Does Generating Examples Aid Proof Production?

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Abstract. Many mathematics education researchers have suggested that asking learners to generate examples of mathematical concepts is an effective way of learning about novel concepts. To date, however, this suggestion has limited empirical support. We asked undergraduate students to study a novel concept by either tackling example generation tasks, or reading worked solutions to these tasks. Contrary to suggestions in the literature, we found no advantage for the example generation group on subsequent proof production tasks. From a second study we found that undergraduate students overwhelmingly adopt a Trial and Error approach to example generation, and suggest that different example generation strategies may result in different learning gains. We conclude by arguing that the teaching strategy of example generation is not yet understood well enough to be a viable pedagogical recommendation.

Keywords: examples, example generation, fine function, proof, undergraduate

1. Introduction

The role that examples play in students' learning has gained increasing attention in recent years. A special issue of this journal was recently dedicated to the role of examples in students' learning, and several articles in that issue discussed example generation (e.g., Watson and Shipman, 2008; Goldenberg and Mason, 2008; Zazkis and Leikin, 2008). Research in this area includes investigations into the richness and structure of students' example spaces (Goldenberg and Mason, 2008), the role of examples as pedagogical tools (Weber et al., 2008) and the pedagogical benefits of asking learners to generate their own examples of mathematical concepts (Watson and Mason, 2005). This paper focuses on the last of these ideas. We explore the pedagogical benefits of asking learners to generate their own examples by reporting two studies. In Study 1 we asked undergraduate mathematics students to study a new mathematical concept either by generating their own examples or by reading pre-prepared examples. By then asking participants to engage in proof-construction tasks, we compared the efficacy of the two learning strategies. In Study 2 we explored the activity of example generation further, by carrying out clinical interviews with mathematics undergraduate students. Our aim was to identify the strategies they use when constructing examples of novel concepts.

2. Background

Following Zaslavsky's (1995) suggestion that asking learners to generate their own examples of mathematical concepts might be an effective teaching strategy, Dahlberg and Housman (1997) gave eleven mathematics undergraduates a definition of a novel mathematical concept and asked them to study it for a few minutes. They reported that students who spontaneously generated their own examples of the concept "learned a significant amount" compared to those who did not, and that they "were best able to identify the correctness of conjectures and provide explanations". Dahlberg and Housman concluded that "it may be beneficial to introduce students to new concepts by requiring them to generate their own examples" (p. 297).

Theoretical arguments have been made for the type of example generation tasks that are most likely to lead to effective learning. Watson and Shipman (2008) linked example generation to variation theory (Marton and Booth, 1997), arguing that it is by the perception of small variations between examples that students can define classes of mathematical objects. They therefore suggested that in order for example generation to be pedagogically viable, students need to be asked to produce examples that present small variations of the mathematical concept under scrutiny. Weber et al. (2008) used the same idea to suggest a sequence of example generation tasks they believed would be an effective way of introducing the concept of a convergent sequence to students. Similar suggestions of example generation tasks that target small variations between examples have also been made by Watson and Mason (2005).

Although Dahlberg and Housman's (1997) findings have been widely cited in the literature and have been used as the basis of pedagogical recommendations for practitioners at both the school and university levels (Mason, 2002; Meehan, 2007; Watson and Mason, 2001; Watson and Mason, 2005; Weber et al., 2008), there are some important limitations to their study. First, they had a relatively small sample of eleven students, of whom only four spontaneously generated examples. Further, conjecture verification tasks in the study indicated that example generation facilitated judgements about conjectures and the production of explanations, but these four students had very mixed success at building a rich understanding of the defined concept. In a later study using the same materials, Housman and Porter (2003) found that none of the three students who spontaneously generated examples of the novel concept (from a total sample of twelve) were successful on subsequent proof tasks. In summary, there appears to be little empirical evidence that having students generate examples will improve their mathematical learning.

Authors who propose example generation as a pedagogical strategy variously indicate its value in terms of improved engagement with formal theory (Meehan, 2007), better constructed understandings of both concepts and proofs (Weber et al., 2008), acting as a basis for expressing generality (Watson and Mason, 2001) and leading to a deeper understanding (Watson and Mason, 2005). Following Dahlberg and Housman (1997) and Moore (1994) who respectively link example generation with explaining and lack of example generation with poor proof production, we explore the question of whether asking students to generate examples of mathematical concepts improves their ability to produce proofs about these concepts.

