Path factors and parallel knock-out schemes of almost claw-free graphs

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Abstract. An $H_1, \{H_2\}$-factor of a graph G is a spanning subgraph of G with exactly one component isomorphic to the graph H_1 and all other components (if there are any) isomorphic to the graph H_2. We completely characterise the class of connected almost claw-free graphs that have a $P_7, \{P_2\}$-factor, where P_7 and P_2 denote the paths on seven and two vertices, respectively. We apply this result to parallel knock-out schemes for almost claw-free graphs. These schemes proceed in rounds in each of which each surviving vertex simultaneously eliminates one of its surviving neighbours. A graph is reducible if such a scheme eliminates every vertex in the graph. Using our characterisation we are able to classify all reducible almost claw-free graphs, and we can show that every reducible almost claw-free graph is reducible in at most two rounds. This leads to a quadratic time algorithm for determining if an almost claw-free graph is reducible (which is a generalisation and improvement upon the previous strongest result that showed that there was a $O(n^{5.376})$ time algorithm for claw-free graphs on n vertices).

Keywords: parallel knock-out schemes, (almost) claw-free graphs, perfect matching, factor

1 Introduction

We denote a graph by $G = (V, E)$. An edge joining vertices u and v is denoted by uv. If not stated otherwise a graph is assumed to be finite, undirected and simple. The *neighbourhood* of $u \in V$, that is, the set of vertices adjacent to u is denoted by $N_G(u) = \{v \mid uv \in E\}$, and the *degree* of u is denoted by $\deg_G(u) = |N_G(u)|$. If no confusion is possible, we omit the subscripts. A set $I \subseteq V$ is called an *independent set* of G if no two vertices in I are adjacent to each other, and α denotes the *independence number* of G, the number of vertices in a maximum size independent set of G. See [3] for other basic graph-theoretic terminology.

A graph $(\{u, v_1, v_2, v_3\}, \{uv_1, uv_2, uv_3\})$ is called a *claw* with *claw centre* u and *leaves* v_1, v_2, v_3. A graph is *claw-free* if it does not contain a claw as a induced

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subgraph. Claw-free graphs form a rich class containing, for example, the class of line graphs and the class of complements of triangle-free graphs. It is a very well-studied graph class, both within structural graph theory and within algorithmic graph theory; see [10] for a survey. We study a generalisation of claw-free graphs, namely *almost claw-free graphs* which were introduced by Ryjáček [22].

Definition 1. *A graph $G = (V, E)$ is* almost claw-free *if the following two conditions hold:*

1. *The set of all vertices that are claw centres of induced claws in G is an independent set in G.*
2. *For all $u \in V$, either $|N(u)| = 1$ or $N(u)$ contains two vertices v_1, v_2 such that $N(u)\backslash\{v_1, v_2\} \subseteq N(v_1) \cup N(v_2)$.*

Claw-free graphs trivially satisfy the first condition, and they also satisfy the second since otherwise they would contain a vertex with three independent neighbours yielding an induced claw. Hence, every claw-free graph is almost claw-free. It is easy to see that there exist almost claw-free graphs that are not claw-free; see, for example, the graph H in Figure 2.

Several papers have generalised results on claw-free graphs to almost claw-free graphs: see [7, 19, 25] for results on hamiltonicity, shortest walks and toughness. A subgraph $M = (V', E')$ of a graph $G = (V, E)$ is called a *matching* of G if every vertex in M has degree one. It is called a *perfect* matching if $V' = V$. We call G *even* if $|V|$ is even, and *odd* otherwise. Las Vergnas [18] and Sumner [23] have independently proven that every even connected claw-free graph $G = (V, E)$ has a perfect matching. The following theorem by Ryjáček [22] generalises this result to almost claw-free graphs.

Theorem 1 ([22]). *Every even connected almost claw-free graph has a perfect matching.*

For an odd graph $G = (V, E)$, the natural analogue of a perfect matching is a *near-perfect* matching: a matching $M = (V\backslash\{v\}, E')$ for some $v \in V$. In this paper we shall prove the following.

Theorem 2. *Every odd connected almost claw-free graph has a near-perfect matching.*

Jünger, Pulleyblank and Reinelt [14] have shown that odd claw-free graphs have near-perfect matchings so Theorem 2 is an extension of this result to *almost* claw-free graphs. In fact, our main result, Theorem 3, is much stronger and more general, but we require some further preliminaries before we can state it.

To capture both even and odd graphs, the notion of a (near-)perfect matching has been generalised in various ways. We consider two such generalisations for almost claw-free graphs, namely *path factors* and *parallel knock-out numbers*, which we relate to each other.

In Section 2, we completely characterise the class of connected almost claw-free graphs that have a spanning subgraph with exactly one component isomorphic to a path on seven vertices while all other components form a matching.

In Section 4 we prove this result and present a polynomial algorithm for finding such a subgraph, but first we apply this result in Section 3 to parallel knock-out schemes for almost claw-free graphs.

These schemes proceed in rounds in each of which each surviving vertex simultaneously eliminates one of its surviving neighbours. A graph is *reducible* if such a scheme eliminates every vertex in the graph. Using our characterisation we are able to classify all reducible almost claw-free graphs, and we can show that every reducible almost claw-free graph is reducible in at most two rounds. This leads to a quadratic time algorithm for determining if an almost claw-free graph is reducible. This is a generalisation and improvement upon the $O(n^{5.376})$ time algorithm for n-vertex claw-free graphs given by Broersma et al. in [6]. Although, in general, determining if a graph is reducible is an NP-complete problem, the new technique that uses (path) factors for this problem might be promising for other graph classes as well. We discuss this in Section 5.

2 Path factors

Let $\mathcal{H} = \{H_1, H_2, \ldots,\}$ be a family of graphs. An $\mathcal{H}$-factor of a graph G is a spanning subgraph of G with each component isomorphic to a graph in $\{\mathcal{H}\}$. Let P_n denote the path on n vertices. A *path factor* of a graph G is a $\{P_1, P_2, \ldots\}$-factor of G. Path factors generalise perfect matchings, which are $\{P_2\}$-factors. Path factors have been the subject of considerable study: see, for example, [24] for a characterisation of bipartite graphs with a $\{P_3, P_4, P_5\}$-factor and [15, 16] for a characterisation of general graphs with a $\{P_3, P_4, P_5\}$-factor. A more recent result [20] shows that the square of any graph on at least six vertices has a $\{P_3, P_4\}$-factor. Connected claw-free graphs with minimum degree d have a $\{P_{d+1}, P_{d+2}, \ldots\}$-factor [1]. In general, obtaining good characterisations of graph classes with path factors might be difficult as it is shown in [11] that the problem of deciding if a given graph has a H-factor is NP-complete for any fixed H with $|V_H| \geq 3$. For a more general survey on factors see [21].

We are interested in another class of path factors. Let H_1, H_2 be graphs. Then an $H_1, \{H_2\}$*-factor* of a graph G is a spanning subgraph of G with exactly one component isomorphic to H_1 and all other components (if there are any) isomorphic to H_2. The components are called H_1*-components* and H_2*-components.* A $P_2, \{P_2\}$-factor of a graph corresponds to a perfect matching, and a $P_1, \{P_2\}$-factor corresponds to a near-perfect matching.

In order to state our main result, we must define two families $\mathcal{F}$ and $\mathcal{G}$ of connected almost claw-free graphs. For an integer $k \geq 0$, let the graph F_k be obtained from the complete graph on $k+1$ vertices $x_0, \ldots, x_k$ by adding a vertex y_i and an edge $x_i y_i$ for $i = 1, \ldots, k$ (note there is no vertex y_0). We say that x_0 is the *root* of F_k. Note that each graph F_k is claw-free. In particular, F_0 is isomorphic to P_1 and F_1 is isomorphic to P_3. For integers $k, \ell \geq 1$, let $F_{k,\ell}$ denote the graph obtained from two vertex-disjoint copies of F_k and F_ℓ after removing their roots and adding a new vertex x^* adjacent to precisely those vertices to which the roots were adjacent in F_k, F_ℓ. We call x^* the *root* of $F_{k,\ell}$. Note that each graph $F_{k,\ell}$ is claw-free. In particular, $F_{1,1}$ is isomorphic to P_5. Finally, for integers $k, \ell \geq 1$, let $F'_{k,\ell}$ denote the graph obtained from $F_{k,\ell}$ with root x^* after adding two new vertices y and z with y adjacent to z and z also adjacent to all vertices in $N_{F_{k,\ell}}(x^*)$. We call x^* the *root* of $F'_{k,\ell}$. Since z is the (only) centre of an induced claw, $F'_{k,\ell}$ is not claw-free. However, it is easy to check that each $F'_{k,\ell}$ is almost claw-free. Let $\mathcal{F} = \{F_0, F_k, F_{k,\ell}, F'_{k,\ell} \mid k, \ell \geq 1\}$. See Figure 1 for some examples of graphs that belong to this family. Let C_n denote the cycle on n vertices. For

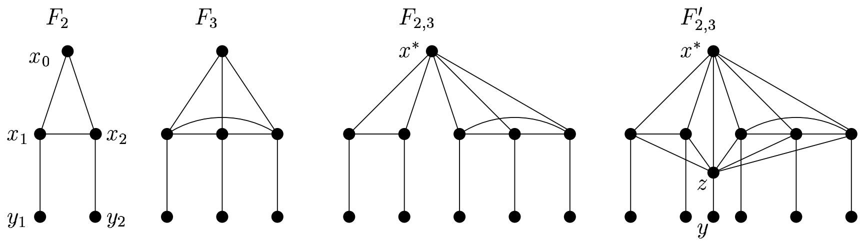

Fig. 1. The graphs $F_2, F_3, F_{2,3}$, and $F'_{2,3}$.

