Multiplicatively badly approximable numbers and generalised Cantor sets

Dzmitry Badziahin [*] Sanju Velani[†]
York York

*Dedicated to Andrew Pollington on hitting 57**

Abstract

Let p be a prime number. The p-adic case of the Mixed Littlewood Conjecture states that $\liminf_{q\to\infty} q\cdot|q|_p\cdot||q\alpha|| = 0$ for all $\alpha\in\mathbb{R}$. We show that with the additional factor of $\log q \log\log q$ the statement is false. Indeed, our main result implies that the set of α for which $\liminf_{q\to\infty} q\cdot\log q\cdot\log\log q\cdot|q|_p\cdot||q\alpha|| > 0$ is of full dimension. The result is obtained as an application of a general framework for Cantor sets developed in this paper.

1 Introduction

The goal of this paper is simple enough. It is an attempt to address the question:

What are the analogues of the classical set of badly approximable numbers within the multiplicative frameworks of Littlewood's conjecture and its mixed counterpart?

1.1 The classical setup and the set Bad

A classical result of Dirichlet states that for any real number α there exist infinitely many $q\in\mathbb{N}$ such that

$$q||q\alpha|| < 1 \ .$$

Here and throughout $||\,.\,||$ denotes the distance to the nearest integer. In general the right hand side of the above inequality cannot be replaced by an arbitrarily small constant. Indeed a result of Jarník [11] and Besicovitch [2] states that the set

$$\mathbf{Bad} := \{\alpha\in\mathbb{R} \ : \ \liminf_{q\to\infty} q||q\alpha|| > 0\}$$

of badly approximable numbers is of maximal Hausdorff dimension; i.e.

$$\dim \mathbf{Bad} = 1.$$

For details regarding Hausdorff dimension the reader is referred to [8]. However, from a measure theoretic point of view the classical theorem of Khintchine [12] enables us to improve

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on the global statement (a statement true for all numbers) of Dirichlet by a logarithm. In particular, for $\lambda \geqslant 0$ let

$$\mathbf{Bad}^{\lambda} := \{\alpha \in \mathbb{R} \ : \ \liminf_{q\to\infty} q \cdot (\log q)^{\lambda} \cdot ||q\alpha|| > 0\} \ .$$

Then, Khintchine's theorem implies that

$$\left|\mathbf{Bad}^{\lambda}\right| = \begin{cases} 0 & \text{if } \lambda \leqslant 1 \\ \text{FULL} & \text{if } \lambda > 1. \end{cases}$$

Here and throughout $|\cdot|$ denotes Lebesgue measure and 'FULL' means that the complement of the set under consideration is of measure zero.

The upshot of the classical setup is that we are able to shave off a logarithm from the measure theoretic 'switch over' set $\mathbf{Bad}^1$ before we precisely hit the set **Bad**. In addition, if we shave off any more (i.e. $(\log q)^{1+\epsilon}$ with $\epsilon > 0$ arbitrary) then the corresponding set becomes empty. This is a theme which we claim reoccurs within the multiplicative framework of Littlewood's conjecture and its mixed counterpart.

1.2 The multiplicative setup and the set Mad

A straightforward consequence of Dirichlet's classical result is that for every $(\alpha, \beta) \in \mathbb{R}^2$, there exist infinitely many $q \in \mathbb{N}$ such that

$$q \cdot ||q\alpha|| \cdot ||q\beta|| < 1.$$

Littlewood conjectured that the right hand side of the above inequality can be replaced by an arbitrarily small constant.

Littlewood Conjecture (LC) *For every* $(\alpha, \beta) \in \mathbb{R}^2$,

$$\liminf_{q\to\infty} q \cdot ||q\alpha|| \cdot ||q\beta|| = 0 \ . \tag{1}$$

Despite concerted efforts over the years this famous conjecture remains open. For background and recent 'progress' concerning this fundamental problem see [6, 15] and references therein.

A consequence of LC is that the set

$$\{(\alpha, \beta) \in \mathbb{R}^2 \ : \ \liminf_{q\to\infty} q \cdot ||q\alpha|| \cdot ||q\beta|| > 0\}$$

is empty and therefore is not a candidate for the multiplicative analogue of **Bad**. Regarding possible candidates, for $\lambda \geqslant 0$ let

$$\mathbf{Mad}^{\lambda} := \{(\alpha, \beta) \in \mathbb{R}^2 \ : \ \liminf_{q\to\infty} q \cdot (\log q)^{\lambda} \cdot ||q\alpha|| \cdot ||q\beta|| > 0\}.$$

¿From a measure theoretic point of view Gallagher's theorem [9] (the multiplicative analogue of Khintchine's theorem) implies that

$$|\mathbf{Mad}^{\lambda}| = \begin{cases} 0 & \text{if } \lambda \leqslant 2 \\ \text{FULL} & \text{if } \lambda > 2. \end{cases}$$

Natural heuristic 'volume' arguments give evidence in favour of the following statement: for every $(\alpha, \beta) \in \mathbb{R}^2$ there exist infinitely many $q \in \mathbb{N}$ such that

$$q \cdot \log q \cdot ||q\alpha|| \cdot ||q\beta|| \ll 1.$$

The results of Peck [14] and Pollington & Velani [15] give solid support to this statement which represents a significant strengthening of Littlewood's conjecture and implies that

[L1] $\quad \mathbf{Mad}^\lambda = \emptyset \;$ if $\; \lambda < 1$.

Moreover, we suspect that the heuristics are sharp and thus $\mathbf{Mad} := \mathbf{Mad}^1$ represents the natural analogue of **Bad** within the multiplicative setup. It is worth emphasizing that **Mad** defined in this manner is precisely the set we hit after shaving off a logarithm from the measure theoretic 'switch over' set $\mathbf{Mad}^2$. Note that this is in keeping with the classical setup. Furthermore, we claim that the analogue of Jarník-Besicovitch theorem is true for **Mad**. In other words,

[L2] $\quad \dim \mathbf{Mad}^\lambda = 2 \;$ if $\; \lambda \geqslant 1$.

Regarding **[L1]**, notice that a counterexample to LC would imply that $\mathbf{Mad}^\lambda$ is non-empty for any $\lambda \geqslant 0$. In principle, it should be easier to give a counterexample to **[L1]**. To date all that is know is the remarkable result of Einsiedler, Katok & Lindenstrauss [6] that states that $\dim \mathbf{Mad}^0 = 0$. The following would be a leap in the right direction towards **[L1]** and would represent a significant strengthening of the Einsiedler-Katok-Lindenstrauss zero dimension result.

[L3] $\quad \dim \mathbf{Mad}^\lambda = 0 \;$ if $\; \lambda < 1$.

To the best of our knowledge, currently we do not even know if $\dim \mathbf{Mad}^\lambda < 2$ for strictly positive $\lambda < 1$.

Regarding **[L2]**, very little beyond the trivial is known. A simple consequence of the 'FULL' statement above is that $\dim \mathbf{Mad}^\lambda = 2$ if $\lambda > 2$. Recently, Bugeaud & Moshchevitin [5] have shown that $\dim \mathbf{Mad}^2 = 2$. Note that this is non-trivial since the set $\mathbf{Mad}^2$ is of measure zero. Surprisingly and somewhat embarrassingly we are unable to show that $\mathbf{Mad}^{2-\epsilon} \neq \emptyset$ let alone

[L4] $\quad \mathbf{Mad}^\lambda \neq \emptyset \;$ if $\; 1 \leqslant \lambda < 2$.

In other words, given our current state of knowledge, we can not rule out the unlikely possibility that LC is actually true with a $(\log q)^{2-\epsilon}$ term inserted in the left hand side of (1) – see also [10, Question 37].

In this paper, we are unable to directly contribute towards the statements **[L1]** – **[L4]**. However, we are able to make a significant contribution towards establishing the analogue of **[L2]** within the framework of the mixed Littlewood conjecture. Thus, if there is a genuine 'dictionary' between the results related to the two conjectures then indirectly our contribution adds weight towards **[L2]**.

1.3 The mixed multiplicative setup and the set $\mathbf{Mad}_{\mathcal{D}}$

Recently, de Mathan & Teulié in [13] proposed the following variant of Littlewood's conjecture. Let $\mathcal{D}$ be a sequence $(d_k)_{k=1}^{\infty}$ of integers greater than or equal to 2 and let

$$D_0 := 1 \quad \text{and} \quad D_n := \prod_{k=1}^{n} d_k \ .$$

For $q \in \mathbb{Z}$ set

$$|q|_{\mathcal{D}} := \inf\{D_n^{-1} \ : \ q \in D_n\mathbb{Z}\}.$$

Mixed Littlewood Conjecture (MLC) *For every real number* α

$$\liminf_{q\to\infty} q\cdot|q|_{\mathcal{D}}\cdot||q\alpha||=0\ . \tag{2}$$

When $\mathcal{D}$ is the constant sequence equal to a prime number p, the norm $|\cdot|_{\mathcal{D}}$ is the usual p-adic norm $|\cdot|_p$. In this particular case, there is a perfect dictionary between the current body of results associated with (p-adic) MLC and LC. The following constitute the main non-trivial entries.

- In [13, Theorem 2.1] de Mathan & Teulié establish the analogue of Peck's cubic result.
- In [13, Section 1] de Mathan & Teulié observe that the ideas within [15] establish the analogue of the Pollington-Velani full dimension result. Also see [3, Theorem 4].
- In [4, Theorem 1] Bugeaud, Haynes & Velani establish the analogue of Gallagher's measure theoretic result.
- In [7, Theorem 1.1] Einsiedler & Kleinbock establish the analogue of the Einsiedler-Katok-Lindenstrauss zero dimension result.
- In [5] Bugeaud & Moshchevitin establish the analogue of their $\dim \mathbf{Mad}^2=2$ result.

Moving away from the p-adic case, the results associated with MLC in the first two items above are valid for any bounded sequence $\mathcal{D}$. In all likelihood, this is also true for the other three items. The biggest challenge of the three seems to lie in generalising the (p-adic) result of Einsiedler & Kleinbock to bounded sequences. We are pretty confident that the other two items can be generalised to bounded $\mathcal{D}$ without too much trouble but stress that we have not carried out the details[1]. The point being made here is that for bounded $\mathcal{D}$ there is reasonably hard evidence in support of a 'LC–MLC' dictionary.

For $\lambda\geqslant 0$ let

$$\mathbf{Mad}^{\lambda}_{\mathcal{D}}:=\left\{\alpha\in\mathbb{R}\ :\ \liminf_{q\to\infty}\ q\cdot(\log q)^{\lambda}\cdot|q|_{\mathcal{D}}\cdot||q\alpha||>0\right\}. \tag{3}$$

For bounded $\mathcal{D}$, in view of the above discussion it is natural to expect that the following statements correspond to the entries **[L1]** and **[L2]** within the 'LC–MLC' dictionary.

[ML1] $\mathbf{Mad}^{\lambda}_{\mathcal{D}}=\emptyset$ if $\lambda<1$.

[ML2] $\dim \mathbf{Mad}^{\lambda}_{\mathcal{D}}=1$ if $\lambda\geqslant 1$.

In short, the upshot for bounded $\mathcal{D}$ is that $\mathbf{Mad}_{\mathcal{D}}:=\mathbf{Mad}^{1}_{\mathcal{D}}$ represents the natural analogue of **Bad** within the 'mixed' multiplicative setup. The assumption that $\mathcal{D}$ is bounded is absolutely necessary – see Theorem 2 below.