3. Generating Examples versus Reading Examples

3.1. Participants and Materials

The study involved 53 undergraduate students from four high ranking universities in the UK and US who were studying for a degree in mathematics or a joint degree with a substantial mathematics component. The students were in the second, third or fourth year of their degree. The participants were volunteers, and were paid a small fee for their time.

The materials used were largely based on Dahlberg and Housman's (1997) study. Participants were randomly allocated into one of two groups: the 'generating' group (N_{gen}=25) and the 'reading' group (N_{read}=28). Each participant was given a two-section booklet of tasks to work through individually. In the first section of the booklet a novel mathematical concept was introduced. The participants were given the definition of a *fine function*:

> Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. Then f is called *fine* if it has a root (zero) at each integer. In other words, f is fine if $n \in \mathbb{Z} \Rightarrow f(n) = 0$.

After having read the definition, participants were asked to read through a series of twelve worked examples of the concept (if they were in the 'reading' group) or tackle a series of twelve example generation tasks (if they were in the 'generating' group). The example generation tasks were constructed following the strategies proposed by Watson and Mason (2005) using small variations between examples, asking students to find several examples of the same type, examples that would satisfy one property but not another and so on. The worked examples in the reading group consisted of solutions to the tasks given to the generating group.

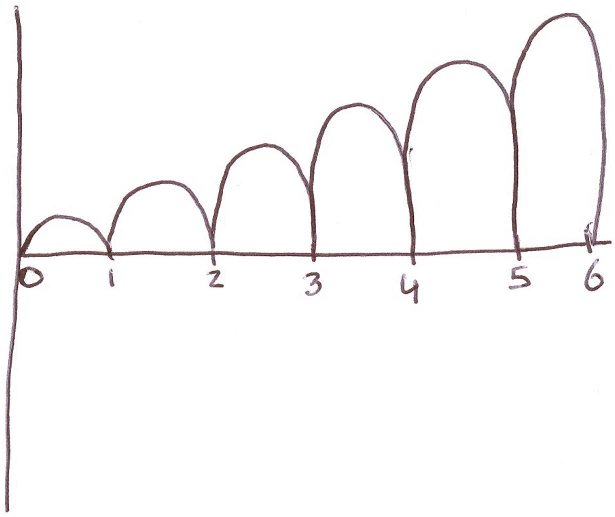

Figure 1. Student's response for the task: Draw a function which is fine but not periodic.

For example, one task the 'generating' group were given was:

Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is fine but not continuous.

The equivalent task for the 'reading' group was:

Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is fine but not continuous.
Example solution: Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Z} \\ x & \text{otherwise} \end{cases}$$

f has zeroes at every integer, but it is not continuous at any integer (other than 0).

The students in the 'generating' group typically produced examples by either drawing the graph of the function required, or by producing an algebraic representation. In Figures 1 and 2 we give two representative examples of students' work.

After twenty minutes participants were asked to move on to the second section of the booklet. This section was identical for both groups. It consisted of four proof production tasks similar to those used by Dahlberg and Housman (1997), and was designed to assess the extent to which example generation tasks had helped students with producing proofs involving the new concept. Twenty minutes were available for participants to work through the

$$f(r) = \begin{cases} 0 & \text{if } r \in \mathbb{Q} \\ 1 & \text{if } r \in \mathbb{R}/\mathbb{Q} \end{cases}$$

Figure 2. Student‘s response for the task: Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is fine but not continuous.

proof production tasks. A full list of example generation and proof production tasks is included in Appendix A. The order in which tasks appeared in both the first and second sections of the booklets was randomised for each participant.

3.2. Results and Discussion

The proof production tasks were marked according to a pre-designed scheme, with a maximum of 5 marks available per task (see Appendix A). The proof-tasks were sufficiently discriminating, with no suggestion of ceiling or floor effects in the distribution of marks.

The mean scores of the two groups are shown in Figure 3. If the hypothesis that engaging students with generating their own examples of mathematical concepts facilitates production of proofs involving these concepts were correct, we would expect to see higher scores between the generating group compared with the reading group. In our data the between-conditions difference did not approach significance, $t(51) = 0.455, p = .651$, and represented a small negative effect size, $d = -0.13$. (95% CI $[-0.66, 0.42]$).

