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AN ANALOGUE OF THE FIELD-OF-NORMS FUNCTOR AND OF THE GROTHENDIECK CONJECTURE

VICTOR ABRASHKIN

Abstract

The paper contains a construction of an analogue of the Fontaine-Wintenberger field-of-norms functor for higher-dimensional local fields. This construction is done completely in terms of the ramification theory of such fields. It is applied to deduce the mixed characteristic case of a local analogue of the Grothendieck conjecture for these fields from its characteristic p case, which was proved earlier by the author.

0. Introduction

Throughout this paper, p is a fixed prime number.

The field-of-norms functor [FW1], [FW2] allows us to identify the Galois groups of some infinite extensions of $\mathbb{Q}_p$ with those of complete discrete valuation fields of characteristic p. This functor is an essential component of Fontaine's theory of φ-Γ-modules—one of most powerful tools in the modern study of p-adic representations. Other areas of very impressive applications are the Galois cohomology of local fields [He], arithmetic aspects of dynamical systems [LMS], explicit reciprocity formulae [Ab2], [Ab3], [Ben], a description of the structure of ramification filtration [Ab7], the proof of an analogue of the Grothendieck conjecture for 1-dimensional local fields [Ab4].

A local analogue of the Grothendieck conjecture establishes an opportunity to recover the structure of a local field from the structure of its absolute Galois group provided with the filtration by ramification subgroups. The study of this situation in the context of higher-dimensional local fields became actual due to a recent development of the ramification theory for such fields [Zh2], [Ab5]. The case of fields, of characteristic $p>2$, has been already considered in [Ab6]. (Notice that the restriction to 2-dimensional fields is not essential in [Ab6]—the method works for any dimension $N\geqslant 2$.) This could lead to the

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proof of the mixed characteristic case of the Grothendieck conjecture if there were a suitable analogue of the field-of-norms functor for higher-dimensional local fields.

The construction of such a functor is suggested in the present paper. In our setting, we replace the appropriate category of infinite arithmetically profinite extensions of $\mathbb{Q}_p$ from [FW1], [FW2] by the category $\mathcal{B}^a(N)$ of infinite increasing field towers $K_0 \subset K_1 \subset \dots \subset K_n \subset \dots$ with restrictions on the upper ramification numbers of the intermediate extensions K_{n+1}/K_n for $n \gg 0$. In order to introduce the set of elements of the corresponding field-of-norms, one cannot use the sequences of norm compatible elements in such towers, but it is still possible to work with the sequences of elements $a_n \in \mathcal{O}_{K_n}$ such that $a_n \equiv a_{n+1}^p \bmod p^c$, where $0 < c \leqslant 1$ is independent on n.

The main difficulty in the realization of this idea comes from the fact that the construction of ramification theory for an N-dimensional local field L depends on the choice of its F-structure, i.e. on the choice of the subfields $L(i)$ of i-dimensional constants, where $1 \leqslant i \leqslant N$. On the other hand, in order to be able to work with elements of L, one should use one or another choice of its local parameters. This choice can be made compatible with a given F-structure only after passing to some finite "semistable" extension of L. This explains why we have a precise analogue of the Fontaine-Wintenberger functor only for a subcategory of "special" towers $\mathcal{B}^{fa}(N)$ in $\mathcal{B}^a(N)$. Nevertheless, the construction of our functor can be extended to the whole category $\mathcal{B}^a(N)$ and can be applied to deduce the mixed characteristic case of the Grothendieck conjecture from its characteristic $p > 2$ case. Notice that another approach to the problem of generalisation of the field-of-norms functor can be found in the papers [And] and [Sch].

We now briefly explain the content of this article.

Section 1 contains preliminaries: definitions and simplest properties of N-dimensional local fields L. We pay special attention to the concept of the P-topology — this is a topology on L, which accumulates properties of N valuation topologies which can be attached to L. Then the Witt-Artin-Schreier duality and the Kummer theory allow us to transfer the P-topological structure to the group $\Gamma_L^{\mathrm{ab}}(p)$, where $\Gamma_L(p)$ is the Galois group of the maximal p-extension of L. This structure gives an opportunity to work with $\Gamma_L(p)$ in terms of generators (cf. [Ab6]).

Section 2 contains a "co-analogue" of Epp's elimination wild ramification. This statement deals with a subfield of $(N-1)$-dimensional constants in an N-dimensional local field. (The most widely known interpretation of Epp's procedure deals with a subfield of 1-dimensional constants.) Our proof establishes an elimination procedure which is similar to the procedure developed in [ZhK], where it was shown that an essential part of such elimination can be done inside a given deeply ramified extension in the sense of [CG]. This elimination procedure is required to justify the main starting point in the construction of the ramification theory for higher-dimensional local fields from [Ab5]. (The original arguments from [Ab5] were not complete (cf. remark in section 2.1).)

Section 3 contains a brief introduction into the ramification theory and contains a version of Krasner's Lemma in the context of higher-dimensional local fields. In Section 4 we introduce and study the categories of special towers $\mathcal{B}^a(N)$ and $\mathcal{B}^{fa}(N)$. These towers play a role of strict arithmetic profinite extensions from the Fontaine-Wintenberger construction of the field-of-norms functor.

In section 5 we explain the construction of the family $X(K_{\bullet})$ of local fields of characteristic p, where $K_{\bullet} \in \mathcal{B}^{fa}(N)$. We prove that all such fields can be identified after (roughly speaking) taking inseparable extensions of constant subfields of lower dimension. These fields will play the role of the field-of-norms attached to a given tower $K_{\bullet} \in \mathcal{B}^{fa}(N)$. In section 6 we apply Krasner's Lemma from section 3 to establish all expected properties of the correspondence $K_{\bullet} \mapsto \mathcal{K} \in X(K_{\bullet})$, where $K_{\bullet} \in \mathcal{B}^{fa}(N)$. In section 7 we use these properties to define the analogue $\mathcal{X}_{K_{\bullet}}$, $K_{\bullet} \in \mathcal{B}^{fa}(N)$, of the field-of-norms functor. In addition, we use the operation of the radical closure to extend this construction to the whole category $\mathcal{B}^a(N)$. In section 8 it is proved that the corresponding identification of the Galois groups $\Gamma_{\widetilde{K}}$ (where $\widetilde{K}$ is the p-adic closure of the composite of all fields from the tower $K_{\bullet}$) and $\Gamma_{\mathcal{K}}$ becomes P-continuous when being restricted to their maximal abelian p-quotients. The proof is based on a higher-dimensional version of the relation between the Witt-Artin-Schreier theory for $\mathcal{K}$ and the Kummer theory for $\widetilde{K}$ from [Ab2]. This relation and the proof of compatibility of the proposed field-of-norms functor with the class field theories for $\mathcal{K}$ and $\widetilde{K}$, leads to another proof of the explicit reciprocity formula from [Vo] (cf. also [Ka]) — the details will appear later elsewhere.

Finally, the P-continuity result from section 8 allows us to prove in section 9 the mixed characteristic case of the Grothendieck conjecture under the restriction $p > 2$. Notice that the construction of the higher-dimensional version of the field-of-norms functor from this paper is especially adjusted to the proof of this conjecture and was motivated by Deligne's paper [De]. It should also be mentioned that there are definite ideological links with methods of the paper [Fu], where the construction of Coleman power series was developed in the context of 2-dimensional local fields with further applications to the construction of p-adic L-functions.

1. Preliminaries

1.1. The concept of higher-dimensional local field. Let K be an N-dimensional local field, where $N \in \mathbb{Z}_{\geqslant 0}$. In other words, if $N = 0$, then K is a finite field and for $N \geqslant 1$, K is a complete discrete valuation field with the residue field $K^{(1)}$, which is an $(N-1)$-dimensional local field. We use the notation $K^{(N)}$ for the last residue field of K. (This field is 0-dimensional by its definition and, therefore, is finite.)

Let $O_K^{(1)}$ be the valuation ring of K with respect to first valuation and let $\alpha : O_K^{(1)} \longrightarrow K^{(1)}$ be a natural projection. Define the valuation ring $\mathcal{O}_K$ of K by setting for $N = 0$, $\mathcal{O}_K = K$ and for $N \geqslant 1$, $\mathcal{O}_K = \alpha^{-1}(\mathcal{O}_{K^{(1)}})$. Recall that a system $t_1, \ldots, t_N \in \mathcal{O}_K$ is a system of local parameters in K if t_1 is a uniformiser in $O_K^{(1)}$ and $\alpha(t_2), \ldots, \alpha(t_N)$ is a system of local parameters in $K^{(1)}$.

In terms of such a system of local parameters, any element $\xi \in K$ can be uniquely presented as a power series of the following form:

$$\xi = \sum_{\bar{a}=(a_1,\ldots,a_N)} \alpha_{\bar{a}} t_1^{a_1} \ldots t_N^{a_N}.$$

Here, all coefficients $\alpha_{\bar{a}}$ are either elements of $K^{(N)}$ if $\operatorname{char} K = p > 0$, or the Teichmüller representatives of those if $\operatorname{char} K = 0$. All indices $a_i \in \mathbb{Z}$ and there are integers (which depend on ξ) A_1, $A_2(a_1), \ldots, A_N(a_1, \ldots, a_{N-1})$ such that $\alpha_{\bar{a}} = 0$ if either $a_1 < A_1$, or $a_2 < A_2(a_1), \ldots$, or $a_N < A_N(a_1, \ldots, a_{N-1})$.

There is an important concept of the P-topology on K which brings into correlation all N valuation topologies related to K. The P-topological structure provides us with a reasonable treatment of morphisms of higher-dimensional local fields. We discuss this structure briefly in section 1.2 below. Notice that if $f : K \longrightarrow L$ is a sequentially P-continuous morphism of higher-dimensional local fields, then $E = f(K)$ is a closed subfield in L (i.e. $O_E^{(1)}$ is closed in $O_L^{(1)}$ with respect to first valuation and $E^{(1)}$ is closed in $L^{(1)}$), for any system $t_1, \ldots, t_N$ of local parameters in K, their images $f(t_1), \ldots, f(t_N)$ are local parameters in E and their knowledge determines the morphism f uniquely.

Our considerations will be limited with local fields K such that $\operatorname{char} K^{(1)} = p$ where p is a fixed prime number (such fields possess the most interesting arithmetic structure). Under this assumption there is the following classification of N-dimensional local fields:

— If $\operatorname{char} K = p$, then $K = k((t_N)) \ldots ((t_1))$ where $k = K^{(N)}$ is the last residue field of K. As a matter of fact, this result is equivalent to the existence of a system of local parameters $t_1, \ldots, t_N$ in K.

— If $\operatorname{char} K = 0$, then $K \supset \mathbb{Q}_p$ and we can introduce a canonical subfield $K(1)$ of 1-dimensional constants in K; this is the algebraic closure of $\mathbb{Q}_p$ in K. Suppose a uniformising element t_1 of $K(1)$ can be included in a system of local parameters $t_1, t_2, \dots, t_N$ of K. Then $K = K(1)\{\{t_N\}\} \dots \{\{t_2\}\}$ and such K is called standard. Otherwise, there is a finite extension E of $K(1)$ such that the composite KE is standard.

The above result concerning the characteristic 0 fields is implied by the following version of Epp's theorem [Epp], which holds for all (not necessarily characteristic 0) higher-dimensional local fields K:

— *Suppose K is an N-dimensional field and $K(1)$ is its subfield of 1-dimensional constants; then there is a finite extension E of $K(1)$ such that the fields KE and E have a common uniformising element (with respect to the first valuation in K).*

1.2. Concept of P-topology. Let K be an N-dimensional local field. Its P-topology can be described explicitly by induction on N in terms of any chosen system $t_1, \dots, t_N$ of local parameters of K by constructing a basis of open 0-neighborhoods $\mathcal{U}_b(K)$ (cf. [Zh1]). We shall consider the following three cases:

(a) $\operatorname{char} K = p$;

(b) $\operatorname{char} K = 0$, $\operatorname{char} K^{(1)} = p$ and t_1 is a local parameter in $K(1)$;

(c) K is a finite extension of a field K_0, which satisfies the above assumptions from (b).

The case (a).

Here $K = k((t_N)) \dots ((t_1))$, where k is a finite field of characteristic p. If $N = 0$, then $\mathcal{U}_b(K)$ contains by definition only one set $\{0\}$. Then the family of all open sets in K consists of all subsets of K. Suppose $N \geqslant 1$. Let $\bar{t}_N, \dots, \bar{t}_2$ be the images of $t_N, \dots, t_2$ in $K^{(1)}$. Then $K^{(1)} = k((\bar{t}_N)) \dots ((\bar{t}_2))$ and we can use the correspondences $\bar{t}_N \mapsto t_N, \dots, \bar{t}_2 \mapsto t_2$ and $\alpha \mapsto \alpha$ for $\alpha \in k$, to define the embedding $h : K^{(1)} \longrightarrow K$. Then $\mathcal{U}_b(K)$ consists of the sets $\sum_{a \in \mathbb{Z}} t_1^a h(U_a)$, where either $U_a \in \mathcal{U}_b(K^{(1)})$ or $U_a = K^{(1)}$ for $a \gg 0$.

The case (b).

Here again the images $\bar{t}_2, \dots, \bar{t}_N$ give a system of local parameters of $K^{(1)}$ and the family of all open subsets of $K^{(1)}$ is already defined by induction. So, we again use the map $h : K^{(1)} \longrightarrow K$, which is determined by the correspondences $\bar{t}_i \mapsto t_i$, $i = 2, \dots, N$, and $\alpha \mapsto \alpha$ for $\alpha \in k$, and proceed along the lines in case (a).

The case (c).

If $[K : K_0] = n$, then the P-topological structure on K comes from any isomorphism of K_0-vector spaces $K \simeq K_0^n$ and the P-topological structure on K_0.

It is well known that K is an additive P-topological group, but the multiplication in K has very bad P-topological properties. Fortunately, the multiplicative structure on K is sequentially P-continuous. We will need this property later when studying the P-continuity of maps between objects obtained from K-spaces by duality. For this reason we shall use the following description of sequentially compact subsets in K. Introduce a basis $\mathcal{C}_b(K)$ of sequentially compact subsets in K. In other words, if $\mathcal{C}_b(K)$ is such a family, then any sequentially compact subset D in K will appear as a closed subset of some $C \in \mathcal{C}_b(K)$. Proceed again by induction on the dimension N of K according to the above assumptions (a)–(c) about K.

In case (a), $\mathcal{C}_b(K)$ will consist of only one set $\{K\}$ if $N = 0$. If $N \geqslant 1$, then in cases (a) and (b) we can use the map $h : K^{(1)} \longrightarrow K$ to define $\mathcal{C}_b(K)$ as the family of subsets $\sum_{a\in\mathbb{Z}} t_1^a h(C_a)$, where $C_a \in \mathcal{C}_b(K^{(1)})$ and $C_a = \{0\}$ for $a \ll 0$. In case (c), we just set $\mathcal{C}_b(K) = \{C^n \mid C \in \mathcal{C}_b(K_0)\}$.

Proposition 1.1. *The above-defined family $\mathcal{C}_b(K)$ is a basis of sequentially compact subsets in K.*

Proof. Proceed by induction on N when K satisfies the assumptions from cases (a) and (b). The case $N = 0$ is clear.

Let $N \geqslant 1$. Prove first that $\mathcal{C}_b(K)$ consists of sequentially compact subsets in K. Suppose $C = \sum t_1^a h(C_a) \in \mathcal{C}_b(K)$. Notice first, that each $h(C_a)$ is sequentially P-compact in K, because h is P-continuous. For any $b \in \mathbb{Z}$, set $C_{\leqslant b} = \sum_{a\leqslant b} t_1^a h(C_a)$. Then $C_{\leqslant b}$ is P-homeomorphic to the product of finitely many sequentially compact sets $h(C_a)$, $a \leqslant b$. Therefore, $C_{\leqslant b}$ is sequentially P-compact. Finally,

$$C = \varprojlim_b C_{\leqslant b}$$

as P-topological sets. So, C is sequentially compact.

Suppose D is a sequentially P-compact subset in K. Take $a_0 \in \mathbb{Z}$ such that $D \subset \sum_{a\geqslant a_0} t_1^a h(K^{(1)})$ (a_0 exists because D is sequentially compact). From the definition of the P-topology, it follows that all projections $\mathrm{pr}_a : D \longrightarrow K^{(1)}$ (where, for any $d \in D$, $d = \sum t_1^a h(\mathrm{pr}_a(d))$) are P-continuous maps. Therefore, all $\mathrm{pr}_a(D)$ are sequentially compact subsets in $K^{(1)}$. By induction there are $C_a \in \mathcal{C}_b(K^{(1)})$ such that $\mathrm{pr}_a(D)$ are closed subsets in C_a. Notice that we can assume $C_a = \{0\}$ for $a \ll 0$. So, D is a subset in the P-compact set $\sum t_1^a h(C_a) \in \mathcal{C}_b(K)$.

Finally, the case (c) follows from the definition of the P-topology as the product topology associated with the P-topology on K_0. The proposition is proved. □

2. Higher-dimensional elimination of wild ramification

2.1. Introduce the category LC of higher-dimensional local fields with a given subfield of constants of codimension 1. The objects in LC are couples (K, E) where K is a local field of dimension $N \geqslant 1$ and E is a topologically closed subfield of dimension $N-1$ which is algebraically closed in K. If $N = 1$ and $\operatorname{char} K = 0$, we shall agree by definition to take as E the maximal unramified extension of $\mathbb{Q}_p$ in K, i.e. in this case a 1-dimensional field will play a role of a subfield of 0-dimensional constants. Morphisms $(K, E) \longrightarrow (K', E')$ in the category LC are given by sequentially P-continuous morphisms of local fields $f : K \longrightarrow K'$ such that $f(E) \subset E'$.

We shall use the notation LC(N) for the full subcategory in LC consisting of (K, E), where K is an N-dimensional field. Notice that LC(1) is equivalent to the usual category of complete discrete valuation fields with finite residue field of characteristic p.

Remark. Suppose $(K, E) \in$ LC. Then there is a natural embedding of the first residue fields $E^{(1)} \subset K^{(1)}$ but $(K^{(1)}, E^{(1)})$ is not generally an object of the category LC$(N-1)$, because $E^{(1)}$ is not generally algebraically closed in $K^{(1)}$. Notice that it is separably closed in $K^{(1)}$; otherwise, E will possess a non-trivial unramified extension in K.

Definition. $(K, E) \in$ LC(N) is standard if there is a system of local parameters $t_1, \dots, t_N$ in K such that $t_1, \dots, t_{N-1}$ is a system of local parameters in E. In other words, if (K, E) is standard, then there is a $t_N \in K$ which extends any system of local parameters in E to a system of local parameters in K. Such an element t_N of K will be called an Nth local parameter in K (with respect to a given subfield of $(N-1)$-dimensional constants E).

One of reasons to introduce the concept of a standard object is that the situation from the above remark will never take place if $(K, E) \in$ LC(N) is standard. In other words, $E^{(1)}$ is algebraically closed in $K^{(1)}$ if (K, E) is standard.

We mention the following simple properties:

(a) For any $(K, E) \in$ LC, there is always a closed subfield K_0 in K containing E such that $(K_0, E) \in$ LC is standard; this field K_0 appears in the form $E\{\{t\}\}$ with a suitably chosen element t of $\mathcal{O}_K$.

(b) If $(\widetilde{K}, E) \in$ LC(N) is standard and K is a closed subfield in $\widetilde{K}$ such that $K \supset E$ and $(K, E) \in$ LC(N), then (K, E) is also standard. (One can see easily, that $[\widetilde{K} : K] < \infty$ and if $\tilde{t}_N$ is an Nth local parameter for $\widetilde{K}$, then $N_{\widetilde{K}/K}\tilde{t}_N$ is an Nth local parameter for K.)

(c) If $(K, E) \in \mathrm{LC}$ is standard, then for any finite extension E' of E, $(KE', E') \in \mathrm{LC}$ is standard. (Any Nth local parameter in K is still an Nth local parameter in KE'.)

(d) Any $(K, E) \in \mathrm{LC}(1)$ is standard.

(e) For any $(K, E) \in \mathrm{LC}(2)$, there is a finite extension E' of E such that $(KE', E') \in \mathrm{LC}(2)$ is standard. (This follows from Epp's Theorem.)

The following property plays a very important role in the construction of ramification theory for higher-dimensional fields.

Proposition 2.1. *Suppose* $(K, E), (L, E) \in \mathrm{LC}(N)$, $L \supset K$ *and* (L, E) *is standard. Then* $\mathcal{O}_L = \mathcal{O}_K[t_N]$, *where* t_N *is an* N*th local parameter in* L.

Proof. Clearly, $\mathcal{O}_K[t_N] \subset \mathcal{O}_L$.

Let $t_1, \dots, t_{N-1}$ be local parameters in E. It will be sufficient to prove that

$$t_1^{a_1} \dots t_{N-1}^{a_{N-1}} t_N^{a_N} \in \mathcal{O}_K[t_N]$$

if $(a_1, \dots, a_{N-1}, a_N) \geqslant \bar{0}_N$.

We can assume that $a_N < 0$ (otherwise, there is nothing to prove).

Notice that $\tilde{t}_N = N_{L/K} t_N$ is an Nth local parameter for K and $\tilde{t}_N t_N^{-1} \in \mathcal{O}_K[t_N]$. Therefore,

$$t_1^{a_1} \dots t_{N-1}^{a_{N-1}} t_N^{a_N} = t_1^{a_1} \dots t_{N-1}^{a_{N-1}} \tilde{t}_N^{a_N} (\tilde{t}_N t_N^{-1})^{-a_N} \in \mathcal{O}_K[t_N]$$

because $t_1^{a_1} \dots t_{N-1}^{a_{N-1}} \tilde{t}_N^{a_N} \in \mathcal{O}_K$. The proposition is proved. □

Definition. If K is an N-dimensional local field, then $F(T) = T^n + a_1 T^{n-1} + \dots + a_n \in \mathcal{O}_K[T]$ is an Nth Eisenstein polynomial if $a_1, \dots, a_n$ belong to the maximal ideal m_K of $\mathcal{O}_K$ and a_n can be taken as Nth local parameter in K. Equivalently, the image of $F(T)$ in $K^{(N-1)}[T]$, where $K^{(N-1)}$ is the prelast residue field of K, is a usual Eisenstein polynomial.

Notice the following simple properties:

(1) If $(K, E), (L, E) \in \mathrm{LC}(N)$, $L \supset K$ and (L, E) is standard, then $L = K(\theta)$, where θ is a root of Nth Eisenstein polynomial.

(2) If $(K, E) \in \mathrm{LC}(N)$ is standard and $L = K(\theta)$, where θ is a root of an Nth Eisenstein polynomial from $\mathcal{O}_K[T]$, then $(L, E) \in \mathrm{LC}(N)$ is standard.

(3) In both of the above situations (1) and (2), the element θ can be taken as Nth local parameter in L.

2.2. The following theorem, in our setting, plays a role of a higher-dimensional version of Epp's Theorem.

Theorem 1. *If* $(K, E) \in \mathrm{LC}(N)$, *then there is a finite separable extension* E' *of* E *such that* $(KE', E') \in \mathrm{LC}(N)$ *is standard.*

Proof of Theorem 1. Use induction on N. □

2.3. If $N = 1$, there is nothing to prove. Notice that the case $N = 2$ follows from Epp's Theorem.

Suppose $N > 1$ and the theorem holds for all local fields of dimension $< N$.

Choose a standard $(K_0, E) \in \mathrm{LC}(N)$ such that $K_0 \subset K$, and denote by $t_1, \dots, t_N$ a system of local parameters in K_0 such that the first $N-1$ of them give a system of local parameters in E. It will be sufficient to prove our theorem for extensions K/K_0 satisfying one of the following conditions (because any finite extension of K_0 can be embedded into a bigger extension obtained as a sequence of such subextensions):

(a_0) There is a finite extension $\widetilde{E}$ of E such that $\widetilde{K} := K\widetilde{E}$ is unramified over $\widetilde{K}_0 := K_0\widetilde{E}$, i.e. such that both fields $\widetilde{K}$ and $\widetilde{K}_0$ have the same first uniformiser and $\widetilde{K}^{(1)}$ is separable over $\widetilde{K}_0^{(1)}$.

(a_1) K/K_0 is a cyclic extension of a prime to p degree m.

(b) K/K_0 is a cyclic extension of degree p such that after arbitrary finite extension of E, the corresponding extension of first residue fields is either trivial or purely inseparable. When considering this case below, we shall treat the subcases separately:

(b_1) $\mathrm{char}\, K = 0$;
(b_2) $\mathrm{char}\, K = p$;
(c) K/K_0 is a purely non-separable extension of degree p.

Following the terminology from [Zh2] we can call (K, E) an almost constant extension of (K_0, E) in the case (a_0) and an infernal elementary extension in the case (b).

2.4. The case (a_0)**.** This case follows from the following observation. Consider the natural field embedding $\widetilde{E}^{(1)} \subset \widetilde{K}_0^{(1)}$. Clearly, $(\widetilde{K}_0^{(1)}, \widetilde{E}^{(1)}) \in \mathrm{LC}(N-1)$ is standard. On the other hand, $\widetilde{E}^{(1)}$ is separably closed in $\widetilde{K}^{(1)}$ (otherwise, $\widetilde{E}$ will have a non-trivial unramified extension in $\widetilde{K}$). This implies that any finite extension E' of $\widetilde{E}^{(1)}$ in $\widetilde{K}^{(1)}$ is either purely inseparable or trivial. Therefore, $E' \subset \widetilde{K}_0^{(1)}$ (because $\widetilde{K}^{(1)}/\widetilde{K}_0^{(1)}$ is separable) and $E' = \widetilde{E}^{(1)}$ (because $\widetilde{E}^{(1)}$ is algebraically closed in $\widetilde{K}_0^{(1)}$). So, $(\widetilde{K}^{(1)}, \widetilde{E}^{(1)}) \in \mathrm{LC}(N-1)$. Therefore, by the inductive assumption there is a finite separable extension E_1 of $\widetilde{E}^{(1)}$ such that (K_1, E_1) is standard, where $K_1 = \widetilde{K}^{(1)}E_1$. Denote by $\bar{t}_2, \dots, \bar{t}_N$ a system of local parameters in K_1 such that $\bar{t}_2, \dots, \bar{t}_{N-1}$ is a system of local parameters of E_1. Let E' be an unramified extension of $\widetilde{E}$ such that ${E'}^{(1)} = E_1$. Notice that if $K' = KE'$, then ${K'}^{(1)} = K_1$. Let $t_2, \dots, t_{N-1}$ be liftings of $\bar{t}_2, \dots, \bar{t}_{N-1}$ to $O_{E'}^{(1)}$ and let t_N be a lifting of $\bar{t}_N$ to $O_{K'}^{(1)}$. If t_1 is a common first uniformiser of $\widetilde{K}$ and $\widetilde{E}$, then $t_1, \dots, t_N$ is a system of local parameters in K' and $t_1, \dots, t_{N-1}$ is a system of local parameters in E'. In other words, $(K', E') \in \mathrm{LC}(N)$ is standard.