Obviously a counterexample to MLC would imply that $\mathbf{Mad}^{\lambda}_{\mathcal{D}}\neq\emptyset$ for any $\lambda\geqslant 0$. In principle, it should be easier to give a counterexample to **[ML1]**. The Einsiedler–Kleinbock result ($\dim \mathbf{Mad}^{0}_{\mathcal{D}}=0$ within the p-adic case) represents the current state of knowledge regarding **[ML1]**. It would be highly desirable to obtaining the following generalization.

[ML3] $\dim \mathbf{Mad}^{\lambda}_{\mathcal{D}}=0$ if $\lambda<1$.

[1] The problem of generalizing the (p-adic) mixed result obtained in [4] to arbitrary sequences $\mathcal{D}$ is particularly interesting since for unbounded $\mathcal{D}$ we suspect that the 'volume' sum is dependant on $\mathcal{D}$.

As far as we are aware, it is not even known if $\dim \mathbf{Mad}_{\mathcal{D}}^{\lambda} < 1$ for strictly positive $\lambda < 1$.

The following contribution towards [**ML2**] constitutes the main result proved in this paper. In our opinion, up to powers of logarithms it is best possible for bounded $\mathcal{D}$.

Theorem 1 *Let $\mathcal{D}$ be a sequence of integers greater than or equal to 2. Then the set of real numbers α such that*

$$\liminf_{q\to\infty} \; q \cdot \log q \cdot \log\log q \cdot |q|_{\mathcal{D}} \cdot ||q\alpha|| > 0. \tag{4}$$

has Hausdorff dimension equal to 1.

A simple consequence of the theorem is the following statement.

Corollary 1 *Let $\mathcal{D}$ be a sequence of integers greater than or equal to 2. For $\lambda > 1$*

$$\dim \mathbf{Mad}_{\mathcal{D}}^{\lambda} = 1\,.$$

Unfortunately, for bounded $\mathcal{D}$ we are unable to deal with the case $\lambda = 1$. In fact, we are unable to show that

[**ML4**] $\quad \mathbf{Mad}_{\mathcal{D}}^{1} \neq \emptyset\,.$

However, for unbounded $\mathcal{D}$ we can do much better in the following sense.

Theorem 2 *Let $\mathcal{D} = \{2^{2^n}\}_{n\in\mathbb{N}}$. Then the set of real numbers α such that*

$$\liminf_{n\to\infty} q \cdot \log\log q \cdot \log\log\log q \cdot |q|_{\mathcal{D}} \cdot ||q\alpha|| > 0 \tag{5}$$

has Hausdorff dimension equal to 1.

A simple consequence of the theorem is the following statement.

Corollary 2 *There exist uncountably many unbounded sequences $\mathcal{D}$ of integers greater than or equal to 2 such that*

$$\dim \mathbf{Mad}_{\mathcal{D}}^{\lambda} = 1 \quad \forall\ \lambda > 0\ . \tag{6}$$

The theorem shows that [**ML1**] is not generally true for unbounded $\mathcal{D}$. It also suggests that if there are counterexamples to MLC then they may be easier to find among rapidly increasing sequences. Furthermore, for unbounded $\mathcal{D}$ it is not generally true that the natural analogue of **Bad** within the 'mixed' multiplicative setup is $\mathbf{Mad}_{\mathcal{D}}^{1}$. This is yet an other reason to why we restrict the 'LC–MLC' dictionary to bounded sequences. Indeed, we can deduce from the proof of Theorem 2 that the analogue of **Bad** for any given unbounded $\mathcal{D}$ is in fact dependant on the growth of $\mathcal{D}$.

2 Preliminaries

To prove Theorems 1 and 2 it will be convenient to work with the 'modified logarithm' function $\log^* : \mathbb{R} \to \mathbb{R}$ defined as follows

$$\log^* x := \begin{cases} 1 & \text{for } x < e \\ \log x & \text{for } x \geqslant e\,. \end{cases}$$

This will guarantee that for small values of x the function $\log^* x$ is well defined.

2.1 The basic strategy

Given a function $f : \mathbb{N} \to \mathbb{R}$ and a sequence $\mathcal{D}$ of integers not smaller than 2, consider the set

$$\mathbf{Mad}_{\mathcal{D}}(f) := \{\alpha \in \mathbb{R} \ : \ \liminf_{q \to \infty} f(q) \cdot q \cdot |q|_{\mathcal{D}} \cdot ||q\alpha|| > 0\}. \tag{7}$$

By definition the set $\mathbf{Mad}_{\mathcal{D}}(f)$ is a subset of $\mathbb{R}$ and therefore

$$\dim \mathbf{Mad}_{\mathcal{D}}(f) \leqslant 1.$$

Thus the proofs of Theorems 1 and 2 are reduced to establishing the following respective statements.

Proposition 1 *Let $\mathcal{D}$ be a sequence of integers greater than or equal to* 2*. Then*

$$\dim \mathbf{Mad}_{\mathcal{D}}(f) \geqslant 1 \quad \textit{with} \ \ f(q) := \log^* q \cdot \log^* \log q. \tag{8}$$

Proposition 2 *Let $\mathcal{D} = \{2^{2^n}\}_{n \in \mathbb{N}}$. Then*

$$\dim \mathbf{Mad}_{\mathcal{D}}(f) \geqslant 1 \quad \textit{with} \ \ f(q) := \log^* \log q \cdot \log^* \log^* \log q. \tag{9}$$

To establish the propositions we make use of the following decomposition of $\mathbf{Mad}_{\mathcal{D}}(f)$. For any constant $c > 0$ define

$$\mathbf{Mad}_{\mathcal{D}}(f, c) := \{\alpha \in \mathbb{R} \ : \ f(q) \cdot q \cdot |q|_{\mathcal{D}} \cdot ||q\alpha|| > c \ \ \forall \ q \in \mathbb{N}\} \, .$$

It is easily verified that

$$\mathbf{Mad}_{\mathcal{D}}(f, c) \subset \mathbf{Mad}_{\mathcal{D}}(f)$$

and

$$\mathbf{Mad}_{\mathcal{D}}(f) \ = \ \bigcup_{c>0} \mathbf{Mad}_{\mathcal{D}}(f, c) \ .$$

Geometrically, the set $\mathbf{Mad}_{\mathcal{D}}(f, c)$ simply consists of points on the real line that avoid all intervals

$$\Delta(r/q) := \left[\frac{r}{q} - \frac{c}{f(q)q^2|q|_{\mathcal{D}}}, \frac{r}{q} + \frac{c}{f(q)q^2|q|_{\mathcal{D}}}\right]$$

centered at rational points r/q with $q \geqslant 1$. Alternatively, points on the real line that lie within any such interval are removed. Given a rational r/q, let

$$H(q) := q^2 |q|_{\mathcal{D}} \tag{10}$$

denote its *height*. Trivially, we have that

$$|\Delta(r/q)| = \frac{2c}{f(q)H(q)}.$$

In order to show that $\dim \mathbf{Mad}_{\mathcal{D}}(f) \geqslant 1$, the idea is to construct a Cantor-type subset $\mathbf{K}_{\mathcal{D}}(f, c)$ of $\mathbf{Mad}_{\mathcal{D}}(f, c)$ such that

$$\dim \mathbf{K}_{\mathcal{D}}(f, c) \geqslant 1$$

for some small constant $c > 0$. Hence, by construction we have that

$$\dim \mathbf{Mad}_{\mathcal{D}}(f) \ \geqslant \ \dim \mathbf{Mad}_{\mathcal{D}}(f, c) \ \geqslant \ \dim \mathbf{K}_{\mathcal{D}}(f, c) \ \geqslant \ 1 \, .$$

Thus, the name of the game is to construct the 'right type' of Cantor set $\mathbf{K}_{\mathcal{D}}(f, c)$. In short, the properties of the desired set fall naturally within a general framework which we now describe.

2.2 A general Cantor framework

The parameters. Let I be a closed interval in $\mathbb{R}$. Let

$$\mathbf{R} := (R_n) \quad \text{with} \quad n \in \mathbb{Z}_{\geqslant 0}$$

be a sequence of natural numbers and

$$\mathbf{r} := (r_{m,n}) \quad \text{with} \quad m, n \in \mathbb{Z}_{\geqslant 0} \ \text{ and } \ m \leqslant n$$

be a two parameter sequence of non-negative real numbers.

The construction. We start by subdividing the interval I into R_0 closed intervals I_1 of equal length and denote by $\mathcal{I}_1$ the collection of such intervals. Thus,

$$\#\mathcal{I}_1 = R_0 \qquad \text{and} \qquad |I_1| = R_0^{-1}\,|\mathrm{I}|\ .$$

Next, we remove at most $r_{0,0}$ intervals I_1 from $\mathcal{I}_1$. Note that we do not specify which intervals should be removed but just give an upper bound on the number of intervals to be removed. Denote by $\mathcal{J}_1$ the resulting collection. Thus,

$$\#\mathcal{J}_1 \geqslant \#\mathcal{I}_1 - r_{0,0}\,. \tag{11}$$

For obvious reasons, intervals in $\mathcal{J}_1$ will be referred to as (level one) survivors. It will be convenient to define $\mathcal{J}_0 := \{J_0\}$ with $J_0 := \mathrm{I}$.

In general, for $n \geqslant 0$, given a collection $\mathcal{J}_n$ we construct a nested collection $\mathcal{J}_{n+1}$ of closed intervals J_{n+1} using the following two operations.

- *Splitting procedure.* We subdivide each interval $J_n \in \mathcal{J}_n$ into R_n closed sub-intervals I_{n+1} of equal length and denote by $\mathcal{I}_{n+1}$ the collection of such intervals. Thus,

$$\#\mathcal{I}_{n+1} = R_n \times \#\mathcal{J}_n \qquad \text{and} \qquad |I_{n+1}| = R_n^{-1}\,|J_n|\ .$$

- *Removing procedure.* For each interval $J_n \in \mathcal{J}_n$ we remove at most $r_{n,n}$ intervals $I_{n+1} \in \mathcal{I}_{n+1}$ that lie within J_n. Note that the number of intervals I_{n+1} removed is allowed to vary amongst the intervals in $\mathcal{J}_n$. Let $\mathcal{I}_{n+1}^n \subseteq \mathcal{I}_{n+1}$ be the collection of intervals that remain. Next, for each interval $J_{n-1} \in \mathcal{J}_{n-1}$ we remove at most $r_{n-1,n}$ intervals $I_{n+1} \in \mathcal{I}_{n+1}^n$ that lie within J_{n-1}. Let $\mathcal{I}_{n+1}^{n-1} \subseteq \mathcal{I}_{n+1}^n$ be the collection of intervals that remain. In general, for each interval $J_{n-k} \in \mathcal{J}_{n-k}$ $(1 \leqslant k \leqslant n)$ we remove at most $r_{n-k,n}$ intervals $I_{n+1} \in \mathcal{I}_{n+1}^{n-k+1}$ that lie within J_{n-k}. Also we let $\mathcal{I}_{n+1}^{n-k} \subseteq \mathcal{I}_{n+1}^{n-k+1}$ be the collection of intervals that remain. In particular, $\mathcal{J}_{n+1} := \mathcal{I}_{n+1}^0$ is the desired collection of (level $n+1$) survivors. Thus, the total number of intervals I_{n+1} removed during the removal procedure is at most $r_{n,n}\#\mathcal{J}_n + r_{n-1,n}\#\mathcal{J}_{n-1} + \ldots + r_{0,n}\#\mathcal{J}_0$ and so

$$\#\mathcal{J}_{n+1} \geqslant R_n\#\mathcal{J}_n - \sum_{k=0}^{n} r_{k,n}\#\mathcal{J}_k. \tag{12}$$

Finally, having constructed the nested collections $\mathcal{J}_n$ of closed intervals we consider the limit set

$$\mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r}) := \bigcap_{n=1}^{\infty} \bigcup_{J \in \mathcal{J}_n} J.$$

The set $\mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r})$ will be referred to as a $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ *Cantor set.*

Remark. We stress that the triple $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ does not uniquely determine the set $\mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r})$. The point is that during the construction we only specify the maximum number of intervals rather than the specific intervals to be removed. Thus the triple $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ gives rise to a family of $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ Cantor sets that reflects the various available choices during the removing procedure.