The results did not support the suggestion that generating examples of fine functions leads to better proof production than reading worked examples. Not only was the difference in scores between the generating and reading groups non-significant, but there was a small and non-significant trend in the opposite direction. This suggests that, even if there is a positive effect, it is at best rather modest.

This is surprising giving the strong arguments made in the mathematics education research literature in support of the benefits of example generation for students' understanding of mathematical concepts. There are a number of ways in which we could account for our data, some of which will be addressed later. However, it could be argued that the notion of example generation is more nuanced than is allowed for in our design. In particular, it may be that students engage in example generation tasks in qualitatively different ways and these may lead to different outcomes.

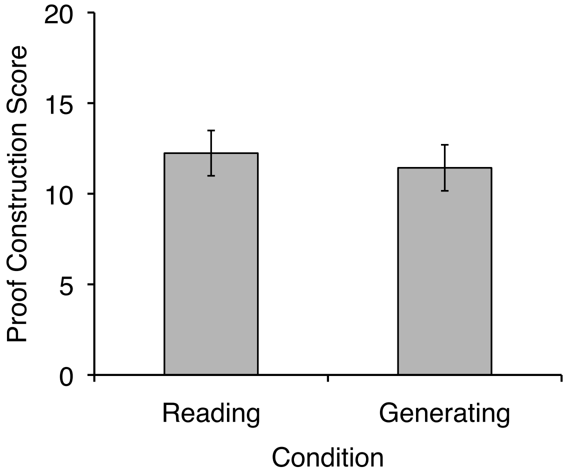

Figure 3. Mean scores of the reading and generating groups on the proof production tasks. Error bars show ± 1 SE of the mean.

At around the same time that we conducted the proof production study, we undertook a separate study designed to address different research questions but which provided some data which give an insight into this issue.

4. How do undergraduate students generate examples?

Antonini (2006) asked mathematics research students to generate examples of four different mathematical concepts (one question, for instance, asked for an example of a real-valued, periodic, non-constant function with no minimum period). He identified three different strategies in use by his participants, which he labelled *Trial and Error*, *Transformation* and *Analysis*.

Students who adopted the *Trial and Error* strategy generated examples which satisfied a given set of criteria by searching a collection of recalled examples of a broader category, and testing each example in turn to see if it met the required criteria. The *Transformation* strategy occurred when participants modified examples that satisfied some of the required properties until it satisfied all of them. Finally the *Analysis* strategy occurred when an individual assumed that the required object existed, and deduced what other properties it must necessarily have until they were able to recall an object with the required properties, or had a procedure for constructing such an object. Antonini (2006) suggested that the analysis strategy was only used by the experts in his study when the other two strategies had failed.

These strategies seem to involve substantially different cognitive activities. Although researchers have suggested that the activity of example generation *per se* aids learning, it might be that this effectiveness is influenced by the strategy adopted by the learner during example generation.

4.1. Participants and Materials

The study involved nine second and third year students on mathematics degrees or degrees with a high mathematics content at a highly-ranked UK university. Each participated in an individual interview. The first part of the interview began with the student being presented with a definition of what was, for them, a novel mathematical concept (adapted from Weber, 2009):

> Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. Let $A \subseteq \mathbb{R}$. Then f is *preserved on* A if and only if $f(A) \subseteq A$. In other words f is preserved on A if and only if $f(a)$ belongs to A for all a in A.

The participants were then given a series of twenty example generation tasks, again designed following the strategies suggested by Watson and Mason (2005). For example the students were asked:

> Let $A = (1,2)$ and $f : \mathbb{R} \to \mathbb{R}$.
>
> (a) Find f such that f is preserved on A.
>
> (b) Find another such f.
>
> (c) Find an f which is strictly decreasing and preserved on A.
>
> (d) Find an f which is strictly increasing and preserved on A.
>
> (e) Find an f which is preserved on A but not continuous on A.

The order in which the example generation tasks were presented was randomised for each participant. The interviews followed the clinical interview procedure described by Ginsburg (1981); the interviewer intervened where necessary to prompt the participant to verbalise their thinking, or to ask for clarification if required. The example generation section of the interview lasted for around 30 minutes. The interviews were video-recorded and transcribed. Each instance of attempted example generation was identified and coded according to Antonini's (2006) scheme and, in addition, was judged as successful provided that the example had each of the required properties.