2.5. The case (a_1)**.** Via the above case a_0) we can assume that the last residue field of E is large enough. This implies that $K = K_0(\sqrt[m]{t_1^{a_1}\dots t_N^{a_N}})$, where $a_1,\dots,a_N \in \mathbb{Z}_{\geqslant 0}$. Let $E' = E(\sqrt[m]{t_1},\dots,\sqrt[m]{t_{N-1}})$, then E' has local parameters $\sqrt[m]{t_1},\dots,\sqrt[m]{t_{N-1}}$ and this system can be extended to a system of local parameters in $K' = KE'$ by adding $\sqrt[m']{t_N}$, where $m' = m/\gcd(m, a_N)$. So, (K', E') is standard.

2.6. Special extensions. For our future targets we need to keep control on the choice of the extension E' of E in Theorem 1. This idea goes back to the paper [ZhK] where it was proved that Epp's elimination of wild ramification for an infernal extension can be done by the use of subextensions of a given deeply ramified extension.

Set $N' = N-1$ and consider an increasing sequence of finite extensions of N'-dimensional local fields

$$E \subset \widetilde{E}_1 \subset E_1 \subset \widetilde{E}_2 \subset E_2 \subset \dots \subset \widetilde{E}_n \subset E_n \subset \dots$$

such that each $\widetilde{E}_n$ and E_n have a system of local parameters $\tilde{t}_{1n},\dots,\tilde{t}_{N'n}$ and, respectively, $t_{1n},\dots,t_{N'n}$, satisfying the following condition:

Condition C. *There is a $c > 0$ such that for all $1 \leqslant i \leqslant N'$ and $n \geqslant 1$,*

$$v^1\left(\frac{t_{in}^p}{\tilde{t}_{in}} - 1\right) \geqslant c,$$

where v^1 is a t_1-adic (1-dimensional) valuation on $\bar{E}$ normalised by the condition $v^1(t_1) = 1$, where t_1 is a first local parameter in E.

Theorem 1 will be implied in cases (b) and (c) by the following statement.

Proposition 2.2. *Suppose K, K_0 and E satisfy the assumptions from cases (b) or (c). Then there is an $n^* \in \mathbb{N}$ (depending only on the extension K/K_0 and the c from the above condition* C*) such that Theorem* 1 *holds with $E' = E_{n^*}$.*

Proof of Proposition 2.2. □

2.7. The case (b_2)**.** In the case (b_2) we have $K = K_0(\theta)$, $\theta^p - \theta = \xi$, where $\xi \in K_0$. Applying the Artin-Schreier equivalence we can replace ξ by an equivalent element $\xi_E \in K_0$ such that its power series

$$\xi_E = \sum_{\bar{a}} \alpha_{\bar{a}} t_1^{a_1}\dots t_N^{a_N}$$

contains only non-zero terms with $\bar{a} \leqslant \bar{0}_N$ and $\bar{a} \not\equiv 0 \bmod p$ if $\bar{a} \neq \bar{0}_N$.

Let $\xi_E = \xi_E' + \xi_E''$, where

$$\xi_E' = \sum_{a_N=0} \alpha_{\bar{a}} t_1^{a_1}\dots t_{N-1}^{a_{N-1}}, \qquad \xi_E'' = \sum_{a_N\neq 0} \alpha_{\bar{a}} t_1^{a_1}\dots t_{N-1}^{a_{N-1}} t_N^{a_N}.$$

Let

$$A = v^1(\xi'_E) = \min\{a_1 \mid \alpha_{\bar{a}} \neq 0, a_N = 0\},$$
$$B = v^1(\xi''_E) = \min\{a_1 \mid \alpha_{\bar{a}} \neq 0, a_N \neq 0\},$$

where v^1 is the t_1-adic valuation from the above condition C.

Notice that the first set can be empty. In this case we set by definition $A = 0$. The second set is never empty; otherwise, K is a composite of an algebraic extension of E and K_0, i.e. E is not algebraically closed in K. For any $s \in \mathbb{Z}_{\geqslant 0}$, let

$$\xi''_{E,s} = \sum_{v_p(a_N)=s} \alpha_{\bar{a}} t_1^{a_1} \dots t_{N-1}^{a_{N-1}} t_N^{a_N}$$

and set $B^{(s)} = v^1(\xi''_{E,s}) = \min\{a_1 \mid \alpha_{\bar{a}} \neq 0, v_p(a_N) = s\}$. (We set $B^{(s)} = 0$ if the corresponding subset of indices is empty.) Clearly, $B = \min\{B^{(s)} \mid s \geqslant 0\}$.

Lemma 2.3. $B < 0$.

Proof. Suppose that $B = 0$. Consider the extension $L' = E(\theta')$, where $\theta'^p - \theta' = \xi'_E$; then $KL' = K_0L'(\theta'')$, where $\theta''^p - \theta'' = \xi''_E$. Clearly, the condition $B = 0$ implies that the first residue field of KL' is a separable extension of the first residue field of K_0L' of degree p. (It is generated by $\bar{\theta}_1$ such that $\bar{\theta}_1^p - \bar{\theta}_1 = \xi''_E \bmod t_1$.) Therefore, we are not in the situation of the case (b_2). The lemma is proved. □

Notice that if we pass from E to its finite extension $\widetilde{E}_1$ (cf. condition C), then $\tilde{t}_{11}, \dots, \tilde{t}_{N-1,1}, t_N$ is a system of local parameters for $K_0\widetilde{E}_1$. Rewrite ξ in terms of these local parameters and apply to this expression the Artin-Schreier equivalence to get rid of all pth powers and terms from the maximal ideal of $O_{K_0\widetilde{E}_1}$. This procedure gives an analogue $\xi_{\widetilde{E}_1}$ of ξ_E for the extension $K\widetilde{E}_1/K_0\widetilde{E}_1$. As earlier, use the t_1-adic valuation v^1 to define the analogues $\widetilde{A}_1$, $\widetilde{B}_1$, $\widetilde{B}_1^{(s)}$ of, respectively, A, B and $B^{(s)}$, $s \geqslant 0$.

Lemma 2.4. (a) $\widetilde{A}_1 \geqslant A$*;*

(b) *for all* $s \geqslant 0$, $\widetilde{B}_1^{(s)} \geqslant \min\left\{\frac{1}{p^u} B^{(s+u)} \mid u \geqslant 0\right\}$.

Proof. It is just an exercise on the Artin-Schreier equivalence. □

Apply the similar procedure to the extensions $E_1, \widetilde{E}_2, E_2, \dots$ to get the elements $\xi_{E_1}, \xi_{\widetilde{E}_2}, \xi_{E_2}, \dots$ and the corresponding invariants $A_1, B_1, B_1^{(s)}$, $\widetilde{A}_2$, $\widetilde{B}_2$, $\widetilde{B}_2^{(s)}$, A_2, B_2, $B_2^{(s)}$,....

Similarly, we have the following property.

Lemma 2.5. *For all $i \in \mathbb{N}$ and $s \geqslant 0$,*
(a) $\widetilde{A}_{i+1} \geqslant A_i$;
(b) $\widetilde{B}_{i+1}^{(s)} \geqslant \min\left\{ \frac{1}{p^u} B_i^{(s+u)} \mid u \geqslant 0 \right\}$.

When passing through the special extensions $E_i/\widetilde{E}_i$, $i \geqslant 1$, we have the following better estimates.

Lemma 2.6. *For all $i \geqslant 1$ and $s \geqslant 1$,*
(a) $A_i \geqslant \min\left\{ \frac{1}{p}\widetilde{A}_i, \widetilde{A}_i + c \right\}$;
(b) $B_i^{(0)} \geqslant \min\left\{ \widetilde{B}_i^{(0)}; \frac{1}{p}\widetilde{B}_i^{(1)}; \frac{1}{p^u}(\widetilde{B}_i^{(u)} + c), u \geqslant 0 \right\}$;
(c) $B_i^{(s)} \geqslant \min\left\{ \frac{1}{p}\widetilde{B}_i^{(s+1)}; \frac{1}{p^u}\left(\widetilde{B}_i^{(s+u)} + c\right), u \geqslant 0 \right\}$.

The above estimates easily imply the following lemma.

Lemma 2.7. (a) $\lim_{i\to\infty} A_i = 0$;
(b) *if $i \in \mathbb{N}$ and $\gamma_i = \min\{B_i^{(s)} \mid s \geqslant 1\}$, then $\lim_{i\to\infty} \gamma_i = 0$.*

In order to study the sequence $B_i^{(0)}, i \in \mathbb{N}$, introduce the invariant $h(\xi) \in \mathbb{Q}^N$ as follows. Denote by v_0 the N-valuation on K_0 uniquely determined by the conditions $v_0(t_1) = (1, 0, \ldots, 0), v_0(t_2) = (0, 1, 0, \ldots, 0), \ldots, v_0(t_N) = (0, \ldots, 0, 1)$. Denote by the same symbol a unique extension of v_0 to the field $K_0\bar{E}$, where $\bar{E}$ is an algebraic closure of E. For any finite extension L' of E in $\bar{E}$ and any system of its local parameters $t'_1, \ldots, t'_{N-1}$ use the local parameters $t'_1, \ldots, t'_{N-1}, t_N$ in K_0L' to define an analogue $\xi_{L't'_1\ldots t'_{N-1}}$ of the above elements $\xi_E, \xi_{\widetilde{E}_1}, \ldots$. Then

$$v_0(\xi_{L't'_1\ldots t'_{N-1}}) := v_0(\xi, L')$$

does not depend on the choice of local parameters $t'_1, \ldots, t'_{N-1}$. Clearly, if L'' is a finite extension of L' in $\bar{E}$, then $v_0(\xi, L') \leqslant v_0(\xi, L'')$.

Set

$$h(\xi) = \sup\left\{ v_0(\xi, L') \mid L' \subset \bar{E}, [L' : E] < \infty \right\}.$$

Lemma 2.8. $h(\xi) = \min\left\{ p^{-s} v_0(\xi''_{E,s}) \mid s \in \mathbb{N} \right\}$.

Proof. Clearly, for any $s \in \mathbb{N}$, $v_0(\xi''_{E,s}) \geqslant v_0(\xi_E) \geqslant v_0(\xi)$.

Therefore, the right-hand side in the statement of our lemma is well defined. Denote its value by $\bar{h}(\xi)$.

Let $M \in \mathbb{N}$ be such that for any $s \geqslant M$, $\dfrac{1}{p^s} v_0(\xi''_{E,s}) > \bar{h}(\xi)$. Then

$$\bar{h}(\xi) = \min\{p^{-s} v_0(\xi''_{E,s}) \mid 0 \leqslant s < M\}.$$

Take $L'=E(t'_1,\dots,t'_{N-1})$, where $t_1'^{p^{M'}}=t_1,\dots,t_{N-1}'^{p^{M'}}=t_{N-1}$ and $M'\geqslant M$ is such that $p^{-M'}v_0(\xi'_E)>\bar h(\xi)$. Then

$$\xi_{L't'_1\dots t'_{N-1}}=\sum_{\bar b\leqslant\bar 0_N}\beta_{\bar b}{t'_1}^{b_1}\dots {t'}_{N-1}^{b_{N-1}}t_N^{b_N},$$

where:

(1) $v_0(\xi,L')=v_0\left({t'_1}^{b_1^0}\dots {t'}_{N-1}^{b_{N-1}^0}t_N^{b_N^0}\right)$, p does not divide b_N^0, but all $b_1^0,\dots,b_{N-1}^0$ are divisible by p;

(2) $v_0(\xi,L')=\bar h(\xi)$.

Notice that the above property (1) implies that for any finite extension L'' of L' we have $v_0(\xi,L')=v_0(\xi,L'')$. The second property implies, clearly, that $h(\xi)\geqslant\bar h(\xi)$.

Suppose that $h(\xi)>\bar h(\xi)$. Then there is a finite extension $\widetilde L'$ of E such that

$$v_0(\xi,\widetilde L')>v_0(\xi,L')=v_0(\xi,\widetilde L'L')\geqslant v_0(\xi,\widetilde L').$$

So, $h(\xi)=\bar h(\xi)$ and the lemma is proved. □

Corollary 2.9. *For any $i\in\mathbb N$,*

$$\mathrm{pr}_1(h(\xi))=\min\{p^{-s}B^{(s)}\mid s\geqslant 0\}=\min\{p^{-s}B_i^{(s)}\mid s\geqslant 0\}.$$

Finally we have the following lemma.

Lemma 2.10. *There is an index n^* such that $A_{n^*}>B_{n^*}$.*

Proof. Let $i\in\mathbb N$ and $\beta_i=\min\left\{\frac{1}{p^s}B_i^{(s)}\mid s\geqslant 0\right\}$. By the above corollary, for all $i\in\mathbb N$, $\beta_i=\beta=\mathrm{pr}_1(h(\xi))<0$ does not depend on i. Then Lemma 2.7(b) implies the existence of an index i^* such that if $i\geqslant i^*$, then $\beta=B_i^{(0)}$ and $B_i^{(0)}<B_i^{(s)}$ for all $s\geqslant 1$. Therefore, for all $i\geqslant i^*$, $B_i=\beta$. So, the lemma follows from Lemma 2.7(a). □

If $n\geqslant n^*$, set $K_n=KE_n$ and $K_{0n}=K_0E_n$. Then $K_n=K_{0n}(\theta_n)$, where

$$\theta_n^p-\theta_n=\sum_{\bar b\neq\bar 0_N}\beta_{\bar b}t_{1n}^{b_1}\dots t_{N-1,n}^{b_{N-1}}t_N^{b_N},$$

where $\min\{(b_1,\dots,b_{N-1},b_N)\mid\beta_{\bar b}\neq 0\}=(b_1^0,\dots,b_{N-1}^0,b_N^0)<\bar 0_N$ is such that $b_1^0,\dots,b_{N-1}^0$ are all divisible by p and b_N^0 is not divisible by p. This easily implies that the system of local parameters $t_{1n},\dots,t_{N-1,n}$ of E_n can be extended to a system of local parameters of K_n by the element

$$(t_{1n}^{-b_1^0/p}\dots t_{N-1,n}^{-b_{N-1}^0/p}\theta_n)^At_N^B$$

where $A,B\in\mathbb Z$ are such that $Ab_N^0+pB=1$.

So, Theorem 1 is proved in the case (b_2).

2.8. The case (c)**.** In this case we have $K = K_0(\theta)$, $\theta^p = \xi_E$, where $\xi_E \in K_0$ is the power series

$$\xi_E = \sum_{\bar a} \alpha_{\bar a} t_1^{a_1} \dots t_N^{a_N}$$

containing non-zero terms only with $\bar a \not\equiv 0 \bmod p$.

Set $\xi_E = \xi'_E + \xi''_E$, where

$$\xi'_E = \sum_{a_N \equiv 0 \bmod p} \alpha_{\bar a} t_1^{a_1} \dots t_{N-1}^{a_{N-1}} t_N^{a_N}, \qquad \xi''_E = \sum_{a_N \not\equiv 0 \bmod p} \alpha_{\bar a} t_1^{a_1} \dots t_{N-1}^{a_{N-1}} t_N^{a_N}.$$

Let

$$A = v^1(\xi'_E) = \min\{a_1 \mid \alpha_{\bar a} \neq 0, a_N \equiv 0 \bmod p\},$$
$$B = v^1(\xi''_E) = \min\{a_1 \mid \alpha_{\bar a} \neq 0, a_N \not\equiv 0 \bmod p\},$$

where v^1 is the t_1-adic valuation from the above condition C.

Notice that the first set can be empty. In this case we set by definition $A = +\infty$. The second set is never empty; otherwise, $\theta \in E(\sqrt[p]{t_1}, \dots, \sqrt[p]{t_{N-1}})$ and E is not algebraically closed in K.

If we pass from E to its finite extension $\widetilde E_i$, $i \in \mathbb{N}$ (cf. condition C), then $\tilde t_{1i}, \dots, \tilde t_{N-1,i}, t_N$ is a system of local parameters for $K_0 \widetilde E_i$. Rewrite ξ_E in terms of these local parameters and take away all pth power terms. This procedure gives an analogue $\tilde\xi_{\widetilde E_i}$ of ξ_E for the extension $K\widetilde E_i / K_0 \widetilde E_i$. As earlier, use the t_1-adic valuation v^1 to define the analogues $\widetilde A_i$ and $\widetilde B_i$ of A and, respectively, B. Similarly, introduce the invariants A_i and B_i, $i \in \mathbb{N}$, when passing in the above procedure from E to E_i.

We have the following estimates.

Lemma 2.11. (a) *$\widetilde A_1 \geqslant A$ and $\widetilde B_1 = B$.*

(b) *For all $i \in \mathbb{N}$, $\widetilde A_{i+1} \geqslant A_i$ and $\widetilde B_{i+1} = B_i$.*

(c) *For all $i \in \mathbb{N}$, $A_i \geqslant \widetilde A_i + c$ and $B_i = \widetilde B_i$.*

This implies that for $n \gg 0$, it holds that $A_n > B_n = B$. Therefore, there is an index n^* such that if $n \geqslant n^*$, $K_n = KE_n$ and $K_{0n} = K_0 E_n$; then $K_n = K_{0n}(\theta_n)$ with

$$\theta_n^p = \sum_{\bar b \neq \bar 0_N} \beta_{\bar b} t_{1n}^{b_1} \dots t_{N-1,n}^{b_{N-1}} t_N^{b_N},$$

where $\min\{(b_1, \dots, b_{N-1}, b_N) \mid \beta_{\bar b} \neq 0\} = (b_1^0, \dots, b_{N-1}^0, b_N^0)$ is such that $b_1^0, \dots, b_{N-1}^0$ are all divisible by p and b_N^0 is not divisible by p. Similarly, to the above case (b_2), this implies that the system of local parameters $t_{1n}, \dots, t_{N-1,n}$ of E_n can be extended to a system of local parameters of K_n.

The case (c) is also considered.

2.9. Characteristic 0 analogue of the Artin-Schreier theory. The characteristic 0 case (b_1) can be treated similarly to the characteristic p case (b_2) due to the characteristic 0 analogue of the Artin-Schreier theory from [Ab1]. This construction can be briefly reviewed as follows.

Suppose L_0 is a complete discreet valuation field of characteristic 0 with the maximal ideal m_{L_0} and the residue field k of characteristic p. Assume that $\zeta_p \in L_0$ (where ζ_p is a primitive pth root of unity) and let $\pi_1 \in L_0$ be such that $\pi_1^{p-1} = -p$.

Proposition 2.12.

(a) *$L = L_0(\sqrt[p]{v})$ with $v \in 1 + \pi_1 \mathrm{m}_{L_0}$ if and only if $L = L_0(\theta)$, where $\theta^p - \theta = w$ with $w \in p^{-1} \mathrm{m}_{L_0}$.*

(b) *With the above notation and assumptions, L admits another presentation $L = L_0(\theta_1)$, where $\theta_1^p - \theta_1 = w_1 \in p^{-1} \mathrm{m}_{L_0}$, if $w_1 = w + \eta^p - \eta$ with $\eta \in L_0$ such that $\eta^p \in p^{-1} \mathrm{m}_{L_0}$.*

Proof. We only sketch the idea of proof.

Let $E(X) = \exp\left(X + X^p/p + \dots + X^{p^n}/p^n + \dots\right) \in \mathbb{Z}_p[[X]]$ be the Artin-Hasse exponential. Then $v = E(\pi_1 V)$ with $V \in \mathrm{m}_{L_0}$ and if $u^p = v$, $u \in L$, then $u = E(U)$ with $U \in \mathrm{m}_L$. Then the equivalence

$$E(X)^p = E(X^p)\exp(pX) \equiv E(X^p + pX) \operatorname{mod}(p^2X, pX^p, X^{p^2})$$

implies that $U^p + pU \equiv \pi_1 V \operatorname{mod}(\pi_1 p \,\mathrm{m}_L)$ (notice that $U^p \in \pi_1 \mathrm{m}_L$).

Divide both sides of the above equivalence by π_1^p and deduce that $L = L_0(\theta)$, where $\theta^p - \theta = w \in p^{-1} \mathrm{m}_{L_0}$ with $\theta \equiv \pi_1^{-1} U \operatorname{mod} \mathrm{m}_L$ and $w \equiv -p^{-1} V \operatorname{mod} \mathrm{m}_{L_0}$. □

2.10. The case (b_1).

2.10.1. Assume first that $\zeta_p \in E$.

For $n \in \mathbb{N}$, set $\widetilde{K}_n = K\widetilde{E}_n$, $K_n = KE_n$, $\widetilde{K}_{0n} = K_0\widetilde{E}_n$ and $K_{0n} = K_0 E_n$. Then for a suitable $\tilde{v}_n \in \widetilde{K}_{0n}$ and $v_n \in K_{0n}$ we have $\widetilde{K}_n = \widetilde{K}_{0n}(\sqrt[p]{\tilde{v}_n})$ and $K_n = K_{0n}(\sqrt[p]{v_n})$.

First reduction: We can assume that all v_n are principal units.

Indeed, suppose

$$\tilde{v}_n = \tilde{t}_{1n}^{c_1} \dots \tilde{t}_{N-1,n}^{c_{N-1}} t_N^{c_N} (1 + \tilde{a}_n),$$

where $\tilde{a}_n \in \mathrm{m}_{\widetilde{K}_{0n}}$ and $c_1, \dots, c_N$ are either zeroes or prime to p natural numbers. Then the condition C from section 2.6 implies that

$$\tilde{v}_n = t_{1n}^{pc_1} \dots t_{N-1,n}^{pc_{N-1}} t_N^{c_N} (1 + a_n),$$

where $a_n \in \mathrm{m}_{K_{0n}}$. This implies that we can take $v_n = \tilde{v}_n (t_{1n}^{c_1} \dots t_{N-1,n}^{c_{N-1}})^{-p} = t_N^{c_N}(1 + a_n)$.

Suppose c_N is a prime to p natural number. Then we can assume that $c_N = 1$. It is easy to see then that $\sqrt[p]{v_n}$ extends a system of local parameters of E_n to a system of local parameters of K_n. So, Proposition 2.2 is proved in this case. Therefore, we can assume that $c_N = 0$ and v_n is a principal unit.

Second reduction: For any $n \in \mathbb{N}$, we cannot choose $v_n \in 1 + p\,\mathrm{m}_{K_{0n}}$.

Indeed, otherwise there is an $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, all v_n and $\tilde{v}_n$ can be chosen from $1 + p\,\mathrm{m}_{K_{0n}}$ and, resp., $1 + p\,\mathrm{m}_{\widetilde{K}_{0n}}$. In particular, all extensions $\widetilde{K}_n/\widetilde{K}_{0n}$ and K_n/K_{0n} can be treated via the analogue of the Artin-Schreier theory from section 2.9. Thus we can apply arguments from section 2.7 to prove Proposition 2.2 in this case.

We can now assume that all v_n cannot be chosen in $1+p\,\mathrm{m}_{K_{0n}}$. Therefore, all $\tilde{v}_n$ also cannot be chosen in $1 + p\,\mathrm{m}_{\widetilde{K}_{0n}}$.

For $n \in \mathbb{N}$, set $\tilde{v}_n \equiv 1 + \tilde{\xi}_n \bmod p\,\mathrm{m}_{\widetilde{K}_{0n}}$ and $v_n \equiv 1 + \xi_n \bmod p\,\mathrm{m}_{K_{0n}}$ with

$$\tilde{\xi}_n = \sum_{\bar{a}} \alpha_{\bar{a}} \tilde{t}_{1n}^{a_1} \dots \tilde{t}_{N-1,n}^{a_{N-1}} t_N^{a_N}, \qquad \xi_n = \sum_{\bar{a}} \alpha_{\bar{a}} t_{1n}^{a_1} \dots t_{N-1,n}^{a_{N-1}} t_N^{a_N},$$

where $\alpha_{\bar{a}} \neq 0$ implies that $\bar{a} > \bar{0}_N$, $\bar{a} \not\equiv 0 \bmod p$ and the corresponding monomial does not belong to $p\,\mathrm{m}_{\widetilde{K}_{0n}}$ or, resp., $p\,\mathrm{m}_{K_{0n}}$. Clearly, such elements $\tilde{\xi}_n$ and ξ_n are determined by $\tilde{v}_n$ and, respectively, v_n uniquely.

Let $\xi_n = \xi'_n + \xi''_n$ and $\tilde{\xi}_n = \tilde{\xi}'_n + \tilde{\xi}''_n$, where ξ'_n, resp. $\tilde{\xi}'_n$, contains only the monomials with $a_N \equiv 0 \bmod p$ and ξ''_n, resp. $\tilde{\xi}''_n$, contains the monomials with $a_N \not\equiv 0 \bmod p$.