As an illustration of the general framework, it is easily seen that the standard middle third Cantor set corresponds to a $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ Cantor set with

$$\mathrm{I} := [0,1], \qquad \mathbf{R} = (3,3,3,\ldots) \quad \text{and} \quad \mathbf{r} = (r_{m,n})$$

where

$$r_{m,n} := \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise.} \end{cases}$$

The results. By definition, if $\mathcal{J}_n$ is empty for some $n \in \mathbb{N}$ then the corresponding set $\mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r})$ is obviously empty. On the other hand, by construction, each closed interval $J_n \in \mathcal{J}_n$ is contained in some closed interval $J_{n-1} \in \mathcal{J}_{n-1}$. Therefore

$$\mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r}) \neq \emptyset \qquad \text{if} \qquad \#\mathcal{J}_n \geqslant 1 \quad \forall\, n \in \mathbb{N}\,.$$

Our first result provides a natural condition that guarantees this cardinality hypothesis and therefore the non-empty statement.

Theorem 3 *Given* $\mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r})$*, let*

$$t_0 := R_0 - r_{0,0} \tag{13}$$

and for $n \geqslant 1$ *let*

$$t_n := R_n - r_{n,n} - \sum_{k=1}^{n} \frac{r_{n-k,n}}{\prod_{i=1}^{k} t_{n-i}}\,. \tag{14}$$

Suppose that $t_n > 0$ *for all* $n \in \mathbb{Z}_{\geqslant 0}$*. Then*

$$\mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r}) \neq \emptyset\ .$$

The proof of Theorem 3 is short and direct and there seems little point in delaying it.

Proof of Theorem 3. We show that a consequence of the construction of $\mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r})$ is that

$$\#\mathcal{J}_n \geqslant t_{n-1} \#\mathcal{J}_{n-1} \quad \forall\, n \in \mathbb{N}\,. \tag{15}$$

This together with the assumption that $t_n > 0$ implies that $\#\mathcal{J}_n \geqslant \prod_{i=0}^{n-1} t_i \,\#\mathcal{J}_0 > 0$ and thereby completes the proof of Theorem 3. To verify (15) we use induction. In view of (11) and (13) the statement is trivially true for $n = 1$. Now suppose that (15) is true for all $1 \leqslant k \leqslant n$. In particular, for any such k we have that

$$\#\mathcal{J}_n \geqslant t_{n-1}\#\mathcal{J}_{n-1} \geqslant \ldots \geqslant \prod_{i=1}^{k} t_{n-i}\#\mathcal{J}_{n-k}.$$

Thus,

$$\begin{aligned}
\#\mathcal{J}_{n+1} &\overset{(12)}{\geqslant} R_n\#\mathcal{J}_n - \sum_{k=0}^{n} r_{n-k,n}\#\mathcal{J}_{n-k} \\
&\geqslant R_n\#\mathcal{J}_n - r_{n,n}\#\mathcal{J}_n - \sum_{k=1}^{n} \frac{r_{n-k,n}\#\mathcal{J}_n}{\prod_{i=1}^{k} t_{n-i}} \\
&\overset{(14)}{=} t_n\#\mathcal{J}_n.
\end{aligned}$$

This completes the induction step and establishes (15) as required. ⊠

Our next result enables us to estimate the Hausdorff dimension of $\mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r})$. It is the key to establishing Propositions 1 & 2.

Theorem 4 *Given $\mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r})$, suppose that $R_n \geqslant 4$ for all $n \in \mathbb{Z}_{\geqslant 0}$ and that*

$$\sum_{k=0}^{n}\left(r_{n-k,n}\prod_{i=1}^{k}\left(\frac{4}{R_{n-i}}\right)\right) \leqslant \frac{R_n}{4}. \tag{16}$$

Then

$$\dim \mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r}) \geqslant \liminf_{n\to\infty}(1 - \log_{R_n} 2).$$

Here we use the convention that the product term in (16) is one when $k = 0$ and by definition $\log_{R_n} 2 := \log 2/\log R_n$. The proof of Theorem 4 is rather involved and constitutes the main substance of §5 and §6. To some extent the raw ideas required to establish Theorem 4 can be found in [1, §7] where a conjecture of W.M. Schmidt regarding the intersection of simultaneously badly approximable sets is proved. Nevertheless we stress that in this paper we develop a general Cantor type framework rather than address a specific problem. As a consequence the key ideas of [1] are foregrounded.

Remark. Although Theorem 4 is more than sufficient for the specific application we have in mind, we would like to point out that we have not attempted to establish the most general or best possible statement. For example, in the case $R_n \to \infty$ as $n \to \infty$, the theorem together with the fact that $\mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r}) \subset \mathbb{R}$ implies that $\dim \mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r}) = 1$. However, we do not claim that condition (16) is optimal for establishing this full dimension result.

In the final section of the paper, we show that the intersection of any finite number of sets $\mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r}_i)$ is yet another $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ Cantor set for some appropriately chosen $\mathbf{r}$. We shall also see in §7 that this enables us to strengthen Theorem 1.

3 Proof of Proposition 1 modulo Theorem 4

Throughout, $f : \mathbb{N} \to \mathbb{R} : q \to f(q) = \log^* q \cdot \log^* \log q$ and $\mathcal{D}$ is a sequence of integers greater than or equal to 2. Let

$$R > e^{12}$$

be an integer. Choose $c_1 = c_1(R) > 0$ sufficiently small so that

$$2e^2 c_1 \frac{\log R + 2}{\log 2} R < 1 \tag{17}$$

and let $c = c(R, c_1) > 0$ be a constant such that

$$c\left(\frac{64R^2(\log R + 2)}{c_1 \log 2} + \frac{16eR^2(\log R + 2)^2}{\log 2}\right) < 1. \tag{18}$$

With reference to §2.1 we now describe the basic construction of the set $\mathbf{K}_{\mathcal{D}}(f, c)$. Let I be any interval of length c_1 contained within the unit interval $[0, 1]$. Denote by $\mathcal{J}_0 := \{J_0\}$ where $J_0 := \mathrm{I}$. The idea is to establish, by induction on n, the existence of a collection $\mathcal{J}_n$ of closed intervals J_n such that $\mathcal{J}_n$ is nested in $\mathcal{J}_{n-1}$; that is, each interval J_n in $\mathcal{J}_n$ is contained in some interval J_{n-1} in $\mathcal{J}_{n-1}$. The length of an interval J_n will be given by

$$|J_n| \ := \ c_1 \, R^{-n} F^{-1}(n) \ ,$$

where

$$F(n) := \prod_{k=1}^{n} k \, [\log^* k] \quad \text{for } n \geqslant 1 \qquad \text{and} \qquad F(0) := 1 \ \text{ for } n \leqslant 0.$$

Moreover, each interval J_n in $\mathcal{J}_n$ will satisfy the condition that

$$J_n \cap \Delta(r/q) \ = \ \emptyset \qquad \forall \ \ r/q \in \mathbb{Q} \ \text{ with } \ H(q) < R^{n-1} F(n-1) \, . \tag{19}$$

In particular, we put

$$\mathbf{K}_{\mathcal{D}}(f, c) := \bigcap_{n=1}^{\infty} \bigcup_{J \in \mathcal{J}_n} J \ .$$

By construction, condition (19) ensures that

$$\mathbf{K}_{\mathcal{D}}(f, c) \subset \mathbf{Mad}_{\mathcal{D}}(f, c) \ .$$

Furthermore, with reference to §2.2 it will be apparent from the construction of the collections $\mathcal{J}_n$ that $\mathbf{K}_{\mathcal{D}}(f, c)$ is in fact a $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ Cantor set with $\mathbf{R} = (R_n)$ given by

$$R_n := R\,(n+1)\,[\log^*(n+1)] \tag{20}$$

and $\mathbf{r} = (r_{m,n})$ given by

$$r_{m,n} \ := \begin{cases} 7 \, \log^2 R \cdot n^2 (\log^* n)^2 & \text{if } \ m = n-1 \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

By definition, note that for any $R > e^9$ we have that the

$$\begin{aligned} \text{l.h.s. of (16)} \quad &= \quad r_{n-1,n} \cdot \frac{4}{R_{n-1}} \ \leqslant \ 7 \cdot 2^3 \cdot \frac{\log^2 R \cdot n \, \log^* n}{R} \\ &\leqslant \quad \frac{7 \cdot 2^6 \log^2 R}{R^2} \cdot \frac{R\,(n+1)\,[\log^*(n+1)]}{4} \\ &\leqslant \quad \frac{R_n}{4} \ = \ \text{r.h.s. of (16)} \, . \end{aligned}$$

Since we are assuming that $R > e^{12}$, it then follows via Theorem 4 that

$$\dim \mathbf{K}_{\mathcal{D}}(f, c) \geqslant \liminf_{n \to \infty} (1 - \log_{R_n} 2) = 1 \, .$$

This completes the proof of Proposition 1 modulo Theorem 4 and the construction of the collections $\mathcal{J}_n$.

3.1 Constructing the collections $\mathcal{J}_n$

For $n = 0$, we trivially have that (19) is satisfied for the interval $J_0 = \mathrm{I}$. The point is that there are no rationals satisfying the height condition $H(q) < 1$ since by definition $H(q) \geqslant 1$. For the same reason (19) with $n = 1$ is trivially satisfied for any interval J_1 obtained by subdividing each J_0 in $\mathcal{J}_0$ into $R_0 = R$ closed intervals of equal length $c_1 R^{-1}$. Denote by $\mathcal{J}_1$ the resulting collection of intervals J_1 and note that $\#\mathcal{J}_1 = R$.

In general, given $\mathcal{J}_n$ satisfying (19) we wish to construct a nested collection $\mathcal{J}_{n+1}$ of intervals J_{n+1} for which (19) is satisfied with n replaced by $n+1$. By definition, any interval J_n in $\mathcal{J}_n$ avoids intervals $\Delta(r/q)$ arising from rationals with height bounded above by the quantity $R^{n-1}F(n-1)$. Since any 'new' interval J_{n+1} is to be nested in some J_n, it is enough to show that J_{n+1} avoids intervals $\Delta(r/q)$ arising from rationals r/q with height satisfying

$$R^{n-1}F(n-1) \leqslant H(q) < R^n F(n) \ . \tag{22}$$

Denote by $C(n)$ the collection of all rationals satisfying this height condition. Formally

$$C(n) := \{ r/q \in \mathbb{Q} \ : \ H(q) \ \text{ satisfies (22)} \}$$

and it is precisely this collection of rationals that comes into play when attempting to construct $\mathcal{J}_{n+1}$ from $\mathcal{J}_n$. We now proceed with the construction.