4.2. Results and Discussion

Between them, the participants attempted a total of 62 example generation tasks ($M = 6.9, SD = 2.8$), of which 30 were successful ($M = 3.3, SD = 2.8$). A breakdown of data for each of the nine participants, together with the strategies that were used, is given in Table I.

Across the sample, the Trial and Error strategy was by far the most common approach, $t(8) = 3.65, p = .006$, being used by all participants and

Table I. Example generation strategies used in the interviews (the number of successful examples generated is shown in parentheses).

Participant	Trial and Error	Transformation	Analysis	Total
1	5(0)	0	0	5(0)
2	8(4)	0	0	8(4)
3	7(5)	2(0)	0	9(5)
4	1(1)	2(0)	0	3(1)
5	3(3)	4(4)	0	7(7)
6	11(1)	1(0)	0	12(1)
7	7(7)	1(1)	0	8(8)
8	4(2)	0	0	4(2)
9	5(1)	1(1)	0	6(2)
Total	51(24)	11(6)	0	62(30)

representing 82% of generation attempts and 80% of successful generation attempts.

When participants used the Trial and Error strategy they always resorted to well known functions, but did not necessarily check that the function they had selected did indeed satisfy the given properties.

For example when Participant 2 was asked to produce a function f from $\mathbb{R}$ to $\mathbb{R}$ that is preserved on $\mathbb{R}\setminus\{0\}$ she reasoned:

> So $f(0)$ cannot equal 0 … Maybe … So… what could that be … when is $f(0)$ not equal zero … Maybe when it is 1 over something … [she writes $f(x)=\frac{1}{x}$] sounds about right. I think … well … maybe not but … it feels all right to me because if you put 0 and then it doesnt exist … that is anything that I can think of.

In this case the participant resorted to a well known example but did not check that it satisfied the required properties.

On the other hand when Participant 1 was asked to give an example of a real valued function f from $\mathbb{R}$ to $\mathbb{R}$ not preserved in $A=(1,2)$ he reasoned:

> So just $f(x)=x+10$ … in the interval $(1,2)$ 1 is 11 and 2 is 12 so that is not preserved on A … yes … 11 and 12 are not in the interval A … it is not preserved.

In this case the participant resorted to the equation of a straight line which seemed appropriate and checked the property that the question required.

The Transformation strategy was used in 18% of the cases and was successful on nearly half of these occasions. In the instances where the participants did not manage to produce a correct example using this strategy, this was caused by problems with algebraic manipulations rather than conceptual difficulties. This is to say that the chosen function could have been successfully transformed to satisfy the required properties, but the students found the algebraic manipulations needed to achieve this difficult. Figure 4 shows a successful example of Transformation Strategy which occurred when Participant 7 was asked to find a function $f : \mathbb{R} \to \mathbb{R}$ such that f has a local minimum in $A = [-1, 0]$ and is preserved in A. In sum, Participant 7 managed to translate successfully the parabola $f(x) = x^2$ to $f(x) = \left(x + \frac{1}{2}\right)^2$ so that it satisfied the required properties.

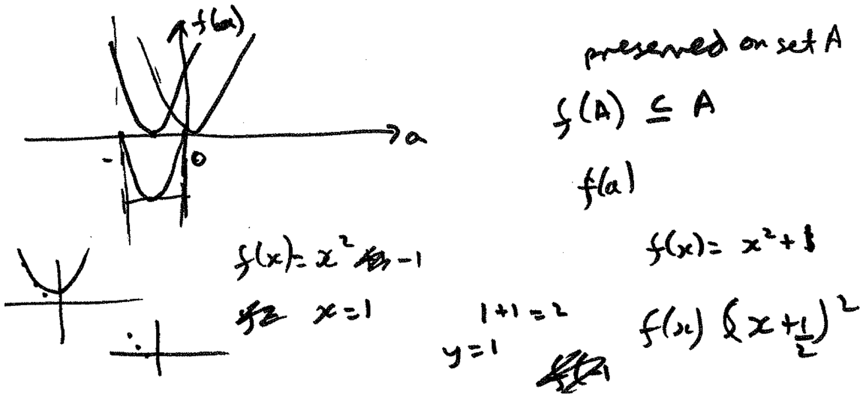

Figure 4. Participant 7's translation of $y = x^2$ to $y = \left(x + \frac{1}{2}\right)^2$.