Set $\widetilde{A}_n = v^1(\tilde{\xi}'_n)$, $A_n = v^1(\xi'_n)$, $\widetilde{B}_n = v^1(\tilde{\xi}''_n)$ and $B_n = v^1(\xi''_n)$. Then these numbers $\widetilde{A}_n$, A_n, $\widetilde{B}_n$ and B_n behave exactly in the same way as the corresponding numbers from case (c). Therefore, they satisfy the properties (a)–(c) from Lemma 2.11 above. This implies that for $n \gg 0$, $A_n > B_n = B_1$ (even more: for all $n \gg 0$, all $A_n = 0$ because the terms $\tilde{\xi}'_n$ and ξ'_n will disappear) and

$$v_n \equiv 1 + \sum_{\bar{a} \geqslant \bar{a}^0} \alpha_{\bar{a}} t_{1n}^{a_1} \dots t_{N-1,n}^{a_{N-1}} t_N^{a_N} \bmod p,$$

where $\alpha_{\bar{a}^0} \neq 0$ and $\bar{a}^0 = (a_1^0, \dots, a_{N-1}^0, a_N^0)$ is such that $a_1^0, \dots, a_{N-1}^0$ are divisible by p, but a_N^0 is not. Then

$$\sqrt[p]{v_n} = 1 + \alpha_{\bar{a}^0}^{1/p} t_{1n}^{a_1^0/p} \dots t_{N-1,n}^{a_{N-1}^0/p} \theta_n,$$

where $\theta_n \in K_n$. Let $v_{K_{0n}}$ be the N-valuation on algebraic closure of K_{0n} uniquely determined by the conditions $v_{K_{0n}}(t_{1n}) = (1, 0, \dots, 0)$, $\dots$, $v_{K_{0n}}(t_{N-1,n}) = (0, \dots, 0, 1, 0)$ and $v_{K_{0n}}(t_N) = (0, \dots, 0, 1)$. Then $v_{K_{0n}}(\theta_n) = \dfrac{1}{p} v_{K_{0n}}(t_N^{a_N^0}) = (0, \dots, 0, a_N^0/p)$. Therefore, $t_N^A \theta_n^B$, where $A, B \in \mathbb{Z}$ are such that $Ap + Ba_N^0 = 1$, can be taken as the Nth parameter for K_n and $(K_n, E_n) \in \mathrm{LC}(N)$ is standard.

2.10.2. Consider the case $\zeta_p \notin E$.

Let $K' = K(\zeta_p)$, $K'_0 = K_0(\zeta_p)$, $E' = E(\zeta_p)$ and $\widetilde{E}'_n = \widetilde{E}_n(\zeta_p)$, $E'_n = E_n(\zeta_p)$ for all $n \in \mathbb{N}$. Then the tower

$$E' \subset \widetilde{E}'_1 \subset E'_1 \subset \dots \subset \widetilde{E}'_n \subset E'_n \subset \dots$$

satisfies condition C from section 2.6 with a suitably chosen parameter $c' > 0$. Therefore, there is an n such that if $K'_n = KE'_n$ and $K'_{0n} = K_0E'_n$, then (K'_n, E'_n) is standard. For such n, let $\Gamma = \mathrm{Gal}(E'_n/E_n)$. Then $|\Gamma|$ divides $p-1$ (therefore, it is prime to p) and Γ can be identified with $\mathrm{Gal}(K'_n/K_n) = \mathrm{Gal}(K'_{0n}/K_{0n})$.

Choose the Nth local parameter t'_{nN} in K'_n with respect to its subfield of $(N-1)$-dimensional constants E'_n. Because the action of Γ on $\mathrm{m}_{K'_n}$ is semisimple, we can assume that for any $\tau \in \Gamma$, $\tau(t'_{nN}) = \chi(\tau)t'_{nN}$, where χ is a character of Γ with values in $\mathbb{F}_p^*$. Notice that this character does not depend on the choice of t'_{nN}.

This implies that $(K_n, E_n) = ({K'_n}^{\Gamma}, {E'_n}^{\Gamma})$ is standard if and only if the character χ is trivial. Indeed, if (K_n, E_n) is standard, then its Nth parameter can be taken as Nth parameter for K'_n and $\chi = \mathrm{id}$. Inversely, if $\chi = \mathrm{id}$, then $t'_{nN} \in K_n$ and $K_n = K'^{\Gamma}_n = E'_n\{\{t'_{nN}\}\}^{\Gamma} = E_n\{\{t'_{nN}\}\}$, i.e. (K_n, E_n) is standard.

Now notice that the norm of t'_{nN} in the extension K'_n/K'_{0n} is an Nth parameter t_{nN} for K'_{0n} such that Γ acts on it via the character $\chi^p = \chi$. But $(K_{0n}, E_{0n}) = (K'^{\Gamma}_{0n}, E'^{\Gamma}_{0n})$ is standard. As we have noticed above, this implies that χ is trivial. Therefore, (K_n, E_n) is also standard.

The Proposition 2.2 together with Theorem 1 are completely proved.

3. Ramification theory and Krasner's lemma

3.1. Category of local fields with F-structure. This category $\mathrm{LF}(N)$ appears as the disjoint union of its two full subcategories $\mathrm{LF}_0(N)$ and $\mathrm{LF}_p(N)$.

The category $\mathrm{LF}_0(N)$.

Choose a simplest N-dimensional local field of characteristic 0 with residue fields of characteristic p, $L_0 = \mathbb{Q}_p\{\{t_N\}\}\dots\{\{t_2\}\}$. Define its F-structure as an increasing sequence of closed subfields $\{L_0(i) \mid 1 \leqslant i \leqslant N\}$ with the system of local parameters $p = t_1, t_2, \dots, t_N$. Choose an algebraic closure $\bar{L}_0$ of L_0. Denote by $\mathbb{C}(N)_p$ the completion of $\bar{L}_0$ with respect to its first (p-adic) valuation. For $1 \leqslant i \leqslant N$, denote by $\mathbb{C}(i)_p$ the completion of the algebraic closure of $L_0(i)$ in $\mathbb{C}(N)_p$. It will be convenient to have a special agreement for $i = 0$. By definition, $\mathbb{C}(0)_p$ is the completion of the maximal unramified extension of $\mathbb{Q}_p$ in $\mathbb{C}(N)_p$ and $L_0(0) = L_0 \cap \mathbb{C}(0)_p = \mathbb{Q}_p$. Notice that $\mathbb{C}(1)_p = \mathbb{C}_p$ is the usual p-adic completion of an algebraic closure of $\mathbb{Q}_p$.

Clearly, the P-topological structure of finite extensions of L_0 induces the P-topological structures on the fields $\mathbb{C}(0)_p \subset \mathbb{C}(1)_p \subset \cdots \subset \mathbb{C}(N)_p$.

The objects of the category $\mathrm{LF}_0(N)$ are finite extensions K of L_0 in $\mathbb{C}(N)_p$ with the induced F-structure. This structure is given by the sequence of algebraically closed and P-closed subfields $\{K(i) \mid 0 \leqslant i \leqslant N\}$, where $K(i) = K \cap \mathbb{C}(i)_p$. Notice that $K(0)$ is the maximal unramified extension of $\mathbb{Q}_p$ in K. We agree to use the notation $\bar{K}$ for the algebraic closure of K in $\mathbb{C}(N)_p$. Notice that $\Gamma_K = \mathrm{Aut}(\bar{K}/K)$ consists of all sequentially P-continuous field automorphisms τ of $\mathbb{C}(N)_p$ such that $\tau|_K = \mathrm{id}$ and for all $0 \leqslant i \leqslant N$, $\tau(\mathbb{C}(i)_p) = \mathbb{C}(i)_p$. It is well known [Hy], that $\mathbb{C}(N)_p^{\Gamma_K} = K$ and, therefore, for all $0 \leqslant i \leqslant N$, $\mathbb{C}(i)_p^{\Gamma_K} = K(i)$.

Suppose $K, L \in \mathrm{LF}_0(N)$. Then the corresponding set of morphisms $\mathrm{Hom}_{\mathrm{LF}_0(N)}(K, L)$ consists of all sequentially P-continuous field morphisms $\varphi : \mathbb{C}(N)_p \to \mathbb{C}(N)_p$ such that for $0 \leqslant i \leqslant N$,

(a) $\varphi(\mathbb{C}(i)_p) = \mathbb{C}(i)_p$;

(b) $\varphi(K) \subset L$.

Notice that any $\varphi \in \mathrm{Hom}_{\mathrm{LF}_0(N)}(K, L)$ transforms the F-structure of K to the F-structure of L.

The category $\mathrm{LF}_p(N)$.

This category consists of fields of characteristic p and can be defined similarly to the above characteristic 0 case. Choose a basic N-dimensional local field $L_p = \mathbb{F}_p((t_N))\dots((t_1))$ and define its F-structure by a sequence of subfields $\{L_p(i) \mid 0 \leqslant i \leqslant N\}$ such that $L_p(0) = \mathbb{F}_p$ and for $1 \leqslant i \leqslant N$, $L_p(i)$ has local parameters $t_1, \dots, t_i$. Choose an algebraic closure $\bar{L}_p$ of L_p. Denote by $\mathcal{C}(N)_p$ the completion of $\bar{L}_p$ with respect to its first valuation. For $0 \leqslant i \leqslant N$, denote by $\mathcal{C}(i)_p$ the completion of the algebraic closure of $L_p(i)$ in $\mathcal{C}(N)_p$. As earlier, the P-topological structure of finite extensions of L_p induces the P-topological structures on the fields $\bar{\mathbb{F}}_p = \mathcal{C}(0)_p \subset \mathcal{C}(1)_p \subset \cdots \subset \mathcal{C}(N)_p$.

The objects of the category $\mathrm{LF}_p(N)$ are finite extensions K of L_p in $\mathcal{C}(N)_p$ with the induced F-structure $\{K(i) \mid 0 \leqslant i \leqslant N\}$, where $K(i) = K \cap \mathcal{C}(i)_p$. Notice that $\mathcal{C}(N)_p^{\Gamma_K} = \mathcal{R}(K)$ — the radical closure (= the completion of the maximal purely non-separable extension) of K in $\mathcal{C}(N)_p$. Similarly for $0 \leqslant i < N$, it holds that $\mathcal{C}(i)_p^{\Gamma_K} = \mathcal{R}(K(i))$. The morphisms in $\mathrm{LF}_p(N)$ are defined also along lines in the above charactersitic 0 case.

3.2. Standard F-structure. We say that the F-structure on $L \in \mathrm{LF}(N)$ is standard if there is a system of local parameters $t_1, \dots, t_N$ in L such that for all $1 \leqslant r \leqslant N$, $t_1, \dots, t_r$ is a system of local parameters for $L(r)$. Clearly, $L \in \mathrm{LF}(N)$ has a standard F-structure if and only if $(L, L(N-1)) \in \mathrm{LC}(N)$

is standard (cf. section 2.1) and $L(N-1) \in \mathrm{LF}(N-1)$ has a standard F-structure. Applying Theorem 1 we obtain the following:

Proposition 3.1. *For any $E \in \mathrm{LF}(N)$, there is a finite separable extension E' of $E(N-1)$ such that EE' has a standard F-structure.*

Remark. The above proposition played a fundamental role in the construction of the higher-dimensional ramification theory in [Ab5], but its proof in [Ab5] was not complete, due to reasons mentioned in the Remark from section 2.1. Notice that the construction of ramification theory (cf. section 3.3 below), needs only the result of Theorem 1.

Note that F-structure allows us to treat higher-dimensional local fields in a very similar way to classical complete discrete valuation fields with finite residue fields. For example, for any finite extension of local fields with F-structure we can introduce:

(a) *a vector ramification index* $\bar{e}(L/K) = (e^1, \dots, e^N)$.

Any finite extension of K in $\bar{K}$ appears with a natural F-structure and a natural P-topology. In particular, if $L \subset M$ are such subfields in $\bar{K}$, then its vector ramification index equals $\bar{e}(M/L) = (e^1, \dots, e^N)$, where for $1 \leqslant r \leqslant N$,

$$e^r = [M(r) : L(r)]/[M(r-1) : L(r-1)] = [M(r) : L(r)M(r-1)].$$

This index plays a role of the usual ramification index in the theory of 1-dimensional local fields. Notice that $e^N = 1$ if and only if M coincides with the composite of L and $M(N-1)$.

(b) *a canonical N-valuation* $v_L : L \longrightarrow \mathbb{Q}^N \cup \{\infty\}$.

If L has a standard F-structure and $t_1, \dots, t_N$ is a corresponding system of local parameters, then v_L is uniquely defined by the conditions $v_L(t_1) = (1, 0, \dots, 0)$, $v_L(t_2) = (0, 1, 0, \dots, 0), \dots, v_L(t_N) = (0, 0, \dots, 0, 1)$. Otherwise, one should use a finite extension L_1 of L with standard F-structure and set $v_L = \bar{e}(L_1/L)^{-1} v_{L_1}$. One can easily verify that v_L does not depend on the choice of L_1. Notice that, as usual, v_L can be extended uniquely to any algebraic extension L' of L. (We shall use the same notation v_L for such extension.) Also notice that if L' is a finite extension of L, then $v_{L'} = \bar{e}(L'/L) v_L$.

3.3. Review of ramification theory (cf. [Ab5]). Suppose $K \in \mathrm{LF}(N)$. Then $\Gamma_K = \mathrm{Aut}(\bar{K}/K)$ has a canonical decreasing filtration by ramification subgroups $\{\Gamma_K^{(j)} \mid j \in J(N)\}$ with the set of indices $J(N) = \coprod_{1 \leqslant r \leqslant N} J_r$. Here $J_r = \{j \in \mathbb{Q}^r \mid j \geqslant \bar{0}_r\}$ with respect to the lexicographic ordering on $\mathbb{Q}^r$, where $\bar{0}_r = (0, \dots, 0) \in \mathbb{Q}^r$. By definition, if $r_1 > r_2$, then any element from J_{r_1} is bigger than any element from J_{r_2}.

The definition of the ramification filtration can be given as follows.

Let E/K be a finite extension in $\bar{K}$ (this is a subfield in $\mathbb{C}(N)_p$ or $\mathcal{C}(N)_p$). Consider the finite set $I_{E/K}$ of all sequentially P-continuous embeddings of E into $\bar{K}$ which are equal to the identity on K. There is a natural filtration of this set

$$I_{E/K} \supset I_{E/K,0} \supset I_{E/K,(0,0)} \supset \cdots \supset I_{E/K,\bar{0}_N}$$

where for $1 \leqslant r \leqslant N$, $I_{E/K,\bar{0}_r}$ are embeddings which are equal to the identity on the subfield of $(r-1)$-dimensional constants $E(r-1)$.

For $1 \leqslant r \leqslant N$ and $j \in J_r$, define the set $I_{E/K,j} \subset I_{E/K,\bar{0}_r}$ as follows.

Take a suitable finite extension E' of $E(r-1)$ in $\bar{K}$ such that if $\widetilde{E}(r) = E'E(r)$ and $\widetilde{K}(r) = K(r)E'$, then $(\widetilde{E}(r), E') \in \mathrm{LC}(r)$ is standard (cf. section 2.1). Then for an rth local parameter θ in $\widetilde{E}(r)$, we have $\mathcal{O}_{\widetilde{E}(r)} = \mathcal{O}_{\widetilde{K}(r)}[\theta]$. (Recall, if $L \in \mathrm{LF}(r)$, then $\mathcal{O}_L = \{l \in L \mid v_L(l) \geqslant \bar{0}_r\}$ and also notice that $v_{E(r)}(\theta) = v_{\widetilde{E}(r)}(\theta) = (0, \ldots, 0, 1) \in \mathbb{Q}^r$.) Then use the natural identification $I_{E/K,\bar{0}_r} = I_{\widetilde{E}(r)/\widetilde{K}(r)}$ to define the ramification filtration of $I_{E/K}$ in lower numbering by setting for every $j \in J_r$,

$$I_{E/K,j} = \{\tau \in I_{\widetilde{E}(r)/\widetilde{K}(r)} \mid v_{E(r)}(\tau(\theta) - \theta) \geqslant v_{E(r)}(\theta) + j\}.$$

The subsets $I_{E/K,j}$, where $j \in J_r$, do not depend on the above choices of the finite extension E' of $E(r-1)$ and the corresponding rth local parameter $\theta \in \widetilde{E}(r)$. The resulting filtration $\{I_{E/K,j} \mid j \in J(N)\}$ does depend on the F-structures on E and K.

We now introduce an analogue of the Herbrand function $\varphi_{E/K} : J(N) \longrightarrow J(N)$ by setting for $1 \leqslant r \leqslant N$ and $j \in J_r$,

$$\varphi_{E/K}(j) = \bar{e}(E(r)/K(r))^{-1} \int_{\bar{0}_r}^{j} |I_{E/K,j}| dj \in J_r.$$

This gives the upper numbering such that for any $j \in J(N)$, $I^{(j)}_{E/K} = I_{E/K,j'}$, where $j' \in J(N)$ is such that $\varphi_{E/K}(j') = j$. As in the classical situation, if $E_2 \supset E_1 \supset K$, then the natural projection $I_{E_2/K} \longrightarrow I_{E_1/K}$ induces for any $j \in J(N)$, an epimorphic map from $I^{(j)}_{E_2/K}$ onto $I^{(j)}_{E_1/K}$ and $\varprojlim_E I^{(j)}_{E/K} = \Gamma^{(j)}_K$ is the ramification subgroup of Γ_K with the upper number j.

As an example, consider the case of an extension E/K in $\mathrm{LF}(N)$ such that $[E:K] = p^N$ and $\bar{e}(E/K) = (p, \ldots, p) \in \mathbb{Q}^N$. Then for $1 \leqslant r \leqslant N$, there are $\alpha_r \in J_r$, $\alpha_r > \bar{0}_r$ such that for all $j \in J_r$,

$$\varphi_{E/K}(j) = \begin{cases} j, & \text{if } j < \alpha_r; \\ \alpha_r + \frac{j - \alpha_r}{p}, & \text{if } j \geqslant \alpha_r. \end{cases}$$

Similarly to the classical case for any finite extension E/K, the Herbrand function $\varphi_{E/K} : J(N) \longrightarrow J(N)$ is a piecewise linear function with finitely many edge points. Define $i(E/K) \in J(N)$ and $j(E/K) \in J(N)$ as the first and the second coordinates of the last edge point of the graph of $\varphi_{E/K}$. Notice that if $1 \leqslant r \leqslant N$ and $j \in J_r$, then j is an edge point iff $\varphi'_-(j) \neq \varphi'_+(j)$, where $\varphi'_-(j)$ and $\varphi'_+(j)$ are slopes of $\varphi_{E/K}$ in the left and right neighbourhoods of j, respectively. (By definition, $\varphi'_-(\bar{0}_r) = g_{r0}\bar{e}(E(r)/K(r))^{-1}$, where $g_{r0} = [E(r) : K(r)E(r-1)]$.)

If $1 \leqslant r \leqslant N$ and $j \in J_r$, then $\varphi'_-(j) = g_-(j)\bar{e}(E(r)/K(r))^{-1}$ and $\varphi'_+(j) = g_+(j)\bar{e}(E(r)/K(r))^{-1}$, where $g_-(j)$ and $g_+(j) \in \mathbb{N}$. We shall call $g_-(j)/g_+(j) := \mathrm{mult}_{E/K}(j)$ — the multiplicity of $\varphi_{E/K}$ in $j \in J_r$. We have:

— $\mathrm{mult}_{E/K}(j) = 1$ if and only if j is not an edge point;

— $\mathrm{mult}_{E/K}(j)$ is prime to p if and only if $j = \bar{0}_r$, $1 \leqslant r \leqslant N$;

— if $j \neq \bar{0}_r$, $1 \leqslant r \leqslant N$, then $\mathrm{mult}_{E/K}(j)$ is a power of p;

— $\prod_{j \in J(N)} \mathrm{mult}_{E/K}(j) = [E : KE(0)]$.

3.4. Krasner's Lemma. Suppose $L, K \in \mathrm{LF}(N)$, $L \supset K$, $L(N-1) = K(N-1)$ and E is a finite extension of $L(N-1)$ such that $(LE, E) \in \mathrm{LC}(N)$ is standard. Then $\mathcal{O}_{\widetilde{L}} = \mathcal{O}_{\widetilde{K}}[\theta]$ where $\widetilde{L} = LE$, $\widetilde{K} = KE$ and θ is an Nth local parameter in $\widetilde{L}$.

Let $F(T) = T^d + a_1T^{d-1} + \cdots + a_d \in \mathcal{O}_{\widetilde{K}}[T]$ be the minimal unitary polynomial for θ over $\widetilde{K}$. Note that $F(T)$ is an Nth Eisenstein polynomial (cf. section 2.1). Denote by $\theta_1 = \theta, \theta_2, \ldots, \theta_d \in \bar{K}$ all roots of $F(T)$. Notice that $v_{\widetilde{L}}(\theta) = v_L(\theta_1) = \cdots = v_L(\theta_d) = (0, \ldots, 0, 1)$. As usual, $\varphi_{L/K}$ is the Herbrand function for L/K.

In this situation the Krasner Lemma can be given by the following proposition.

Proposition 3.2. *If $\alpha \in \bar{K}$ is such that $v_K(F(\alpha)) = A + (0, \ldots, 0, 1)$ with $A > \bar{0}_N$, then*

(1) *there is an index $1 \leqslant l_0 \leqslant d$ such that $v_L(\alpha - \theta_{l_0}) = a + (0, \ldots, 0, 1)$, where $\varphi_{L/K}(a) = A$;*

(2) *if $A > j(L/K)$, then the above index l_0 is unique.*

Proof. Choose an index l_0, $1 \leqslant l_0 \leqslant d$, such that

$$v_L(\alpha - \theta_{l_0}) = \max\{v_L(\alpha - \theta_l) \mid 1 \leqslant l \leqslant d\}.$$

Let $a \in J_N$ be such that $v_L(\alpha - \theta_{l_0}) = a + (0, \ldots, 0, 1)$.

Lemma 3.3. $v_K(F(\alpha)) = \varphi_{L/K}(a) + (0, \ldots, 0, 1)$.

Proof of lemma. Let $i_1 < i_2 < \cdots < i_s$ be the lower indices which correspond to all jumps of the ramification filtration on $I_{L/K}$. Notice that due to the assumption $L(N-1) = K(N-1)$, all ramification jumps $i_1, \ldots, i_s \in J_N$. Then, for some integers, $d = g_0 > g_1 > \cdots > g_{s-1} > g_s = 1$ and all $2 \leqslant i \leqslant d$, $v_L(\theta - \theta_i)$ takes $g_0 - g_1$ times the value $i_1 + v_L(\theta)$, $\ldots$, $g_{s-1} - g_s$ times the value $i_s + v_L(\theta)$. Notice that $i_s = i(L/K)$, $\bar{e}(L/K) = \bar{e}(\widetilde{L}/\widetilde{K}) = (1, \dots, 1, d)$ and if $i_t \leqslant a < i_{t+1}$ for some $0 \leqslant t \leqslant s$ (with the agreements $i_0 = \bar{0}_N$ and $i_{s+1} = \infty$), then

$$\varphi_{L/K}(a) = \bar{e}(L/K)^{-1} \left(g_0 i_1 + \cdots + g_{t-1}(i_t - i_{t-1}) + g_t(a - i_t)\right).$$

Clearly, for all $1 \leqslant l \leqslant d$, $v_L(\alpha - \theta_l) = \min\{v_L(\alpha - \theta_{l_0}), v_L(\theta_{l_0} - \theta_l)\}$. This implies

$$\begin{gathered} v_L(F(\alpha)) = \sum_{1 \leqslant l \leqslant d} v_L(\alpha - \theta_l) \\ = (g_0 - g_1)(i_1 + v_L(\theta_{l_0})) + \cdots + (g_{t-1} - g_t)(i_t + v_L(\theta_{l_0})) + g_t(a + v_L(\theta_{l_0})) \\ g_0 v_L(\theta_{l_0}) + g_0 i_1 + g_1(i_2 - i_1) + \cdots + g_{t-1}(i_t - i_{t-1}) + g_t(a - i_t) \\ = \bar{e}(L/K) \left(v_L(\theta_{l_0}) + \varphi_{L/K}(a)\right). \end{gathered}$$

The lemma is proved, because $v_K = \bar{e}(L/K)^{-1} v_L$. □

It remains to prove part (2) of our proposition.

Suppose θ_{l_1} is a root of F with the same property $v_L(\alpha - \theta_{l_1}) = a + (0, \dots, 0, 1)$. Then $v_L(\theta_{l_1} - \theta_{l_0}) \geqslant a + (0, \dots, 0, 1)$. But if $A > j(L/K)$, then $a > i(L/K)$ and $\theta_{l_1} = \theta_{l_2}$.

The proposition is proved. □

Corollary 3.4. *With the above assumption and notation*

$$v_L(\delta(F)) = (1, \dots, 1, d) j(L/K) - i(L/K) + (0, \dots, 0, d-1)$$

where $\delta(F)$ is the different ideal of F.

Proof. We have $\delta(F) = F'(\theta) = (\theta - \theta_2) \dots (\theta - \theta_d)$. Then

$$\begin{aligned} v_L(\delta(F)) &= \sum_{2 \leqslant i \leqslant d} v_L(\theta - \theta_i) \\ &= (g_0 - g_1)(i_1 + v_L(\theta)) + \cdots + (g_{s-1} - g_s)(i_s + v_L(\theta)) \\ &= \bar{e}(L/K) \varphi_{L/K}(i_s) - i_s + (d-1) v_L(\theta). \end{aligned}$$

It remains to note that $\bar{e}(L/K) = (1, \dots, 1, d)$, $i_s = i(L/K)$, $\varphi_{L/K}(i_s) = j(L/K)$ and $v_L(\theta) = (0, \dots, 0, 1)$. □

Corollary 3.5. $j(L/K) \leqslant 2 v_K(\delta(F))$.

Proof. Notice that

$$i(L/K) = \max\{v_L(\theta_1 - \theta_i) \mid 2 \leqslant i \leqslant d\} - (0, \dots, 0, 1) < v_L(\delta(F)).$$

Then Corollary 3.4 implies that

$$(1, \dots, 1, d) j(L/K) \leqslant 2 v_L(\delta(F)) = 2(1, \dots, 1, d) v_K(\delta(F))$$

and we can cancel by $(1, \dots, 1, d)$. □

4. Families of increasing towers

In this section we work with local fields of characteristic 0 from $\mathrm{LF}_0(N)$.

4.1. The category $\mathcal{B}(N)$. The objects of $\mathcal{B}(N)$ are increasing sequences $K_{\bullet}=\{K_n \mid n\geqslant 0\}$ of $K_n\in \mathrm{LF}_0(N)$. If $K_{\bullet}, L_{\bullet}\in\mathcal{B}(N)$, then $\mathrm{Hom}_{\mathcal{B}(N)}(K_{\bullet}, L_{\bullet})$ consists of field automorphisms $f:\mathbb{C}(N)_p\longrightarrow \mathbb{C}(N)_p$ such that

— f is sequentially P-continuous;

— f is compatible with F-structure;

— $f(K_n)\subset L_n$ for all $n\gg 0$.