Assume that $n \geqslant 1$. We subdivide each J_n in $\mathcal{J}_n$ into

$$R_n \overset{(20)}{=} R(n+1)\,[\log^*(n+1)]$$

closed intervals I_{n+1} of equal length $c_1 R^{-(n+1)} F^{-1}(n+1)$ and denote by $\mathcal{I}_{n+1}$ the collection of such intervals. Thus,

$$|I_{n+1}| = c_1 R^{-(n+1)} F^{-1}(n+1)$$

and

$$\#\mathcal{I}_{n+1} \ = \ R(n+1)\ [\log^*(n+1)] \ \times \ \#\mathcal{J}_n \, .$$

It is obvious that the construction of $\mathcal{I}_{n+1}$ corresponds to the splitting procedure associated with the construction of a $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ Cantor set.

In view of the nested requirement, the collection $\mathcal{J}_{n+1}$ which we are attempting to construct will be a sub-collection of $\mathcal{I}_{n+1}$. In other words, the intervals I_{n+1} represent possible candidates for J_{n+1}. The goal now is simple — it is to remove those 'bad' intervals I_{n+1} from $\mathcal{I}_{n+1}$ for which

$$I_{n+1} \cap \Delta(r/q) \ \neq \emptyset \quad \text{for some} \ \ r/q \in C(n) \ . \tag{23}$$

The sought after collection $\mathcal{J}_{n+1}$ consists precisely of those intervals that survive. Formally, for $n \geqslant 1$ we let

$$\mathcal{J}_{n+1} := \{ I_{n+1} \in \mathcal{I}_{n+1} \ : \ I_{n+1} \cap \Delta(r/q) = \emptyset \ \text{ for any } r/q \in C(n) \}.$$

For any interval $J_{n-1} \in \mathcal{J}_{n-1}$ and any integer $R \geqslant e^{12}$, we claim that

$$\#\{ I_{n+1} \in \mathcal{I}_{n+1} \ : \ J_{n-1} \cap I_{n+1} \cap \Delta(r/q) \neq \emptyset \ \text{ for some } \ r/q \in C(n) \}$$

$$\leqslant \ \ 7\ \log^2 R n^2 (\log^* n)^2 \, . \tag{24}$$

It then follows from the definition of $r_{m,n}$ that

$$\#\{ I_{n+1} \in \mathcal{I}_{n+1} \backslash \mathcal{J}_{n+1} \ : \ I_{n+1} \subset J_{n-1} \} \leqslant 7 \log^2 R \cdot n^2 (\log^* n)^2 \overset{(21)}{=} r_{n-1,n}$$

and therefore the act of removing 'bad' intervals from $\mathcal{I}_{n+1}$ is exactly in keeping with the removal procedure associated with the construction of a $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ Cantor set. The goal now is to justify (24).

3.1.1 Counting removed intervals

Stage 1. Let $r/q \in C(n)$. Then there exists a non-negative integer k and an integer $\bar{q}$ such that

$$q = D_k \cdot \bar{q} \qquad \text{and} \qquad q \notin D_{k+1}\mathbb{Z}. \tag{25}$$

Then,

$$H(q) := D_k \cdot \bar{q}^2 \,.$$

Since all the terms d_k of $\mathcal{D}$ are greater than or equal to two, we have that

$$D_k \geqslant 2^k \,. \tag{26}$$

Next, note that $q^2 \geqslant H(q) \geqslant R^{n-1}F(n-1)$. Thus, for any $R > e^9$ it follows that

$$\begin{aligned} f(q) &\geqslant \tfrac{1}{2}\log^*(R^{n-1}F(n-1))\log^*\tfrac{1}{2}\log(R^{n-1}F(n-1)) \\ &\geqslant \tfrac{1}{2}\, n(\log^* n)^2 \,. \end{aligned} \tag{27}$$

To see this first observe that (27) for $n = 1$ is clearly true. For $n \geqslant 2$, by Stirling formula we have that

$$R^{n-1}F(n-1) \;\geqslant\; R^{n-1}(n-1)! \;>\; (8n)^n \qquad \text{for any} \;\; R > e^9 \,.$$

Therefore the left hand side of (27) is bigger than

$$\tfrac{1}{2}n\log^*(8n)\cdot\log^*\left(\tfrac{1}{2}n\log(8n)\right) \;>\; \tfrac{1}{2}n(\log^* n)^2.$$

Stage 2. We subdivide the collection $C(n)$ of rationals into various 'workable' sub-collections. In the first instance, for any integer $k \geqslant 0$, let $\mathcal{C}(n,k) \subset \mathcal{C}(n)$ denote the collection of rationals satisfying the additional condition (25). Formally,

$$C(n,k) := \{r/q \in C(n) \;:\; q \text{ satisfies } (25)\}\,. \tag{28}$$

For any $r/q \in C(n,k)$ we have that $H(q) = D_k \cdot \bar{q}^2$ and thus in view of (22) and (26) it follows that

$$\begin{aligned} 0 \;\leqslant\; k \;&\leqslant\; [\log_2(R^nF(n))] \;<\; n\log_2 R + n\log_2 n + n\log_2\log^* n \\ &<\; c_2\, n\log^* n, \end{aligned} \tag{29}$$

where $c_2 := (\log R + 2)/\log 2$ is an absolute constant independent on n. The upshot is that for fixed n the number of (non-empty) collections $C(n,k)$ is at most $c_2 n\log^* n$.

Next, for any integer $l \geqslant 0$, let $\mathcal{C}(n,k,l) \subset \mathcal{C}(n,k)$ denote the collection of rationals satisfying the additional condition that

$$e^l R^{n-1}F(n-1) \;\leqslant\; H(q) \;<\; e^{l+1}R^{n-1}F(n-1)\,. \tag{30}$$

Formally,

$$C(n,k,l) := \{r/q \in C(n,k) \;:\; q \text{ satisfies } (30)\}\,.$$

In view of (22) we have that

$$e^l < Rn\log^* n$$

and thus it follows that

$$\begin{aligned} 0 \;\leqslant\; l \;&\leqslant\; [\log(Rn\log^* n)] \;<\; \log R + 2\log^* n \\ &<\; c_3\log^* n \end{aligned} \tag{31}$$

where $c_3 := 2 + \log R$. The upshot is that for fixed n and k the number of (non-empty) collections $C(n,k,l)$ is at most $c_3 \log^* n$. Notice that within any collection $C(n,k,l)$ we have extremely tight control on the height.

Stage 3. Fix an interval $J_{n-1} \in \mathcal{J}_{n-1}$. Recall, that our goal is establish (24). This we will do by estimating the quantity

$$\#\{I_{n+1} \in \mathcal{I}_{n+1} \ : \ J_{n-1} \cap I_{n+1} \cap \Delta(r/q) \neq \emptyset \ \text{ for some } \ r/q \in C(n,k,l)\}$$

and then summing over all possible values of k and l. With this in mind, consider a rational $r/q \in C(n,k,l)$ and assume that $R > e^9$. Then

$$\begin{aligned}
\#\{I_{n+1} \in \mathcal{I}_{n+1} \ : \ I_{n+1} \cap \Delta(r/q) \neq \emptyset\} &\leqslant \frac{|\Delta(r/q)|}{|I_{n+1}|} + 2 \\
&= \frac{2cR^{n+1}F(n+1)}{c_1 f(q) H(q)} + 2 \\
&\overset{(30)}{\leqslant} \frac{2cR^2 n(n+1)\,[\log^* n]\,[\log^*(n+1)]}{c_1 f(q) e^l} + 2 \\
&\overset{(27)}{<} \frac{8cR^2(n+1)}{c_1 e^l} + 2\,. \qquad (32)
\end{aligned}$$

Next, consider two rationals $r_1/q_1,\, r_2/q_2 \in C(n,k,l)$. By definition, there exit integers $\bar{q}_1$, $\bar{q}_2$ so that

$$q_1 = D_k \bar{q}_1 \quad \text{and} \quad q_2 = D_k \bar{q}_2.$$

Thus $(q_1, q_2) \geqslant D_k$ and we have that

$$\left|\frac{r_1}{q_1} - \frac{r_2}{q_2}\right| \geqslant \frac{1}{D_k \bar{q}_1 \bar{q}_2} = (H(q_1)H(q_2))^{-1/2} \overset{(30)}{>} e^{-l-1} R^{-n+1} F^{-1}(n-1).$$

It is easily verified that $2\,|\Delta(r/q)|$ is less than the right hand side of the above inequality – this makes use of the fact that $4ec < 1$ which is true courtesy of (18). Therefore,

$$\Delta(r_1/q_1) \cap \Delta(r_2/q_2) = \emptyset$$

and it follows that

$$\begin{aligned}
&\#\{r/q \in C(n,k,l) : \ J_{n-1} \cap \Delta(r/q) \neq \emptyset\} \\
&\qquad \leqslant \ 2 + \frac{|J_{n-1}|}{e^{-l-1} R^{-n+1} F^{-1}(n-1)} \ = \ 2 + c_1 e^{l+1}. \qquad (33)
\end{aligned}$$

The upshot of the cardinality estimates (32) and (33) is that

$$\begin{aligned}
&\#\{I_{n+1} \in \mathcal{I}_{n+1} : \ J_{n-1} \cap I_{n+1} \cap \Delta(r/q) \neq \emptyset \ \text{ for some } r/q \in C(n,k,l)\} \\
&\qquad \leqslant \ \left(2 + c_1 e^{l+1}\right)\left(2 + \frac{8cR^2(n+1)}{c_1 e^l}\right) \\
&\qquad = \ 4 + 2c_1 e^{l+1} + \frac{16cR^2(n+1)}{c_1 e^l} + 8ecR^2(n+1).
\end{aligned}$$

By summing over l satisfying (31) we find that

$$\#\{I_{n+1} \in \mathcal{I}_{n+1} \ : \ J_{n-1} \cap I_{n+1} \cap \Delta(r/q) \neq \emptyset \ \text{ for some } r/q \in C(n,k)\}$$

$$\leqslant \sum_{e^l < Rn\log^* n} 2c_1 e^{l+1} + \sum_{l=0}^{c_3 \log^* n} \frac{16cR^2(n+1)}{c_1 e^l} + c_3 \log^* n(4 + 8ecR^2(n+1))$$

$$\leqslant c_4 n \log^* n$$

where

$$c_4 := 2e^2 c_1 R + \frac{64c}{c_1} R^2 + (\log R + 2)(16e\, c\, R^2 + 4).$$

By summing over k satisfying (29) we find that

$$\#\{I_{n+1} \in \mathcal{I}_{n+1} \ : \ J_{n-1} \cap I_{n+1} \cap \Delta(r/q) \neq \emptyset \ \text{ for some } r/q \in C(n)\}$$

$$\leqslant c_2 c_4 n^2 (\log^* n)^2 \,. \tag{34}$$

In view of (17) and (18), for any $R > e^{12}$ the right hand side of (34) is bounded by

$$\left(2 + \frac{4(\log R + 2)^2}{\log 2}\right) n^2 (\log^* n)^2 \ < \ 7 \log^2 R \cdot n^2 (\log^* n)^2 \,.$$

This establishes (24) as required.