$k \geq 0$, the graph G_k is obtained from F_k by adding two new vertices a and b that are adjacent to the root of F_k and to each other. Note that G_0 is isomorphic to C_3; see Figure 2 for some other examples. The family $\mathcal{G}$ contains the graphs G_k, $k \geq 0$, and also all other connected graphs on five vertices that have a $C_3, \{P_2\}$-factor. There are eleven such graphs which are depicted in Figure 3 together with the graph G_1. Note that each graph in $\mathcal{G}$ is claw-free and contains a $C_3, \{P_2\}$-factor. Let $H = (\{u_1, u_2, u_3, u_4, u_5\}, \{u_1u_2, u_1u_3, u_1u_4, u_2u_4, u_3u_4, u_4u_5\})$ be the almost claw-free graph in Figure 2. Note that the only connected almost claw-free graphs on five vertices not in $\mathcal{G}$ are $F_2, F_{1,1}, C_5$, and H.

Theorem 3 below states that to check whether a graph G on n vertices has a $P_7, \{P_2\}$-factor can be done by checking whether or not $G \in \mathcal{F} \cup \mathcal{G} \cup \{C_5, H\}$ (and this can clearly be done in time $O(|V|^2)$). The theorem also states that finding such a factor takes $O(|V|^{3.5})$ time. This is a major improvement upon the trivial brute-force algorithm that checks for every 7-tuple of vertices $\{v_1, \ldots, v_7\}$ whether the graph obtained after removing $\{v_1, \ldots v_7\}$ contains a perfect matching.

Theorem 3. *Let $G = (V, E)$ be an odd connected almost claw-free graph. If $G \notin \mathcal{F} \cup \mathcal{G} \cup \{C_5, H\}$ then G has a $P_7, \{P_2\}$-factor, which we can find in $O(|V|^{3.5})$ time.*

Note that Theorem 3 implies Theorem 2. We prove Theorem 3 in Section 4. There we describe an algorithm that computes a $P_7, \{P_2\}$-factor in $O(|V|^{3.5})$ time. The running time of the algorithm on an input graph $G = (V, E)$ depends on the running time of a subalgorithm that is performed $O(|V|)$ times and that finds a perfect matching in at most two subgraphs of G and then attempts to transform these perfect matchings into a $P_7, \{P_2\}$-factor of G. As such a transformation

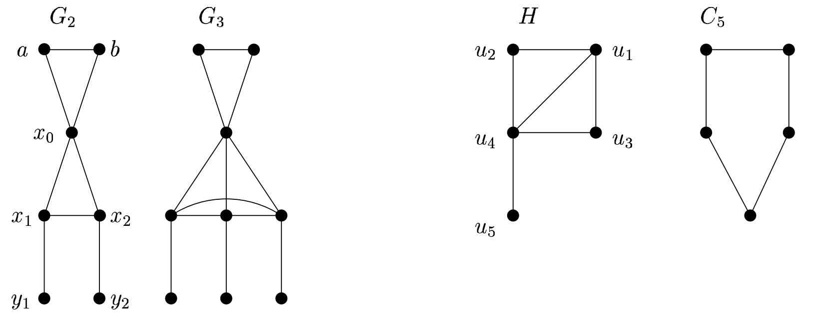

Fig. 2. The graphs G_2, G_3, H and C_5.

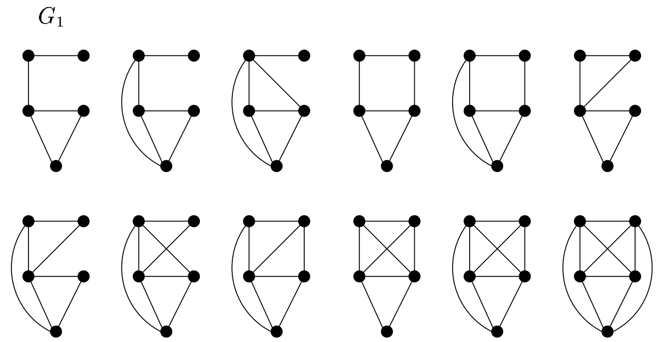

Fig. 3. All connected 5-vertex graphs with a $C_3, \{P_2\}$-factor.

already requires $\Omega(|V|^2)$ time for some almost claw-free graphs, we did not aim to bring down the running time of the $O(|V|^{0.5}|E|) = O(|V|^{2.5})$ time algorithm of Blum that computes a maximum matching for general graphs [2].

3 Parallel knock-out schemes

3.1 Definitions and Observations

In this section we continue the study on *parallel knock-out schemes* for finite undirected simple graphs begun in [17] and continued in [4–6]. Such a scheme proceeds in rounds. In the first round each vertex in the graph selects exactly one of its neighbours, and then all the selected vertices are eliminated simultaneously. In subsequent rounds this procedure is repeated in the subgraph induced by those vertices not yet eliminated. The scheme continues until there are no vertices left, or until an isolated vertex is obtained (since an isolated vertex will never be eliminated).

More formally, for a graph $G = (V, E)$, a *KO-selection* is a function $f : V \to V$ with $f(v) \in N(v)$ for all $v \in V$. If $f(v) = u$, we say that vertex v *fires at* vertex u, or that vertex u *is knocked out* by vertex v. For a KO-selection f, we define the corresponding *KO-successor* of G as the subgraph of G that is induced by the vertices in $V \setminus f(V)$; if G' is the KO-successor of G we write $G \rightsquigarrow G'$. Note that every graph without isolated vertices has at least one KO-successor. A graph G is called *KO-reducible*, if there exists a *KO-reduction scheme*, that is, a finite sequence

$$G \rightsquigarrow G^1 \rightsquigarrow G^2 \rightsquigarrow \cdots \rightsquigarrow G^r,$$

where G^r is the null graph $(\emptyset, \emptyset)$. A single step in this sequence is called a *round*, and the parallel knock-out number of G, pko(G), is the smallest number of rounds of any KO-reduction scheme. If G is not KO-reducible, then pko$(G) = \infty$.

Note that pko(P_1) = pko(P_3) = pko$(P_5) = \infty$, as in each case there is at least one isolated vertex after the first round of any parallel knock-out scheme, and pko$(P_{2k}) = 1$, for $k \geq 1$, and pko$(C_k) = 1$, for $k \geq 3$, as we can define a first round firing along the perfect matching and cycle edges, respectively. Finally, pko$(P_{2k+1}) = 2$ for $k \geq 3$. To see this, consider a KO-reduction scheme for a path $p_1p_2 \cdots p_{2k+1}$ such that in the first round p_{2i-1} and p_{2i} fire at each other for $i = 1, \ldots, k-2$, p_{2k-3} fires at p_{2k-4}, p_{2k-2} fires at p_{2k-3}, p_{2k-1} fires at p_{2k}, and p_{2k} and p_{2k+1} fire at each other. Then, after round one, p_{2k-2} and p_{2k-1} are the only two vertices left and they fire at each other in round two. This yields the following observation which explains our interest in $P_7, \{P_2\}$-factors; note that the reverse implication is not true.

Observation 4 *Let G be a graph. If G has a perfect matching or a $C_k, \{P_2\}$-factor for some $k \geq 3$, then pko$(G) = 1$. If G has a $P_{2k+1}, \{P_2\}$-factor for some $k \geq 3$, then pko$(G) \leq 2$.*

The paper [6] shows that a KO-reducible n-vertex graph G has

$$\text{pko}(G) \leq \min\left\{-\frac{1}{2} + \sqrt{2n - \frac{7}{4}},\ \frac{1}{2} + \sqrt{2\alpha - \frac{7}{4}}\right\},$$

(recall that α is the independence number). This bound is asymptotically tight due to the existence of a family of graphs in [4] whose knock-out numbers grow proportionally to the square root of the number of vertices (and to the square root of the independence number as these graphs are bipartite). KO-reducible claw-free graphs, however, can be knocked out in at most two rounds [4]. Connected claw-free graphs with minimum degree $d \geq 5$ have a $\{P_6, P_7, \ldots\}$-factor [1]: this implies they are KO-reducible in at most two rounds by Observation 4. Using Theorem 3 we can strengthen and generalise the result on parallel knock-out numbers for claw-free graphs to almost claw-free graphs. First, note that every graph $F \in \mathcal{F}$ is not KO-reducible as in the first round of any KO-reduction scheme all neighbours of the root x of F must fire at their neighbour of degree one, and vice versa. So, in the next round, x would be the only remaining vertex which is not possible in a KO-reduction scheme. We find that pko$(H) = 2$ as u_1 can fire at u_2, while u_2 and u_3 fire at u_4, and u_4 and u_5 fire at each other in the first round, and then u_1 and u_3 fire at each other in the second round. By Observation 4, pko$(G) = 1$ if $G \in \mathcal{G} \cup \{C_5\}$. If G is an even connected almost claw-free graph, then G has a perfect matching by Theorem 1 and consequently pko$(G) = 1$ by Observation 4. Hence we have the following result.