Clearly, if $K_{\bullet}=\{K_n \mid n\geqslant 0\}\in\mathcal{B}(N)$, then for any $1\leqslant r\leqslant N$, the subfields of constants of dimension r $\{K_n(r)\mid n\geqslant 0\}$, give an object of the category $\mathcal{B}(r)$. This object will be usually denoted by $K_{\bullet}(r)$.

Notice that two towers $K_{\bullet}$ and $L_{\bullet}$ can be naturally identified if $K_n=L_n$ for all $n\gg 0$ (all sufficiently large n). Such towers will be called almost equal.

Let $K_{\bullet}, L_{\bullet}\in\mathcal{B}(N)$. Then by definition $K_{\bullet}\subset L_{\bullet}$ or $L_{\bullet}$ is an extension of $K_{\bullet}$ if for all $m\gg 0$, $K_m\subset L_m$. $L_{\bullet}$ is a finite extension of $K_{\bullet}$ of degree $d=d(L_{\bullet}/K_{\bullet})$ if for all $m\gg 0$, $[L_m:K_m]=d$. Clearly, if $L_{\bullet}/K_{\bullet}$ and $M_{\bullet}/L_{\bullet}$ are finite extensions, then $M_{\bullet}/K_{\bullet}$ is also finite and $d(M_{\bullet}/K_{\bullet})=d(L_{\bullet}/K_{\bullet})d(M_{\bullet}/L_{\bullet})$.

An extension $L_{\bullet}/K_{\bullet}$ will be called separable if there is an index m_0 and an algebraic extension E of K_{m_0} such that $L_{\bullet}$ is almost equal to $EK_{\bullet}:=\{EK_m\mid m\geqslant 0\}$. Clearly, if $L_{\bullet}/K_{\bullet}$ and $M_{\bullet}/L_{\bullet}$ are separable, then $M_{\bullet}/K_{\bullet}$ is also separable. Notice also, that the composite of finitely many separable extensions of $K_{\bullet}$ is again separable over $K_{\bullet}$. Therefore, any finite extension $L_{\bullet}/K_{\bullet}$ contains a "unique" maximal separable over $K_{\bullet}$ subextension $L_{\bullet}^{(s)}$ (i.e. any another maximal separable subextension is almost equal to $L_{\bullet}^{(s)}$).

An extension $L_{\bullet}/K_{\bullet}$ will be called purely inseparable if for any $n\geqslant 0$, there is an $m=m(n)\geqslant 0$ such that $L_n\subset K_m$. The simplest example of a purely inseparable extension of $K_{\bullet}$ is $K'_{\bullet}$ such that for all m, $K'_m=K_{m+1}$.

Suppose $L_{\bullet}\supset K_{\bullet}$ is a finite extension in $\mathcal{B}(N)$ of degree $d=d(L_{\bullet}/K_{\bullet})$. Let $\widetilde{L}$ and $\widetilde{K}$ be the p-adic completions of the $\bigcup_{m\geqslant 0} L_m$ and, resp., $\bigcup_{m\geqslant 0} K_m$. Suppose that $[\widetilde{L}:\widetilde{K}]=\tilde{d}$. Then there are the following simple properties:

(1) $\tilde{d}\leqslant d$;

(2) $\tilde{d}=d$ iff $L_{\bullet}$ is separable over $K_{\bullet}$;

(3) $\tilde{d}=1$ iff $L_{\bullet}$ is purely inseparable over $K_{\bullet}$;

(4) if $m_0\geqslant 0$ is such that $L_{m_0}\widetilde{K}=\widetilde{L}$, then $L_{\bullet}^{(s)}=L_{m_0}K_{\bullet}$ and $L_{\bullet}$ is purely inseparable over $L_{\bullet}^{(s)}$;

(5) if $L_{\bullet}^{(i)}:=\{L_m\cap\widetilde{K}\mid m\geqslant 0\}$, then $L_{\bullet}^{(i)}$ is the maximal purely inseparable extension of $K_{\bullet}$ in $L_{\bullet}$ and $L_{\bullet}$ is separable over $L_{\bullet}^{(i)}$.

Remark. The above property (2) implies also that a separable $L_\bullet/K_\bullet$ can be decomposed into a sequence of separable extensions $K_\bullet \subset K_\bullet L_\bullet(0) \subset \cdots \subset K_\bullet L_\bullet(r) \subset \cdots \subset K_\bullet L_\bullet(N-1) \subset L_\bullet$, and for any $1 \leqslant r < N$, $L_\bullet(r)$ is separable over $K_\bullet(r)$. In addition, the vector index $\bar{e}(L_m/K_m)$ becomes stable for $m \gg 0$. We shall denote it as $\bar{e}(L_\bullet/K_\bullet)$ below.

4.2. The category $\mathcal{B}^a(N)$, $N \in \mathbb{Z}_{\geqslant 0}$.

Definition. $\mathcal{B}^a(N)$ is a full subcategory in $\mathcal{B}(N)$ consisting of $K_\bullet \in \mathcal{B}(N)$ such that there is an index $n^* = n^*(K_\bullet)$ and a positive real number $c^* = c^*(n^*, K_\bullet) > 0$ such that for all $n \geqslant n^*$,

(a) $[K_{n+1} : K_n] = p^N$ and $\bar{e}(K_{n+1}/K_n) = (p, \ldots, p) \in \mathbb{Z}^N$;

(b) if $1 \leqslant r \leqslant N$, then $j_{rn} := j(K_{n+1}(r)/K_n(r)) \in J_r$ and $\mathrm{pr}_1(j_{rn}) \geqslant p^n c^*$. (As usual, $\mathrm{pr}_1(j)$ denotes the first coordinate of $j \in J_r \subset \mathbb{Q}^r$.)

Remarks. (1) If $K_\bullet \in \mathcal{B}^a(N)$, then for $n \geqslant n^*(K_\bullet)$, all K_n have the same last residue field.

(2) With the above notation, $K_\bullet \in \mathcal{B}^a(N)$ will be called a tower with the index parameter n^* and the ramification parameter c^*; notice that any $n'^* \geqslant n^*$ and $0 < c' \leqslant c^*$ can also be taken as such parameters for $K_\bullet$.

(3) For $n \geqslant n^*(K_\bullet)$, the condition (a) implies the equality of valuations pv_{K_n} and $v_{K_{n+1}}$. We shall use this to define the N-valuation $v_{K_\bullet} := \lim_{n\to\infty} p^{-n} v_{K_n}$ below. Due to the above remark (2), we shall also be able to assume that the parameter $c^* = c^*(n^*, K_\bullet)$ satisfies the restriction $\mathrm{pr}_1\, v_{K_\bullet}(p) \geqslant c^*/p$. The number c^*/p will be denoted below as $c_1^*(n^*, K_\bullet)$.

(4) From condition (b) it follows that if $m \geqslant n^*(K_\bullet)$, $1 \leqslant r \leqslant N$ and $j \in J_r$ is such that $\mathrm{pr}_1(j) < p^m c^*$, then $\varphi_{K_{m+1}/K_m}(j) = j$. Therefore, the composition property of the Herbrand function implies that for such j and all $n \geqslant m$, $\varphi_{K_n/K_0}(j) = \varphi_{K_m/K_0}(j)$. Therefore, there is a limit function $\varphi_{K_\bullet} := \lim_{m\to\infty} \varphi_{K_m/K_0}$.

Proposition 4.1. *Suppose $K_\bullet, L_\bullet \in \mathcal{B}(N)$ and $L_\bullet$ is a separable extension of $K_\bullet$. If $K_\bullet \in \mathcal{B}^a(N)$, then $L_\bullet \in \mathcal{B}^a(N)$.*

Proof. Suppose $K_\bullet$ has parameters $n^* = n^*(K_\bullet)$ and $c^* = c^*(n^*, K_\bullet)$.

If $L_\bullet = \{L_m \mid m \geqslant 0\}$, then we can assume that there is an $m_0 \geqslant n^*$ such that for all $m \geqslant m_0$ and $1 \leqslant r \leqslant N$, $L_{m+1}(r) = L_m(r)K_{m+1}(r)$ and $[L_m(r) : K_m(r)] = d(L_\bullet(r)/K_\bullet(r))$ is independent on m. This implies that for $m \geqslant m_0$, $[L_{m+1} : L_m] = p^N$ and $\bar{e}(L_{m+1}/L_m) = (p, \ldots, p)$. In other words, $L_\bullet$ satisfies the requirement (a) of the above definition of objects in $\mathcal{B}^a(N)$.

Prove that $L_\bullet$ satisfies condition (b) from the definition of objects from $\mathcal{B}^a(N)$.

First, consider the case $K_\bullet(N-1) = L_\bullet(N-1)$.

We must prove that if $j'_m = j(L_{m+1}/L_m)$ and $m_0 \gg 0$, then there is a $c'^* > 0$ such that for all $m \geqslant m_0$, $\mathrm{pr}_1(j'_m) \geqslant p^m c'^*$.

Let $\alpha_m = j(L_m/K_m)$ and $j_m = j(K_{m+1}/K_m)$. Notice that $\alpha_m, j_m \in J_N$.

Lemma 4.2. *Suppose $m \geqslant m_0$. Then*

(1) $\alpha_{m+1} \leqslant \max\{p\alpha_m - (p-1)j_m, \alpha_m\}$;

(2) if $\alpha_m < j_m$, then $\alpha_m = \alpha_{m+1}$.

Proof. By the composition property of Herbrand's function we have

$$\varphi_{L_{m+1}/K_m}(j) = \varphi_{K_{m+1}/K_m}\left(\varphi_{L_{m+1}/K_{m+1}}(j)\right) \tag{1}$$

for any $j \in J(N)$. Looking at the last edge point we obtain

$$j(L_{m+1}/K_m) = \max\left\{\varphi_{K_{m+1}/K_m}(\alpha_{m+1}), j_m\right\}.$$

On the other hand, $L_{m+1} = L_m K_{m+1}$ implies that $j(L_{m+1}/K_m) = \max\{\alpha_m, j_m\}$.

Therefore,

— if $\alpha_m \geqslant j_m$, then $\varphi_{K_{m+1}/K_m}(\alpha_{m+1}) \leqslant \alpha_m$;

— if $\alpha_m < j_m$, then α_m and $\varphi_{K_{m+1}/K_m}(\alpha_{m+1})$ coincide because they both appear as second coordinates of the pre-last edge point of φ_{L_{m+1}/K_m}.

It remains only to notice that for $j \in J_N$ (cf. example in section 3.3),

$$\varphi_{K_{m+1}/K_m}(j) = \begin{cases} j, & \text{if } j \leqslant j_m; \\ j_m + \frac{1}{p}(j - j_m), & \text{if } j \geqslant j_m. \end{cases}$$

The lemma is proved. □

Lemma 4.3. *If $m \geqslant m_0$ and $\alpha_m < j_m$, then $\varphi_{L_m/K_m} = \varphi_{L_{m+1}/K_{m+1}}$.*

Proof. Notice first that $j(L_{m+1}/K_m) = \max\{\alpha_m, j_m\} = j_m$ and $\alpha_{m+1} = \alpha_m < j_m$.

Then (1) implies that the largest edge point of φ_{L_{m+1}/K_m} comes from the edge point of φ_{K_{m+1}/K_m} and they both have the same multiplicity p. Therefore, all edge points of φ_{L_{m+1}/K_m}, apart from the largest one, coincide with the edge points of $\varphi_{L_{m+1}/K_{m+1}}$ counting multiplicities.

Let $j'_m = j(L_{m+1}/L_m)$. Then for all $j \in J(N)$,

$$\varphi_{L_{m+1}/K_m}(j) = \varphi_{L_m/K_m}(\varphi_{L_{m+1}/L_m}(j)) \tag{2}$$

implies that

$$j_m = j(L_{m+1}/K_m) = \max\{\alpha_m, \varphi_{L_m/K_m}(j'_m)\} = \varphi_{L_m/K_m}(j'_m).$$

Again, the largest edge point of φ_{L_{m+1}/K_m} comes from the edge point of φ_{L_m/K_m}. So, all edge points of φ_{L_{m+1}/K_m}, apart from the largest one, coincide with the edge points of φ_{L_m/K_m} counting multiplicities.

So, $\varphi_{L_m/K_m} = \varphi_{L_{m+1}/K_{m+1}}$ because they have the same edge points counting multiplicities.

The lemma is proved. □

We continue the proof of our proposition.

If $m \geqslant m_0$, then $\mathrm{pr}_1(j_m/p^m) \geqslant c^*$. By taking, if necessary, a bigger m_0 we can assume that for all $m \geqslant m_0$, $\alpha_m < j_m$. Indeed, Lemma 4.2 implies that

$$\frac{\alpha_{m+1}}{p^{m+1}} \leqslant \max\left\{\frac{\alpha_m}{p^m} - \left(1 - \frac{1}{p}\right)(c^*, 0, \dots, 0), \frac{\alpha_m}{p^{m+1}}\right\},$$

therefore, α_m/p^m tends to 0. Hence for a sufficiently large m_0 we have that $\mathrm{pr}_1(\alpha_m/p^m) < c^*$ and $\alpha_m < j_m$ if $m \geqslant m_0$. Then by Lemma 4.3 the Herbrand functions of the extensions L_m/K_m with $m \geqslant m_0$ coincide. Denote this function by $\varphi_{L_\bullet/K_\bullet}$ and notice that $\varphi_{L_\bullet/K_\bullet}(j'_m) = j_m$, where $j'_m = j(L_{m+1}/L_m)$ (cf. the proof of Lemma 4.3).

It remains to notice that $\varphi_{L_\bullet/K_\bullet}$ is a piecewise linear function and from its definition it follows that for any $j \in J_N$, the number $\mathrm{pr}_1(\varphi_{L_\bullet/K_\bullet})(j)$ depends only on $\mathrm{pr}_1(j)$. This gives a piecewise convex linear function $\alpha \mapsto \varphi^1(\alpha) = \mathrm{pr}_1 \varphi_{L_\bullet/K_\bullet}((\alpha, 0, \dots, 0))$ on $\mathbb{R}_{\geqslant 0}$ with the last slope $1/e^1 = 1/d(L_\bullet/K_\bullet)$ if $N = 1$ and $e^1 = 1$ if $N > 1$. (Note that we are considering the case where $L_\bullet(N-1) = K_\bullet(N-1)$.) So, for any given $0 < \gamma < 1$, there is an m_0 such that for all $m \geqslant m_0$, the conditions $\mathrm{pr}_1(j_m) \geqslant p^m c^*$ and $\varphi^1(\mathrm{pr}_1(j'_m)) = \mathrm{pr}_1(j_m)$ imply that $\mathrm{pr}_1(j'_m) \geqslant p^m e^1 c^* \gamma$. Taking $\gamma = 1/2$ we obtain that for a sufficiently large index parameter $n^* = n^*(L_\bullet)$, $L_\bullet$ has the ramification parameter $c^*(n^*, L_\bullet) = e^1 c^*/2$.

Consider the case of a general separable extension $L_\bullet/K_\bullet$, where $K_\bullet \in \mathcal{B}^a(N)$ with parameters $n^* = n^*(K_\bullet)$ and $c^* = c^*(n^*, K_\bullet)$. Then our proposition is implied by the following lemma.

Lemma 4.4. *If $e^1 = \mathrm{pr}_1(\bar{e}(L_\bullet/K_\bullet))$, then for a sufficiently large parameter $m^* = n^*(L_\bullet)$, one can take for $L_\bullet$ the ramification parameter $c^*(m^*, L_\bullet) = e^1 c^*/2^N$.*

Proof. Apply induction on $N \geqslant 1$.

The case $N = 1$ follows from the above considerations.

Let $N \geqslant 2$. Then we have two separable extensions $K_\bullet \subset E_\bullet \subset L_\bullet$, where for any $m \geqslant 0$, $E_m = K_m L_m(N-1)$.

Prove that for $n \gg 0$, one can take $c^*(n, E_\bullet) = e^1 c^*/2^{N-1}$.

Indeed, if $1 \leqslant r < N$, then

$$\mathrm{pr}_1\, j(E_{m+1}(r)/E_m(r)) = \mathrm{pr}_1\, j(L_{m+1}(r)/L_m(r)) \geqslant e^1 c^*/2^r \geqslant e^1 c^*/2^{N-1}$$

by the inductive assumption. Compare the values of the Herbrand functions of E_{m+1}/E_m and K_{m+1}/K_m for $j \in J_N$. In both cases the definition of the Herbrand function uses two ingredients:

— the canonical valuations $v_{E_{m+1}}$ and $v_{K_{m+1}}$, which differ by the factor $(e^1, \dots, e^{N-1}, 1)$, where $\bar{e}(L_{\centerdot}/K_{\centerdot}) = (e^1, \dots, e^{N-1}, e^N)$.

— the ramification indices $\bar{e}(E_{m+1}/E_m)$ and $\bar{e}(K_{m+1}/K_m)$, which are both equal to $(p, \dots, p)$ for $m \gg 0$.

This implies that for any $j \in J_N$,

$$\varphi_{E_{m+1}/E_m}(j) = (e^1, \dots, e^{N-1}, 1)\varphi_{K_{m+1}/K_m}((e^1, \dots, e^{N-1}, 1)^{-1} j).$$

Therefore, $j(E_{m+1}/E_m) = (e^1, \dots, e^{N-1}, 1) j(K_{m+1}/K_m)$ and $\mathrm{pr}_1\, j(E_{m+1}/E_m) \geqslant p^m e^1 c^*$.

So, for $n \gg 0$, $c^*(n, E_{\centerdot}) = \min\{e^1 c^*, e^1 c^*/2^{N-1}\} = e^1 c^*/2^{N-1}$. Finally, because $E_{\centerdot}(N-1) = L_{\centerdot}(N-1)$ we have $L_{\centerdot} \in \mathcal{B}^a(N)$ and for $n \gg 0$, $c^*(n, L_{\centerdot}) = \mathrm{pr}_1(1, \dots, 1, e^N) c^*(n, E_{\centerdot})/2 = e^1 c^*/2^N$.

The lemma is proved. □

Remark. Suppose $K_{\centerdot}, L_{\centerdot} \in \mathcal{B}^a(N)$ and $L_{\centerdot}$ is separable over $K_{\centerdot}$. Then the above arguments give the equality of the Herbrand functions φ_{L_m/K_m} for $m \gg 0$. This function will be denoted below as $\varphi_{L_{\centerdot}/K_{\centerdot}}$.

4.3. The category $\mathcal{B}^{fa}(N)$.

4.3.1. Suppose $K_{\centerdot} \in \mathcal{B}^a(N)$ with an index parameter $n^* = n^*(K_{\centerdot})$ and a ramification parameter $c^*(n^*, K_{\centerdot})$.

Definition. If indices $u_1, \dots, u_N$ are such that $n^* \leqslant u_N \leqslant u_{N-1} \leqslant \dots \leqslant u_1$, then $K_{u_1 \dots u_N} := K_{u_1}(1) \dots K_{u_{N-1}}(N-1) K_{u_N}$. We shall denote this field with its natural F-structure below also as $K_{\bar{u}}$, where $\bar{u} = (u_1, \dots, u_N)$.

Definition. $\mathcal{B}^{fa}(N)$ is the full subcategory of all $K_{\centerdot} \in \mathcal{B}^a(N)$ such that for some vector index parameter $\bar{u}^0 = \bar{u}^0(K_{\centerdot})$, $K_{\bar{u}^0}$ has a standard F-structure.

Remark. If $\bar{u}^0 = \bar{u}^0(K_{\centerdot})$ is the above vector index parameter, then we always assume that the corresponding index parameter $n^*(K_{\centerdot})$ equals u^0_N. $K_{\centerdot}$ will be called a tower with the vector index parameter $\bar{u}^0 = \bar{u}^0(K_{\centerdot})$ and the ramification parameter $c^* = c^*(\bar{u}^0, K_{\centerdot})$. Note that we use the notation $c^*_1 = c^*_1(\bar{u}^0, K_{\centerdot}) := c^*/p$ and assume that $\mathrm{pr}_1(v_{K_{\centerdot}}(p)) \geqslant c^*_1$.

A tower extension $L_{\centerdot} \supset K_{\centerdot}$ will be called finite Galois if it is finite separable and there is an index m_0 such that for all $m \geqslant m_0$, all $L_m \supset K_m$ are Galois. Equivalently, there is a finite Galois field extension L of K_{m_0} such that for all $m \geqslant m_0$, $L_m = L K_m$.

Proposition 4.5. *Suppose $L_{\centerdot} \supset K_{\centerdot}$ is a separable extension in $\mathcal{B}^a(N)$. Then there is a finite Galois extension $\widetilde{L}_{\centerdot}$ of $K_{\centerdot}$ such that $\widetilde{L}_{\centerdot} \supset L_{\centerdot}$ and $\widetilde{L}_{\centerdot} \in \mathcal{B}^{fa}(N)$.*

Proof. Let $n^* = n^*(L_\centerdot) = n^*(K_\centerdot)$. Choose a finite Galois extension E of K_{n^*} such that $E_\centerdot = EK_\centerdot \supset L_\centerdot$. Then $E_\centerdot \in \mathcal{B}^a(N)$ (cf. section 4.2) and we can assume that $n^* = n^*(E_\centerdot)$. Take a finite extension F of $E_{n^*}(N-1)$ such that $(E_{n^*}F, F)$ is standard in the category $\mathrm{LC}(N)$.

Let $F_\centerdot = FK_\centerdot(N-1)$. We can assume that $m^* := n^*(F_\centerdot) = n^*(K_\centerdot(N-1)) \geqslant n^*$. By induction there is a finite Galois extension H of $K_{m^*}(N-1)$ such that $H_\centerdot = HK_\centerdot(N-1) \supset F_\centerdot$ and $H_\centerdot \in \mathcal{B}^{fa}(N-1)$. Then $(E_{n^*}H, H) \in \mathrm{LC}(N)$ is still standard and, therefore, $HE_\centerdot \in \mathcal{B}^{fa}(N)$. At the same time, $HE_\centerdot$ is Galois over $K_\centerdot$ as a composite of Galois extensions.

The proposition is proved. □

Remark. The above proposition shows that for a given $K_\centerdot \in \mathcal{B}^a(N)$, the family of all its Galois extensions in $\mathcal{B}^{fa}(N)$ is cofinal in the family of all its separable extensions in $\mathcal{B}^a(N)$.

4.3.2. The following proposition (or more precisely, its applications below) plays an important role in the construction of an analogue of the field-of-norms functor.

Proposition 4.6. *Suppose $K_\centerdot \in \mathcal{B}^{fa}(N)$. Then for any $u \geqslant u_N^0(K_\centerdot)$, there is a $v = v(u) \geqslant u$ such that $(K_uK_v(N-1), K_v(N-1)) \in \mathrm{LC}(N)$ is standard.*

In sections 4.3.3–4.3.6 below we assume that this proposition is proved and consider its applications. We need these applications later on in our construction of the field-of-norms functor. We also need them in dimension $< N$, when proving the above Proposition 4.6 by induction on N in section 4.4.

Notice, if Proposition 4.6 holds with a function $v(u)$, then this proposition also holds with any function $v_1(u)$ such that $v_1(u) \geqslant v(u)$ for all $u \geqslant u_N^0(K_\centerdot)$.

4.3.3. Structural functions m_r, $1 \leqslant r < N$.

Proposition 4.7. *Suppose $K_\centerdot \in \mathcal{B}^{fa}(N)$ with the index parameter $\bar{u}^0(K_\centerdot) = (u_1^0, \dots, u_N^0)$. Then for $1 \leqslant r < N$, there are non-decreasing functions $m_r : \mathbb{Z}_{\geqslant u_{r+1}^0} \longrightarrow \mathbb{Z}_{\geqslant u_r^0}$ such that for any $u \geqslant u_{r+1}^0$, $m_r(u) \geqslant u$ and for any $u_1, \dots, u_N$, $u_{N-1} \geqslant m_{N-1}(u_N)$, $\dots$, $u_1 \geqslant m_1(u_2)$, $K_{u_1u_2\dots u_N}$ has a standard F-structure.*

Proof. Use induction on N.

If $N = 1$, there is nothing to prove.

Assume that $N > 1$. Then $K_\centerdot(N-1) \in \mathcal{B}^{fa}(N-1)$ and there are functions $m_r : \mathbb{Z}_{\geqslant u_{r+1}^0} \longrightarrow \mathbb{Z}_{\geqslant u_r^0}$, where $1 \leqslant r \leqslant N-2$, such that if $u_{N-1} \geqslant u_{N-1}^0$, $u_{N-2} \geqslant m_{N-2}(u_{N-1}), \dots, u_1 \geqslant m_1(u_2)$, then $K(N-1)_{u_1\dots u_{N-1}}$ has a standard F-structure.

If $u \geqslant u_N^0$, take $v = v(u) \geqslant u_{N-1}^0$ from Proposition 4.6. Then define $m_{N-1} : \mathbb{Z}_{\geqslant u_N^0} \longrightarrow \mathbb{Z}_{\geqslant u_{N-1}^0}$ by the relation

$$m_{N-1}(u) = \max\{v(u') \mid u_N^0 \leqslant u' \leqslant u\}.$$

Then this collection of functions m_r, $1 \leqslant r < N$, satisfies the requirements of our proposition. □

Remark. With the above notation, suppose the indices $(v_1^0, \dots, v_N^0)$ are such that $v_1^0 \geqslant \dots \geqslant v_N^0$ and the functions $n_r : \mathbb{Z}_{\geqslant v_{r+1}^0} \longrightarrow \mathbb{Z}_{\geqslant v_r^0}$, $1 \leqslant r < N$, are such that $v_{r+1}^0 \geqslant u_{r+1}^0$ and $n_r(u) \geqslant m_r(u)$ for all $u \geqslant v_{r+1}^0$. Then the proposition holds also with the new indices $v_1^0, \dots, v_N^0$ and the new functions $n_{N-1}, \dots, n_1$. In particular, we can assume (if necessary) that the functions m_r from our proposition are strictly increasing. For a similar reason, if $L_{\bullet}, K_{\bullet} \in \mathcal{B}^{fa}(N)$, then we can always choose a common vector parameter $\bar{u}^0(L_{\bullet}) = \bar{u}^0(K_{\bullet})$ and common corresponding functions $m_1, \dots, m_{N-1}$ such that Proposition 4.7 holds for both $K_{\bullet}$ and $L_{\bullet}$.