4 Proof of Proposition 2 modulo Theorem 4

The proof of Proposition 2 follows the same structure and ideas as the proof of Proposition 1. In view of this it is really only necessary to point out the key differences.

Throughout, $f : \mathbb{N} \to \mathbb{R} : q \to f(q) = \log^* \log q \cdot \log^* \log^* \log q$ and $\mathcal{D} := \{2^{2^n}\}_{n \in \mathbb{N}}$. Note that by definition

$$D_k \geqslant 2^{2^k} \,. \tag{35}$$

With R, c_1 and c as in the proof of Proposition 1, the basic construction of

$$\mathbf{K}_{\mathcal{D}}(f,c) := \bigcap_{n=1}^{\infty} \bigcup_{J \in \mathcal{J}_n} J \ \subset \ \mathbf{Mad}_{\mathcal{D}}(f,c)$$

remains pretty much unchanged apart from the fact that the function F is given by

$$F(n) := \prod_{k=1}^{n} [\log^* k \cdot \log^* \log k] \quad \text{for } n \geqslant 1 \qquad \text{and} \qquad F(0) := 1 \ \text{ for } n \leqslant 0.$$

Also, it becomes apparent from the construction of the collections $\mathcal{J}_n$ that $\mathbf{K}_{\mathcal{D}}(f,c)$ is a $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ Cantor set with $\mathbf{R} = (R_n)$ given by

$$R_n := R\,[\log^*(n+1) \log^* \log(n+1)]$$

and $\mathbf{r} = (r_{m,n})$ given by

$$r_{m,n} \ := \begin{cases} 7 \ \log^2 R (\log^* n)^2 (\log^* \log n)^2 & \text{if } \ m = n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Then, it is easily verified that (16) is valid for any $R > e^9$ and so Proposition 2 follows via Theorem 4.

Regarding the construction of the collections $\mathcal{J}_n$, the induction procedure is precisely as in §3.1. The upshot is that the proof of Proposition 2 reduces to establishing the following analogue of (24). For any interval $J_{n-1} \in \mathcal{J}_{n-1}$ and any integer $R > e^{12}$, we have that

$$\begin{aligned} \#\{I_{n+1} \in \mathcal{I}_{n+1} \, : \, J_{n-1} \cap I_{n+1} \cap \Delta(r/q) \neq \emptyset \;\text{ for some }\; r/q \in C(n)\} \\ \leqslant \; 7 \, \log^2 R (\log^* n)^2 (\log^* \log n)^2 \, . \end{aligned} \tag{36}$$

This implies that act of removing 'bad' intervals from $\mathcal{I}_{n+1}$ when constructing $\mathcal{J}_{n+1}$ from $\mathcal{J}_n$ is exactly in keeping with the removal procedure associated with the construction of a $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ Cantor set. In order to establish (36) we follow the arguments set out in §3.1.1. For completeness and ease of comparison we briefly describe the analogue of the key estimates.

Stage 1. The analogue of (27) is the statement that for any $R > e^4$

$$\begin{aligned} f(q) \;&\geqslant\; \log^* \tfrac{1}{2} \log(R^{n-1} F(n-1)) \cdot \log^* \log^* \tfrac{1}{2} \log(R^{n-1} F(n-1)) \\ &\geqslant\; \log^* n \log^* \log n . \end{aligned} \tag{37}$$

This makes use of the fact that for $n \geqslant 2$

$$R^{n-1} F(n-1) \geqslant e^{2n} \quad \text{ for any } \; R > e^4 .$$

Stage 2. In view of (35), it follows that the analogue of (29) is that

$$0 \;\leqslant\; k \;\leqslant\; [\log_2 \log_2 (R^n F(n))] \;<\; \tilde{c}_2 \log^* n \tag{38}$$

where

$$\tilde{c}_2 := \frac{1}{\log 2} \left(2 + \log \frac{\log R + 2}{\log 2} \right) < c_2$$

Note that $\tilde{c}_2 < c_2$ is valid since $R \geqslant 6$. Next, in view of (22) we have that

$$e^l \;<\; R \log^* n \log^* \log n$$

and thus it follows that

$$0 \;\leqslant\; l \;\leqslant\; c_3 \log^* \log n \, . \tag{39}$$

Stage 3. Fix an interval $J_{n-1} \in \mathcal{J}_{n-1}$. Then (33) remains unchanged and the analogue of (32) is as follows. Consider a rational $r/q \in C(n, k, l)$ and assume that $R > e^4$. Then

$$\begin{aligned} \#\{I_{n+1} \in \mathcal{I}_{n+1} \, : \, I_{n+1} \cap \Delta(r/q) \neq \emptyset\} \;&\leqslant\; \frac{2cR^{n+1} F(n+1)}{c_1 f(q) H(q)} + 2 \\ &\overset{(30)}{\leqslant}\; \frac{2cR^2 \, [\log^* n \log^* \log n] \, [\log^*(n+1) \log^* \log(n+1)]}{c_1 f(q) e^l} + 2 \\ &\overset{(37)}{<}\; \frac{8cR^2 \log(n+1) \log^* \log(n+1)}{c_1 e^l} + 2 \, . \end{aligned}$$

The upshot is that

$$\#\{I_{n+1}\in\mathcal{I}_{n+1}:\ J_{n-1}\cap I_{n+1}\cap\Delta(r/q)\neq\emptyset\ \text{ for some } r/q\in C(n,k,l)\}$$
$$\leqslant\ \left(2+c_1e^{l+1}\right)\left(2+\frac{8cR^2\log^*(n+1)\log^*\log(n+1)}{c_1e^l}\right).$$

By summing up over l satisfying (39) we find that

$$\#\{I_{n+1}\in\mathcal{I}_{n+1}\ :\ J_{n-1}\cap I_{n+1}\cap\Delta(r/q)\neq\emptyset\ \text{ for some } r/q\in C(n,k)\}$$
$$\leqslant\ \sum_{e^l<R\log^*n\log^*\log n}2c_1e^{l+1}\ +\ \sum_{l=0}^{c_3\log^*\log n}\frac{16cR^2\log^*(n+1)\log^*\log(n+1)}{c_1e^l}$$
$$+\ c_3\log^*\log n(4+8e\,c\,R^2\log^*(n+1)\log^*\log(n+1))$$
$$\leqslant\ c_4\log^*n(\log^*\log^*n)^2\,.$$

By summing up over k satisfying (38) we find that

$$\#\{I_{n+1}\in\mathcal{I}_{n+1}\ :\ J_{n-1}\cap I_{n+1}\cap\Delta(r/q)\neq\emptyset\ \text{ for some } r/q\in C(n)\}$$
$$\leqslant\ c_2c_4(\log^*n)^2\cdot(\log^*\log n)^2.$$

In view of (17) and (18), for any $R>e^{12}$ the right hand side of the above inequality is bounded by

$$\left(2+\frac{4(\log R+2)^2}{\log 2}\right)(\log^*n)^2\cdot(\log^*\log n)^2\quad<\quad 7\log^2R\cdot(\log^*n)^2\cdot(\log^*\log n)^2\,.$$

This establishes (36) and thereby completes the proof of Proposition 2.

5 Preliminaries for Theorem 4

The overall strategy is simple enough. We show that under the hypothesis of the theorem, any given set $\mathbf{K}(\mathrm{I},\mathbf{R},\mathbf{r})$ contains a 'local' subset $\mathbf{LK}(\mathrm{I},\mathbf{R},\mathbf{s})$ satisfying the desired lower bound inequality for the Hausdorff dimension. A general and classical method for obtaining a lower bound for the Hausdorff dimension of an arbitrary set is the following mass distribution principle – see [8, pg. 55].

Mass Distribution Principle *Let μ be a probability measure supported on a subset X of $\mathbb{R}$. Suppose there are positive constants a, s and l_0 such that*

$$\mu(B)\leqslant\ a\ |B|^s\,, \tag{40}$$

for any interval B with length $|B|\leqslant l_0$. Then, $\dim X\geqslant s$.

As we shall soon see, the construction of the local set alluded to above is much simpler than that of $\mathbf{K}(\mathrm{I},\mathbf{R},\mathbf{r})$ and enables us to exploit the mass distribution principle.

5.1 Local Cantor sets

A $(\mathrm{I},\mathbf{R},\mathbf{r})$ Cantor set $\mathbf{K}(\mathrm{I},\mathbf{R},\mathbf{r})$ is said to be local if $r_{m,n}=0$ whenever $m\neq n$. Furthermore, we write $\mathbf{LK}(\mathrm{I},\mathbf{R},\mathbf{s})$ for $\mathbf{K}(\mathrm{I},\mathbf{R},\mathbf{r})$ where

$$\mathbf{s}:=(s_n)_{n\in\mathbb{Z}_{\geqslant 0}} \quad \text{and} \quad s_n:=r_{n,n}.$$

The set $\mathbf{LK}(\mathrm{I},\mathbf{R},\mathbf{s})$ will be referred to as a $(\mathrm{I},\mathbf{R},\mathbf{s})$ *local Cantor set.*

In a nutshell, the removing procedure associated with the construction of a local Cantor set has no 'memory' – it depends only on the level under consideration. More formally, given the collection $\mathcal{J}_n$ of level n survivors, the construction of $\mathcal{J}_{n+1}$ is completely independent of the previous level k $(<n)$ survivors. Indeed the construction is totally local within each interval $J_n\in\mathcal{J}_n$. It is this fact that is utilized when attempting to establish the following dimension result for the associated local Cantor set. Note that in view of Theorem 3, any local set $\mathbf{LK}(\mathrm{I},\mathbf{R},\mathbf{s})$ is non-empty if $R_n-s_n>0$ for all $n\in\mathbb{Z}_{\geqslant 0}$.

Lemma 1 *Given* $\mathbf{LK}(\mathrm{I},\mathbf{R},\mathbf{s})$*, suppose that*

$$t_n:=R_n-s_n>0 \quad \forall\ n\in\mathbb{Z}_{\geqslant 0}\,.$$

Furthermore, suppose there are positive constants s and n_0 such that for all $n>n_0$

$$R_n^s\leqslant t_n\,. \tag{41}$$

Then

$$\dim\mathbf{LK}(\mathrm{I},\mathbf{R},\mathbf{s})\geqslant s.$$

Proof. We start by constructing a probability measure μ supported on

$$\mathbf{LK}(\mathrm{I},\mathbf{R},\mathbf{s}):=\bigcap_{n=1}^{\infty}\bigcup_{J\in\mathcal{J}_n}J.$$

in the standard manner. For any $J_n\in\mathcal{J}_n$, we attach a weight $\mu(J_n)$ defined recursively as follows.