In contrast, we did not observe any instance of a student clearly using the Analysis strategy and we did not find any strategy in use that was substantially different to those described by Antonini (2006).

The findings of this study suggest that amongst the type of students who took part in our experiments, the Trial and Error strategy is by far the most dominant method of generating examples. It may be that a Trial and Error approach, especially when used without subsequent checking of the properties of the example produced, is not one which leads to the types of learning gains or enriched concept images which those who champion example generation might expect.

5. General discussion

Many mathematics education researchers have suggested that asking students to generate their own examples of concepts is an effective learning strategy. In order to empirically test this hypothesis we took achievement in proof production tasks related to the learnt concept to stand as a measure of learning (note that the students in our studies were all undergraduate students enrolled in a degree course in mathematics (or one with a large mathematics component) and thus have extensive experience of proving). We investigated this hypothesis by asking mathematics undergraduates to either generate or read examples related to Dahlberg and Housman's (1997) fine function concept. Our findings did not support the suggestion that generating examples is an effective preparation for proof production tasks. We found no significant difference between the proof production success of students who generated examples compared to those who studied worked examples. In fact we found a non-significant trend in the opposite direction.

There are several ways of accounting for our failure to find the predicted advantage for the generating group, while retaining belief in the general proposition that asking learners to generate examples is a more effective teaching strategy than asking learners to read examples.

A first possibility is that our example generation tasks were poorly designed. However, the tasks were produced using the methods advocated by Watson and Mason (2005) and Weber et al. (2008). While it is hard to refute accusations of poor task design, if we were unsuccessful at creating an appropriately nuanced series of example generation tasks then we believe that there is currently insufficient guidance available to educational practitioners on how to accomplish this task effectively.

A second possible account for our data is that the proof production tasks were not sufficiently discriminatory to detect an effect; that is, there might have been an effect, but it was too small to be detected. Assuming that our tasks are a valid test of proof production, then equivalently, we could argue that the effect is so small that the importance that some appear to ascribe to it is somewhat exaggerated. In other words, while it may be the case that asking students to generate their own examples is a good teaching strategy – better than asking students to read pre-prepared examples – or that example generation facilitates proof production, the gains may be so small as to be negligible and very difficult to detect.

A third possibility is that example generation does indeed facilitate understanding of a new concept but does not facilitate proof production (as implied by Moore, 1994, and Dahlberg and Housman, 1997). For example it is possible that example generation might yield more long-term learning gains. However, we note that there is no empirical evidence to date that supports this claim.

A fourth account for the data is that the undergraduate students in our study were not engaging in the example generation tasks in the manner predicted by earlier researchers. Although we have evidence that the participants in the generating group did engage with the example generation tasks (i.e. no participant left this section blank), we have no information about how they tackled the tasks. It is possible that undergraduates, when asked to generate examples, typically use strategies that are not effective at developing the rich concept images which support proof production.

Our second study showed that, when generating examples, undergraduates seem to use a more restricted range of strategies than expert mathematicians. We found the overwhelming majority of example generation attempts used the Trial and Error strategy. This relies upon recalling examples from memory and testing the recalled object against the mathematical criteria required. However we observed that the participants in Study 2 sometimes did not carry out this test, and in this case the strategy seems to be substantially less sophisticated than either the Transformation or Analysis strategies. Both of these latter two strategies require an active analysis and logical manipulation of the required properties of the to-be-constructed example. The relative levels of sophistication of the three strategies found by Antonini suggest that learners who use different example generation strategies might well develop concept images of differing levels of sophistication.

This idea that strategy choice is related to the richness of the resulting concept image suggests a natural account both for why the suggestion that example generation is an effective way of developing concept images has gained such currency in the mathematics education community, and for why this appeared not to be the case in our data. It may be the case that teachers and curriculum designers intuitively base some initial pedagogic suggestions on their own (relatively successful) practice. Indeed, Antonini (2006) suggests some sophisticated example generation methods which may indeed prove to be useful in learning a novel concept. However, in examining the efficacy of example generation in supporting concept formation, we may need to focus on the type of example generation the learners undertake. Merely using the term "example generation" may mask mechanisms of different efficacy. When sophisticated mathematicians generate examples, they may do so in such a way that they do indeed make gains in learning about a novel concept. Our studies may indicate that the forms of example generation which less sophisticated learners undertake do not lead to such gains.

