \frac{(\sqrt{1+r^2}-1)\sqrt{1-r^2}}{\sqrt{1+r^2}}.\end{aligned}$$

Using elementary calculus we see that as r varies between 0 and 1, this function attains its maximum when $r^2 = 2^{2/3} - 1$. The maximum value is $(2^{1/3} - 1)^{3/2} = 0.132514\ldots$.

We have $\rho_0(p, B^{-1}(\infty)) = r_B|\lambda|^{-1}$ and $\rho_0(p, B(\infty)) = r_B|\lambda|$ from parts (4) and (5) of [**7**, proposition 2.6]. Taking these with Lemma 2·4, we obtain

$$\begin{aligned}\rho_0(AB^{-1}(\infty), B^{-1}(\infty)) &\leqslant \rho_0(A(p), p) + \rho_0(p, B^{-1}(\infty))\sqrt{2}|e^{i\theta} - 1|^{1/2}\\ &= \rho_0(A(p), p) + r_B|\lambda|^{-1/2}\sqrt{2}|e^{i\theta} - 1|^{1/2},\\ \rho_0(AB(\infty), B(\infty)) &\leqslant \rho_0(A(p), p) + \rho_0(p, B(\infty))\sqrt{2}|e^{i\theta} - 1|^{1/2}\\ &= \rho_0(A(p), p) + r_B|\lambda|^{1/2}\sqrt{2}|e^{i\theta} - 1|^{1/2}.\end{aligned}$$

Therefore, using $|e^{i\theta} - 1| < 3/16$, we have

$$\begin{aligned}&\frac{\rho_0(AB^{-1}(\infty), B^{-1}(\infty))\rho_0(AB(\infty), B(\infty))}{{r_B}^2}\\ &\leqslant \frac{\rho_0(A(p), p)^2}{{r_B}^2} + \frac{\rho_0(A(p), p)}{r_B}(|\lambda|^{1/2} + |\lambda|^{-1/2})\sqrt{2}|e^{i\theta} - 1|^{1/2} + 2|e^{i\theta} - 1|\\ &< (2^{1/3} - 1)^3 + (2^{1/3} - 1)^{3/2}\frac{5}{\sqrt{6}}\sqrt{2}\frac{\sqrt{3}}{4} + \frac{3}{8}\\ &= (2^{1/3} - 1)^3 + (2^{1/3} - 1)^{3/2}\frac{5}{4} + \frac{3}{8}\\ &< \frac{9}{16}\\ &< \left(\frac{1 + \sqrt{1 - 4|e^{i\theta} - 1|}}{2}\right)^2.\end{aligned}$$

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