Corollary 1. *Let G be a connected almost claw-free graph. Then G is KO-reducible if and only if pko$(G) \leq 2$ if and only if $G \notin \mathcal{F}$.*

Note that odd paths on at least seven vertices are examples of (almost) claw-free graphs with parallel knock-out number two. We observe that Corollary 1 restricted to claw-free graphs states that a connected claw-free graph G is KO-reducible if and only if pko$(G) \leq 2$ if and only if G is not isomorphic to some F_k or $F_{k,\ell}$. This characterisation of claw-free graphs is new. A further implication is the following corollary.

Corollary 2. *Let G be a 2-connected almost claw-free graph. Then pko$(G) \leq 2$.*

3.2 Running Times

In [4], a polynomial time algorithm is given that determines the parallel knock-out number of any tree. For general bipartite graphs, however, the problem of finding the parallel knock-out number is NP-hard [5]. In fact, even the problem of deciding if pko$(G) \leq 2$ for a given bipartite graph G is NP-complete. On the positive side, a polynomial time algorithm for finding a KO-reduction scheme for general claw-free graphs was presented in [6]. Corollary 1 provides us with an $O(|V|^2)$ algorithm for checking if an *almost* claw-free graph $G = (V, E)$ is KO-reducible as it takes $O(|V|^2)$ time to verify that each component of G does not belong to $\mathcal{F}$. This is a considerable improvement upon the polynomial time algorithm for claw-free graphs in [6] which we briefly describe now as its running time was not previously analysed.

The algorithm first checks if pko$(G) = 1$ by determining if G has a *[1,2]-factor* (a spanning subgraph in which every component is either a cycle or an edge). The problem of deciding if $G = (V, E)$ contains a [1,2]-factor is a folklore problem appearing in many standard books on combinatorial optimisation. It is solved as follows. Let $V = \{v_1, v_2, \ldots, v_n\}$. Define the *product graph* of G as the bipartite graph $G' = (V', E')$ with vertex set $V' = \{u_1, u_2, \ldots, u_n,\ w_1, w_2, \ldots, w_n\}$ in which $u_i w_j \in E'$ and $u_j w_i \in E'$ if and only if $v_i v_j \in E$. A [1,2]-factor in G corresponds to a perfect matching in G'. The fastest known algorithms for checking if a bipartite graph $G = (V, E)$ has a perfect matching have running time $O(|V|^{0.5}|E|)$ [9, 12] or $O(|V|^{2.376})$ [13].

If pko$(G) \neq 1$, the algorithm checks if pko$(G) = 2$ by using a result (also proved in [6]) that any connected claw-free graph G with pko$(G) = 2$ allows a KO-reduction scheme in which only two vertices x, y remain in the second round such that

1. x knocks out a vertex w in the first round that is not knocked out by any other vertex and that fires at a vertex that is knocked out by some other vertex as well.

2. y knocks out a vertex in the first round that is knocked out by some other vertex as well.

The algorithm simply checks all possibilities for x, y, w. After guessing these three vertices, it checks if the remaining graph has parallel knock-out number one. Thus the algorithm of [6] takes $O(|V|^{5.376})$ time if we use the algorithm of [13] and $O(|V|^{3.5}|E|)$ time if we use the algorithms in [9, 12] for finding a perfect matching in a bipartite graph. (We have not examined if the algorithms in [9, 12, 13] can be improved if the bipartite graph under consideration is the product graph of a claw-free graph.) Note that our new algorithm ***finds*** a KO-reduction scheme for the class of almost claw-free graphs in $O(|V|^{3.5})$ time. This can be seen as follows. We first check in $O(|V|^2)$ time if our input graph $G = (V, E)$ that is almost claw-free belongs to $\mathcal{G} \cup \{C_5, H\}$. If so, then we can immediately deduce a KO-reduction scheme. We then check in $O(|V|^2)$ time if G belongs to $\mathcal{F}$. If so, then $\text{pko}(G) = \infty$. If not then G contains a $P_7, \{P_2\}$-factor which we can find in $O(|V|^{3.5})$ time by Theorem 3. This $P_7, \{P_2\}$-factor immediately provides us with a KO-reduction scheme of G.

We summarise what we have proved:

Corollary 3. *Let $G = (V, E)$ be an almost claw-free graph. Deciding whether G is KO-reducible or has pko$(G) \leq 2$, respectively, can be done in $O(|V|^2)$ time. The problem of finding a KO-reduction scheme for G can be done in $O(|V|^{3.5})$ time.*

4 Proof of Theorem 3

4.1 Definitions and Lemmas

In this section we prove Theorem 3 after first introducing some additional notation and preliminary results. The subgraph of a graph $G = (V, E)$ induced by a set $U \subseteq V$ is denoted by $G[U]$. A set $U \subseteq V$ is a *dominating set* of G if each vertex in V is in U or adjacent to a vertex in U. If $U = \{u\}$ we call u a *dominating vertex* of G and if $U = \{u_1, u_2\}$ we call u_1 and u_2 a *dominating pair.* Note that condition 2 of Definition 1 is equivalent to: "for all $v \in V$, $G[N(v)]$ must contain a dominating vertex or dominating pair". We denote the set of vertices in a graph G that have degree i by V_i and all vertices that have degree at least i by $V_{\geq i}$. We denote by $V'_{\geq 2}$ the subset of $V_{\geq 2}$ containing vertices that do not have neighbours of degree 1. For convenience, we sometimes use the notation $|G|$ to denote the number of vertices in G.

The following fact is a complicating factor in the proof of Theorem 3: removing a vertex x from an almost claw-free graph does not automatically result in a new almost claw-free graph. Note that claw-free graphs do satisfy such a property. An example is the almost claw-free graph H: if we remove u_1 from H then we obtain a claw, which does not satisfy condition 2 of Definition 1. Hence, one of the conditions in Lemma 5 below, namely that $G[V \backslash \{x\}]$ is almost claw-free, is not satisfied by every almost claw-free graph (if it were, then Lemma 5 alone would imply Theorem 3). The next lemma tells us about the structure of a graph obtained by removing a single vertex from an almost claw-free graph.

Lemma 1. *Let x be a vertex of an almost claw-free graph $G = (V, E)$ such that $G[V\backslash\{x\}]$ is not almost claw-free. Let Y be the subset of $V\backslash\{x\}$ such that $G[N(y)\backslash\{x\}]$ does not contain a dominating pair. Then the following holds:*

(i) Y is an independent set with $|Y| \in \{1, 2\}$.
(ii) Each $y \in Y$ is adjacent to x.
(iii) For each $y \in Y$ there exist vertices $a, b \in N(x)$ and $c \notin N(x) \cup \{x\}$ such that y is the centre of an induced claw with edges ya, yb, yc.

Proof. Let x be a vertex of an almost claw-free graph $G = (V, E)$ and let $G' = G[V\backslash\{x\}]$. Suppose G' is not almost claw-free. If G' violates condition 1 of Definition 1, then G would violate this condition as well. Hence G' violates condition 2 of Definition 1. Then there exists a vertex y^*, such that $G'[N_{G'}(y^*)] = G[N(y^*)\backslash\{x\}]$ does not contain a dominating pair. As G is almost claw-free, x is in any dominating pair of $G[N(y^*)]$. Then $y^* \in Y$ and $xy^* \in E$. This proves $|Y| \geq 1$ and (ii).

Let x, c be a dominating pair of $G[N(y)]$ for some $y \in Y$. Since $G[N(y)\backslash\{x\}]$ does not contain a dominating pair, x has a neighbour $a \in N(y)\backslash\{x, c\}$ not adjacent to c. Because $\{a, c\}$ is not a dominating pair of $G[N(y)\backslash\{x\}]$, x has a neighbour $b \in N(y)\backslash\{a, x, c\}$ neither adjacent to a nor to c. We note that y is the centre of an induced claw in G with edges ya, yb, yc. Then, by condition 1 of Definition 1, x is not the centre of an induced claw. We then deduce that $xc \notin E$. This proves (iii).

Because each $y \in Y$ is the centre of an induced claw, Y is an independent set of G due to condition 1 of Definition 1. To finish the proof of (i), suppose $Y = \{y_1, \ldots, y_r\}$ with $r \geq 3$. Because $\{y_1, y_2, y_3\}$ is an independent set in $G[N(x)]$, we then find that x is the centre of an induced claw with edges xy_1, xy_2, xy_3. We already observed x is not the centre of an induced claw. Hence we conclude that $r \leq 2$. This completes the proof of Lemma 1. □

The following lemmas are used in the proof of Theorem 3. They are proved in Section 4.3.