Whereas researchers have previously suggested that the task of example generation is an effective way of developing rich concept images of novel concepts, this account suggests that the task *per se* is not effective, but rather that only certain approaches to tackling the task may be effective. If this hypothesis is correct then if example generation is to be adopted in teaching practice, students might require explicit instruction in how to generate examples using strategies that are likely to develop their concept images.

A fifth and final way of accounting for our findings that cannot be dismissed is that asking students to generate examples of a new mathematical concept does not lead to more learning than simply reading examples that satisfy the definition of that concept. We note the empirical support for the hypothesis that example generation aids in concept understanding appears to come from just seven students (across two studies) who spontaneously generated examples, and these students collectively had limited success on subsequent proof production tasks (Dahlberg and Housman, 1997; Housman and Porter, 2003).

6. Concluding Remarks

Drawing pedagogical implications from theoretical analyses and qualitative studies is complex. The results of our paper illustrate this complexity. We applied instructional recommendations that are commonplace in the mathematics education literature and discovered, to our surprise, that these recommendations did not lead to the predicted improvements. To avoid misinterpretation, our studies do not imply that example generation must necessarily fail as a pedagogical technique to increase students' understandings of mathematical concepts. However, they do demonstrate that simply asking students to generate examples about a concept may not substantially improve their abilities to write proofs about that concept; at least not more so than providing students with examples to read. This suggests that if example generation is to be a useful pedagogical strategy, more nuance is needed in its implementation. There is clearly a need for further empirical research in this area if we are to determine whether and how example generation tasks can lead to significant learning gains. Such studies would be useful, both for providing empirical support for a widely cited pedagogical recommendation and for working out the instructional details needed for that recommendation to yield positive results.

7. Acknowledgment

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Appendix

A. Material used for the proof production study

A.1. Example Generation Tasks

The following definition is about a mathematical concept you have probably not seen before. Please spend a few minutes reading it carefully:

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. Then f is called *fine* if it has a root (zero) at each integer. In other words, f is fine if $n \in \mathbb{Z} \Rightarrow f(n) = 0$.

You will have 20 minutes to answer a series of questions about this concept. In each question you will be asked to provide examples relevant to the concept of fine function. Each question appears on a new page. Write your answers in the space below each question. If you get completely stuck, move on to the next question. Try to answer as many questions as you can in the 20 minutes available.

1. A periodic function is one which repeats itself values after a certain period. That is to say that $f : \mathbb{R} \to \mathbb{R}$ is periodic if there exists a $P \neq 0$ such that $f(x+P) = f(x)$ for every $x \in \mathbb{R}$. P is called the function's period.
   a) Find an f which is fine and periodic with period $\frac{1}{2}$.
   b) Find another such f.
2. Let $f(x) = \sin(kx)$, where $k \in \mathbb{R}$. Give two examples of values of k for which f is fine.
3. Let $f(x) = \cos(ax+b)$. Give an example of a pair of values a, b for which f is fine.
4. a) Draw a function which is fine but not periodic.
   b) Draw another such function.
5. a) Find $f : \mathbb{R} \to \mathbb{R}$ such that f is fine and periodic with period 1.
   b) Find another such f.
6. Let $f(x) = k$, where $k \in \mathbb{R}$. Give an example of a value of k for which f is fine.
7. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is fine but not continuous.
8. Draw a fine function that takes no negative values.
9. Draw a fine function that always takes values between $-\frac{3}{2}$ and 1.

A.2. Proof production Tasks

In this section there are four questions, each on a new page. Each question involves proving a statement about fine functions.

1. Let $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$. Prove that f is not fine.

2. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be fine functions. Prove that $g \circ f$ is fine. [Here $(g \circ f)(x) = g(f(x))$].
3. Let $f, g : \mathbb{R} \to \mathbb{R}$ be fine functions. Prove that $f + g$ is a fine function. [Here $(f + g)(x) = f(x) + g(x)$].
4. Let $f : \mathbb{R} \to \mathbb{R}$ be a fine function. Let $g : \mathbb{R} \to \mathbb{R}$ be a function defined by $g(x) = f(x - k)$ for some $k \in \mathbb{Z}$. Prove that g is a fine function.