4.3.4. Local parameters. Suppose $K_{\bullet} \in \mathcal{B}^{fa}(N)$ and for $1 \leqslant r < N$, $m_r : \mathbb{Z}_{\geqslant u_{r+1}^0} \longrightarrow \mathbb{Z}_{\geqslant u_r^0}$ are corresponding functions from Proposition 4.7. We always assume in this situation that $n^*(K_{\bullet}) = u_N^0$ and $m_r(u_{r+1}^0) = u_r^0$ for all $1 \leqslant r < N$.

For $1 \leqslant r \leqslant N$ and $v_r \geqslant u_r^0$, fix a choice of an rth local parameter $t_{v_r}^{(r)}$ in the field $K_{v_1}(1) \dots K_{v_r}(r)$, where $v_{r-1} = m_{r-1}(v_r), \dots, v_1 = m_1(v_2)$.

Proposition 4.8. *For any indices $u_1, \dots, u_N$ such that $u_N \geqslant u_N^0$, $u_{N-1} \geqslant m_{N-1}(u_N), \dots, u_1 \geqslant m_1(u_2)$, the above introduced elements $t_{u_1}^{(1)}, \dots, t_{u_N}^{(N)}$ give a system of local parameters in the field $K_{\bar{u}} = K_{u_1}(1) \dots K_{u_{N-1}}(N-1) K_{u_N}$.*

Proof. If $N = 1$, there is nothing to prove.

If $N > 1$ we can assume by induction that $t_{u_1}^{(1)}, \dots, t_{u_{N-1}}^{(N-1)}$ is a system of local parameters in $E = K_{u_1}(1) \dots K_{u_{N-1}}(N-1)$.

Let $v_{N-1} = m_{N-1}(u_N)$, $v_{N-2} = m_{N-2}(v_{N-1}), \dots, v_1 = m_1(v_2)$. Let $E' = K_{v_1}(1) \dots K_{v_{N-1}}(N-1)$, then $K_{\bar{v}} = E' K_{u_N}$ with $\bar{v} = (v_1, \dots, v_{N-1}, u_N)$. Clearly, $E' \subset E$ and $(K_{\bar{v}}, E') \in \mathrm{LC}(N)$ is standard. Therefore, $(K_{\bar{v}} E, E)$ is also standard and $t_{u_N}^{(N)}$ extends the system of local parameters $t_{u_1}^{(1)}, \dots, t_{u_{N-1}}^{(N-1)}$ of E to a system of local parameters of $K_{\bar{u}} = K_{\bar{v}} E$.

The proposition is proved. □

4.3.5. Construction of special extensions. Assume that $K_{\bullet} \in \mathcal{B}^{fa}(N)$ is given via the above notation. Assume, in addition, that the functions m_r, $1 \leqslant r < N$, are strictly increasing. Under these assumptions, for any $n \in \mathbb{N}$, set $v_N^n = u_N^0 + n - 1$ and define the vector $\bar{v}^n = (v_1^n, \dots, v_N^n)$ by the relations $v_{N-1}^n = m_{N-1}(v_N^n + 1), \dots, v_1^n = m_1(v_2^n + 1)$. Notice that for any indices $w_1, \dots, w_N$ such that $v_r^n \leqslant w_r \leqslant v_r^n + 1$ with $1 \leqslant r \leqslant N$, the field $K_{w_1 \dots w_N}$ has a standard F-structure. Indeed, for any $1 \leqslant r < N$, we have the inequalities $m_r(w_{r+1}) \leqslant m_r(v_{r+1}^n + 1) = v_r^n \leqslant w_r$.

Set for all $n \in \mathbb{N}$, $\bar{u}^n = (v_1^n+1, \dots, v_N^n+1)$, $\mathcal{O}_{\bar{v}^n} = \mathcal{O}_{K_{\bar{v}^n}}$ and $\mathcal{O}_{\bar{u}^n} = \mathcal{O}_{K_{\bar{u}^n}}$. Notice that we have natural embeddings

$$\mathcal{O}_{\bar{u}^0} \subset \mathcal{O}_{\bar{v}^1} \subset \mathcal{O}_{\bar{u}^1} \subset \dots \subset \mathcal{O}_{\bar{v}^n} \subset \mathcal{O}_{\bar{u}^n} \subset \dots .$$

Indeed, the embedding $O_{\bar{u}^0} \subset \mathcal{O}_{\bar{v}^1}$ exists because $v_N^1 = u_N^0$ and for $1 \leqslant r < N$, $v_{r+1}^1 \geqslant u_{r+1}^0$ implies that $v_r^1 = m_r(v_{r+1}^1 + 1) > m_r(u_{r+1}^0) = u_r^0$. The existence of the embeddings $\mathcal{O}_{\bar{v}^n} \subset \mathcal{O}_{\bar{u}^n}$ for $n \in \mathbb{N}$ is obvious, because for any $1 \leqslant r \leqslant N$, $v_r^n < u_r^n = v_r^n + 1$. In order to prove the existence of the embeddings $\mathcal{O}_{\bar{u}^n} \subset \mathcal{O}_{\bar{v}^{n+1}}$ for $n \in \mathbb{N}$, compare $\bar{u}^n$ and $\bar{v}^{n+1}$. Clearly, $u_N^0 = v_N^1$ and for $n \geqslant 1$, $u_N^n = v_N^n + 1 = u_N^0 + n = v_N^{n+1}$. Suppose $1 \leqslant r < N$ and $u_{r+1}^n \leqslant v_{r+1}^{n+1}$. Then $u_r^n = v_r^n + 1 = m_r(v_{r+1}^n + 1) + 1 = m_r(u_{r+1}^n) + 1 \leqslant m_r(v_{r+1}^{n+1}) + 1 \leqslant m_r(v_{r+1}^{n+1} + 1) = v_r^{n+1}$.

For any $u \geqslant u_N^0$, let v_{K_u} be the canonical N-valuation associated with K_u (cf. section 3.2(b)). Then $v_{K_\bullet} := v_{K_u}/p^u$ does not depend on the choice of u (cf. Remark (3) in section 4.2). Introduce the 1-valuations $v_{K_u}^1 := \mathrm{pr}_1\, v_{K_u}$ and $v_{K_\bullet}^1 = \mathrm{pr}_1\, v_{K_\bullet}$. For any $c > 0$, set

$$\mathrm{m}_{K_\bullet}^1(c) = \{o \in \mathcal{O}_{\mathbb{C}(N)_p} \mid v_{K_\bullet}^1(o) \geqslant c\}.$$

For any subring O in $\mathcal{O}_{\mathbb{C}(N)_p}$, agree to denote by $O \bmod \mathrm{m}_{K_\bullet}^1(c)$ the image of O in $\mathcal{O}_{\mathbb{C}(N)_p} \bmod \mathrm{m}_{K_\bullet}^1(c)$. Then for any $n \geqslant 0$, there are natural inclusions $\mathcal{O}_{\bar{u}^n} \bmod \mathrm{m}_{K_\bullet}^1(c) \subset \mathcal{O}_{\bar{v}^{n+1}} \bmod \mathrm{m}_{K_\bullet}^1(c) \subset \mathcal{O}_{\bar{u}^{n+1}} \bmod \mathrm{m}_{K_\bullet}^1(c)$.

Proposition 4.9. *Let $c_1^* = c^*(u_N^0, K_\bullet)/p$. If $c_1^* \leqslant v_{K_\bullet}^1(p)$ (cf. Remark* (3) *in section* 4.2*), then for all $n \in \mathbb{N}$, the pth power map induces a ring epimorphism*

$$\mathcal{O}_{\bar{u}^n} \bmod \mathrm{m}_{K_\bullet}^1(c_1^*) \longrightarrow \mathcal{O}_{\bar{v}^n} \bmod \mathrm{m}_{K_\bullet}^1(c_1^*).$$

Proof. Note that for $n \in \mathbb{N}$, $\bar{u}^n = (u_1^n, \dots, u_N^n) = (v_1^n + 1, \dots, v_N^n + 1)$. Let $1 \leqslant r \leqslant N$ and let $t_{u_r^n}^{(r)}$ be the rth local parameter for $K_{\bar{u}^n}(r)$ from section 4.3.4. It will be sufficient to prove that its pth power is congruent modulo $\mathrm{m}_{K_\bullet}^1(c_1^*)$ to some rth local parameter of the field $K_{\bar{v}^n}(r)$. By induction we can assume that $r = N$.

Let $E = K_{\bar{v}^n}$, $E' = K_{\bar{v}^n}(t_{u_N^n}^{(N)}) \subset K_{\bar{u}^n}$. Then $[E' : E] = p$ and both these fields have a standard F-structure. If $\tau \in I_{E'/E}$ and $\tau \neq \mathrm{id}$, then

$$v_{K_{u_N^n}}^1 \left(\tau t_{u_N^n}^{(N)} - t_{u_N^n}^{(N)} \right) \geqslant p^{v_N^n} c^*$$

(cf. the definition of objects of $\mathcal{B}^a(N)$ in section 4.2 and use the Herbrand function from section 3.3). This implies that all conjugates to $t_{u_N^n}^{(N)}$ over E are congruent modulo $\mathrm{m}_{K_\bullet}^1(c^*/p) = \mathrm{m}_{K_\bullet}^1(c_1^*)$. Therefore, the pth power of $t_{u_N^n}^{(N)}$ is congruent modulo $\mathrm{m}^1_{K_\bullet}(c_1^*)$ to the norm $N_{E'/E}\left(t^{(N)}_{u^n_N}\right)$, which is an Nth local parameter in $K_{\bar v^n}$.

The proposition is proved. □

Corollary 4.10. *With the above notation and assumptions the field tower*

$$K_{\bar u^0} \subset K_{\bar v^1} \subset K_{\bar u^1} \subset K_{\bar v^2} \subset \dots \subset K_{\bar v^n} \subset K_{\bar u^n} \subset \dots$$

satisfies condition C from section 2.6 *with the parameter* $c = c_1^*(\bar u^0, K_\bullet) = c^*(\bar u^0, K_\bullet)/p$.

4.3.6. A modified system of local parameters. As earlier, we have $K_\bullet \in \mathcal{B}^{fa}(N)$ together with the corresponding strictly increasing functions $m_r : \mathbb{Z}_{\geqslant u^0_{r+1}} \longrightarrow \mathbb{Z}_{\geqslant u^0_r}$ for $1 \leqslant r < N$.

For $1 \leqslant r < N$, define $U(m_1, \dots, m_r) \subset \mathbb{Z}^{r+1}$ as the set of $\bar u = (u_1, \dots, u_{r+1})$ such that $u_{r+1} \geqslant u^0_{r+1} + 1$, $u_r \geqslant m_r(u_{r+1}) + 1, \dots, u_1 \geqslant m_1(u_2) + 1$.

Notice that $\bar u = (u_1, \dots, u_N) \in U(m_1, \dots, m_{N-1})$ if and only if $\bar v = (v_1, \dots, v_N) := (u_1 - 1, \dots, u_N - 1)$ satisfies the restrictions $v_N \geqslant u^0_N$, $v_{N-1} \geqslant m_{N-1}(v_N + 1), \dots, v_1 \geqslant m_1(v_2 + 1)$. In other words, the vectors $\bar u \in U(m_1, \dots, m_{N-1})$ can be taken as the vectors $\bar u^n$, where $n \in \mathbb{N}$, in the towers from Corollary 4.10. In particular, the pth power map induces a ring epimorphism

$$\mathcal{O}_{K_{\bar u}} \bmod \mathrm{m}^1_{K_\bullet}(c_1^*) \longrightarrow \mathcal{O}_{K_{\bar v}} \bmod \mathrm{m}^1_{K_\bullet}(c_1^*).$$

Proposition 4.11. *For all* $1 \leqslant r \leqslant N$ *and* $u \geqslant u^0_r$, *there are* $\tau^{(r)}_u \in \mathbb{C}(r)_p$ *such that:*

(a) $\tau^{(1)}_{u^0_1}, \dots, \tau^{(N)}_{u^0_N}$ *is a given system of local parameters in* $K_{\bar u^0}$*;*

(b) $\tau^{(r)p}_{u+1} \equiv \tau^{(r)}_u \bmod m^1_{K_\bullet}(c_1^*)$*;*

(c) *if* $\bar u = (u_1, \dots, u_r) \in U(m_1, \dots, m_{r-1})$, *then* $\tau^{(1)}_{u_1}, \dots, \tau^{(r)}_{u_r}$ *is a system of local parameters in* $K_{\bar u}(r)$.

Proof. Use induction on $0 \leqslant r \leqslant N$. If $r = 0$, there is nothing to prove. So, it will be sufficient to define $\tau^{(N)}_u$ with $u \geqslant u^0_N$.

Set $\tau^{(N)}_{u^0_N} = t^{(N)}_{u^0_N}$ (cf. section 4.3.4).

Then use induction on $n \geqslant 0$, where $u = u^0_N + n$. We can assume that $\tau^{(N)}_{u^0_N+n-1} = \tau^{(N)}_{v^n_N} \in \mathcal{O}_{\bar v^n}$ has already been constructed. By Proposition 4.9 we can take $\tau^{(N)}_{u^0_N+n} = \tau^{(N)}_{u^n_N} \in \mathcal{O}_{\bar u^n}$ such that

$$\tau^{(N)p}_{u^n_N} \equiv \tau^{(N)}_{v^n_N} \bmod \mathrm{m}^1_{K_\bullet}(c_1^*).$$

Clearly, this is an Nth local parameter in $K_{\bar u^n}$.

It remains to prove the property (c) for $r = N$. We must prove that if $\bar u = (u_1, \dots, u_N) \in U(m_1, \dots, m_{N-1})$ and $\bar u^n = (u^n_1, \dots, u^n_N)$ is such that $u^n_N = u_N$, then $u_r \geqslant u^n_r$ for all $1 \leqslant r \leqslant N$.

Indeed, it holds with $r=N$. Suppose $1\leqslant r<N$ and $u_{r+1}\geqslant u^n_{r+1}$. Then

$$u_r\geqslant m_r(u_{r+1})+1\geqslant m_r(u^n_{r+1})+1=m_r(v^n_{r+1}+1)+1=v^n_r+1=u^n_r.$$

The proposition is proved. □

4.4. Proof of Proposition 4.6. Notice that there is nothing to prove if $N=1$ and use induction on N by assuming that the proposition holds in dimensions $<N$.

Therefore, we can use the result of Corollary 4.10 in dimensions $<N$. It remains to note that if $K_{\bar v^n}$ is F-standard, then $K_{v^n_N+1}K_{\bar v^n}$ is elementary infernal over $K_{\bar v^n}$, due to the condition on the upper ramification numbers from the definition of objects of the category $\mathcal{B}^a$ from section 4.2. So, Proposition 4.6 follows from case (b) of the procedure of elimination of wild ramification from section 2.3 via an analogue of the tower from Corollary 4.10 constructed for $K_{\bullet}(N-1)$.

5. Family of fields $X(K_{\bullet})$, $K_{\bullet}\in\mathcal{B}^{fa}(N)$

5.1. Fontaine's field $R_0(N)$. Recall that objects $K\in\mathrm{LF}_0(N)$ are realised as subfields in $\mathbb{C}(N)_p$. They are closed subfields with induced F-structure and P-topology. Any $K\in\mathrm{LF}_0(N)$ has a canonical valuation v_K of rank N.

Notice that if $K'\in\mathrm{LF}_0(N)$, then $v_{K'}=\bar\alpha v_K$ with some $\bar\alpha\in\mathbb{Q}^N_{>0}$, and therefore, all such valuations belong to the same class of equivalent valuations. If $K\in\mathrm{LF}_0(N)$ and v_K is the extension of its canonical valuation of rank N to $\mathbb{C}(N)_p$, then $\mathcal{O}_{\mathbb{C}(N)_p}=\{o\in\mathbb{C}(N)_p\mid v_K(o)\geqslant\bar 0_N\}$. For any $c>0$, $\mathrm{m}^1_K(c):=\{o\in\mathbb{C}(N)_p\mid v^1_K(o)\geqslant c\}$ is an ideal in $\mathcal{O}_{\mathbb{C}(N)_p}$ (as earlier, $v^1_K:=\mathrm{pr}_1(v_K)$).

Set $R(N)=\varprojlim_n(\mathcal{O}_{\mathbb{C}(N)_p}\operatorname{mod}p)_n$ where the connecting morphisms are induced by the pth power map. Then $R(N)$ is an integral domain and its fraction field $R_0(N)$ is a perfect field of characteristic p. The F-structure on $\mathbb{C}(N)_p$ induces the F-structure on $R_0(N)$ given by the decreasing sequence of the subfields

$$R_0(N)\supset R_0(N-1)\supset\dots\supset R_0(1)\supset R_0(0).$$

In addition, the field $R_0(0)$ consists of the sequences $\{\alpha^{p^{-n}}\}_{n\geqslant 0}$, where $\alpha\in\bar{\mathbb{F}}_p$. The map $\{\alpha^{p^{-n}}\}_{n\geqslant 0}\mapsto\alpha$ identifies $R_0(0)$ with $\bar{\mathbb{F}}_p$, in particular, any finite field of characteristic p can be embedded naturally into $R_0(N)$.

Notice that $R=R(1)$ and $\operatorname{Frac}R=R_0(1)$ is the original notation introduced for the corresponding 1-dimensional objects by J.-M. Fontaine.

Let $K_{\centerdot} \in \mathcal{B}^a(N)$. It determines the N-valuation on $\mathbb{C}(N)_p$ given by the formula $v_{K_{\centerdot}} = \lim\limits_{n\to\infty}(v_{K_n}/p^n)$. Define the N-valuation $v_{R,K_{\centerdot}}$ on $R_0(N)$. If $\bar{r} = (r_n)_{n\geqslant 0} \in R(N)$, then

$$v_{R,K_{\centerdot}}(\bar{r}) = \lim_{n\to\infty} p^n v_{K_{\centerdot}}(\hat{r}_n) = \lim_{n\to\infty} v_{K_n}(\hat{r}_n)$$

where $\hat{r}_n \in \mathcal{O}_{\mathbb{C}(N)_p}$ is such that $\hat{r}_n \bmod p = r_n$. (For $n \gg 0$, $v_{K_n}(\hat{r}_n) < v_{K_n}(p)$ and then $v_{K_{n+1}}(\hat{r}_{n+1}) = p^{-1}v_{K_{n+1}}(\hat{r}^p_{n+1}) = v_{K_n}(\hat{r}_n)$.)

If $L_{\centerdot} \in \mathcal{B}^a(N)$, then $v_{R,L_{\centerdot}} = \bar{\alpha} v_{R,K_{\centerdot}}$ with $\bar{\alpha} \in \mathbb{Q}^N_{>0}$. Therefore, the equivalence class of valuations $v_{R,K_{\centerdot}}$ does not depend on the choice of $K_{\centerdot}$.

For $c > 0$, set $\mathrm{m}^1_{R,K_{\centerdot}}(c) = \{o \in R(N) \mid v^1_{R,K_{\centerdot}}(o) \geqslant c\}$, where $v^1_{R,K_{\centerdot}} = \mathrm{pr}_1\, v_{R,K_{\centerdot}}$.

The following proposition is just an easy consequence of the above definitions.

Proposition 5.1. *Suppose $c > 0$ is such that $p \in \mathrm{m}^1_{K_{\centerdot}}(c)$ (or, equivalently, $v^1_{K_{\centerdot}}(p) \geqslant p$). Then:*

(a) $R(N) = \varprojlim\limits_n(\mathcal{O}_{\mathbb{C}(N)_p} \bmod \mathrm{m}^1_{K_{\centerdot}}(c))_n$, *where all connecting morphisms are induced by the pth power map;*

(b) *for any $u \geqslant 0$, the uth projection $\mathrm{pr}_u : R(N) \longrightarrow \mathcal{O}_{\mathbb{C}(N)_p} \bmod \mathrm{m}^1_{K_{\centerdot}}(c)$ induces a ring isomorphism of $R(N) \bmod \mathrm{m}^1_{R,K_{\centerdot}}(p^u c)$ and $\mathcal{O}_{\mathbb{C}(N)_p} \bmod \mathrm{m}^1_{K_{\centerdot}}(c)$.*

Proof. Let $\bar{r} = (r_n \bmod p)$. Take $n_0 \geqslant 0$ such that $p^{n_0}c \geqslant v^1_{K_{\centerdot}}(p)$. Consider the map $\iota : R(N) \longrightarrow \varprojlim\limits_n(\mathcal{O}_{\mathbb{C}(N)_p} \bmod \mathrm{m}^1_{K_{\centerdot}}(c))_n$ given by the correspondence $r \mapsto (r_n \bmod \mathrm{m}^1_{K_{\centerdot}}(c))_{n\geqslant 0}$. Then:

— ι is injective.

Indeed, suppose for all $n \geqslant 0$, $r_n \in \mathrm{m}^1_{K_{\centerdot}}(c)$. Then for $n \geqslant 0$, $r_n \equiv r^{p^{n_0}}_{n+n_0} \equiv 0 \bmod p$.

— ι is surjective.

Indeed, suppose $u = (u_n \bmod \mathrm{m}^1_{K_{\centerdot}}(c)) \in \varprojlim\limits_n(\mathcal{O}_{\mathbb{C}(N)_p} \bmod \mathrm{m}^1_{K_{\centerdot}}(c))_n$. Then $r = (u^{p^{n_0}}_{n+n_0} \bmod p)_n \in R(N)$ and $\iota(r) = u$. □

Remark. $\mathcal{O}_{\mathbb{C}(N)_p}$ is equipped with the P-topology induced by the inductive limit of the P-topologies on all fields $K \in \mathrm{LF}_0(N)$. This topology induces the P-topology on $R(N)$ and $R_0(N)$. With respect to this topology the arithmetic operations in $R_0(N)$ are sequentially P-continuous.

5.2. The family of fields $X(K_{\centerdot})$. Suppose $K_{\centerdot} \in \mathcal{B}^{fa}(N)$ with the vector index parameter $\bar{u}^0(K_{\centerdot}) = (u^0_1, \ldots, u^0_N)$ and the ramification parameter $c^* = c^*(u^0_N, K_{\centerdot})$. As earlier in section 4.3.3, choose for all $1 \leqslant r < N$, the corresponding strictly increasing functions $m_r : \mathbb{Z}_{\geqslant u^0_{r+1}} \longrightarrow \mathbb{Z}_{\geqslant u^0_r}$ and elements $\tau^{(r)}_u$, where $u \geqslant u^0_r$, such that $\tau^{(r)}_{u_r}$ is the rth local parameter in $K_{\bar{u}}(r)$ if $\bar{u} = (u_1, \ldots, u_r) \in U(m_1, \ldots, m_{r-1})$.

For $c_1^*=c^*/p$, set

$$\tau^{(r)}=(\tau_u^{(r)}\operatorname{mod}\mathrm{m}^1_{K_\centerdot}(c_1^*))_{u\geqslant u_r^0}\in\varprojlim_u(\mathcal{O}_{\mathbb{C}(N)_p}\operatorname{mod}\mathrm{m}^1_{K_\centerdot}(c_1^*))_u=R(N).$$

Let $k=k(K_\centerdot)$ be the last residue field of $K_{u_N^0}$ (this is also the residue field for all K_u with $u\geqslant u_N^0$). As mentioned in section 5.1, k can be naturally identified with a subfield in $R_0(0)\subset R_0(N)$.

Proposition 5.2. *The correspondences $T_1\mapsto\tau^{(1)},\dots,T_N\mapsto\tau^{(N)}$ determine a unique P-continuous identification ι of the N-dimensional local field $\mathcal{K}=k((T_N))\dots((T_1))$ and the N-dimensional local subfield in $R_0(N)$ with the system of local parameters $\tau^{(1)},\dots,\tau^{(N)}$ and the residue field k.*

Proof. We need the following obvious lemma.

Lemma 5.3. *Suppose $L\in\mathrm{LF}_0(N)$ has a standard F-structure, which is compatible with given local parameters $t_1,\dots,t_N$. If $c>0$ is such that $p\in\mathrm{m}^1_L(c)$, then any $o\in\mathcal{O}_L$ can be uniquely presented modulo $\mathrm{m}^1_L(c)$ in the form*

$$\sum_{a_1<c}\alpha_{\bar a}t_1^{a_1}\dots t_N^{a_N}.$$

Remark. The coefficients $[\alpha_{\bar a}]$ are the Teichmüller representatives of the elements of the last residue field of L and satisfy the standard restrictions from the beginning of section 1.1.

Continue the proof of Proposition 5.2.

We first prove that the power series

$$\sum_{\bar a\geqslant\bar 0_N}\alpha_{\bar a}\tau^{(1)a_1}\dots\tau^{(N)a_N}\tag{3}$$

converges in $R(N)$ if its coefficients $\alpha_{\bar a}$ satisfy the restrictions described in section 1.1. This is equivalent to the fact that for all $u>u_N^0$, the series

$$\sum_{\bar a\geqslant\bar 0_N}\alpha_{\bar a}^{p^{-u}}\tau_u^{(1)a_1}\dots\tau_u^{(N)a_N}\tag{4}$$

converge to elements $f_u\in\mathcal{O}_{\mathbb{C}(N)_p}$ such that $f_{u+1}^p\equiv f_u\operatorname{mod}\mathrm{m}^1_{K_\centerdot}(c_1^*)$.

Let $\bar v(u)=(u_1,\dots u_{N-1},u_N)\in U(m_1,\dots,m_{N-1})$ be such that $u_N=u$. Then for $1\leqslant r\leqslant N$, it holds $u\leqslant u_r$ and

$$\tau_u^{(r)}\equiv\tau_{u_r}^{(r)p^{u_r-u}}\operatorname{mod}\mathrm{m}^1_{K_\centerdot}(c_1^*).$$

This means that the above series (4) can be expressed in terms of local parameters of the field $K_{\bar v(u)}$, its coefficients $[\alpha_{\bar a}]^{p^{-u}}$ satisfy the restrictions from section 1.1 and, therefore, this series converges in $\mathcal{O}_{\bar v(u)}\subset\mathcal{O}_{\mathbb{C}(N)_p}$. Denote this limit by $f_{\bar v(u)}$. Clearly, the elements $f_{\bar v(u)}$ do not depend modulo $\mathrm{m}^1_{K_\centerdot}(c_1^*)$ on the choice of $\bar v(u)$.