For $n=0$,

$$\mu(J_0)\ :=\frac{1}{\#\mathcal{J}_0}=1$$

and for $n\geqslant 1$,

$$\mu(J_n)\ :=\ \frac{\mu(J_{n-1})}{\#\{J\in\mathcal{J}_n\ :\ J\subset J_{n-1}\}} \tag{42}$$

where $J_{n-1}\in\mathcal{J}_{n-1}$ is the unique interval such that $J_n\subset J_{n-1}$. This procedure thus defines inductively a mass on any interval appearing in the construction of $\mathbf{LK}(\mathrm{I},\mathbf{R},\mathbf{s})$. In fact a lot more is true — μ can be further extended to all Borel subsets F of $\mathbb{R}$ to determine $\mu(F)$ so that μ constructed as above actually defines a measure supported on $\mathbf{LK}(\mathrm{I},\mathbf{R},\mathbf{s})$. We now state this formally.

> *Fact.* The probability measure μ constructed above is supported on $\mathbf{LK}(\mathrm{I},\mathbf{R},\mathbf{s})$ and for any Borel set F
>
> $$\mu(F):=\mu(F\cap\mathbf{LK}(\mathrm{I},\mathbf{R},\mathbf{s}))\ =\ \inf\sum_{J\in\mathcal{J}}\mu(J)\ .$$
>
> The infimum is over all coverings $\mathcal{J}$ of $F\cap\mathbf{LK}(\mathrm{I},\mathbf{R},\mathbf{s})$ by intervals $J\in\{\mathcal{J}_n:n\in\mathbb{Z}_{\geqslant 0}\}$.

For further details see [8, Prop. 1.7]. It remains to show that μ satisfies (40). Firstly, notice that for any interval $J_n \in \mathcal{J}_n$ we have that

$$\mu(J_n) \overset{(42)}{\leqslant} t_{n-1}^{-1}\ \mu(J_{n-1}) \leqslant \prod_{i=0}^{n-1} t_i^{-1} \tag{43}$$

Next, let δ_n denote the length of a generic interval $J_n \in \mathcal{J}_n$. In view of the splitting procedure associated with the construction of $\mathbf{LK}(\mathrm{I}, \mathbf{R}, \mathbf{s})$, we find that

$$\delta_n = |I| \cdot \prod_{i=0}^{n-1} R_i^{-1}\,. \tag{44}$$

Consider an arbitrary interval $B \subset [0,1]$ with length $|B| < \delta_{n_0}$. Then there exists an integer $n \geqslant n_0$ such that

$$\delta_{n+1} \ \leqslant\ |B| \ <\ \delta_n\ . \tag{45}$$

It follows that

$$\begin{aligned}
\mu(B) \quad &\leqslant \sum_{\substack{J\in\mathcal{J}_{n+1}:\\ J\cap B\neq\emptyset}} \mu(J) \quad \overset{(43)}{\leqslant} \quad \left\lceil \frac{|B|}{\delta_{n+1}} \right\rceil \prod_{i=0}^{n} t_i^{-1} \\
&\overset{(44)}{\leqslant} \quad 2\,\frac{|B|}{|I|} \prod_{i=0}^{n} \frac{R_i}{t_i} \quad = \quad 2\,\frac{|B|}{|I|}^{1-s} \quad \prod_{i=0}^{n} \frac{R_i}{t_i} \ \cdot\ |B|^s \\
&\overset{(45)}{<} \quad 2\,\frac{\delta_n^{1-s}}{|I|} \quad \prod_{i=0}^{n} \frac{R_i}{t_i} \ \cdot\ |B|^s \\
&\overset{(44)}{<} \quad 2|I|^{-s} \prod_{i=0}^{n} \frac{R_i^s}{t_i} \ \cdot\ |B|^s \\
&\overset{(41)}{\leqslant} \quad 2\,|I|^{-s} \prod_{i=0}^{n_0} \frac{R_i^s}{t_i} \ \cdot\ |B|^s\,.
\end{aligned}$$

In other words, (40) is valid with

$$a := 2\,|I|^{-s} \prod_{i=0}^{n_0} \frac{R_i^s}{t_i}$$

and on applying the mass distribution principle we obtain the desired statement.

⊠

In view of Lemma 1, the proof of Theorem 4 reduces to establishing the following key statement.

Proposition 3 *Let* $\mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r})$ *be as in Theorem 4. Then there exists a local Cantor type set*

$$\mathbf{LK}(\mathrm{I}, \mathbf{R}, \mathbf{s}) \subset \mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r})$$

where

$$\mathbf{s} := (s_n)_{n\in\mathbb{Z}_{\geqslant 0}} \quad \textit{with} \quad s_n := \tfrac{1}{2}\, R_n\,.$$

Indeed, by Proposition 3 we have that

$$\dim \mathbf{K}(\mathrm{I},\mathbf{R},\mathbf{r}) \geqslant \dim \mathbf{LK}(\mathrm{I},\mathbf{R},\mathbf{s}).$$

Now fix some positive $s < \liminf_{n\to\infty}(1-\log_{R_n} 2)$. Then, there exists an integer n_0 such that

$$s \ < \ 1-\log_{R_n} 2 \quad \text{ for all } \quad n > n_0 \, .$$

Also note that

$$t_n = R_n - s_n = \frac{R_n}{2}$$

and

$$R_n^s < \frac{R_n}{2} = t_n \quad \text{ for all } \quad n > n_0 \, .$$

Therefore, Lemma 1 implies that

$$\dim \mathbf{LK}(\mathrm{I},\mathbf{R},\mathbf{s}) \geqslant s \, .$$

This inequality is true for any $s < \liminf_{n\to\infty}(1-\log_{R_n} 2)$ and hence completes the proof of Theorem 4 modulo Proposition 3.

Before moving on to the proof of the proposition, it is useful to first investigate the distribution of intervals within each collection $\mathcal{J}_n$ associated with $\mathbf{K}(\mathrm{I},\mathbf{R},\mathbf{r})$.

5.2 The distribution of intervals within $\mathcal{J}_n$

In this section, the set

$$\mathbf{K}(\mathrm{I},\mathbf{R},\mathbf{r}) := \bigcap_{n=1}^{\infty} \bigcup_{J\in\mathcal{J}_n} J$$

and the sequence $\mathbf{s}$ are as in Proposition 3.

Let $\mathcal{T}_0 := \{\mathrm{I}\}$. For $n \geqslant 1$, let $\mathcal{T}_n$ denote a generic collection of intervals obtained from $\mathcal{T}_{n-1}$ via the splitting and removing procedures associated with a $(\mathrm{I},\mathbf{R},\mathbf{R}-\mathbf{s})$ local Cantor set. Here $\mathbf{R}-\mathbf{s}$ is the sequence $(R_n - s_n)$. Then, clearly

$$\#\mathcal{T}_{n+1} \ \geqslant \ \#\mathcal{T}_n \times s_n \quad \forall \ n \in \mathbb{Z}_{\geqslant 0} \, .$$

Loosely speaking, the following result shows that the intervals J_n from $\mathcal{J}_n$ are ubiquitous within the interval I.

Lemma 2 *For R sufficiently large,*

$$\mathcal{T}_n \cap \mathcal{J}_n \neq \emptyset \qquad \forall \ \ n \in \mathbb{Z}_{\geqslant 0} \, . \tag{46}$$

Proof. For an integer $n \geqslant 0$, let $h(n)$ denote the cardinality of the set $\mathcal{T}_n \cap \mathcal{J}_n$. Trivially, $h(0) = 1$ and lemma would follow on showing that

$$h(n+1) \geqslant \frac{R_n}{4}\, h(n) \, . \tag{47}$$

for all $n \in \mathbb{Z}_{\geqslant 0}$. This we now do via induction. Consider the set $\mathcal{T}_n \cap \mathcal{J}_n$. By the construction of $\mathcal{T}_{n+1}$ and the splitting procedure associated with $\mathbf{K}(\mathrm{I},\mathbf{R},\mathbf{r})$, each of the $h(n)$ intervals in

$\mathcal{T}_n \cap \mathcal{J}_n$ gives rise to at least s_n intervals I_{n+1} in $\mathcal{T}_{n+1} \cap \mathcal{I}_{n+1}$. By the the removing procedure associated with $\mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r})$, for each interval $J_k \in \mathcal{T}_k \cap \mathcal{J}_k$ $(0 \leqslant k \leqslant n)$ we remove at most $r_{k,n}$ intervals $I_{n+1} \in \mathcal{T}_{n+1} \cap \mathcal{I}_{n+1}$ that lie within J_k. The upshot of this is that

$$h(n+1) \geqslant s_n h(n) - \sum_{k=0}^{n} r_{k,n}\, h(k). \tag{48}$$

For $n = 0$ this inequality is transformed to

$$h(1) \geqslant \frac{R_0}{2} h(0) - r_{0,0} h(0) \overset{(16)}{\geqslant} \frac{R_0}{4} h(0)$$

as required. Now assume that (47) is valid for all $1 \leqslant m \leqslant n$. In particular, it means that

$$h(m) \leqslant \frac{4}{R_m}\, h(m+1) \leqslant \ldots \leqslant \prod_{k=1}^{n-m} \frac{4}{R_{n-k}}\, h(n)$$

which together with (48) implies that

$$\begin{aligned} h(n+1) \quad &\geqslant \quad \frac{R_n}{2} h(n) - \left(\sum_{k=0}^{n} \left(r_{n-k,n} \prod_{i=1}^{k} \frac{4}{R_{n-i}} \right) \right) h(n) \\ &\overset{(16)}{\geqslant} \quad \frac{R_n}{4}\, h(n)\,. \end{aligned}$$

This completes the induction step and thus establishes the desired inequality (47) for all $n \in \mathbb{Z}_{\geqslant 0}$. ⊠

6 Proof of Proposition 3

By definition, the set $\mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r})$ is the intersection of closed intervals J_n lying within nested collections $\mathcal{J}_n$. For each integer $n \geqslant 0$, the aim is to construct a nested collection $\mathcal{L}_n \subseteq \mathcal{J}_n$ that complies with the construction of a $(\mathrm{I}, \mathbf{R}, \mathbf{s})$ local Cantor set. Then, it would follow that

$$\bigcap_{n=0}^{\infty} \bigcup_{J \in \mathcal{L}_n} J$$

is precisely the desired set $\mathbf{LK}(\mathrm{I}, \mathbf{R}, \mathbf{s})$.

6.1 Construction the collection $\mathcal{L}_n$

For any integer $n \geqslant 0$, the goal of this section is to construct the desired nested collection $\mathcal{L}_n \subseteq \mathcal{J}_n$ alluded to above. This will involve constructing auxiliary collections $\mathcal{L}_{m,n}$ and $\mathcal{R}_{m,n}$ for integers m, n satisfying $0 \leqslant m \leqslant n$. For a fixed n, let

$$\mathcal{J}_0\,,\ \mathcal{J}_1\,,\ \ldots,\ \mathcal{J}_n$$

be the collections arising from the construction of $\mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r})$. We will require $\mathcal{L}_{m,n}$ to satisfy the following conditions.

C1. For any $0 \leqslant m \leqslant n$, $\mathcal{L}_{m,n} \subseteq \mathcal{J}_m$.