Lemma 2. *If $G = (V, E)$ is an odd connected almost claw-free graph not in $\mathcal{F} \cup \mathcal{G} \cup \{C_5, H\}$, then $|V| \geq 7$, $V'_{\geq 2} \neq \emptyset$. Furthermore all vertices in $V'_{\geq 2}$ have a neighbour in $V'_{\geq 2}$.*

Lemma 3. *Let $G = (V, E) \notin \mathcal{G}$ be a connected almost claw-free graph with a $C_3, \{P_2\}$-factor. Then G has a $P_7, \{P_2\}$-factor. Moreover, given a $C_3, \{P_2\}$-factor of G, there is an algorithm that finds a $P_7, \{P_2\}$-factor of G in $O(|V|^2)$ time.*

Lemma 4. *Let $G = (V, E)$ with $|V| \geq 7$ be a connected almost claw-free graph that has a $C_5, \{P_2\}$-factor or an $H, \{P_2\}$-factor. Then G has a $P_7, \{P_2\}$-factor. Moreover, given a $C_5, \{P_2\}$-factor or $H, \{P_2\}$-factor of G, there is an algorithm that finds a $P_7, \{P_2\}$-factor of G in $O(|V|^2)$ time.*

Lemma 5. *Let $G = (V, E) \notin \mathcal{F} \cup \mathcal{G} \cup \{C_5, H\}$ be an odd connected almost claw-free graph. If $G[V \backslash \{x\}]$ is almost claw-free for some $x \in V'_{\geq 2}$, then G has a $P_7, \{P_2\}$-factor. Moreover, given such a vertex x, there is an algorithm that finds a $P_7, \{P_2\}$-factor of G in $O(|V|^{2.5})$ time.*

Lemma 6. *Let $G = (V, E)$ be an odd connected almost claw-free graph not in $\mathcal{F} \cup \mathcal{G}$ such that $G[V \backslash \{x\}]$ is not almost claw-free for all $x \in V'_{\geq 2}$. Then, for each $x \in V'_{\geq 2}$, there exist two vertices $\{c, y\}$ with $y \in N(x)$ and $c \in N(y) \cap V_1$ such that $G^* = G[V \backslash \{c, y\}]$ is either in $\mathcal{G} \cup \{C_5, H\}$ or else G^* is an odd connected almost claw-free graph not in $\mathcal{F}$ such that $G^*[V_{G^*} \backslash \{x\}]$ is almost claw-free.*

4.2 The Algorithm

We restate Theorem 3 before presenting the algorithm that provides a proof.

Theorem 3 *Let $G = (V, E)$ be an odd connected almost claw-free graph. If $G \notin \mathcal{F} \cup \mathcal{G} \cup \{C_5, H\}$ then G has a $P_7, \{P_2\}$-factor, which we can find in $O(|V|^{3.5})$ time.*

Outline of the algorithm. Let $G = (V, E)$ be an odd connected almost claw-free graph. Suppose $G \notin \mathcal{F} \cup \mathcal{G} \cup \{C_5, H\}$. We show how to find a $P_7, \{P_2\}$-factor of G in $O(|V|^{3.5})$ time.

Step 1. Determine the set $V'_{\geq 2}$.

This takes time $O(|V|^2)$ time, and, by Lemma 2, the set is nonempty. (In fact Lemma 2 says more than this as it is used in the proofs of later lemmas.)

Step 2. For each vertex $x \in V'_{\geq 2}$, run the algorithm of Lemma 5.

If $G[V \backslash \{x\}]$ is almost claw-free, then, by Lemma 5, we will find a $P_7, \{P_2\}$-factor of G. If, after trying all possible choices for x, we still have not found a $P_7, \{P_2\}$-factor of G, then we know that $G[V \backslash \{x\}]$ is not almost claw-free for all $x \in V'_{\geq 2}$. Step 2 takes time $|V'_{\geq 2}| O(|V|^{2.5}) = O(|V|^{3.5})$.

Step 3. Choose an arbitrary vertex $x \in V_{\geq 2}$. Find all edges cy where $c \in V_1$, $y \in N(x)$ and $N(y) \backslash \{c\}$ is dominated by x.

After Step 3 we have obtained a set of p edges $c_1 y_1, \ldots, c_p y_p$ with $c_i \in N(y) \cap V_1$ and $y_i \in N(x)$ with $N(y_i) \backslash \{c_i\} \subseteq N(x)$ for each $i = 1, \ldots, p$. Note that $p \leq |V|$. Step 3 takes time $O(|V|^3)$.

Step 4. For each i, consider the graph $G_i^* = G[V \backslash \{c_i, y_i\}]$. Check whether $G_i^* \in \mathcal{G} \cup \{C_5, H\}$.

Step 4a. If $G_i^* \in \mathcal{G}$, then find a $C_3, \{P_2\}$-factor of G_i^* (this is easy). Extend this factor with the P_2-component $c_i y_i$ to obtain a $C_3, \{P_2\}$-factor of G. Use the algorithm of Lemma 3 to obtain a $P_7, \{P_2\}$-factor of G.

We can use the algorithm of Lemma 3 since $G \notin \mathcal{G}$. Step 4a takes time $O(|V|^2)$.

Step 4b. If G_i^* is isomorphic to C_5 or H, then find a $C_5, \{P_2\}$-factor or $H, \{P_2\}$-factor of G (by adding the edge $c_i y$). Then use the algorithm of Lemma 4 to find a $P_7, \{P_2\}$-factor of G.

Step 4b takes time $O(|V|^2)$. If we have still not found a $P_7, \{P_2\}$-factor of G at the end of Step 4, then we have taken $p \cdot O(|V|^2) = O(|V|^3)$ time to find that $G_i^* \notin \mathcal{G} \cup \{C_5, H\}$ for each i.

Step 5. Apply the algorithm of Lemma 5 to G_i^* and x for each i.

By Lemma 6, there must exist an i such that $G_i^* \notin \mathcal{F} \cup \mathcal{G} \cup \{C_5, H\}$ and both G_i^* and $G_i^*[V_{G_i^*} \backslash \{x\}]$ are almost claw-free. Hence we obtain a $P_7, \{P_2\}$-factor of some G_i^* in $p \cdot O(|V|^{2.5}) = O(|V|^{3.5})$ time. We extend this $P_7, \{P_2\}$-factor to a $P_7, \{P_2\}$-factor of G by adding the P_2-component $c_i y_i$. □

4.3 Proofs

Proof of Lemma 2. Let $G = (V, E)$ be an odd connected almost claw-free graph not in $\mathcal{F} \cup \mathcal{G}$. We first prove the following claim.

Claim 1. Each vertex in V has at most one neighbour in V_1.

Let $u \in V$ have two neighbours u' and u'' in V_1. As $G \notin \mathcal{F}$, we know that G is not isomorphic to $F_1 = P_3$. Hence u has a neighbour $v \notin \{u', u''\}$. Thus each dominating set of $G[N(u)]$ contains u', u'' and at least one other vertex. This violates condition 2 of Definition 1, and Claim 1 is proved.

If G has only one or three vertices, then, since it is connected, it is $P_1 = F_0$, $P_3 = F_1$ or $C_3 = G_0$, contradicting our assumption that $G \notin \mathcal{F} \cup \mathcal{G}$. Thus $|V| \geq 5$ and, by the connectedness of G, $V_{\geq 2} \neq \emptyset$. Suppose $|V| = 5$. If G has a $C_3, \{P_2\}$-factor then $G \in \mathcal{G}$ by definition. The only four remaining connected almost claw-free graphs on five vertices are $F_2, F_{1,1}, C_5$, and H. All these four graphs are excluded. Hence $|V| \geq 7$. Suppose $V'_{\geq 2} = \emptyset$, that is, all vertices in $V_{\geq 2}$ are adjacent to a vertex in V_1. By Claim 1, each vertex in $V_{\geq 2}$ has exactly one neighbour in V_1. This means that G has a perfect matching and contradicts the assumption that G is odd. Hence we find that $V'_{\geq 2} \neq \emptyset$.

We now prove the second statement of the lemma by contradiction. Suppose x is a vertex in $V'_{\geq 2}$ such that $N(y) \cap V_1 \neq \emptyset$ for all $y \in N(x)$. We first show that this implies that $V = \{x\} \cup N(x) \cup N'(x)$, where $N'(x)$ denotes the set of vertices of degree one that are at distance two from x. If $V \neq \{x\} \cup N(x) \cup N'(x)$ then there exists a vertex $w \in N(x)$ that has a neighbour w^* not in $\{x\} \cup N(x) \cup N'(x)$. Let w' be the neighbour of w in V_1 (so $w' \in N'(x)$). Note that $\{w', w^*, x\}$ is an independent set in $G[N(w)]$. Due to condition 2 in Definition 1, $G[N(w)]$ must have a dominating pair. Hence w^* and x must have a common neighbour z in $G[N(w)]$. Then $z \in V_{\geq 2} \cap N(x)$, and z must have a neighbour z' in V_1. Thus w is the centre of an induced claw in G with edges ww^*, ww', wx, and z is the centre of an induced claw in G with edges zw^*, zx, zz'. This is in contradiction to condition 1 of Definition 1, as z and w are adjacent. Hence we may indeed conclude that if there exists $x \in V'_{\geq 2}$ with no neighbour in $V'_{\geq 2}$, then $V = \{x\} \cup N(x) \cup N'(x)$.

We need to distinguish two cases according to whether or not x has a neighbour that dominates all others. When both cases lead to a contradiction, the lemma is proved.

Case 1. x has a neighbour y that is adjacent to all vertices in $N(x)\backslash\{y\}$.