For $u > u_N^0$, prove the congruences $f_{u+1}^p \equiv f_u \bmod \mathrm{m}_{K_\bullet}^1(c_1^*)$.

Choose $\bar v(u+1) \in U(m_1,\dots,m_{N-1})$. Then we can take $\bar v(u) := \bar v(u+1) - (1,\dots,1) \in U(m_1,\dots,m_{N-1})$. Then the congruences from 4.10(a) imply that

$$f_{u+1}^p \equiv f_{\bar v(u+1)}^p \equiv f_{\bar v(u)} \equiv f_u \bmod \mathrm{m}_{K_\bullet}^1(c_1^*).$$

So, the map $\iota|_{\mathcal{O}_\mathcal{K}} : \mathcal{O}_\mathcal{K} \longrightarrow R(N)$ is well defined.

Then the uniqueness property from Lemma 5.3 implies that any element from $R(N)$ can be presented in at most one way as a sum of the series (3). Indeed, suppose

$$\sum_{\bar a \geqslant \bar 0_N} \alpha_{\bar a} \tau^{(1)a_1} \dots \tau^{(N)a_N} = 0$$

in $R(N)$. Then for all $u \gg 0$ and $\bar v(u) = (u_1,\dots,u_{N-1},u) \in U(m_1,\dots,m_{N-1})$,

$$\sum_{\bar a \geqslant \bar 0_N} \alpha_{\bar a}^{p^{-u}} \tau_{u_1}^{(1)p^{u_1-u}a_1} \dots \tau_{u_{N-1}}^{(N-1)p^{u_{N-1}-u}a_{N-1}} \tau_u^{(N)a_N} \in \mathrm{m}_{K_\bullet}^1(c_1^*).$$

In other words, if $\bar \alpha = (a_1,\dots,a_{N-1},a_N)$ is such that $\alpha_{\bar a} \neq 0$, then $\tau_{u_1}^{(1)p^{u_1-u}a_1} \in \mathrm{m}_{K_\bullet}^1(c_1^*)$ and $a_1 p^{-u} v_{K_\bullet}^1(\tau_{u_1}^{(1)p^{u_1}}) \geqslant c_1^*$. But this is impossible because $v_{K_\bullet}^1(\tau_{u_1}^{(1)p^{u_1}}) = v_{K_{u_1}}^1(\tau_{u_1}^{(1)}) = 1$ does not depend on u.

So, all $\alpha_{\bar a} = 0$ and the image $\iota(\mathcal{K})$ is an N-dimensional local field with the set of local parameters $\tau^{(1)},\dots,\tau^{(N)}$.

The proposition is proved. □

Notice that the above fields $\iota(\mathcal{K}) \subset R_0(N)$ are not uniquely determined by a given $K_\bullet \in \mathcal{B}^{fa}(N)$. They do depend on the choice of the structural functions $m_1,\dots,m_{N-1}$ and on the next choice of the modified system of local parameters $\{\tau_u^{(r)}\}_{u \geqslant u_r^0}$, $1 \leqslant r \leqslant N$. Denote by $X(K_\bullet; m_1,\dots,m_{N-1})$ the family of all fields $\mathcal{K}$ which can be constructed for a given tower $K_\bullet$ by the use of a given vector-index $\bar u^0(K_\bullet)$ and the ramification invariant $c^*(\bar u^0, K_\bullet)$ together with an appropriate choice of strictly increasing structural functions $m_1,\dots,m_{N-1}$. Notice that taking a bigger vector-index, and a smaller ramification invariant together with the contraction of the domain of definition of functions $m_1,\dots,m_{N-1}$ doesn't affect this family. For a fixed $K_\bullet$, all $X(K_\bullet, m_1,\dots,m_{N-1})$ form an inductive system. Its limit we shall denote by $X(K_\bullet)$.

5.3. The categories $\mathrm{LF}_R(N)$ and $\widetilde{\mathrm{LF}_R(N)}$. Consider the category $\mathrm{LF}_R(N)$ of all N-dimensional closed subfields $\mathcal{K}$ in $R_0(N)$ together with the induced F-structure given by the subfields of r-dimensional constants $\mathcal{K}(r) = R_0(r) \cap \mathcal{K}$, $0 \leqslant r \leqslant N$. If $\mathcal{K}, \mathcal{L} \in \mathrm{LF}_R(N)$, then $\mathrm{Hom}_{\mathrm{LF}_R(N)}(\mathcal{K}, \mathcal{L})$ consists of sequentially P-continuous morphisms $f : R_0(N) \longrightarrow R_0(N)$, which are compatible with F-structure and such that $f(\mathcal{K}) \subset \mathcal{L}$.

Suppose that v^1 is a 1-dimensional valuation coinciding with one of equivalent valuations $\mathrm{pr}_1\, v_{R,K_{\bullet}}$, where $K_{\bullet} \in \mathcal{B}^{fa}(N)$. For a subfield $\mathcal{L}$ in $R_0(N)$ denote by $\mathcal{R}(\mathcal{L})$ the v^1-adic closure of the maximal inseparable extension of $\mathcal{L}$ in $R_0(N)$.

Definition. If $\mathcal{K}, \mathcal{L} \in \mathrm{LF}_R(N)$, then $\mathcal{K} \sim \mathcal{L}$ if for $1 \leqslant r \leqslant N$, $\mathcal{K}(r)\mathcal{R}(\mathcal{K}(r-1)) = \mathcal{L}(r)\mathcal{R}(\mathcal{L}(r-1))$, where the composite is taken in the category of v^1-adic closed subfields of $R_0(N)$.

Clearly, the above-defined relation $\sim$ is an equivalence relation. Denote by $\widetilde{\mathrm{LF}}_R(N)$ the category whose objects are the equivalence classes $\mathrm{cl}(\mathcal{K})$ of all $\mathcal{K} \in \mathrm{LF}_R(N)$ and for any $\mathrm{cl}(\mathcal{K}), \mathrm{cl}(\mathcal{L}) \in \widetilde{\mathrm{LF}}_R(N)$, $\mathrm{Hom}_{\widetilde{\mathrm{LF}}_R(N)}(\mathrm{cl}(\mathcal{K}), \mathrm{cl}(\mathcal{L}))$ consists of sequentially P-continuous field morphisms $f : R_0(N) \longrightarrow R_0(N)$ which are compatible with F-structure and such that for any $1 \leqslant r \leqslant N$, $f(\mathcal{K}(r)) \subset \mathcal{L}(r)\mathcal{R}(\mathcal{L}(r-1))$.

We shall need below the following property, which is an easy consequence of the (usual 1-dimensional) Krasner Lemma.

Proposition 5.4. *Suppose $\mathcal{L}_1, \mathcal{L} \in \mathrm{LF}_R(N)$, $\mathcal{L}_1 \supset \mathcal{L}$ is a separable extension of degree $m \in \mathbb{N}$. If $\mathcal{L}' \in \mathrm{LF}_R(N)$, $\mathcal{L}' \sim \mathcal{L}$, then there is a unique $\mathcal{L}'_1 \in \mathrm{LF}_R(N)$ such that $\mathcal{L}'_1$ is a separable extension of $\mathcal{L}'$ of degree m and $\mathcal{L}'_1 \sim \mathcal{L}_1$.*

Remark. By this proposition we can use below the concept of a finite separable extension in the category $\widetilde{\mathrm{LF}}_R(N)$.

Proof. We can proceed by induction on N and, therefore, can assume that $\mathcal{L}_1(N-1) = \mathcal{L}(N-1)$. Then $\mathcal{L}' \sim \mathcal{L}$ implies that $\mathcal{L} \subset \mathcal{L}\mathcal{R}(\mathcal{L}(N-1)) = \mathcal{L}'\mathcal{R}(\mathcal{L}'(N-1))$.

Suppose $\mathcal{L}_1 = \mathcal{L}(\alpha)$, where α is a root of a separable polynomial $G(T) \in \mathcal{L}[T]$, $\deg G = m$. By Krasner's Lemma there is a finite extension E' of $\mathcal{L}'(N-1)$ in $\mathcal{R}(\mathcal{L}'(N-1))$ and $\widetilde{G}(T) \in \mathcal{L}'E'[T]$ such that $\deg \widetilde{G} = m$ and $\mathcal{L}_1\mathcal{L}'\mathcal{R}(\mathcal{L}'(N-1)) = \mathcal{L}'\mathcal{R}(\mathcal{L}'(N-1))(\beta)$, where β is a root of $\widetilde{G}(T)$. ($\widetilde{G}$ is a sufficiently nice v^1-adic approximation of G.) Notice that E' is a purely inseparable extension of $\mathcal{L}'(N-1)$.

Let $\widetilde{\mathcal{L}}_1 = \mathcal{L}'E'(\beta)$. Then $\widetilde{\mathcal{L}}_1$ is a separable extension of $\mathcal{L}'E'$ of degree m. Let $\mathcal{L}'_1$ be a separable closure of $\mathcal{L}'$ in $\widetilde{\mathcal{L}}_1$. Then

(α) $\mathcal{L}'_1$ is a separable extension of $\mathcal{L}'$ of degree m;

(β) $\mathcal{L}'_1(N-1) = \mathcal{L}'(N-1)$, because $\mathcal{L}'_1(N-1)$ is a separable extension of $\mathcal{L}'(N-1)$ inside E';

(γ) $\mathcal{L}'_1E' = \widetilde{\mathcal{L}}_1$ and this implies that $\mathcal{L}'_1\mathcal{R}(\mathcal{L}'_1(N-1)) = \widetilde{\mathcal{L}}_1\mathcal{R}(\widetilde{\mathcal{L}}_1(N-1))$.

Finally, notice that (β) and (γ) imply that $\mathcal{L}'_1 \sim \widetilde{\mathcal{L}}_1 \sim \mathcal{L}_1$.

The proposition is proved. □

5.4. The class $\mathrm{cl}(K_\centerdot)\in\widetilde{\mathrm{LF}}_R(N)$, $K_\centerdot\in\mathcal{B}^{fa}(N)$**.**

Proposition 5.5. *Suppose* $K_\centerdot\in\mathcal{B}^{fa}(N)$. *Then all fields from* $X(K_\centerdot)$ *are equivalent in* $\widetilde{\mathrm{LF}}_R(N)$.

Proof. Let $\bar u^0(K_\centerdot)=(u_1^0,\dots,u_N^0)$ and $c_1^*=c^*(\bar u^0(K_\centerdot),K_\centerdot)/p$ be parameters of $K_\centerdot$.

Suppose $\mathcal{K}\in X(K_\centerdot,m_1,\dots,m_{N-1})$ is obtained via a choice of strictly increasing functions $m_r:\mathbb{Z}_{\geqslant u^0_{r+1}}\longrightarrow\mathbb{Z}_{\geqslant u^0_r}$, and a special system of local parameters $\tau_u^{(r)}$, where $1\leqslant r\leqslant N$ and $u\geqslant u_r^0$, from Proposition 4.11.

Take some $u>u_N^0$ and choose $\bar u=(u_1,\dots,u_{N-1},u)\in U(m_1,\dots,m_{N-1})$.

Set $\mathcal{K}_{\bar u}=\mathcal{K}\left(\sigma^{u-u_{N-1}}\mathcal{K}(N-1)\right)\dots(\sigma^{u-u_1}\mathcal{K}(1))$, where σ is, as usual, the pth power map. Then

$$\sigma^{u-u_1}(\tau^{(1)}),\dots,\sigma^{u-u_{N-1}}(\tau^{(N-1)}),\tau^{(N)}$$

is a system of local parameters in $\mathcal{K}_{\bar u}$ which is compatible with a given standard F-structure of $\mathcal{K}_{\bar u}$. In particular, this will imply the equality $v_{\mathcal{K}}=(p^{u-u_1},\dots,p^{u-u_{N-1}},1)v_{\mathcal{K}_{\bar u}}$.

For $1\leqslant r\leqslant N$, the correspondences $\sigma^{u-u_r}\tau^{(r)}\mapsto\tau^{(r)}_{u_r}$ give the ring identification

$$\psi_{\bar u}:\mathcal{O}_{\mathcal{K}_{\bar u}}\operatorname{mod}\mathrm{m}^1_{R,K_\centerdot}(p^uc_1^*)\simeq\mathcal{O}_{K_{\bar u}}\operatorname{mod}\mathrm{m}^1_{K_\centerdot}(c_1^*).$$

Notice that this identification transforms $v_{\mathcal{K}_{\bar u}}$ to $v_{K_{\bar u}}$. Therefore, $\psi_{\bar u}$ transforms $v_{\mathcal{K}}$ to $v_{K_u}=p^uv_{K_\centerdot}$ for $u>u_N^0$.

If $\bar u'=(u_1',\dots,u_{N-1}',u)\in U(m_1,\dots,m_{N-1})$ is such that $u_r'\geqslant u_r$ for all $1\leqslant r<N$, then $\psi_{\bar u}$ and $\psi_{\bar u'}$ are compatible via the natural inclusions $\mathcal{K}_{\bar u}\subset\mathcal{K}_{\bar u'}$ and $\mathcal{O}_{K_{\bar u}}\subset\mathcal{O}_{K_{\bar u'}}$. Therefore, the uth projection $\mathrm{pr}_u:R(N)\longrightarrow\mathcal{O}_{\mathbb{C}(N)_p}\operatorname{mod}m^1_{K_\centerdot}(c_1^*)$ induces the identification

$$\psi_u:\mathcal{O}_{\mathcal{K}\mathcal{R}(\mathcal{K}(N-1))}\operatorname{mod}\mathrm{m}^1_{R,K_\centerdot}(p^uc_1^*)\longrightarrow\mathcal{O}^{(u)}\operatorname{mod}m^1_{K_\centerdot}(c_1^*)$$

where $\mathcal{O}^{(u)}$ is the valuation ring of the composite of all $K_{\bar u}$ with $\bar u$ running over the set of all $\bar u=(u_1,\dots,u_{N-1},u_N)\in U(m_1,\dots,m_{N-1})$ such that $u_N=u$.

In order to understand the relation between different ψ_u, notice that if $\bar v(u+1)=(u_1,\dots,u_{N-1},u+1)\in U(m_1,\dots,m_{N-1})$, then $\bar v(u)=(u_1-1,\dots,u_{N-1}-1,u)\in U(m_1,\dots,m_{N-1})$ and $\mathcal{K}_{\bar v(u)}=\mathcal{K}_{\bar v(u+1)}$. This implies that $\psi_{\bar v(u)}$ and $\psi_{\bar v(u+1)}$ fit into a commutative diagram via the natural projection

$$\mathcal{O}_{\mathcal{K}_{\bar v(u+1)}}\operatorname{mod}\mathrm{m}^1_{R,K_\centerdot}(p^{u+1}c_1^*)\longrightarrow\mathcal{O}_{\mathcal{K}_{\bar v(u)}}\operatorname{mod}\mathrm{m}^1_{R,K_\centerdot}(p^uc_1^*)$$

and the restriction of the transition morphism of the projective system $\mathcal{O}_{\mathbb{C}(N)_p}\operatorname{mod}\mathrm{m}^1_{K_\centerdot}(c_1^*)$ from the definition of $R(N)$. Therefore, $\varprojlim_u\psi_u$ identifies $\mathcal{O}_{\mathcal{K}\mathcal{R}(\mathcal{K}(N-1))}$ with $\varprojlim_u(\mathcal{O}^{(u)}\operatorname{mod}\mathrm{m}^1_{K_\centerdot}(c_1^*))_u\subset R(N)$. In particular,

$\mathcal{K}\mathcal{R}(\mathcal{K}(N-1))$ does not depend on the choice of functions $m_1, \dots, m_{N-1}$ and the corresponding system of modified local parameters. For similar reasons we also have the similar property for all $\mathcal{K}(r)\mathcal{R}(\mathcal{K}(r-1))$ with $1 \leqslant r < N$.

The proposition is proved. □

If $K_{\bullet} \in \mathcal{B}^{fa}(N)$, then set $\mathrm{cl}(K_{\bullet}) := \mathrm{cl}(\mathcal{K}) \in \widetilde{\mathrm{LF}}_R(N)$, where $\mathcal{K} \in X(K_{\bullet})$. So, $\mathrm{cl}(K_{\bullet})$ denotes the class of equivalence of fields $\mathcal{K} \in X(K_{\bullet})$ in the category $\mathrm{LF}_R(N)$. Notice that if $\mathcal{K} \in \mathrm{cl}(K_{\bullet})$, then $v_{\mathcal{K}} = v_{R,K_{\bullet}}$. Indeed, $\psi_{\bar{u}}$ transforms $v_{\mathcal{K}}$ to $p^u v_{K_{\bullet}}$ (cf. the proof of the above proposition) and, therefore, can be recovered as $\varprojlim_u (p^u v_{K_{\bullet}})_u = v_{R,K_{\bullet}}$.

For future references point out the following corollary of the above considerations.

Corollary 5.6. *If $u > u_N^0$ and $\bar{v}(u) \in U(m_1, \dots, m_{N-1})$ has Nth coordinate u, then the uth projection pr_u from $R(N) = \varprojlim_u (\mathcal{O}_{\mathbb{C}(N)_p} \bmod \mathrm{m}^1_{K_{\bullet}}(c_1^*))_u$ to $\mathcal{O}_{\mathbb{C}(N)_p} \bmod \mathrm{m}^1_{K_{\bullet}}(c_1^*)$ induces the identification*

$$\psi_{\bar{v}(u)} : \mathcal{O}_{\mathcal{K}_{\bar{v}(u)}} \bmod \mathrm{m}^1_{\mathcal{K}}(p^u c_1^*) \longrightarrow \mathcal{O}_{K_{\bar{v}(u)}} \bmod \mathrm{m}^1_{K_u}(p^u c_1^*)$$

and this identification transforms $v_{\mathcal{K}}$ to v_{K_u}.

5.5. If $K_{\bullet}, L_{\bullet} \in \mathcal{B}^{fa}(N)$ denote by $\widetilde{K}$ and, resp., $\widetilde{L}$ the p-adic completions of the fields $\bigcup_{m \geqslant 0} K_m$ and, resp., $\bigcup_{m \geqslant 0} L_m$.

Proposition 5.7. *With the above notation, if $\widetilde{K} = \widetilde{L}$, $\mathcal{K} \in \mathrm{cl}(K_{\bullet})$ and $\mathcal{L} \in \mathrm{cl}(L_{\bullet})$, then $\mathcal{R}(\mathcal{K}) = \mathcal{R}(\mathcal{L})$.*

Proof. Suppose $K_{\bullet}, L_{\bullet} \in \mathcal{B}^{fa}(N)$ are such that $\widetilde{K} = \widetilde{L}$. By induction on N we can assume that for $\mathcal{K}' \in \mathrm{cl}(K_{\bullet}(N-1))$ and $\mathcal{L}' \in \mathrm{cl}(L_{\bullet}(N-1))$, it holds that $\mathcal{R}(\mathcal{K}') = \mathcal{R}(\mathcal{L}')$. We can assume also that $\bar{u}^0(K_{\bullet}) = \bar{u}^0(L_{\bullet}) := \bar{u}^0$ and the ramification parameters $c^*(\bar{u}^0, K_{\bullet})$ and $c^*(\bar{u}^0, L_{\bullet})$ are such that $\mathrm{m}^1_{K_{\bullet}}(c_1^*) = \mathrm{m}^1_{L_{\bullet}}(d_1^*) := \mathrm{m}^1_0$, where $c_1^* = c^*(\bar{u}^0, K_{\bullet})/p$ and $d_1^* = c^*(\bar{u}^0, L_{\bullet})/p$. We can also assume that $\mathcal{K} \in X(K_{\bullet}, m_1, \dots, m_{N-1})$ and $\mathcal{L} \in X(L_{\bullet}, m_1, \dots, m_{N-1})$ with the same strictly increasing functions $m_r : \mathbb{Z}_{\geqslant u^0_{r+1}} \longrightarrow \mathbb{Z}_{\geqslant u^0_r}$, $1 \leqslant r < N$.

For any $u > u_N^0$, choose a vector $\bar{v}(u) \in U(m_1, \dots, m_{N-1})$ with Nth coordinate u. Then there is a $\bar{w}(u) \in U(m_1, \dots, m_{N-1})$ and an embedding

$$\mathcal{O}_{K_{\bar{v}(u)}} \bmod \mathrm{m}^1_0 \subset \mathcal{O}_{L_{\bar{w}(u)}} \bmod \mathrm{m}^1_0$$

induced by the embeddings $K_{\bar{v}(u)} \subset \widetilde{K}$, $L_{\bar{w}(u)} \subset \widetilde{L}$ and the identification $\widetilde{K} = \widetilde{L}$.

If u' is an Nth coordinate of $\bar{w}(u)$, then by Corollary 5.6 we obtain the embeddings

$$\delta_u : \mathcal{O}_{\mathcal{K}_{\bar{v}(u)}} \bmod \mathrm{m}^1_{\mathcal{K}}(p^u c_1^*) \longrightarrow \sigma^{u-u'} \mathcal{O}_{\mathcal{L}_{\bar{w}(u)}} \bmod \mathrm{m}^1_{\mathcal{K}}(p^u c_1^*)$$

(notice that $\mathrm{m}^1_{\mathcal{K}}(p^u c_1^*) = \mathrm{m}^1_{\mathcal{L}}(p^u d_1^*)$). The embedding δ_u, $u > u_N^0$, is induced by the identity morphism of $R(N) \bmod \mathrm{m}^1_{\mathcal{K}}(p^u c_1^*)$ and natural embeddings of $\mathcal{K}$ and $\mathcal{L}$ into $R_0(N)$. Therefore, the projective limit of all δ_u induces the embedding of $\mathcal{O}_{\mathcal{K}}$ into $\mathcal{O}_{\mathcal{R}(\mathcal{L})}$ and this embedding is compatible with the natural embeddings of $\mathcal{K}$ and $\mathcal{L}$ into $R_0(N)$.

This proves that $\mathcal{R}(\mathcal{K}) \subset \mathcal{R}(\mathcal{L})$. By symmetry, we also have the opposite embedding. The proposition is proved. □

6. Separable extensions in $\mathcal{B}^{fa}(N)$ and $\widetilde{\mathrm{LF}}_R(N)$

6.1. In this subsection we prove that the correspondence $K_{\bullet} \mapsto \mathrm{cl}(K_{\bullet})$, transforms finite separable extensions in $\mathcal{B}^{fa}(N)$ to finite separable extensions of the same degree in $\widetilde{\mathrm{LF}}_R(N)$ (cf. section 5.3).

Proposition 6.1. *Suppose $L_{\bullet}, K_{\bullet} \in \mathcal{B}^{fa}(N)$ and $L_{\bullet} \supset K_{\bullet}$ is separable of degree $d(L_{\bullet}/K_{\bullet}) = d$. If $\mathcal{K} \in \mathrm{cl}(K_{\bullet})$ and $\mathcal{L} \in \mathrm{cl}(L_{\bullet})$, then $\mathrm{cl}(\mathcal{L})$ is a separable extension of $\mathrm{cl}(\mathcal{K})$ of degree d.*

Proof. We can assume that:

— $\bar{u}^0(K_{\bullet}) = \bar{u}^0(L_{\bullet}) = (u_1^0, \dots, u_N^0)$, $L_{\bullet} = L_{u_N^0} K_{\bullet}$ and $d(L_{\bullet}/K_{\bullet}) = [L_{u_N^0} : K_{u_N^0}]$;

— $K_{\bullet}$ and $L_{\bullet}$ have common strictly increasing structural functions $m_r : \mathbb{Z}_{\geqslant u_{r+1}^0} \longrightarrow \mathbb{Z}_{\geqslant u_r^0}$, where $1 \leqslant r < N$, such that $m_r(u_{r+1}^0) = u_r^0$;

— $c_1^*(\bar{u}^0, L_{\bullet}) = e^1 c_1^*(\bar{u}^0, K_{\bullet})$, where $e^1 = \mathrm{pr}_1(\bar{e}(L_{\bullet}/K_{\bullet}))$; in other words, $\mathrm{m}^1_{K_{\bullet}}(c_1^*(\bar{u}^0, K_{\bullet})) = \mathrm{m}^1_{L_{\bullet}}(c_1^*(\bar{u}^0, L_{\bullet}))$ (this ideal will be denoted below by m_0^1);

— for all $u \geqslant u_N^0$, the Herbrand functions of extensions L_u/K_u coincide and are equal to $\varphi_{L_{\bullet}/K_{\bullet}}$ and $\mathrm{pr}_1(i(L_u/K_u)) + \delta_{1N} < p^u c_1^*(\bar{u}^0, L_{\bullet})$.

We must prove that if $\mathcal{L} \in X(K_{\bullet}, m_1, \dots, m_{N-1})$, $\mathcal{K} \in X(K_{\bullet}, m_1, \dots, m_{N-1})$, then $\mathcal{L}$ is equivalent in the category $\mathrm{LF}_R(N)$ to a separable extension of $\mathcal{K}$ of degree d.

Notice that $\mathcal{K}(N-1) \in X(K_{\bullet}(N-1), m_1, \dots, m_{N-2})$ and $\mathcal{L}(N-1) \in X(L_{\bullet}(N-1), m_1, \dots, m_{N-2})$. Therefore, by induction on N we can assume that $L_{\bullet}(N-1) = K_{\bullet}(N-1)$ and $\mathcal{K}(N-1) = \mathcal{L}(N-1)$.

Consider the corresponding sequence of multi-indices $\bar{u}^0, \bar{v}^1, \bar{u}^1, \dots, \bar{v}^n, \bar{u}^n, \dots$ from section 4.3.5 and the corresponding field towers:

$$L_{\bar{u}^0} \subset L_{\bar{v}^1} \subset L_{\bar{u}^1} \subset L_{\bar{v}^2} \subset \dots \subset L_{\bar{v}^n} \subset L_{\bar{u}^n} \subset \dots$$
$$K_{\bar{u}^0} \subset K_{\bar{v}^1} \subset K_{\bar{u}^1} \subset K_{\bar{v}^2} \subset \dots \subset K_{\bar{v}^n} \subset K_{\bar{u}^n} \subset \dots$$

For any $u \geqslant u_N^0$, set $n = n(u) = u - u_N^0$. So, if $u > u_0^N$, then $\bar{u}^n = (u_1^n, \dots, u_{N-1}^n, u) \in U(m_1, \dots, m_{N-1})$.