A.3. PROOF MARKING SCHEME

Question 1

1. $f(x) = ax^2 + bx + c$ is a quadratic equation, which has at most two real roots (by the Fundamental Theorem of Algebra). [2]
2. Consequently, there must exist integers n such that $f(n) \neq 0$ [2]
3. So f is not fine. [1]

Question 2

1. Let $k \in \mathbb{Z}$. Then $f(k) = 0$, as f is fine. [2]
2. So $g(f(k)) = g(0) = 0$ as g is fine. [2]
3. So, $g \circ f$ is fine. [1]

Question 3

1. Let $n \in \mathbb{Z}$. then $f(n) = g(n) = 0$ as f and g are fine. [2]
2. So $f(x) + g(x) = 0 + 0 = 0$. [2]
3. So $f + g$ is fine. [1]

Question 4

1. Let $n \in \mathbb{Z}$. Then $n - k \in \mathbb{Z}$ as $k \in \mathbb{Z}$. [1]
2. So $f(n - k) = 0$ as f is fine. [2]
3. So $g(n) = f(n - k) = 0$. [1]

4. In other words $g(x) = 0$ whenever $x \in \mathbb{Z}$. [1]

B. Instrument used in the example generation study

The following definition is about a mathematical concept you have probably not seen before. Please spend a few minutes reading it carefully:

> Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. Let $A \subseteq \mathbb{R}$. Then f is *preserved on A* if and only if $f(A) \subseteq A$. In other words f is preserved on A if and only if $a \in A \Rightarrow f(a) \in A$.

You will have 20 minutes to answer a series of questions about this concept. In each question you will be asked to provide examples relevant to the concept of function preservation. Each question appears on a new page. Write your answers in the space below each question. If you get completely stuck, move on to the next question. Try to answer as many questions as you can in the 20 minutes available.

1. Let A be the open interval $(1,2)$.
   a) Find $f : \mathbb{R} \to \mathbb{R}$ such that f is preserved on A.
   b) Find another such f.
   c) Find an f which is strictly decreasing and preserved on A.
   d) Find an f which is strictly increasing and preserved on A.
   e) Find an f which is preserved on A but not continuous on A.
2. Let A be the open interval $(1,2)$ and B be the open interval $(2,3)$. Find an $f : \mathbb{R} \to \mathbb{R}$ such that f is preserved on A but not on B.
3. Let A be the open interval $(1,2)$, B be the open interval $(2,3)$ and C be the open interval $(3,4)$. Find an $f : \mathbb{R} \to \mathbb{R}$ such that f is preserved on A and C, but not on B.
4. Let A be the closed interval $[-1,0]$. Find an $f : \mathbb{R} \to \mathbb{R}$ such that f has a local minimum in A and is preserved on A.
5. Let B be the open interval $(1,2)$. Find $f : \mathbb{R} \to \mathbb{R}$ such that f has a local minimum in B and is preserved on B.
6. Let A be the open interval $(1,2)$. Find an $f : \mathbb{R} \to \mathbb{R}$ which is not preserved on A.
7. Find $f : \mathbb{R} \to \mathbb{R}$ such that f is preserved on $\mathbb{N}$ but not on the set of negative numbers (i.e. not on the set $\{x \in \mathbb{R} \,|\, x < 0\}$).

8. Find $f : \mathbb{R} \to \mathbb{R}$ such that f is preserved on $\mathbb{R} \setminus \{0\}$ but not preserved on $\{0\}$.
9. Let f be a step-function defined by $f(x) = \max\{n \in \mathbb{Z} \,|\, n \leq x\}$.
   a) Find a set A such that A has five members and f is preserved on A.
   b) Find $a, b \in \mathbb{R}$ such that $a \neq b$ and f is preserved on $[a, b]$.
10. Let $f(x) = \sin x$.
    a) Find $a \in \mathbb{R}$ such that f is preserved on $\{a\}$.
    b) Find $a, b \in \mathbb{R}$ such that $a \neq b$ and f is preserved on $[a, b]$.
11. Let $f(x) = x^{13}$.
    a) Find a set A such that A has two members and f is preserved on A.
    b) Find the largest $a \in \mathbb{R}$ such that f is preserved on $[-\frac{1}{2}, a]$.
12. Let $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} & \text{if } x \neq 0 \end{cases}$.
    a) Find a set A such that A has five members and f is preserved on A.
    b) Find $m \in \mathbb{R}$ such that f is preserved on $\left[\frac{1}{2}, m\right]$.

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