Let y' be the neighbour of y in V_1. As $x \in V'_{\geq 2} \subseteq V_{\geq 2}$, we have $|N(x)\backslash\{y\}| \geq 1$. Suppose $G[N(x)\backslash\{y\}]$ is connected. If $G[N(x)\backslash\{y\}]$ is not a complete graph, then $G[N(x)\backslash\{y\}]$ contains two non-adjacent vertices s and t. Let $P = u_1u_2\cdots u_p$ be a shortest (and consequently induced) path from $s = u_1$ to $t = u_p$ in $G[N(x)\backslash\{y\}]$. Then $p \geq 3$ and $u_1u_3 \notin E$. By our assumption, u_2 has a neighbour u'_2 in V_1. Hence, y is the centre of an induced claw with edges yy', yu_1, yu_3, and u_2 is the centre of an induced claw with edges $u_2u'_2, u_2u_1, u_2u_3$. However, y is adjacent to u_2. This is not possible as condition 1 of Definition 1 is violated. Hence we find that $G[N(x)\backslash\{y\}]$, and consequently, $G[N(x)]$ is a complete graph. Recall that $V = \{x\} \cup N(x) \cup N'(x)$. By Claim 1 and our assumption on x, every vertex in $N(x)$ has exactly one neighbour in $N'(x)$. This would mean that G is isomorphic to $F_{|N(x)|}$, which contradicts our assumption that $G \notin \mathcal{F}$. Hence, $G[N(x)\backslash\{y\}]$ is not connected.

Let $D_1, \ldots D_q$ be the $q \geq 2$ components of $G[N(x)\backslash\{y\}]$. Suppose $q \geq 3$. Then x is the centre of an induced claw in G with edges xd_i for some $d_i \in V_{D_i}$ for $i = 1, 2, 3$. Also y is the centre of an induced claw with edges yd_i for $i = 1, 2, 3$. As $xy \in E$, condition 1 of Definition 1 is again violated. Hence $q = 2$.

If D_1 is not a complete graph, then D_1 contains two vertices a and b with $ab \notin E$. Let $c \in D_2$. Then x and y are adjacent centres of induced claws with edges xa, xb, xc and ya, yb, yc respectively. By condition 1 of Definition 1, this is not possible. Hence D_1, and, by the same argument, D_2, is a complete graph. Recall that $V = \{x\} \cup N(x) \cup N'(x)$. Then G is isomorphic to $F'_{|D_1|,|D_2|}$. This contradicts our assumption that $G \notin \mathcal{F}$. We conclude that Case 1 cannot occur.

Case 2. $N(x)$ does not contain a vertex adjacent to all vertices in $N(x)$.

By condition 2 of Definition 1, $N(x)$ contains a dominating pair y_1 and y_2. First suppose $y_1y_2 \in E$. By our assumption, y_1 is not adjacent to some vertex $z_1 \in N(x)$, and y_2 is not adjacent to some vertex $z_2 \in N(x)$. As y_1, y_2 form a dominating pair, we deduce that y_1z_2 and y_2z_1 are edges of G. Let y'_1 be the neighbour of y_1 in V_1 and let y'_2 be the neighbour of y_2 in V_1. Then y_1 is the centre of an induced claw in G with edges $y_1y'_1, y_1y_2, y_1z_2$, and y_2 is the centre of an induced claw in G with edges $y_2y_1, y_2y'_2, y_2z_1$. This violates condition 1 of Definition 1, because y_1 and y_2 are adjacent. Hence we find that $y_1y_2 \notin E$.

Let $D_1, \ldots, D_p$ denote the components of $G[N(x)]$. Suppose $p \geq 3$. We may without loss of generality assume $\{y_1, y_2\} \subseteq V_{D_1} \cup V_{D_2}$. Then $\{y_1, y_2\}$ does not dominate D_i for $i \geq 3$. Hence $p \leq 2$. Suppose $p = 1$ and let $P = u_1u_2\cdots u_r$ be a shortest (and consequently induced) path from $u_1 = y_1$ to $u_r = y_2$ in $G[N(x)]$. Let u'_i be the neighbour of u_i in V_1 for $i = 1, \ldots, r$. As $y_1y_2 \notin E$ and P is an induced path, we find that $r \geq 3$. Suppose $r \geq 4$. Then u_2, u_3 are adjacent centres of induces claws in G with edges $u_2u_1, u_2u'_2, u_2u_3$ and $u_3u_2, u_3u'_3, u_3u_4$ respectively. As this is not possible by condition 1 of Definition 1, we find that $r = 3$. Because u_2 cannot be a dominating vertex of $G[N(x)]$ due to our Case 2 assumption, there exists a vertex $z \in N(x)$ not adjacent to u_2. Since $\{u_1, u_3\} = \{y_1, y_2\}$ is a dominating pair of $G[N(x)]$, we have u_1z or u_3z in E. We may without loss of generality assume $u_1z \in E$. Then u_1 and u_2 are adjacent centres of induced claws in G with edges u_1u_1', u_1u_2, u_1z and u_2u_1, u_2u_2', u_2u_3, respectively. This is not possible due to condition 1 of Definition 1.

Hence $p = 2$. We assume without loss of generality that y_1 belongs to D_1 and y_2 to D_2 (if y_1, y_2 are in the same component, say D_1, they will not dominate the vertices in D_2). Suppose D_1 is not a complete graph. Then there exist vertices a, b in D_1 with $ab \notin E$. Let y_1' be the neighbour of y_1 in V_1. Then x and y_1 are adjacent centres of induced claws with edges xa, xb, xy_2 and y_1a, y_1b, y_1y_1' respectively. By condition 1 of Definition 1, this is not possible. Hence D_1, and by the same arguments, D_2 are complete graphs. Recall that $V = \{x\} \cup N(x) \cup N'(x)$. Hence G is isomorphic to $F_{|D_1|,|D_2|}$. This contradicts our assumption that $G \notin \mathcal{F}$. We conclude that Case 2 does not occur. This completes the proof of Lemma 2. □

Proof of Lemma 3. Let $G = (V, E)$ be a connected almost claw-free graph not in $\mathcal{G}$ that has a $C_3, \{P_2\}$-factor L. Let $C = abca$ be the C_3-component of L. We shall show how we can combine C with P_2-components of L to obtain a P_7, which together with the remaining edges in L, forms a $P_7, \{P_2\}$-factor of G. As we only need to check the P_2-components in L this process takes $O(|E|) = O(|V|^2)$ time.

First note that $|V|$ is odd. If $|V| = 3$, then G is isomorphic to $C_3 \in \mathcal{G}$, which is not possible. Since by definition all connected 5-vertex graphs with a $C_3, \{P_2\}$-factor belong to $\mathcal{G}$, $|V| \neq 5$ either. So, from now on we can suppose $|V| \geq 7$.

We consider two cases according to the number of vertices in C that have neighbours not in C.

Case 1. At least two vertices of C are adjacent to vertices not in C.

Let us assume that a and b are adjacent to vertices r and s respectively. Suppose $r \neq s$. Let $rr^* \in E_L$ and $ss^* \in E_L$. If $r^* = s$, (and so $s^* = r$), then $acbsra$ is a cycle, and as $|V| \geq 7$, there exists an edge $tt^* \in E_L$ with t adjacent to a vertex on this cycle. Thus $G[\{a, b, c, s, r, t, t^*\}]$ has a P_7 as a subgraph, which forms, together with the remaining edges in L, a $P_7, \{P_2\}$-factor of G. If $r^* \neq s$ (so $s^* \neq r$), then the path $r^*racbss^*$, together with the remaining edges in L, forms a $P_7, \{P_2\}$-factor of G.

Now suppose $r = s$ and $r^* = s^*$. Since $|V| \geq 7$ and G is connected, there exists a P_2-component $tt^* \in L$ with $tt^* \neq rr^*$ such that at least one of the vertices in tt^*, say t, is adjacent to a vertex in $\{a, b, c, r, r^*\}$. If t is adjacent to a vertex in $\{a, b, c, r^*\}$ then we immediately obtain a $P_7, \{P_2\}$-factor of G. Suppose $\{at, bt, ct, r^*t\} \cap E = \emptyset$. Then $rt \in E$. By symmetry, we may assume $\{at^*, bt^*, ct^*, r^*t^*\} \cap E = \emptyset$ as well. If $\{ar^*, br^*, cr^*\} \cap E \neq \emptyset$ then we immediately find a $P_7, \{P_2\}$-factor of G. Suppose $\{ar^*, br^*, cr^*\} \cap E = \emptyset$ Then $\{a, r^*, t^*\}$ is an independent set. By condition 2 of Definition 1, $G[(N(r)]$ must contain a dominating pair. Due to all the forbidden edges, this requires that there exist a P_2-component uu^* in L with $uu^* \notin \{rr^*, tt^*\}$ such that at least one of the vertices in $\{u, u^*\}$, say u, is adjacent to r and at least two vertices in $\{a, t, r^*\}$, so to at least one vertex in $\{a, r^*\}$. If u is adjacent to a we find the path $u^*uacbrr^*$, and if u is adjacent to r^* we find the path u^*ur^*rabc. Hence, both cases yield a $P_7, \{P_2\}$-factor of G.

Case 2. Exactly one vertex in C has a neighbour not in C.

Assume that a has a neighbour outside C, so $N(b) = \{a, c\}$, and $N(c) = \{a, b\}$. Then $G[N(a)\backslash\{b, c\}]$ contains a dominating vertex d, due to condition 2 of Definition 1. Assume $G[N(a)\backslash\{b, c, d\}]$ is not a complete graph. Let v, w be two nonadjacent vertices in $N(a)\backslash\{b, c, d\}$. Let $vv^*, ww^* \in E_L$. Note that v, v^*, w, w^* are four different vertices. First, suppose $d = v^*$ or $d = w^*$, say $d = v^*$. Then the path $w^*wdvabc$ together with the remaining edges in L forms a $P_7, \{P_2\}$-factor of G. Second, suppose $d \notin \{v^*, w^*\}$. Then $dd^* \in E_L$ for some $d^* \notin \{v, w\}$. Let $vv^* \in E_L$. If d^* is adjacent to v or w, then we obtain a path v^*vd^*dabc or w^*wd^*dabc, respectively, and this immediately leads to a $P_7, \{P_2\}$-factor of G. In the remaining case, we find that a, d are adjacent centres of induced claws in G with edges ab, av, aw and dd^*, dv, dw, respectively. By condition 1 of Definition 1 this is not possible.