For $1\leqslant r\leqslant N$ and $u\geqslant u_r^0$, let $\tau_u^{(r)}\in\mathbb{C}(r)_p$ be the modified local parameters from section 4.3.6 used to construct the field $\mathcal{L}\in X(L_\centerdot,m_1,\dots,m_{N-1})$. They satisfy the conditions of Proposition 4.11 with $K_\centerdot$ replaced by $L_\centerdot$.

If $\varprojlim_u \tau_u^{(r)}=\tau_r\in R(N)$ and k is the last residue field of $\mathcal{L}$ (and of $\mathcal{K}$, as well), then $\mathcal{L}=k((\tau_N))\dots((\tau_1))$. For $u\geqslant u_N^0$, let τ'_u be the norm of $\tau_u^{(N)}$ in the extension $L_{\bar u^n}/K_{\bar u^n}$. This gives a system of Nth local parameters τ'_u for $K_{\bar u^n}$ such that $\varprojlim_u \tau'_u=\tau'\in R(N)$ and $\mathcal{K}\sim\mathcal{K}_1=k((\tau'))((\tau_{N-1}))\dots((\tau_1))\in X(K_\centerdot,m_1,\dots,m_{N-1})$.

For any $u\geqslant u_N^0$, $\eta_u:=\tau_u^{(N)}$ belongs to $\mathcal{O}_{L_{\bar u^n}}$ and is a root of an Nth Eisenstein polynomial $F_u(T)=T^d+a_{1u}T^{N-1}+\dots+a_{id}\in\mathcal{O}_{K_{\bar u^n}}[T]$.

Notice that for any $n\geqslant 0$, $L_{\bar u^n}=L_{u_N^0}K_{\bar u^n}$ and we have a natural identification of the sets $I_{L_{\bar u^n}/K_{\bar u^n}}$ of all isomorphic embeddings of $L_{\bar u^n}$ into $\mathbb{C}(N)_p$, which are the identity on $K_{\bar u^n}$, with the set $I_{L_{u_N^0}/K_{u_N^0}}$. Then, for any $u\geqslant u_N^0$, another root of F_u appears in the form $g(\eta_u)$, where $g\in I_{L_{u_N^0}/K_{u_N^0}}$, $g\neq\mathrm{id}$. Therefore, the conditions $\eta_{u+1}^p\equiv\eta_u \bmod \mathrm{m}_0^1$, imply that for all $1\leqslant i\leqslant d$, $a_{i,u+1}^p\equiv a_{iu}\bmod \mathrm{m}_0^1$. It then follows from Corollary 5.6 that for all $1\leqslant i\leqslant d$, $\varprojlim_u(a_{iu}\bmod \mathrm{m}_0^1)_u:=\alpha_i\in\mathcal{O}_{\widetilde{\mathcal{K}}}$, where $\widetilde{\mathcal{K}}=\mathcal{K}\mathcal{R}(\mathcal{K}(N-1))=\mathcal{K}_1\mathcal{R}(\mathcal{K}_1(N-1))$.

Therefore, $F(T):=T^n+\alpha_1T^{N-1}+\dots+\alpha_d\in\mathcal{O}_{\widetilde{\mathcal{K}}}[T]$ is the Nth Eisenstein polynomial and $\eta:=\tau_N\in\mathcal{L}$ is its root. Clearly, $\widetilde{\mathcal{L}}=\widetilde{\mathcal{K}}(\eta)$, where $\widetilde{\mathcal{L}}=\mathcal{L}\mathcal{R}(\mathcal{L}(N-1))$.

Prove that $\widetilde{\mathcal{L}}$ is separable over $\widetilde{\mathcal{K}}$. Notice that for any $g\in I_{L_{u_N^0}/K_{u_N^0}}$, $\varprojlim_u g(\eta_u)$ is a root of $F(T)$ in $R(N)$. It remains to prove that these roots are different for different elements $g\in I_{L_{u_N^0}/K_{u_N^0}}$.

Suppose $g_1,g_2\in I_{L_{u_N^0}/K_{u_N^0}}$ and $\varprojlim_u g_1(\eta_u)=\varprojlim_u g_2(\eta_u)$. Then for any $u\geqslant u_N^0$, $g_1(\eta_u)=g_2(\eta_u)\bmod \mathrm{m}_0^1$. Because $\mathrm{m}_0^1=\mathrm{m}_{L_u}^1(p^uc_1^*(\bar u^0,L_\centerdot))$ this implies for $g=g_2^{-1}g_1$,

$$v_{L_u}(g(\eta_u)-\eta_u)>i(L_u/K_u)+v_{L_u}(\eta_u).$$

So, by the definition of the biggest lower ramification number $i(L_u/K_u)$, we have $g=\mathrm{id}$, i.e. $g_1=g_2$.

Therefore, $F(T)$ has d distinct roots in $R(N)$ and $\widetilde{\mathcal{L}}$ is separable over $\widetilde{\mathcal{K}}$.

Finally, applying arguments from the proof of Proposition 5.4 we obtain the existence of $\mathcal{L}_1\sim\mathcal{L}$ in the category $\mathrm{LF}_R(N)$, which is separable of degree d over $\mathcal{K}$. □

Corollary 6.2. *In addition to the assumptions of the above proposition assume that $\mathcal{L}_1 \in \mathrm{cl}(\mathcal{L})$ is a separable extension of $\mathcal{K}$ of degree d. Then:*

(a) *there is a natural identification of the set $I_{L_\bullet/K_\bullet}$ of all isomorphic embeddings ι of $L_\bullet$ into $\mathbb{C}(N)_p$ such that $\iota|_{K_\bullet} = \mathrm{id}$ and the set $I_{\mathcal{L}_1/\mathcal{K}}$ of all isomorphic embeddings $\iota : \mathcal{L}_1 \longrightarrow R_0(N)$ such that $\iota|_{\mathcal{K}} = \mathrm{id}$;*

(b) $\varphi_{L_\bullet/K_\bullet} = \varphi_{\mathcal{L}_1/\mathcal{K}}$.

Proof. We can assume that $K_\bullet(N-1) = L_\bullet(N-1)$. From the proof of the above proposition we obtain a natural identification of the set $I_{L_\bullet/K_\bullet}$ and the set $I_{\widetilde{\mathcal{L}}/\widetilde{\mathcal{K}}}$ of all field embeddings of $\widetilde{\mathcal{L}}$ into $R_0(N)$ which are identical on $\widetilde{\mathcal{K}}$ (notice that $\mathcal{K}(N-1) = \mathcal{L}_1(N-1)$). Then Proposition 5.4 gives the identification of the sets $I_{\widetilde{\mathcal{L}}/\widetilde{\mathcal{K}}}$ and $I_{\mathcal{L}_1/\mathcal{K}}$.

So, for large u, we have a natural identification of the sets $I_{\mathcal{L}_1/\mathcal{K}}$ and I_{L_u/K_u} and this identification transforms $v_{\mathcal{L}_1}$ to v_{L_u}. This implies that the identification of sets from section (a) is compatible with the ramification filtration in lower numbering and, as a result, we obtain the equality of the Herbrand functions $\varphi_{L_\bullet/K_\bullet} = \varphi_{L_u/K_u} = \varphi_{\mathcal{L}_1/\mathcal{K}}$. □

6.2. With the above notation we are going to now prove that for a sufficiently large separable extension $E_\bullet$ of $K_\bullet$, the appropriate $\mathcal{E} \in \mathrm{cl}(E_\bullet)$ contains any given separable extension of $\mathcal{K}$ in $R_0(N)$. By induction on N this will be implied by the following proposition.

Proposition 6.3. *Suppose $K_\bullet \in \mathcal{B}^{fa}(N)$, $\mathcal{K} \in X(K_\bullet)$ and $\mathcal{L}$ is a finite separable extension of $\mathcal{K}$ of degree $d > 1$ with a standard F-structure such that $\mathcal{K}(N-1) = \mathcal{L}(N-1)$. Then there is an $L_\bullet \in \mathcal{B}^{fa}(N)$ and a field embedding $\iota : \mathcal{L} \longrightarrow R_0(N)$ such that:*

(a) *$L_\bullet$ is a separable extension of $K_\bullet$ of degree d;*

(b) *$\iota(\mathcal{L}) \in \mathrm{cl}(L_\bullet)$.*

Proof. We can assume that:

— there are parameters $\bar{u}^0(K_\bullet) = (u_1^0, \dots, u_N^0)$, $c_1^*(u_N^0, K_\bullet) = c_1^*$ and strictly increasing structural functions $m_r : \mathbb{Z}_{\geqslant u_{r+1}^0} \mapsto \mathbb{Z}_{\geqslant u_r^0}$, where $1 \leqslant r < N$, such that $\mathcal{K} \in X(K_\bullet; m_1, \dots, m_{N-1})$;

— $\mathcal{O}_{\mathcal{L}} = \mathcal{O}_{\mathcal{K}}[\theta]$ where θ is a root of the Nth Eisenstein polynomial $\mathcal{F}(T) = T^d + \alpha_1 T^{d-1} + \dots + \alpha_d \in \mathcal{O}_{\mathcal{K}}[T]$;

— if $v_{\mathcal{K}}^1(D(\mathcal{F})) = D^1$ and $u \geqslant u_N^0$, where $D(\mathcal{F})$ is the discriminant of $\mathcal{F}$, then $2D^1 < p^u c_1^* - 2(d-1)\delta_{1N}$.

Consider the sequence $\bar{u}^0 = \bar{u}^0(K_\bullet), \bar{u}^1, \dots, \bar{u}^n, \dots$ and set $u = n + u_N^0$; this is the Nth coordinate of $\bar{u}^n$. For $u \geqslant u_N^0$, introduce the polynomials

$$F_u(T) = T^d + a_{1u}T^{d-1} + \dots + a_{du} \in \mathcal{O}_{K_{\bar{u}^n}}[T]$$

where for $1 \leqslant u \leqslant d$, $a_{iu} \in \mathcal{O}_{K_{\bar{u}^n}}$ are such that $a_{iu} \operatorname{mod} \mathrm{m}^1_{K_\bullet}(c_1^*) = \mathrm{pr}_u(\alpha_i)$. Recall that the projection $\mathrm{pr}_u : R(N) \longrightarrow \mathcal{O}_{\mathbb{C}(N)_p} \operatorname{mod} m^1_{K_\bullet}(c_1^*)$ induces identification of $\mathcal{O}_{\mathcal{K}_{\bar{u}^n}} \operatorname{mod} \mathrm{m}^1_{\mathcal{K}}(p^u c_1^*)$ and $\mathcal{O}_{K_{\bar{u}^n}} \operatorname{mod} \mathrm{m}^1_{K_u}(p^u c_1^*)$. This identification transfers the valuation $v_{\mathcal{K}}$ to the valuation v_{K_u} (cf. Corollary 5.6). So, for all $u \geqslant u_N^0$, F_u are Nth Eisenstein polynomials and their discriminants $D(F_u)$ satisfy the following congruences

$$D(F_u) \equiv D(F_{u+1})^p \operatorname{mod} \mathrm{m}^1_{K_\bullet}(c_1^*)$$

and

$$D(F_u) \operatorname{mod} \mathrm{m}^1_{K_u}(p^u c_1^*) = D(\mathcal{F}) \operatorname{mod} \mathrm{m}^1_{\mathcal{K}}(p^u c_1^*).$$

From the above restriction on $D^1 = v^1_{\mathcal{K}}(D(\mathcal{F}))$ it follows that for all $u \geqslant u_N^0$, $D^1 < p^u c_1^*$. Therefore, $D(\mathcal{F}) \notin \mathrm{m}^1_{\mathcal{K}}(p^u c_1^*)$ and $v_{K_u}(D(F_u)) = v_{\mathcal{K}}(D(\mathcal{F}))$.

Choose a root $\eta_{u_N^0} \in \mathcal{O}_{\mathbb{C}(N)_p}$ of $F_{u_0^N}$ and set $L_{u_N^0} = K_{u_N^0}(\eta_{u_N^0})$. Then $\eta_{u_N^0}$ is Nth local parameter in $L_{u_N^0}$, $[L_{u_N^0} : K_{u_N^0}] = d$, $L_{u_N^0}(N-1) = K_{u_N^0}(N-1)$ and $\bar{e} := \bar{e}(L_{u_N^0}/K_{u_N^0}) = (1, \dots, 1, d)$. Notice that $e^1 = \mathrm{pr}_1(\bar{e})$ is 1 if $N \neq 1$ and is d if $N = 1$.

For $u \geqslant u_N^0$, we want to prove the existence of roots $\eta_u \in \mathcal{O}_{\mathbb{C}(N)_p}$ of $F_u(T)$ such that $\eta_{u_N^0}$ is the above chosen root of $F_{u_N^0}$, and if $M_u = K_{\bar{u}^n}(\eta_u)$, then:

(1) η_u is the Nth local parameter in M_u;

(2) $M_u = L_{u_N^0} K_{\bar{u}^n}$;

(3) $\eta_u - \eta_{u+1}^p \in \mathrm{m}^1_{M_u}(e^1 e_{1u} p^u c_1^*/2)$, where $e_{1u} = \mathrm{pr}_1(\bar{e}(K_{\bar{u}^n}/K_u))$.

Notice that $M_{u_N^0} = L_{u_N^0}$ and the above properties (1)–(3) imply that $\mathrm{m}^1_{M_u}(e^1 e_{1u} p^u c_1^*/2) = \mathrm{m}^1_{K_u}(p^u c_1^*/2) = \mathrm{m}^1_{K_\bullet}(c_1^*/2)$ does not depend on $u \geqslant u_N^0$.

Suppose $u \geqslant u_N^0$ and such roots $\eta_{u_N^0}, \dots, \eta_u$ have already been constructed.

Let $\theta_{u+1} \in \mathcal{O}_{\mathbb{C}(N)_p}$ be a root of $F_{u+1}(T)$. Then

$$F_u(\theta_{u+1}^p) \in \mathrm{m}^1_{K_u}(p^u c_1^*) = \mathrm{m}^1_{K_{\bar{u}^n}}(e_{1u} p^u c_1^*)$$

and, therefore, $v_{K_{\bar{u}^n}}(F_u(\theta_{u+1}^p)) = j_u + (0, \dots, 0, 1)$ with $\mathrm{pr}_1(j_u) + \delta_{1N} \geqslant e_{1u} p^u c_1^*$.

Lemma 6.4. $j_u > j(M_u/K_{\bar{u}^n})$.

Proof of lemma. From Corollary 3.5 we have

$$\begin{aligned} \mathrm{pr}_1(j(M_u/K_{\bar{u}^n})) &\leqslant 2 v^1_{K_{\bar{u}^n}}(\delta(F_u)) = \frac{2}{d} v^1_{K_{\bar{u}^n}}(D(F_u)) = \frac{2 e_{1u}}{d} D^1 \\ &\leqslant 2 e_{1u} D^1 < e_{1u} p^u c_1^* - 2(d-1)\delta_{1N} \leqslant e_{1u} p^u c_1^* - \delta_{1N} \leqslant \mathrm{pr}_1(j_u). \end{aligned}$$

The lemma is proved. □

Continue the proof of our proposition.

The above lemma together with Krasner's Lemma from section 3.4 imply the existence of a unique root θ_u of F_u such that

$$v_{M_u}(\theta_{u+1}^p - \theta_u) = i_u + (0, \dots, 0, 1)$$

where $\varphi_{M_u/K_{\bar{u}^n}}(i_u) = j_u$. From Lemma 6.4 and the definition of the Herbrand function it follows

$$\frac{j_u - j(M_u/K_{\bar{u}^n})}{i_u - i(M_u/K_{\bar{u}^n})} = \bar{e}^{-1}(M_u/K_{\bar{u}^n})$$

where $\bar{e}(M_u/K_{\bar{u}^n}) = (1, \dots, 1, d)$. This formula together with the formula from Corollary 3.4 gives

$$i_u = (1, \dots, 1, d)j_u - v_{M_u}(\delta(F_u)) + (0, \dots, 0, d-1).$$

Notice that

$$v^1_{M_u}(\delta(F_u)) = \frac{e^1}{d} e_{1u} D^1 < \frac{e^1}{2} e_{1u} p^u c_1^* - e_{1u}(d-1)\delta_{1N}$$

(use our assumption about D^1).

Therefore,

$$\mathrm{pr}_1(i_u) + \delta_{1N} = e^1 pr_1(j_u) - v^1_{M_u}(\delta(F_u)) + d\delta_{1N}$$
$$\geqslant e^1(e_{1u}p^u c_1^* - \delta_{1N}) - \frac{e^1}{2} e_{1u} p^u c_1^* + e_{1u}(d-1)\delta_{1N} + d\delta_{1N}$$
$$= \frac{e^1}{2} e_{1u} p^u c_1^* + e_{1u}(d-1)\delta_{1N} \geqslant \frac{1}{2} e^1 e_{1u} p^u c_1^*.$$

In other words, for any root θ_{u+1} of F_{u+1}, there is a unique root θ_u of F_u such that $\theta_u - \theta^p_{u+1} \in \mathrm{m}^1_{M_u}(e^1 e_{1u} p^u c_1^*/2)$.

Suppose two different roots θ_{u+1} and θ'_{u+1} of F_{u+1} satisfy this condition with the same root θ_u of F_u. Then θ^p_{u+1} is congruent to θ'^p_{u+1} modulo the ideal

$$\mathrm{m}^1_{M_u}(e^1 e_{1u} p^u c_1^*/2) = \mathrm{m}^1_{K_{\bullet}}(c_1^*/2) = \mathrm{m}^1_{K_{u+1}}(p^{u+1} c_1^*/2).$$

Therefore, $\theta_{u+1} - \theta'_{u+1} \in \mathrm{m}^1_{K_{u+1}}(p^u c_1^*/2)$.

But

$$v^1_{K_{u+1}}(\theta_{u+1} - \theta'_{u+1}) \leqslant v^1_{K_{u+1}}(D(F_{u+1})) = D^1 < p^u c_1^*/2.$$

Contradiction.

So, the above correspondence $\theta_{u+1} \mapsto \theta_u$ is a one-to-one correspondence between roots of $F_{u+1}(T)$ and $F_u(T)$. This correspondence is stable under the action of any sequentially P-continuous automorphism of $\mathbb{C}(N)_p$, which is the identity on $K_{\bar{u}^{n+1}}$. So, if η_{u+1} is the root of F_{u+1}, which corresponds to η_u, then $K_{\bar{u}^{n+1}}(\eta_u) = K_{\bar{u}^{n+1}}(\eta_{u+1})$. In other words, $M_{u+1} = M_u K_{\bar{u}^{n+1}} = L_{u^0_N} K_{\bar{u}^{n+1}}$.

The existence of the sequence η_u, $u \geqslant u^0_N$, which satisfies the above requirements (1)–(3), is proved.

Consider the tower $L_{\bullet} = L_{u_N^0} K_{\bullet}$. Then one can easily verify the following:

— $L_{\bullet} \in \mathcal{B}^{fa}(N)$ and has the parameters $\bar{u}^0(K_{\bullet})$ and $e^1 c^*(u_N^0, K_{\bullet})/2$;

— for suitable structural functions $m'_1, \dots, m'_{N-1}$, $\mathcal{K}_{\bar{u}^0}(\eta)$ belongs to the family $X(L_{\bullet}; m'_1, \dots, m'_{N-1})$, where $\eta = \varprojlim \eta_u$ is a root of $\mathcal{F}(T)$ in $R_0(N)$;

— the choice of this root η of $\mathcal{F}(T)$ determines a field isomorphism ι of $\mathcal{L}$ and $\mathcal{K}(\eta)$, which induces the identity on $\mathcal{K}$.

The proposition is proved. □

Corollary 6.5. *Suppose $K_{\bullet} \in \mathcal{B}^{fa}(N)$ with the parameters $\bar{u}^0(K_{\bullet})$ and $c^*(\bar{u}^0, K_{\bullet})$. Suppose that $\mathcal{K} \in X(K_{\bullet})$ and $\mathcal{L}/\mathcal{K}$ is a finite separable extension in $R_0(N)$ with standard F-structure. Then there is an $L_{\bullet} \in \mathcal{B}^{fa}(N)$ such that*

(a) *$L_{\bullet}$ is a separable extension of $K_{\bullet}$;*

(b) *$\mathcal{L} \in X(L_{\bullet})$;*

(c) *for $m \gg 0$, $c^*(m, L_{\bullet}) = e^1 c^*(u_N^0, K_{\bullet})/2^N$, where $e^1 = \mathrm{pr}_1(\bar{e}(\mathcal{L}/\mathcal{K}))$.*

Proof. Apply the construction from the proof of the above proposition to the sequence of extensions $\mathcal{K} \subset \mathcal{K}\mathcal{L}(0) \subset \mathcal{K}\mathcal{L}(1) \subset \cdots \subset \mathcal{K}\mathcal{L}(N) = \mathcal{L}$. This gives $L_{\bullet} \in \mathcal{B}^{fa}(N)$ such that $\mathcal{L} \in X(L_{\bullet})$. Then use that $\bar{e}(\mathcal{L}/\mathcal{K}) = \bar{e}(L_{\bullet}/K_{\bullet})$ and apply Lemma 4.4 from section 4.2. □

Remark. Notice that the ideal $\mathrm{m}^1_{L_{\bullet}}(c_1^*(\bar{v}^0, L_{\bullet})) = \mathrm{m}^1_{K_{\bullet}}(c_1^*(\bar{u}^0, K_{\bullet})/2^N)$ does not depend on $L_{\bullet}$.

Corollary 6.6. *The correspondence $K_{\bullet} \mapsto \mathrm{cl}(K_{\bullet}) \in \widetilde{\mathrm{LF}}_R(N)$, where $K_{\bullet} \in \mathcal{B}^{fa}(N)$, induces the identification of the absolute Galois groups $\psi : \Gamma_{\widetilde{K}} \longrightarrow \Gamma_{\mathcal{K}}$ (here $\widetilde{K}$ is the p-adic closure of the $\bigcup_{m \geqslant 0} K_m$). This identification is compatible with ramification filtrations, i.e. for any $j \in J(N)$, $\Gamma_{K_0}^{(\varphi_{K_{\bullet}}(j))} \cap \Gamma_{\widetilde{K}} = \Gamma_{\mathcal{K}}^{(j)}$, where $\varphi_{K_{\bullet}} = \lim_{m \to \infty} \varphi_{K_m/K_0}$ is the function from Remark* (4) *in section* 4.2.

Proof. Suppose $\mathcal{L}$ is a finite Galois extension of $\mathcal{K}$ in $R_0(N)$. By Corollary 6.2, there is a Galois extension $L_{\bullet} \in \mathcal{B}^{fa}(N)$ of $K_{\bullet}$ and a natural identification of Galois groups $\Gamma_{\mathcal{L}/\mathcal{K}} = \Gamma_{L_u/K_u}$, where $u \gg 0$. This identification is compatible with the ramification filtration in lower numbering, i.e. for any $j \in J(N)$, it holds that

$$\Gamma_{\mathcal{L}/\mathcal{K}, j} = \Gamma_{L_u/K_u, j} = \Gamma_{L_u/K_0, j} \cap \Gamma_{\widetilde{K}}.$$

Also, for $u \gg 0$, we have $\varphi_{\mathcal{L}/\mathcal{K}} = \varphi_{L_u/K_u}$. Suppose $j_1 = \varphi_{\mathcal{L}/\mathcal{K}}(j)$. Then for $u \gg 0$,

$$\varphi_{L_u/K_0}(j) = \varphi_{K_u/K_0}(\varphi_{L_u/K_u}(j)) = \varphi_{K_u/K_0}(j_1) = \varphi_{K_{\bullet}}(j_1).$$

So, for any $j_1 \in J(N)$, we have $\Gamma_{\mathcal{L}/\mathcal{K}}^{(j_1)} = \Gamma_{L_u/K_0}^{(\varphi_{K_{\bullet}}(j_1))} \cap \Gamma_{\widetilde{K}}$ and we obtain the statement of our corollary by taking projective limit on $\mathcal{L}$. □

6.3. The above results show that if $\mathcal{K} \in X(K_{\bullet})$ with $K_{\bullet} \in \mathcal{B}^{fa}(N)$, then $R_0(N)$ contains a separable closure of $\mathcal{K}$. Because $R_0(N)$ is perfect, the algebraic closure of $\mathcal{K}$ in $R_0(N)$ is algebraically closed. Even more, $R_0(N)$ is $v^1_{\mathcal{K}}$-complete, therefore, $R_0(N)$ contains the $v^1_{\mathcal{K}}$-completion $\mathcal{R}(\bar{\mathcal{K}})$ of $\bar{\mathcal{K}}$.

Proposition 6.7. *$R_0(N) = \mathcal{R}(\bar{\mathcal{K}})$.*

Proof. Suppose $K_{\bullet}$ has the index parameter $n^*(K_{\bullet})$ and the ramification parameter $c^* = c^*(n^*, K_{\bullet})$. Let $d = c^*(n^*, K_{\bullet})/p2^N$. Consider the identification $R(N) = \varprojlim_u (\mathcal{O}_{\mathbb{C}(N)_p} \operatorname{mod} \mathrm{m}^1_{K_{\bullet}}(d))_u$ and take any $r = (r_w)_{w \geqslant 0} \in R(N)$. Fix $w \geqslant 0$.