C2. For any $0 \leqslant m < n$, the collections $\mathcal{L}_{m,n}$ are nested; that is

$$\bigcup_{J \in \mathcal{L}_{m+1,n}} J \quad \subset \quad \bigcup_{J \in \mathcal{L}_{m,n}} J.$$

C3. For any $0 \leqslant m < n$ and $J_m \in \mathcal{L}_{m,n}$, there are at least $R_m - s_m$ intervals $J_{m+1} \in \mathcal{L}_{m+1,n}$ contained within J_m; that is

$$\#\{J_{m+1} \in \mathcal{L}_{m+1,n} \ : \ J_{m+1} \subset J_m\} \ \geqslant \ R_m - s_m \ .$$

In addition, define $\mathcal{R}_{0,0} := \emptyset$ and for $n \geqslant 1$

$$\mathcal{R}_{n,n} := \{I_n \in \mathcal{I}_n \backslash \mathcal{J}_n \ : \ I_n \subset J_{n-1} \text{ for some } J_{n-1} \in \mathcal{L}_{n-1,n-1}\} \ . \tag{49}$$

Furthermore, for $0 \leqslant m < n$ define

$$\mathcal{R}_{m,n} := \mathcal{R}_{m,n-1} \cup \{J_m \in \mathcal{L}_{m,n-1} \ : \ \#\{J_{m+1} \in \mathcal{R}_{m+1,n} \ : \ J_{m+1} \subset J_m\} \geqslant s_m \ \} \ . \tag{50}$$

Loosely speaking and with reference to condition (C3), the collections $\mathcal{R}_{m,n}$ are the 'dumping ground' for those intervals $J_m \in \mathcal{L}_{m,n-1}$ which do not contain enough sub-intervals J_{m+1}. Note that for n fixed, the collections $\mathcal{R}_{m,n}$ are defined in descending order with respect to m. In other words, we start with $\mathcal{R}_{n,n}$ and finish with $\mathcal{R}_{0,n}$.

The construction is as follows.

Stage 1. Let $\mathcal{L}_{0,0} := \mathcal{J}_0$ and $\mathcal{R}_{0,0} := \emptyset$.

Stage 2. Let $0 \leqslant t \leqslant n$. Suppose we have constructed the desired collections

$$\mathcal{L}_{0,t} \subseteq \mathcal{J}_0, \ \mathcal{L}_{1,t} \subseteq \mathcal{J}_1, \ldots, \mathcal{L}_{t,t} \subseteq \mathcal{J}_t$$

and

$$\mathcal{R}_{0,t}, \ldots, \mathcal{R}_{t,t} \, .$$

We now construct the corresponding collections for $t = n+1$.

Stage 3. Define

$$\mathcal{L}'_{n+1,n+1} := \{J_{n+1} \in \mathcal{J}_{n+1} \ : \ J_{n+1} \subset J_n \text{ for some } J_n \in \mathcal{L}_{n,n}\}$$

and let $\mathcal{R}_{n+1,n+1}$ be given by (49) with $n+1$ instead of n. Thus the collection $\mathcal{L}'_{n+1,n+1}$ consists of 'good' intervals from $\mathcal{J}_{n+1}$ that are contained within some interval from $\mathcal{L}_{n,n}$. Our immediate task is to construct the corresponding collections $\mathcal{L}'_{u,n+1}$ for each $0 \leqslant u \leqslant n$. These will be constructed together with the 'complementary' collections $\mathcal{R}_{u,n+1}$ in descending order with respect to u.

Stage 4. With reference to Stage 3, suppose we have constructed the collections $\mathcal{L}'_{u+1,n+1}$ and $\mathcal{R}_{u+1,n+1}$ for some $0 \leqslant u \leqslant n$. We now construct $\mathcal{L}'_{u,n+1}$ and $\mathcal{R}_{u,n+1}$. Consider the collections $\mathcal{L}_{u,n}$ and $\mathcal{R}_{u,n}$. Observe that some of the intervals J_u from $\mathcal{L}_{u,n}$ may contain less than $R_u - s_u$ sub-intervals from $\mathcal{L}'_{u+1,n+1}$ (or in other words, at least s_u intervals from $\mathcal{R}_{u+1,n+1}$). Such intervals J_u fail the counting condition (C3) for $\mathcal{L}_{u,n+1}$ and informally speaking are moved out of $\mathcal{L}_{u,n}$ and into $\mathcal{R}_{u,n}$. The resulting sub-collections are $\mathcal{L}'_{u,n+1}$ and $\mathcal{R}_{u,n+1}$ respectively. Formally,

$$\mathcal{L}'_{u,n+1} := \{J_u \in \mathcal{L}_{u,n} \ : \ \#\{J_{u+1} \in \mathcal{R}_{u+1,n+1} \ : \ J_{u+1} \subset J_u\} < s_u \ \}$$

and $\mathcal{R}_{u,n+1}$ is given by (50) with m replaced by u and n replaced by $n+1$.

Stage 5. By construction the collections $\mathcal{L}'_{u,n+1}$ satisfy conditions (C1) and (C3). However, for some $J_{u+1} \in \mathcal{L}'_{u+1,n+1}$ it may be the case that J_{u+1} is not contained in any interval $J_u \in \mathcal{L}'_{u,n+1}$ and thus the collections $\mathcal{L}'_{u,n+1}$ are not necessarily nested. The point is that during Stage 4 above the interval $J_u \in \mathcal{J}_u$ containing J_{u+1} may be 'moved' into $\mathcal{R}_{u,n+1}$. In order to guarantee the nested condition (C2) such intervals J_{u+1} are removed from $\mathcal{L}'_{u+1,n+1}$. The resulting sub-collection is the required auxiliary collection $\mathcal{L}_{u+1,n+1}$. Note that $\mathcal{L}_{u+1,n+1}$ is constructed via $\mathcal{L}'_{u+1,n+1}$ in ascending order with respect to u. Formally,

$$\mathcal{L}_{0,n+1} := \mathcal{L}'_{0,n+1}$$

and for $1 \leqslant u \leqslant n+1$

$$\mathcal{L}_{u,n+1} := \{J_u \in \mathcal{L}'_{u,n+1} \ : \ J_u \subset J_{u-1} \text{ for some } J_{u-1} \in \mathcal{L}_{u-1,n+1}\}.$$

With reference to Stage 2, this completes the induction step and thereby the construction of the auxiliary collections.

For any integer $n \geqslant 0$, it remains to construct the sought after collection $\mathcal{L}_n$ via the auxiliary collections $\mathcal{L}_{m,n}$. Observe that since

$$\mathcal{L}_{m,m} \supset \mathcal{L}_{m,m+1} \supset \mathcal{L}_{m,m+2} \supset \dots$$

and the cardinality of each collection $\mathcal{L}_{m,n}$ with $m \leqslant n$ is finite, there exists some integer $N(m)$ such that

$$\mathcal{L}_{m,n} \ = \ \mathcal{L}_{m,n'} \qquad \forall \quad n, n' \geqslant N(m) \ .$$

Now simply define

$$\mathcal{L}_n := \mathcal{L}_{n,N(n)} \ .$$

Unfortunately, there remains one slight issue. The collection $\mathcal{L}_n$ defined in this manner could be empty.

The goal now is to show that $\mathcal{L}_{m,n} \neq \emptyset$ for any $m \leqslant n$. This clearly implies that $\mathcal{L}_n \neq \emptyset$ and thereby completes the construction.

6.2 The collection $\mathcal{L}_{m,n}$ is non-empty

Lemma 3 *For any $m, n \in \mathbb{N}, m \leqslant n$, the set $\mathcal{L}_{m,n}$ is nonempty.*

Proof. Suppose the contrary: $\mathcal{L}_{m,n} = \emptyset$ for some integers satisfying $0 \leqslant m \leqslant n$. In view of the construction of $\mathcal{L}_{m,n}$ every interval in $\mathcal{L}_{m-1,n}$ contains at least $R_{m-1} - s_{m-1} > 0$ sub-intervals from $\mathcal{L}_{m,n}$. Therefore each of the collections $\mathcal{L}_{m-1,n}, \mathcal{L}_{m-2,n}, \dots, \mathcal{L}_{0,n}$ are also empty and it follows that $\mathcal{R}_{0,n} = \mathcal{J}_0$.

Now consider the set $\mathcal{R}_{m,n}$. By the construction we have the chain of nested sets

$$\mathcal{R}_{m,n} \supseteq \mathcal{R}_{m,n-1} \supseteq \dots \supseteq \mathcal{R}_{m,m}$$

and in view of (49) the elements of $\mathcal{R}_{m,m}$ are intervals from $\mathcal{I}_m \backslash \mathcal{J}_m$. Consider any interval $J_m \in \mathcal{R}_{m,n} \backslash \mathcal{R}_{m,m}$. Take $m < n_0 \leqslant n$ such that $J_m \in \mathcal{R}_{m,n_0}$ but $J_m \notin \mathcal{R}_{m,n_0-1}$. Then J_m was added to $\mathcal{R}_{m,n_0}$ on stage 4 of the construction. Hence I_m should have at least s_m sub-intervals from $\mathcal{R}_{m+1,n_0}$ and therefore from $\mathcal{R}_{m+1,n}$. The upshot of this is the following:

for any interval I_m from $\mathcal{R}_{m,n}$ either $I_m \in \mathcal{I}_m \backslash \mathcal{J}_m$ or I_m contains at least s_m sub-intervals $I_{m+1} \in \mathcal{R}_{m+1,n}$.

Next we exploit Lemma 2. Choose an interval J_0 from $\mathcal{R}_{0,n} = \mathcal{J}_0$ and define $\mathcal{T}_0 := \{J_0\}$. For $0 \leqslant m < n$, we define inductively nested collections

$$\mathcal{T}_{m+1} := \{I_{m+1} \in \mathcal{T}(I_m) \ : \ I_m \in \mathcal{T}_m\}$$

with $\mathcal{T}(I_m)$ given by one of the following three scenarios.

- $I_m \in \mathcal{R}_{m,n}$ and I_m contains at least s_m sub-intervals I_{m+1} from $\mathcal{R}_{m+1,n}$. Let $\mathcal{T}(I_m)$ be the collection consisting of these sub-intervals. Note that when $m = n-1$ we have $\mathcal{T}(I_m) \subset \mathcal{R}_{n,n} \subset \mathcal{I}_n \backslash \mathcal{J}_n$. Therefore $\mathcal{T}(I_{n-1}) \cap \mathcal{J}_n = \emptyset$.

- $I_m \in \mathcal{R}_{m,n}$ and I_m contains strictly less than s_m sub-intervals I_{m+1} from $\mathcal{R}_{m+1,n}$. Then the interval $I_m \in \mathcal{I}_m \backslash \mathcal{J}_m$ and we subdivide I_m into R_m closed intervals I_{m+1} of equal length. Let $\mathcal{T}(I_m)$ be the collection consisting of all of these sub-intervals. Note that $\mathcal{T}(I_m) \cap \mathcal{J}_{m+1} = \emptyset$.

- $I_m \notin \mathcal{R}_{m,n}$. Then the interval I_m does not intersect any interval from $\mathcal{J}_m$ and we subdivide I_m into R_m closed intervals I_{m+1} of equal length. Let $\mathcal{T}(I_m)$ be any collection consisting of all such sub-intervals. Note that $\mathcal{T}(I_m) \cap \mathcal{J}_{m+1} = \emptyset$.

The upshot is that

$$\#\mathcal{T}_{m+1} \ \geqslant \ \#\mathcal{T}_m \times s_m \qquad \forall\, 0 < m \leqslant n$$

and that

$$\mathcal{T}_n \cap \mathcal{J}_n \ = \ \emptyset \ .$$

However, in view of Lemma 2 the latter is impossible and therefore the starting premise that $\mathcal{L}_{m,n} = \emptyset$ is false. This completes the proof of Lemma 3 and therefore Proposition 3.