We now assume that $G[N(a)\backslash\{b, c, d\}]$, and consequently, $G[N(a)\backslash\{b, c\}]$ is a complete graph. Suppose L has a P_2-component vv^* with $v, v^* \in N(a)\backslash\{b, c\}$. Since $|V| \geq 7$ and G is connected, L has a P_2-component $zz^* \neq vv^*$, such that one of the vertices in $\{z, z^*\}$, say z, is adjacent to $\{a, v, v^*\}$. If z is adjacent to a then $zv, zv^* \in E$, since $G[N(a)\backslash\{b, c\}]$ is complete. Hence z is adjacent to at least one of the vertices in $\{v, v^*\}$, say to v. Then the path z^*zvv^*abc together with the remaining edges in L form a $P_7, \{P_2\}$-factor of G.

Suppose $G[N(a)\backslash\{b, c\}]$ does not contain edges of L. Let $N(a)\backslash\{b, c\} = \{v_1, \ldots, v_p\}$ for some $p \geq 1$. Then each vertex $v_i \in N(a)\backslash\{b, c\}$ has a unique neighbour $v_i^* \notin N(a)$ such that $v_iv_i^*$ is a P_2-component $v_iv_i^*$ of L. Suppose $\{v_1^*, \ldots, v_p^*\}$ is not an independent set, say $v_i^*v_j^* \in E$. Then the path $v_iv_i^*v_j^*v_jabc$ together with the remaining edges in L form a $P_7, \{P_2\}$-factor of G.

Suppose $\{v_1^*, \ldots, v_p^*\}$ is an independent set. Then G contains a subgraph G' induced by $N(a) \cup \{a, v_1^*, \ldots, v_p^*\}$ that is isomorphic to $G_p \in \mathcal{G}$. By our assumption that $G \notin \mathcal{G}$, we have $G \neq G'$. As G is connected, L then contains a P_2-component rr^* with both r, r^* not in $V_{G'}$ such that at least one of the vertices in $\{r, r^*\}$, say r, is adjacent to a vertex in $V_{G'}$. If r is adjacent to a, then r is adjacent to all vertices in $N(a)\backslash\{b, c\}$ as $G[N(a)\backslash\{b, c\}]$ is complete. Then $r \in \{v_1, \ldots, v_p\} \subset V_{G'}$, which is not possible. Hence $ar \notin E$. If r is adjacent to a vertex v_i^*, then the path $r^*rv_i^*v_iabc$ together with the remaining edges in L forms a $P_7, \{P_2\}$-component of G, and we are done. Suppose r is not adjacent to a vertex in $\{v_1^*, \ldots, v_p^*\}$. Since $N(b) = \{a, c\}$ and $N(c) = \{a, b\}$ we then find that r is adjacent to some vertex v_i. As we already deduced that $av_i^* \notin E$, we obtain that $\{a, r, v_i^*\}$ is an independent set. We claim that v_i is the only vertex of G' that is adjacent to r. In order to see this, suppose r is adjacent to some other vertex in G'. By the same arguments as above, we find that this vertex must be some v_j with $j \neq i$ and that $\{a, r, v_j^*\}$ is an independent set. Then v_i, v_j are adjacent centres of induced claws with edges $v_ia, v_ir, v_iv_i^*$ and $v_ja, v_jr, v_jv_j^*$, respectively. This contradicts condition 1 of Definition 1 and shows that v_i is indeed the only vertex of G' adjacent to r.

We note that $\{a, r, v_i^*\} \subseteq N(v_i)$ is an independent set. By condition 2 of Definition 1, $G[N(v_i)]$ must contain a dominating pair. Hence there exists a vertex $s \notin \{a, r, v_i^*\}$ that is adjacent to v_i and to at least two vertices in $\{a, r, v_i^*\}$. If s is adjacent to a, then $s = v_j$ for some $j \neq i$. Since G' is an induced subgraph of G, we find that $sv_i^* \notin E$. As v_i is the only vertex of G' adjacent to r, we find that $sr \notin E$ either. Hence s cannot be adjacent to a, and consequently, s must be adjacent to both r and v_i^*. Because $s \neq v_i$ is adjacent to v_i^*, we obtain $s \notin V_{G'}$. Let $ss^* \in E_L$. Then $s^* \notin V_{G'}$, because $s \notin V_{G'}$ and there are no edges in E_L with exactly one end vertex in G'. Hence, we obtain a $P_7, \{P_2\}$-factor by taking the path $s^*sv_i^*v_iabc$ together with the remaining edges of L. This completes the proof of Lemma 3. □

Proof of Lemma 4. Let $G = (V, E)$ be a connected almost claw-free graph on at least seven vertices. that has a $C_5, \{P_2\}$-factor or $H, \{P_2\}$-factor L. Let C be the C_5-component or H-component of L. Below we show how we can combine C with one P_2-component of L to obtain a P_7, which together with the remaining edges in L, forms a $P_7, \{P_2\}$-factor of G. As we only need to check the P_2-components in L this process takes $O(|E|) = O(|V|^2)$ time.

First suppose L is a $C_5, \{P_2\}$-factor, so C is isomorphic to C_5. Since $|V| \geq 7$ and G is connected, L has a P_2-component xy such that at least one of the vertices x, y, say x, is adjacent to C. We use C and xy to obtain a P_7. We combine this P_7 with the remaining edges in L to obtain a $P_7, \{P_2\}$-factor of G.

Second suppose L is a $H, \{P_2\}$-factor, so C is isomorphic to H. Let $C = (\{a, d, x, y, z\} \cup \{xy, xz, yz, za, ya, zd\})$. Since G is connected and $|V| \geq 7$, there exists a P_2-component $qq^* \in E_L$ such that at least one of the vertices q, q^*, say q, has a neighbour in $\{a, d, x, y, z\}$. If q is adjacent to a, d or x we find the path $q^*qayxzd$, $q^*qdzayx$, or $q^*qxyazd$, respectively. We take this P_7 together with the remaining P_2-components in L to form a $P_7, \{P_2\}$-factor of G. Suppose q and, similarly, q^* are not adjacent to a vertex in $\{a, d, x\}$. If q is adjacent to y, then y and z are adjacent centres of induced claws with edges ya, yq, yx and za, zd, zx. This violates condition 1 of Definition 1. By the same argument we find that q^* is not adjacent to y. Hence at least one of the vertices q or q^*, say q again, is adjacent to z.

By condition 2 of Definition 1, $G[N(z)]$ has a dominating pair s, t. Because $\{d, q, x\}$ is an independent set in $G[N(z)]$, at least one of the vertices s and t, say s, is adjacent to two vertices of $\{d, q, x\}$, and consequently to at least one vertex of $\{d, x\}$. Then $s \notin V_C$. Let ss^* be the P_2-component of L that contains s. If $sx \in E$ we obtain the path $s^*sxyazd$ and if $sd \in E$ we obtain the path $s^*sdzayx$. In both cases we find a P_7, and we take this P_7 together with the remaining edges of L to obtain a $P_7, \{P_2\}$-factor of G. This completes the proof of Lemma 4. □

Proof of Lemma 5. Let $G = (V, E)$ be an odd connected almost claw-free graph that is not in $\mathcal{F} \cup \mathcal{G} \cup \{C_5, H\}$. Assume that $G[V \backslash \{x\}]$ is almost claw-free for some $x \in V'_{\geq 2}$. Denote the components of $G[V \backslash \{x\}]$ by $Q_1, \ldots, Q_l$. If $l \geq 3$, then

$G[N(x)]$ does not have a dominating pair. This is not possible by condition 2 of Definition 1. Hence $l \leq 2$. We distinguish two subcases.

Case 1. $l = 1$, or $l = 2$ and Q_1 and Q_2 are both even.

We first compute a perfect matching M of $G[V \backslash \{x\}]$ as follows. Suppose $l = 1$. Since $|V|$ is odd, Q_1 is even. Since Q_1 is almost claw-free and connected, by Theorem 1, Q_1 has a perfect matching. We define M as the perfect matching that we compute in $O(|V|^{0.5}|E|) = O(|V|^{2.5})$ time by Blum's algorithm [2]. Suppose $l = 2$, and since Q_1 and Q_2 are even, almost claw-free and connected, both Q_1 and Q_2 have a perfect matching, by Theorem 1. We can compute these perfect matchings M_1 and M_2, respectively, in $O(|V|^{2.5})$ time by Blum's algorithm and define $M := (V_{M_1} \cup V_{M_2}, E_{M_1} \cup E_{M_2})$.

We show how we can obtain a $P_7, \{P_2\}$-factor of G from M in $O(|V|^2)$ time.

By Lemma 2, x has a neighbour $y \in V'_{\geq 2}$. We can find y in $O(|V|^2)$ time. Let $ay \in E_M$. If $ax \in E$, then G has a $C_3, \{P_2\}$-factor with components $axya$ and the remaining matching edges of M. Since $G \notin \mathcal{G}$, we use Lemma 3 to find a $P_7, \{P_2\}$-factor of G in $O(|V|^2)$ extra time. Suppose $ax \notin E$. As $x \in V_{\geq 2}$, x is adjacent to some vertex $z \neq y$. Since y does not have degree one neighbours, a has at least two neighbours.