⊠

7 Intersecting Cantor sets

With reference to §2.2, fix the interval I and the sequence $\mathbf{R} := (R_n)$. Let $k \in \mathbb{N}$ and consider the two parameter sequences

$$\mathbf{r}_i := (r^{(i)}_{m,n}) \qquad 1 \leqslant i \leqslant k \, .$$

The following result shows that the intersection of any finite number of $(\mathrm{I}, \mathbf{R}, \mathbf{r}_i)$ Cantor sets is yet another $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ Cantor set.

Theorem 5 *For each integer $1 \leqslant i \leqslant k$, suppose we are given a set $\mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r}_i)$. Then*

$$\bigcap_{i=1}^{k} \mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r}_i)$$

is a $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ Cantor set where

$$\mathbf{r} := (r_{m,n}) \quad \textit{with} \quad r_{m,n} := \sum_{i=1}^{k} r^{(i)}_{m,n} \, .$$

Proof. Loosely speaking we need to show that there exists a $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ Cantor set that simultaneously incorporates the splitting and removing procedures associated with the sets

$$\mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r}_i) := \bigcap_{n=1}^{\infty} \bigcup_{J \in \mathcal{J}_n^{(i)}} J \qquad (1 \leqslant i \leqslant k) .$$

For each $n \in \mathbb{Z}_{\geqslant 0}$, consider the collection

$$\mathcal{J}_n := \bigcap_{i=1}^{k} \mathcal{J}_n^{(i)} .$$

We claim that $\mathcal{J}_n$ complies with the construction of a $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ Cantor set. If true, then we are done since

$$\mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r}) := \bigcap_{n=1}^{\infty} \bigcup_{J \in \mathcal{J}_n} J = \bigcap_{n=1}^{\infty} \bigcap_{i=1}^{k} \bigcup_{J \in \mathcal{J}_n^{(i)}} J = \bigcap_{i=1}^{k} \bigcap_{n=1}^{\infty} \bigcup_{J \in \mathcal{J}_n^{(i)}} J := \bigcap_{i=1}^{k} \mathbf{K}(\mathrm{I}, \mathbf{R}, \mathbf{r}_i) .$$

Firstly note that the claim is true for $n = 0$ since $\mathcal{J}_0 := \{\mathrm{I}\}$. Now assume that the claim is true for some fixed $n \in \mathbb{Z}_{\geqslant 0}$. Consider an arbitrary interval $J_n \in \mathcal{J}_n$. By definition, $J_n \in \mathcal{J}_n^{(i)}$ for each i. By construction, every interval in $\mathcal{J}_n^{(i)}$ gives rise to R_n intervals $I_{n+1} \in \mathcal{I}_{n+1}^{(i)}$. Thus, for each $J_n \in \mathcal{J}_n$ the collection

$$\mathcal{I}_{n+1} := \bigcap_{i=1}^{k} \mathcal{I}_{n+1}^{(i)}$$

contains exactly R_n intervals I_{n+1} that lie within J_n. This coincides precisely with the splitting procedure associated with a $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ Cantor set. We now turn our attention to the removing procedure. By construction, for each interval $J_n \in \mathcal{J}_n^{(i)}$ we remove at most $r_{n,n}^{(i)}$ intervals $I_{n+1} \in \mathcal{I}_{n+1}^{(i)}$ that lie within J_n. Thus for any $J_n \in \mathcal{J}_n$ there are at most

$$r_{n,n} := \sum_{i=1}^{k} r_{n,n}^{(i)}$$

intervals $I_{n+1} \subset J_n$ that are removed from $\mathcal{I}_{n+1}$. In general, for each $0 \leqslant m \leqslant n$ and each interval $J_m \in \mathcal{J}_m$ there are at most

$$r_{m,n} := \sum_{i=1}^{k} r_{m,n}^{(i)}$$

additional intervals $I_{n+1} \subset J_m$ that are removed from $\mathcal{I}_{n+1}$. This coincides precisely with the removing procedure associated with a $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ Cantor set. The upshot is that $\mathcal{J}_{n+1}$ complies with the construction of a $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ Cantor set. This completes the induction step and thereby completes the proof of Theorem 5.

⊠

• *An application.* We now describe a simple application of Theorem 5 which enables us to deduce a non-trivial strengthening of Theorem 1. In the course of establishing Proposition 1 we show that the set $\mathbf{Mad}_{\mathcal{D}}(f)$ contains the Cantor-type set $\mathbf{K}(\mathrm{I},\mathbf{R},\mathbf{r})$ where $\mathbf{R}=(R_n)$ and $\mathbf{r}=(r_{m,n})$ are given by (20) and (21) respectively; namely, for any fixed integer $R>e^{12}$

$$R_n := R\,(n+1)\,[\log^*(n+1)] \quad \text{and} \quad r_{m,n} \;:=\; 7\,\log^2\!R\cdot n^2(\log^*n)^2$$

if $m=n-1$ and zero otherwise. Note that although these quantities are dependent on the actual value of R the statement that $\mathbf{K}(\mathrm{I},\mathbf{R},\mathbf{r})\subset\mathbf{Mad}_{\mathcal{D}}(f)$ is not.

Now for each $1\leqslant i\leqslant k$, let $\mathcal{D}_i$ be a sequence of integers greater than or equal to two and let f be as in Proposition 1. Then, with $\mathbf{R}$ and $\mathbf{r}$ as above, Theorem 5 implies that

$$\bigcap_{i=1}^{k}\mathbf{Mad}_{\mathcal{D}_i}(f)\supset\mathbf{K}(\mathrm{I},\mathbf{R},k\mathbf{r}) \qquad \text{where} \qquad k\,(r_{m,n}):=(kr_{m,n}).$$

It is easily verified that for $R>k\,e^{9}$

$$\begin{aligned}\text{l.h.s. of (16)} \;&=\; k\cdot r_{n-1,n}\cdot\frac{4}{R_{n-1}}\;\leqslant\; k\cdot 7\cdot 2^3\cdot\frac{\log^2\!R\cdot n\,\log^*n}{R}\\ &\leqslant\; \frac{k\cdot 7\cdot 2^6\log^2\!R}{R^2}\cdot\frac{R\,(n+1)\,[\log^*(n+1)]}{4}\\ &\leqslant\; \frac{R_n}{4} \;=\; \text{r.h.s. of (16)}\,.\end{aligned}$$

Hence, for any fixed $R>k\,e^{12}$, Theorem 4 implies that

$$\dim\left(\bigcap_{i=1}^{k}\mathbf{Mad}_{\mathcal{D}_i}(f)\right)\geqslant\liminf_{n\to\infty}(1-\log_{R_n}2)=1.$$

The complementary upper bound inequality for the dimension is trivial. Thus we have established the following strengthening of Theorem 1.

Theorem 6 *For each $1\leqslant i\leqslant k$, let $\mathcal{D}_i$ be a sequence of integers greater than or equal to 2 and let f be as in Proposition 1. Then*

$$\dim\left(\bigcap_{i=1}^{k}\mathbf{Mad}_{\mathcal{D}_i}(f)\right)=1\,.$$

• *What about other intersections?* There are two natural problems that arise in relation to Theorem 5. Firstly, to generalise the statement so as to incorporate any finite number of sequences $\mathbf{R}_i:=(R_n^{(i)})$. Secondly, to establish the analogue of Theorem 5 for countable intersections. This is more challenging than the first and in all likelihood will involve imposing extra conditions on the sequences $\mathbf{R}$ and $\mathbf{r}$. A direct consequence of the 'correct' countable version of Theorem 5 would be the statement that

$$\dim\left(\bigcap_{i=1}^{\infty}\mathbf{Mad}_{\mathcal{D}_i}(f)\right)=1\,.$$

Note that establishing the countable analogue of Theorem 6 remains an open problem.

• *A more general Cantor framework.* The Cantor framework of §2.2 and indeed of this section is one-dimensional. Naturally it would be interesting to develop the analogous n–dimensional Cantor framework in which intervals are replaced by balls. Establishing the higher dimensional generalisation of Theorem 4 and indeed Theorem 5 will almost certainly make use of standard covering arguments from geometric measure theory; for example, the '$5r$' and Besicovitch covering lemmas. Beyond higher dimensions, it would be highly desirable to develop an analogue of the framework of §2.2 within the context of 'reasonable' metric spaces – such as a (locally) compact metric space equipped with an Ahlfors regular measure. A generalisation of this type would enhance the scope of potential applications.

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References

[1] D. Badziahin, A. Pollington and S. Velani: On a problem in simultaneously Diophantine approximation: Schmidt's conjecture. *Ann. of Math.* to appear. *Pre-print: arXiv:1001.2694* (2010), 1–43.

[2] A.S. Besicovitch: Sets of fractional dimension (IV); on rational approximation to real numbers. *Jou. L.M.S.* **9** (1934), 126–131.

[3] Y. Bugeaud, M. Drmota and B. de Mathan: On a mixed Littlewood conjecture in Diophantine approximation. *Acta Arith.* **2** 128 (2007), 107–124.

[4] Y. Bugeaud, A. Haynes and S. Velani: Metric considerations concerning the mixed Littlewood conjecture. *International Journal of Number Theory*, **3** (2011), 593–609.

[5] Y. Bugeaud and N. Moshchevitin: Badly approximable numbers and Littlewood-type problems. *Math. Proc. Cambridge Philos. Soc.* **2** 150 (2011), 215–226.

[6] M. Einsiedler, A. Katok and E. Lindenstrauss: Invariant measures and the set of exceptions to Littlewood's conjecture. *Ann. of Math.* **164** (2006), 513–560

[7] M. Einsiedler and D. Kleinbock: Measure rigidity and p-adic Littlewood-type problems. *Compositio Math.* **143** (2007), 689-702.

[8] K. Falconer: *Fractal Geometry: Mathematical Foundations and Applications.* John Wiley & Sons, (1990).

[9] P.X. Gallagher: Metric Simultaneous Diophantine Approximation. *Jou. L.M.S.* **37** (1962), 387–390.

[10] A. Gorodnik: Open problems in dynamics and related fields. *Journal of Modern Dynamics* **1** (2007), 1–35.

[11] I. Jarník: Zur metrischen Theorie der diophantischen Appoximationen. *Proc. Mat. Fyz.* **36** (1928), 91–106.

[12] A. I. Khintchine: Einige Sätze über Kettenbruche, mit Anwendungen auf die Theorie der Diophantischen Approximationen. *Math. Ann.* **92** (1924), 115–125.

[13] B. de Mathan and O. Teulié: Problèmes diophantiens simultanés. *Monatsh. Math.* **143** (2004), 229–245.

[14] L.G. Peck: Simultaneous rational approximations to algebraic numbers. *Bull. A.M.S.* **67** (1961), 197–201.

[15] A. Pollington and S. Velani: On a problem in simultaneously Diophantine approximation: Littlewood's conjecture. *Acta Math.* **66** (2000), 29–40.

Dzmitry A. Badziahin: Department of Mathematics, University of York,
Heslington, York, YO10 5DD, England.
e-mail: db528@york.ac.uk

Sanju L. Velani: Department of Mathematics, University of York,
Heslington, York, YO10 5DD, England.
e-mail: slv3@york.ac.uk