Suppose a has a neighbour $b \notin \{y, z\}$. Since $ax \notin E$, $b \neq x$. Let $bc \in E_M$. If $c = z$, we obtain a $C_5, \{P_2\}$-factor L of G with components $abzxya$ and the remaining edges in M. By Lemma 2, $|V| \geq 7$, and we can find a $P_7, \{P_2\}$-factor of G in $O(|V|^2)$ time, by Lemma 4. Hence $c \neq z$. Note that $c \notin \{a, b, x, y\}$ either. Let $zd \in E_M$. Then $d \notin \{a, b, c, x, y, z\}$. Hence we have found a $P_7, \{P_2\}$-factor of G with components $dzxyabc$ and the remaining edges in M. We can check this case in $O(|V|^2)$ time.

In the remaining case, a has exactly two neighbours, namely y and z. Again, let $dz \in M$. If $dx \in E$, then again we find a $C_3, \{P_2\}$-factor of G, and consequently, we find a $P_7, \{P_2\}$-factor of G in $O(|V|^2)$ time, by Lemma 3. Suppose $dx \notin E$. Note that $ad \notin D$ since $N(a) = \{y, z\}$. Hence z is the centre of induced claw with edges za, zd, zx. By condition 2 of Definition 1, there exists a vertex p adjacent to z and at least two vertices in $\{a, x, d\}$, and so to at least one vertex in $\{a, d\}$.

First assume that $p = y$ (meaning that $yz \in E$). If $yd \in E$, then G contains two adjacent centres, namely y, z, of induced claws with edges ya, yd, yx and za, zd, zx, respectively. This is not possible due to condition 1 of Definition 1. Hence $yd \notin E$. However, then $G[\{a, d, x, y, z\}]$ is isomorphic to H. Recall that $|V| \geq 7$. Then, by Lemma 4, we find a $P_7, \{P_2\}$-factor of G in $O(|V|^2)$ time.

Now suppose $p \neq y$. Let $pq \in E_M$. Note that $q \notin \{a, d, p, x, y, z\}$. Assume that p is adjacent to a. We find a path $qpayxzd$ on seven vertices in G. This path together with the remaining edges in M forms a $P_7, \{P_2\}$-factor of G. If $ap \notin E$, then $dp \in E$ and we find a path $qpdzxya$ on seven vertices in G. So, also in this case, which we can check in $O(|V|^2)$ time, we have found a $P_7, \{P_2\}$-factor of G. This finishes Case 1.

Case 2. $l = 2$ but either Q_1 or Q_2 is odd.

As $|V|$ is odd, we find that both $|Q_1|$ and $|Q_2|$ are odd, and consequently $G_1 = G[V_{Q_1} \cup \{x\}]$ and $G_2 = G[V_{Q_2} \cup \{x\}]$ are even. Then G_1 and G_2 are almost claw-free, as otherwise G would not be almost claw-free. Since G_1 and G_2 are almost claw-free and connected as well, they have a perfect matching M_1, M_2, respectively, due to Theorem 1. By Using Blum's algorithm [2], we can find M_1 and M_2 in $O(|V|^{0.5}|E|) = O(|V|^{2.5})$ time. Let xu_1 be an edge in M_1 and xu_2 an edge in M_2. Since $x \in V'_{\geq 2}$, u_1 and u_2 are in $V_{\geq 2}$ by definition. Let $u_1^* \neq x$ be a neighbour of u_1 (in Q_1) and let $u_2^* \neq x$ be a neighbour of u_2 (in Q_2). Let $w_i u_i^* \in E_{M_i}$ for $i = 1, 2$. We note that $|\{u_1, u_1^*, u_2, u_2^*, w_1, w_2, x\}| = 7$. Hence we found a $P_7, \{P_2\}$-factor of G with components $w_1 u_1^* u_1 x u_2 u_2^* w_2$ and the remaining edges in M_1 and M_2. This finishes Case 2 and completes the proof of Lemma 5. □

Proof of Lemma 6. Let $G = (V, E)$ be an odd connected almost claw-free graph not in $\mathcal{F} \cup \mathcal{G}$ such that $G[V \backslash \{x\}]$ is not almost claw-free for all $x \in V'_{\geq 2}$. Let $x \in V'_{\geq 2}$. Let $G' = G[V \backslash \{x\}]$. Let Y be the set of vertices such that $G'[\bar{N}_{G'}(y)]$ does not contain a dominating pair for each $y \in Y$.

Suppose there exists a vertex $y \in Y$ that has no neighbour of degree one in G. Since $y \in V_{\geq 2}$ by definition of Y, we then obtain $y \in V'_{\geq 2}$. By our assumption, $G[V \backslash \{y\}]$ is not almost claw-free. Then, by Lemma 1 (i), there exists a vertex z' such that $G[N(z') \backslash \{y\}]$ does not contain a dominating pair. Then, by Lemma 1 (ii) and (iii), z' is the centre of an induced claw adjacent to y. Since, by Lemma 1 (iii), y is also a centre of an induced claw, we obtain a contradiction with condition 1 of Definition 1. Hence, each vertex $y_i \in Y$ has a neighbour $c_i \in V_1$.

By Lemma 1 (i), $1 \leq |Y| \leq 2$ holds. We will show by contradiction that $|Y| = 1$. Suppose that $Y = \{y_1, y_2\}$. By Lemma 1 (iii) and by definition of c_i, each y_i is the centre of an induced claw with leaves $y_i a_i, y_i b_i, y_i c'_i$ for some $a_i, b_i \in N(x)$. Since y_1 is the centre of an induced claw and $y_1 x \in E$, we find that x is not the centre of an induced claw by condition 1 of Definition 1. By Lemma 1 (i), $y_1 y_2 \notin E$. Then at least one of the edges $a_1 y_2, b_1 y_2$, say $a_1 y_2$, exists (as otherwise x is the centre of an induced claw with edges xa_1, xb_1, xy_2). Clearly, $a_1 \in V_{\geq 2}$. If a_1 has a neighbour d of degree one, then a_1 is the centre of an induced claw in G with edges $a_1 d, a_1 y_1, a_1 y_2$. As a_1 is adjacent to y_1 and y_1 is the centre of an induced claw, this is not possible due to condition 1 of Definition 1. Hence $a_1 \in V'_{\geq 2}$. Then, by our assumption, $G[V \backslash \{a_1\}]$ is not almost claw-free. Then, by Lemma 1 (i), there exists a vertex b' such that $G[N(b') \backslash \{a_1\}]$ does not contain a dominating pair. As G is almost claw-free, $\{x, c_i\}$ forms a dominating pair of $G[N(y_i)]$ for $i = 1, 2$. So, x is adjacent to all vertices in $N(y_i) \backslash \{c_i\}$ for $i = 1, 2$. This means that $b' \notin \{y_1, y_2\}$, as otherwise $\{x, c_1\}$ or $\{x, c_2\}$ would be a dominating pair for $G[N(b') \backslash \{a_1\}]$. By Lemma 1 (iii), b' is the centre of an induced claw. Since we already deduced that x is not the centre of an induced claw in G, we obtain $b' \neq x$. By Lemma 1 (ii), $a_1 b' \in E$. Hence $b' \in N(a_1) \backslash \{x, y_1, y_2\}$. If $b' \notin N(y_1) \cup N(y_2)$ then a_1 and y_1 are two adjacent centres of induced claws in G with edges $a_1 b', a_1 y_1, a_1 y_2$ and $y_1 a_1, y_1 b_1, y_1 c_1$, respectively. This violates condition 1 of Definition 1. Hence $b' \in N(y_1) \cup N(y_2)$.

Since x is adjacent to all vertices in $N(y_1) \cup N(y_2) \backslash \{c_1, c_2\}$, we then obtain $xb' \in E$. As G is almost claw-free, a_1 is in any dominating pair $\{a_1, w\}$ of $G[N(b')]$. Let $b'' \in N(b')$ be adjacent to a_1. Then, by using the same arguments as above, $b'' \in N(y_1) \cup N(y_2)$, and consequently, $b'' \in N(x)$. Hence $\{x, w\}$ is a dominating pair of $G[N(b')]$ (or x is a dominating vertex of $G[N(b')]$ if $x = w$), and consequently, of $G[N(b') \backslash \{a_1\}]$. This contradiction shows that $|Y| = 1$ must hold.

From now on we write $y := y_1$ and $c := c_1$. We define $G^* := G[V \backslash \{c, y\}]$. Suppose G^* is not isomorphic to a graph in $\mathcal{G} \cup \{C, H\}$. We first show by contradiction that $G^* \notin \mathcal{F}$. Suppose $G^* \in \mathcal{F}$. Let r be the root of G^*. Obviously, G^* is not isomorphic to $F_0 = P_1$.

Suppose G is isomorphic to F_k for some $k \geq 1$. If x has degree one in G^* then x has degree two in G. Hence $G[V \backslash \{x\}]$ is almost claw-free, which is not possible by our assumption. Suppose x is a neighbour of r. Let x' be the degree one neighbour of x in G^*. If $x'y \notin E$ then $x \notin V'_{\geq 2}$ as x' will then be in V_1. If $x'y \in E$ then $x' \in V_2$ and hence $G[V \backslash \{x'\}]$ is not almost claw-free. As $x' \in V'_{\geq 2}$ as well, this is not possible, again by our assumption on vertices in $V'_{\geq 2}$. In the remaining case, $x = r$. As x dominates $G[N(y) \backslash \{c\}]$, y is not adjacent to a vertex of degree one in G^*. Then $G[V \backslash \{x\}]$ is almost claw-free. Hence, G is not isomorphic to F_k.

Suppose G is isomorphic to $F_{k,\ell}$ for some $k, \ell \geq 1$. By the same arguments as in the previous case, x neither has degree one in G^* nor is a neighbour of r. Suppose $x = r$. Then y is not adjacent to a vertex of degree one in G^*. Denote the two components of $G^*[N_{G^*}(x)]$ by A and B. If y is adjacent to all vertices in $V_A \cup V_B$, then G is isomorphic to $F'_{k,\ell}$. This is not possible. Suppose y is adjacent to no vertex of one set in $\{V_A, V_B\}$, say V_A. Then $G[N \backslash \{x\}]$ is almost claw-free, which is not possible. Hence, we may without loss of generality assume that y is adjacent to $z_1 \in V_A$ and to $z_2 \in V_A$ while y is not adjacent to $z_3 \in V_A$. Let z'_1 denote the neighbour of z_1 in G^* that has degree one in G^*. As z'_1 is not adjacent to c or y in G either, we obtain $z'_1 \in V_1$. However, then y and z_1 are adjacent centres of induced claws with edges yc, yz_1, yz_2 and $z_1y, z_1z'_1, z_1z_2$. This violates condition 1 of Definition 1. Hence, G^* is not isomorphic to $F_{k,\ell}$.

Suppose G^* is isomorphic to $F'_{k,\ell}$ for some $k, \ell \geq 0$. Let s be the (unique) vertex in G^* that is adjacent to r and all vertices in $N_{G^*}(r) \backslash \{s\}$. Let s' be the degree one neighbour of s in G^*. By exactly the same arguments as in the previous cases, x neither has degree one in G^* nor is in $N_{G^*}(r)$ (so $x \neq s$ is not possible either). Suppose $x = r$. Then $ys \notin E$ as otherwise $G[V \backslash \{x\}]$ is almost claw-free. Let A and B denote the components of $G^*[N_{G^*}(x) \backslash \{s\}]$. As $G[V \backslash \{x\}]$ is not almost claw-free, y is adjacent to a neighbour $v \in N(x) \backslash \{s\}$, say $v \in V_A$. Let v' be the degree one neighbour of v in G^*. Let $w \in V_B$. Then v and s are adjacent centres of induced claws with edges vv', vy, vs and sv, sw, ss'. This violates condition 1 of Definition 1. Hence x is not the root of G^*. So, we have shown that $G^* \notin \mathcal{F}$.

We now show that G^* is almost claw-free. If it is not, then G^* contains a vertex t such that $G^*[N_{G^*}(t)]$ does not contain a dominating pair. Since G is almost claw-free, y is then in any dominating pair of $G[N(t)]$. Let $\{y, u\}$ be a dominating pair of $G[N(t)]$. Since x is adjacent to all vertices in $N(y)\backslash\{c\}$, we may replace y by x in $\{y, u\}$. We then find a dominating pair $\{x, u\}$ (or dominating vertex x if $x = u$) of $G^*[N_{G^*}(t)]$. Hence, G^* is almost claw-free.

Finally, we show that $G^*[V_{G^*}\backslash\{x\}] = G[V\backslash\{c, x, y\}]$ is almost claw-free. If it is not, then, by Lemma 1 (i), G^* contains a vertex y^* such that $G^*[N_{G^*}(y^*)\backslash\{x\}]$ does not have a dominating pair. By Lemma 1 (ii), y^* is adjacent to x, and by Lemma 1 (iii), y^* is the centre of an induced claw in G^*, and consequently in G. Since $y^* \notin Y$, we obtain $yy^* \in E$. Then G contains two adjacent centres of induced claws (namely y and y^*). This violates condition 1 of Definition 1. Hence, $G^*[V_{G^*}\backslash\{x\}]$ is indeed almost claw-free. This completes the proof of Lemma 6.

□

5 Conclusions

We completely characterised the class of connected almost claw-free graphs that have a $P_7, \{P_2\}$-factor. Using this characterisation we were able to classify all KO-reducible almost claw-free graphs, and we could show that every reducible almost claw-free graph is reducible in at most two rounds. This lead to a quadratic time algorithm for determining if an almost claw-free graph is KO-reducible.

The following open questions are interesting. Can we characterise all (almost) claw-free graphs that have a $P_{2k+1}, \{P_2\}$-factor for $k \geq 4$? Let $K_{1,r}$ denote the *star* on $r+1$ vertices, that is, the complete bipartite graph with partition classes X and Y with $|X| = 1$ and $|Y| = r$. Can we characterise all KO-reducible $K_{1,r}$-free graphs for $r \geq 4$? This already seems to be a difficult question for $r = 4$, since there exist $K_{1,4}$-free graphs with parallel knock-out number equal to three. In contrast with Corollary 2, there are 2-connected $K_{1,4}$-free graphs that are not reducible; for example the graph obtained from K_4 by subdividing each edge with a single vertex. Hence, the family of forbidden subgraphs seems considerably more difficult to characterise.

References

1. K. Ando, Y. Egawa, A. Kaneko, K. Kawarabayashi, and H. Matsuda (2002). Path factors in claw-free graphs, *Discrete Mathematics 243*, 195-200.
2. N. Blum (1990). A new approach to maximum matching in general graphs, *Proceedings of the 17th International Colloquium on Automata, Languages and Programming (ICALP 1990)*, Lecture Notes in Computer Science 443, 586–597.
3. J.A. Bondy and U.S.R. Murty (1976). *Graph Theory with Applications.* Macmillan, London and Elsevier, New York.
4. H.J. Broersma, F.V. Fomin, R. Královič, and G.J. Woeginger (2007). Eliminating graphs by means of parallel knock-out schemes, *Discrete Applied Mathematics 155*, 92–102.
5. H.J. Broersma, M. Johnson, D. Paulusma, and I.A. Stewart (2008). The computational complexity of the parallel knock-out problem, *Theoretical Computer Science 393*, 182–195.
6. H.J. Broersma, M. Johnson, and D. Paulusma (2008). Upper bounds and algorithms for parallel knock-out numbers, to appear in *Theoretical Computer Science.*

7. H.J. Broersma, Z. Ryjáček, and I. Schiermeyer (1996). Toughness and hamiltonicity in almost claw-free graphs, *Journal of Graph Theory 21,* 431–439.
8. O. Favaron, E. Flandrin, H. Li, and Z. Ryjáček (1996). Shortest walks in almost claw-free graphs, *Ars Combinatoria 42*, 223–232.
9. T. Feder and R. Motwani (1995). Clique partitions, graph compression and speeding-up algorithms, *Journal of Computer and System Sciences 51*, 261–272.
10. R. Faudree, E. Flandrin, and Z. Ryjáček (1997). Claw-free graphs-a survey, *Discrete Mathematics 164*, 87–147.
11. P. Hell and D.G. Kirkpatrick (1983). On the complexity of general graph factor problems, *SIAM Journal on Computing 12*, 601-609.
12. J. Hopcroft and R.M. Karp (1973). An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs, *SIAM Journal on Computing 2*, 225–231.
13. O.H. Ibarra and S. Moran (1981). Deterministic and probabilistic algorithms for maximum bipartite matching via fast matrix multiplication, *Information Processing Letters 13*, 12–15.
14. S. Jünger, W.R. Pulleyblank, and G. Reinelt (1985). On partitioning the edges of graphs into connected subgraphs, *Journal of Graph Theory 9*, 539–549.
15. A. Kaneko (2003). A necessary and sufficient condition for the existence of a path factor every component of which is a path of length at least two, *Journal of Combinatorial Theory, Series B 88*, 195-218.
16. M. Kano, G.Y. Katona, and Z. Király (2004). Packing paths of length at least two, *Discrete Mathematics 283*, 129-135.
17. D.E. Lampert and P.J. Slater (1998). Parallel knockouts in the complete graph, *American Mathematical Monthly 105*, 556–558.
18. M. Las Vergnas (1975). A note on matchings in graphs, *Cahiers Centre Études Recherche Oprationelle 17*, 257-260.
19. M. Li (2001). Hamiltonian Cycles in Almost Claw-Free Graphs, *Graphs and Combinatorics 17*, 687–706.
20. X. Li and Z. Zhang (2008). Path factors in the square of a tree, *Graphs and Combinatorics 24*, 107–111.
21. M.D. Plummer (2007). Graph factors and factorization: 1985-2003: A survey, *Discrete Mathematics 307*, 791–821.
22. Z. Ryjáček (1994). Almost claw-free graphs, *Journal of Graph Theory 18*, 469–477.
23. D.P. Sumner (1976). 1-factors and antifactor sets, *Journal of the London Mathematical Society 13*, 351-359.
24. H. Wang (1994). Path factors of bipartite graphs, *Journal of Graph Theory 18*, 161-167.
25. M. Zhan (2002). Neighborhood intersections and Hamiltonicity in almost claw-free graphs, *Discrete Mathematics 243*, 171–185.