Let L be a finite extension of K_0 in $\mathbb{C}(N)_p$ such that $r_w \in \mathcal{O}_L \operatorname{mod} \mathrm{m}^1_{K_{\bullet}}(d)$. We can assume that L is such that $L_{\bullet} := LK_{\bullet} \in \mathcal{B}^{fa}(N)$. We know that for $m \gg 0$, we can take the ramification invariant $c^*(m, L_{\bullet})$ for $L_{\bullet}$ such that

$$c^*(m, L_{\bullet})/p = c^*_1(m, L_{\bullet}) = e^1 c^*/p2^N = e^1 d,$$

where $e^1 = \mathrm{pr}_1 \bar{e}(L_{\bullet}/K_{\bullet})$. Notice that $\mathrm{m}^1_{L_{\bullet}}(e^1 d) = \mathrm{m}^1_{K_{\bullet}}(d)$ does not depend on $L_{\bullet}$. By Corollary 5.6 after an appropriate choice of the structural functions $m_1, \ldots, m_{N-1}$ we can find an index parameter $\bar{u} \in U(m_1, \ldots, m_{N-1})$ such that pr_u induces an identification of $\mathcal{O}_{\mathcal{L}_{\bar{u}}} \operatorname{mod} \mathrm{m}^1_{R,K_{\bullet}}(p^u d)$ and $\mathcal{O}_{L_{\bar{u}}} \operatorname{mod} \mathrm{m}^1_{K_{\bullet}}(d)$. Because, $L_{\bar{u}} \supset L$, there is an $l(w) \in \mathcal{O}_{\mathcal{L}_{\bar{u}}} \subset R(N)$ such that $\mathrm{pr}_u(l(w)) = r_w = \mathrm{pr}_u(\sigma^{u-w} r)$.

So, $\sigma^{u-w}(r) - l(w) \in \mathrm{m}^1_{R,K_{\bullet}}(p^u d)$ and $r \equiv \sigma^{w-u}(l(w)) \operatorname{mod} \mathrm{m}^1_{R,K_{\bullet}}(p^w d)$.

Because $\sigma^{w-u}(l(w)) \in \mathcal{R}(\mathcal{L})$ and w can be taken arbitrarily large, this implies that $\bar{\mathcal{K}}$ is $v^1_{R,K_{\bullet}}$-adically dense in $R_0(N)$.

The proposition is proved. □

Remark. Corollary 5.6 implies that the above identification $R_0(N) = \mathcal{R}(\bar{\mathcal{K}})$ is compatible with the P-topological structures. Also, this identification is compatible with the natural structures of $\Gamma_{\widetilde{K}}$-module on $R_0(N)$ and $\Gamma_{\mathcal{K}}$-module on $\mathcal{R}(\bar{\mathcal{K}})$ via the identification ψ from Corollary 6.6.

7. The functors $\mathcal{X}$ and $\mathcal{X}_{K_{\bullet}}$

7.1. The functor $\mathcal{X}_{K_{\bullet}}$, $K_{\bullet} \in \mathcal{B}^{fa}(N)$.

Let $K_{\bullet} \in \mathcal{B}^{fa}(N)$ and let $\mathcal{B}^a_{K_{\bullet}}(N)$ be the category of separable extensions $L_{\bullet}$ of $K_{\bullet}$ in $\mathcal{B}^a(N)$. Morphisms in $\mathcal{B}^a_{K_{\bullet}}(N)$ are isomorphisms f in the category $\mathcal{B}^a(N)$ such that $f|_{K_{\bullet}} = \mathrm{id}$.

Let $\widetilde{\mathrm{LF}}_R(N)_{K_{\bullet}}$ be the category of finite separable extensions of $\mathrm{cl}(K_{\bullet}) \in \widetilde{\mathrm{LF}}_R(N)$, where morphisms come from isomorphisms in $\mathrm{LF}_R(N)$, which are the identity on $\mathcal{K}$. In this section we use results of section 6 about the correspondence $E_{\bullet} \mapsto \mathrm{cl}(E_{\bullet})$, where $E_{\bullet} \in \mathcal{B}^{fa}(N)$, to construct an equivalence of the categories $\mathcal{B}^a_{K_{\bullet}}(N)$ and $\widetilde{\mathrm{LF}}_R(N)_{K_{\bullet}}$.

Let $L_{\bullet}$ be a separable extension of $K_{\bullet} \in \mathcal{B}^{fa}(N)$ in $\mathcal{B}(N)$. Then $L_{\bullet} \in \mathcal{B}^a(N)$ (cf. section 4.2). Choose a finite Galois extension $E_{\bullet}$ of $K_{\bullet}$ such that $E_{\bullet} \in \mathcal{B}^{fa}(N)$ and $E_{\bullet} \supset L_{\bullet}$ (cf. Proposition 4.5). If $\mathcal{K} \in \mathrm{cl}(K_{\bullet})$, then there is a unique separable extension $\mathcal{E}$ of $\mathcal{K}$ in $R_0(N)$ such that $\mathcal{E} \in \mathrm{cl}(E_{\bullet})$ and $[\mathcal{E} : \mathcal{K}] = [E_{\bullet} : K_{\bullet}]$ (cf. section 6). Therefore, $G = \mathrm{Gal}(E_{\bullet}/L_{\bullet})$ (by definition it equals $\mathrm{Gal}(E_u/L_u)$ for $u \gg 0$) acts on $\mathcal{E}$ and we can set $\mathcal{L} = \mathcal{E}^H$, where $H \subset G$ is such that $E_{\bullet}^H = L_{\bullet}$.

Proposition 7.1. *With the above notation,* $\mathrm{cl}(\mathcal{L}) \in \widetilde{\mathrm{LF}}_R(N)$ *does not depend on the choice of* $\mathcal{K} \in X(K_{\bullet})$ *and* $E_{\bullet} \in \mathcal{B}^{fa}(N)$.

The proof is straightforward.

With the notation from the above proposition set $\mathcal{X}_{K_{\bullet}}(L_{\bullet}) = \mathrm{cl}(\mathcal{L})$.

Suppose $L_{\bullet}, L'_{\bullet} \in \mathcal{B}^a_{K_{\bullet}}(N)$ and $f : L_{\bullet} \longrightarrow L'_{\bullet}$ is a morphism in $\mathcal{B}^a_{K_{\bullet}}(N)$. In other words, f is sequentially P-continuous and compatible with the corresponding F-structures automorphism of $\mathbb{C}(N)_p$ such that $f(L_m) = L'_m$ for $m \gg 0$ and $f|_{K_{\bullet}} = \mathrm{id}$.

Choose $E_{\bullet} \in \mathcal{B}^{fa}(N)$ such that $E_{\bullet} \supset L_{\bullet}$ and $E_{\bullet}$ is finite Galois over $K_{\bullet}$. Let $E'_{\bullet} = f(E_{\bullet})$, $G = \mathrm{Gal}(E_{\bullet}/K_{\bullet})$, $G' = \mathrm{Gal}(E'_{\bullet}/K_{\bullet})$, $H = \mathrm{Gal}(E_{\bullet}/L_{\bullet})$ and $H' = \mathrm{Gal}(E'_{\bullet}/L'_{\bullet})$.

Let $\mathcal{K} \in \mathrm{cl}(K_{\bullet})$ and let $\mathcal{E}$ be its Galois extension from $\mathrm{cl}(E_{\bullet})$ of degree $[E_{\bullet} : K_{\bullet}]$. Then $\mathcal{E}^H = \mathcal{L} \in \mathcal{X}_{K_{\bullet}}(L_{\bullet})$. Let f_R be an automorphism of $R_0(N)$ induced by f. Then f_R is sequentially P-continuous and compatible with F-structures, $f_R(\mathcal{E}) \in \mathrm{cl}(E'_{\bullet})$ and $f_R(\mathcal{E})^{H'} = f_R(\mathcal{E}^H) = f_R(\mathcal{L}) = \mathcal{L}' \in \mathcal{X}_{K_{\bullet}}(L'_{\bullet})$.

So, $f_R \in \mathrm{Hom}_{\widetilde{\mathrm{LF}}_R(N)_{K_{\bullet}}}(\mathcal{X}_{K_{\bullet}}(L_{\bullet}), \mathcal{X}_{K_{\bullet}}(L'_{\bullet}))$. Clearly, if we set $f_R = \mathcal{X}_{K_{\bullet}}(f)$, then we get a functor $\mathcal{X}_{K_{\bullet}}$ from $\mathcal{B}^a_{K_{\bullet}}(N)$ to $\widetilde{\mathrm{LF}}_R(N)_{K_{\bullet}}$.

Summarizing the results of section 6 we obtain the following principal result of this paper.

Theorem 2. (a) *The above defined functor* $\mathcal{X}_{K_{\bullet}}$*, where* $K_{\bullet} \in \mathcal{B}^{fa}(N)$*, is an equivalence of the categories* $\mathcal{B}^a_{K_{\bullet}}(N)$ *and* $\widetilde{\mathrm{LF}}_R(N)_{K_{\bullet}}$.

(b) $\mathcal{X}_{K_{\bullet}}$ *induces an identification* $\psi_{K_{\bullet}}$ *of groups* $\Gamma_{\widetilde{K}} = \mathrm{Gal}(\bar{K}/\widetilde{K})$ *and* $\Gamma_{\mathcal{K}} = \mathrm{Gal}(\mathcal{K}_{\mathrm{sep}}/\mathcal{K})$*, where* $\mathcal{K} \in \mathrm{cl}(K_{\bullet})$ *and* $\mathcal{K}_{\mathrm{sep}}$ *is the separable closure of* $\mathcal{K}$ *in* $R_0(N)$.

(c) *The identification* $\psi_{K_{\bullet}}$ *is compatible with ramification filtrations on* $\Gamma_{\widetilde{K}}$ *and* $\Gamma_{\mathcal{K}}$*, i.e. for any* $j \in J(N)$*,* $\psi_{K_{\bullet}}$ *identifies the groups* $\Gamma_{\widetilde{K}} \cap \Gamma_{K_0}^{(\varphi_{K_{\bullet}}(j))}$ *and* $\Gamma_{\mathcal{K}}^{(j)}$*, where* $\varphi_{K_{\bullet}}$ *is the function from Remark* (4) *in section* 4.2.

Remark. If $N = 1$, then $\mathrm{cl}(K_{\bullet})$ consists of only one field $\mathcal{K}$. So, we obtain the functor from $\mathcal{B}^a_{K_{\bullet}}(1)$ to the category of finite separable extensions of $\mathcal{K}$ in $R_0(1)$. Even more, we can treat all $K_{\bullet} \in \mathcal{B}^a(1)$ with the same field $\bigcup_u K_u := \widetilde{K}^0$ by introducing the family $\mathcal{E}$ of all finite extensions E of $\mathbb{Q}_p$ in $\widetilde{K}^0$ such that $\widetilde{K}^0$ is a p-extension of E. Then $\mathcal{O}_{\mathcal{K}}$ can be identified with the family of all $\left(\alpha_E \operatorname{mod} \mathrm{m}^1_{K_{\bullet}}(c_1^*)\right)_{E\in\mathcal{E}}$, where $\alpha_E \in \mathcal{O}_E$, such that if $E_1 \supset E$ is an extension in $\mathcal{E}$ and $[E_1 : E] = p^d$, then

$$\alpha_{E_1}^{p^d} \equiv \alpha_E \operatorname{mod} \mathrm{m}^1_{K_{\bullet}}(c_1^*).$$

This description of the elements of the field-of-norms, which is attached to the infinite extension $\widetilde{K}^0$, was used in [FW1, FW2] to prove all the basic properties of the field-of-norms functor in the case of 1-dimensional local fields.

7.2. The functor $\mathcal{X} : \mathcal{B}^a(N) \longrightarrow \mathrm{RLF}_{\mathcal{R}}(N)$. Let $\mathrm{RLF}_R(N)$ be the category of sequentially P-closed perfect subfields in $R_0(N)$. These subfields are considered with their natural F-structure and P-topology. Morphisms are sequentially P-continuous isomorphisms of such fields, which are compatible with corresponding F-structures.

If $K_{\bullet} \in \mathcal{B}^a(N)$, choose $L_{\bullet} \in \mathcal{B}^{fa}(N)$ such that $L_{\bullet}/K_{\bullet}$ is a finite Galois extension. If $\mathcal{L} \in \mathrm{cl}(L_{\bullet})$, then $G = \mathrm{Gal}(L_{\bullet}/K_{\bullet})$ acts on $\mathcal{R}(\mathcal{L})$. Indeed, for any $g \in G$ and any $\mathcal{L} \in \mathrm{cl}(L_{\bullet})$, the action of g on $L_{\bullet}$ induces a field isomorphism $g : \mathcal{L} \longrightarrow \mathcal{L}'$, where $\mathcal{L}' \in X(L_{\bullet})$ and we have a natural identification $\mathcal{R}(\mathcal{L}) = \mathcal{R}(\mathcal{L}')$ (cf. Proposition 5.7). With the above notation set $\mathcal{X}(K_{\bullet}) = \mathcal{R}(\mathcal{L})^G \in \mathrm{RLF}_R(N)$.

Proposition 7.2. *$\mathcal{X}(K_{\bullet})$ does not depend on a choice of $L_{\bullet} \in \mathcal{B}^{fa}(N)$.*

Proof. Suppose $L'_{\bullet} \in \mathcal{B}^{fa}(N)$ is such that $L'_{\bullet}/K_{\bullet}$ is a finite Galois extension with the Galois group G'. Choose $M_{\bullet} \in \mathcal{B}^{fa}(N)$ such that $M_{\bullet} \supset L_{\bullet}$, $M_{\bullet} \supset L'_{\bullet}$ and $M_{\bullet}$ is a finite Galois extension of $K_{\bullet}$ with the Galois group S.

Let $H = \mathrm{Gal}(M_{\bullet}/L_{\bullet})$, $H' = \mathrm{Gal}(M_{\bullet}/L'_{\bullet})$. If $\mathcal{L} \in \mathrm{cl}(L_{\bullet})$, $\mathcal{L}' \in \mathrm{cl}(L'_{\bullet})$, then there are $\mathcal{M} \in X(M_{\bullet})$ and $\mathcal{M}' \in X(M_{\bullet})$ such that $\mathcal{M}/\mathcal{L}$ and $\mathcal{M}'/\mathcal{L}'$ are Galois extensions with Galois groups H and H', respectively. Then $\mathcal{R}(\mathcal{M}) = \mathcal{R}(\mathcal{M}')$ and $\mathcal{R}(\mathcal{L}')^{G'} = \mathcal{R}(\mathcal{M}')^S = \mathcal{R}(\mathcal{M})^S = \mathcal{R}(\mathcal{L})^G$.

The proposition is proved. □

Suppose $K_{\bullet}, K'_{\bullet} \in \mathcal{B}^a(N)$ and $f \in \mathrm{Hom}_{\mathcal{B}^a(N)}(K_{\bullet}, K'_{\bullet})$ is an isomorphism, i.e. $f : \mathbb{C}(N)_p \longrightarrow \mathbb{C}(N)_p$ is a sequentially P-continuous and compatible with F-structures field automorphism such that $f(K_{\bullet}) = K'_{\bullet}$. As earlier, denote by f_R the automorphism of $R_0(N)$ which is induced by f.

Choose $L_{\bullet} \in \mathcal{B}^{fa}(N)$ such that $L_{\bullet}/K_{\bullet}$ is a finite Galois extension with the group G. Then $L'_{\bullet} = f(L_{\bullet})$ is a Galois extension of $K'_{\bullet}$ with the group G' which is conjugate to G via the automorphism f of $\mathbb{C}(N)_p$. If $\mathcal{L} \in \mathrm{cl}(L_{\bullet})$, then $f_R(\mathcal{L}) = \mathcal{L}' \in \mathrm{cl}(L'_{\bullet})$ and $f_R(\mathcal{X}(K_{\bullet})) = f_R(\mathcal{R}(\mathcal{L})^G) = \mathcal{R}(\mathcal{L}')^{G'} = \mathcal{X}(K'_{\bullet})$.

So, $f_R \in \mathrm{Hom}_{\mathrm{RLF}_R(N)}(\mathcal{X}(K_{\bullet}), \mathcal{X}(K'_{\bullet}))$ and $\mathcal{X} : \mathcal{B}^a(N) \longrightarrow \mathrm{RLF}_R(N)$ is a functor. The following property follows directly from the above definitions.

Proposition 7.3. (a) *$\mathcal{X}$ is a faithful functor;*

(b) *if $L_{\bullet}, K_{\bullet} \in \mathcal{B}^{fa}(N)$ and $L_{\bullet}$ is a separable extension of $K_{\bullet}$, then* $\mathcal{R}(\mathcal{X}_{K_{\bullet}}(L_{\bullet})) = \mathcal{X}(L_{\bullet})$.

7.3. Let $\varepsilon = (\varepsilon^{(n)} \bmod p)_{n\geqslant 0} \in R(1) \subset R(N)$, where $\varepsilon^{(0)} = 1$, $\varepsilon^{(1)} \neq 1$ and $\varepsilon^{(n+1)p} = \varepsilon^{(n)}$ for all $n \geqslant 0$, be Fontaine's element. Let $\langle\varepsilon\rangle = \varepsilon^{\mathbb{Z}_p} \subset R(1)^*$ be the multiplicative subgroup of all Fontaine's elements. Notice that if $f : \mathbb{C}(N)_p \longrightarrow \mathbb{C}(N)_p$ is a field automorphism, then $f_R(\langle\varepsilon\rangle) = \langle\varepsilon\rangle$, where f_R is induced by f.

Proposition 7.4. (a) *The correspondence $f \mapsto f_R$ identifies* $\operatorname{Aut}\mathbb{C}(N)_p$ *and the subgroup* $\operatorname{Aut}' R_0(N)$ *of $g \in \operatorname{Aut} R_0(N)$ such that $g(\langle\varepsilon\rangle) = \langle\varepsilon\rangle$.*

(b) *If f is sequentially P-continuous (respectively, compatible with F-structures), then so is f_R.*

Proof. We have noticed already that for any $f \in \operatorname{Aut}\mathbb{C}(N)_p$, $f_R(\langle\varepsilon\rangle) = \langle\varepsilon\rangle$.

Suppose $g \in \operatorname{Aut} R_0(N)$ and $g(\langle\varepsilon\rangle) = \langle\varepsilon\rangle$, i.e. $g(\varepsilon) = \varepsilon^a$ with $a \in \mathbb{Z}_p^*$.

Notice that $g : R(N) \longrightarrow R(N)$ induces the automorphism $W(g) : W(R(N)) \longrightarrow W(R(N))$, where W is the functor of Witt vectors. Consider Fontaine's map (cf. [Fo]), $\gamma : W(R(N)) \longrightarrow \mathcal{O}_{\mathbb{C}(N)_p}$ given by the correspondence

$$(r_0, r_1, \dots, r_n, \dots) \mapsto r_0^{(0)} + pr_1^{(1)} + \dots + p^n r_n^{(n)} + \dots,$$

where for any $r = (r_m \bmod p)_{m\geqslant 0} \in R(N)$ and $n \geqslant 0$, $r^{(n)} = \lim_{m\to\infty} r_{m+n}^{p^m} \in \mathcal{O}_{\mathbb{C}(N)_p}$. This map is a surjective morphism of p-adic algebras and its kernel J is a principal ideal generated by $1 + [\varepsilon]^{1/p} + \dots + [\varepsilon]^{(p-1)/p}$. Therefore, $W(g)(J) = J$ and $W(g)$ induces an automorphism $f = W(g) \bmod J$ of $\mathbb{C}(N)_p$. Clearly, $f_R = g$.

From the above description of the correspondence $f \mapsto f_R$ it is clear that the compatibility of f_R with F-structures is implied by the same property for f.

Suppose f is sequentially P-continuous. Because arithmetic operations in $\mathbb{C}(N)_p$ and $R_0(N)$ are sequentially P-continuous, it will be sufficient to prove that for any $M \geqslant 0$, the map

$$\alpha : R(N) \longrightarrow \mathcal{O}_{\mathbb{C}(N)_p} \bmod p^{M+1}$$

such that for $r = (r_m \bmod p)_{m\geqslant 0} \in R(N)$, $\alpha(r) = \gamma([r]) \bmod p^{M+1} = r_M^{p^M} \bmod p^{M+1}$ is sequentially P-continuous. But the map $r \mapsto r_M \bmod p$ is sequentially P-continuous by the definition of the P-topological structure on $R(N)$ and the map $r_M \bmod p \mapsto r_M^{p^M} \bmod p^{M+1}$ is sequentially p-adically continuous and, therefore, sequentially P-continuous. The proposition is proved. □

Remark. Suppose $K_{\bullet} \in \mathcal{B}^{fa}(N)$, $\widetilde{K}$ is the p-adic closure of the union of all K_u, $u \geqslant 0$, and $\mathcal{K} \in \operatorname{cl}(K_{\bullet})$. Then the Fontaine map induces the ring epimorphism $\gamma : W(\mathcal{O}_{\mathcal{R}(\mathcal{K})}) \longrightarrow \mathcal{O}_{\widetilde{K}}$. This follows from the basic properties of the construction of $\mathcal{K}$, e.g. from Corollary 5.6. This map also transports the $\Gamma_{\mathcal{K}}$-module structure on the left to the $\Gamma_{\widetilde{K}}$-structure on the right via the identification of $\Gamma_{\mathcal{K}}$ and $\Gamma_{\widetilde{K}}$ from Corollary 6.6.

7.4. Introduce the following definition.

Definition. A subfield $\widetilde{K}$ of $\mathbb{C}(N)_p$ is an SAPF-field if there is a $K_{\bullet} \in \mathcal{B}^a(N)$ such that $\widetilde{K}$ is the p-adic closure of $\bigcup_{n\geqslant 0} K_n$.

Remark. The above defined SAPF-fields are higher-dimensional analogues of strict arithmetic profinite extensions introduced in [FW1, FW2].

Denote by SAPF(N) the category of SAPF-fields in $\mathbb{C}(N)_p$ such that if $\widetilde{K}, \widetilde{K}' \in$ SAPF(N), then $\mathrm{Hom}_{\mathrm{SAPF}(N)}(\widetilde{K}, \widetilde{K}')$ consists of sequentially P-continuous and compatible with F-structures $f \in \mathrm{Aut}\,\mathbb{C}(N)_p$ such that $f(\widetilde{K}) = \widetilde{K}'$.

Let $\widetilde{K} \in$ SAPF(N). Set $\widetilde{\mathcal{X}}(\widetilde{K}) = \mathcal{X}(K_{\bullet})$, where $K_{\bullet} \in \mathcal{B}^a(N)$ is such that $\widetilde{K}$ is the p-adic closure of $\bigcup_{n\geqslant 0} K_n$ and $\mathcal{X}$ is the functor from section 7.2.

Lemma 7.5. *The above defined $\widetilde{\mathcal{X}}(\widetilde{K})$ does not depend on the choice of $K_{\bullet} \in \mathcal{B}^a(N)$.*

Proof. The proof follows directly from the construction of the functor $\mathcal{X}$ and Proposition 5.7. □

The correspondence $\widetilde{K} \mapsto \widetilde{\mathcal{X}}(\widetilde{K})$ can be naturally extended to the functor $\widetilde{\mathcal{X}} : \mathrm{SAPF}(N) \longrightarrow \mathrm{RLF}_R(N)$. Taking together the above results about the functor $\mathcal{X}$ we obtain the following theorem.

Theorem 3. *Suppose $K_{\bullet} \in \mathcal{B}^a(N)$ and $\widetilde{K}$ is the p-adic closure of $\bigcup_{n\geqslant 0} K_n$. Then the functor $\widetilde{\mathcal{X}}$ induces the identification $\iota : \Gamma_{\widetilde{K}} \longrightarrow \Gamma_{\widetilde{\mathcal{K}}}$ where $\widetilde{\mathcal{K}} = \mathcal{X}(K_{\bullet})$. If $K_{\bullet} \in \mathcal{B}^{fa}(N)$ and $\mathcal{K} \in \mathrm{cl}(K_{\bullet})$, then $\mathcal{R}(\mathcal{K}) = \widetilde{\mathcal{K}}$ and under a natural identification $\Gamma_{\mathcal{K}} = \Gamma_{\widetilde{\mathcal{K}}}$, the identification ι is compatible with ramification filtrations, i.e. for any $j \in J(N)$,*

$$\Gamma_{\widetilde{K}} \cap \Gamma_{K_0}^{(\varphi_{K_{\bullet}}(j))} = \Gamma_{\mathcal{K}}^{(j)}.$$

Remark. If $N = 1$ we can relate all fields from $X(K_{\bullet})$ with a given field $\widetilde{K}$ without using the operation of radical closure (cf. the above remark to Theorem 2).

8. A property of the P-continuity for the functor $\mathcal{X}$

8.1. Suppose $\mathcal{K} \in \mathrm{LF}_p(N)$.

Let $\Gamma_{\mathcal{K}}^{\mathrm{ab}}(p)$ be the Galois group of the maximal abelian p-extension of $\mathcal{K}$.

For any $M \geqslant 1$, consider the Witt-Artin-Schreier duality

$$\Gamma^{\text{ab}}_{\mathcal{K}}(p)/p^M \times W_M(\mathcal{K})/(\sigma - \text{id})W_M(\mathcal{K}) \longrightarrow W_M(\mathbb{F}_p)$$

where σ is the Frobenius endomorphism of the additive group $W_M(\mathcal{K})$ of Witt vectors of length M with coefficients in $\mathcal{K}$. This allows us to provide $\Gamma^{\text{ab}}_{\mathcal{K}}(p)/p^M$ with the P-topological structure. Its basis of open 0-neighborhoods consists of the annihilators of the sequentially P-compact subsets of $W_M(\mathcal{K})/(\sigma - \text{id})W_M(\mathcal{K})$. By the results of section 1.2 the basis of such compact subsets consists of the images in $W_M(\mathcal{K})/(\sigma - \text{id})W_M(\mathcal{K})$ of all subsets of the form

$$W_M(D) = \{(a_0, \dots, a_{M-1}) \in W_M(\mathcal{K}) \mid a_0, \dots, a_{M-1} \in D\}$$

where $D \in \mathcal{C}_b(\mathcal{K})$ is the basis of P-sequentially compact subsets in $\mathcal{K}$.

Finally, the P-topology on $\Gamma^{\text{ab}}_{\mathcal{K}}(p)$ appears as the projective limit topology of the projective system of P-topological groups $\Gamma^{\text{ab}}_{\mathcal{K}}(p)/p^M$.

8.2. Suppose $K \in \text{LF}_0(N)$ and K contains a primitive p^Mth root of unity ζ_{p^M}. Then the P-topological structure on K^* induces the P-topological structure on $\Gamma^{\text{ab}}_K(p)/p^M$, where $\Gamma_K(p)$ is the Galois group of the maximal abelian p-extension of K. This structure is defined similarly to the characteristic p case by the use of the Kummer duality

$$\Gamma^{\text{ab}}_K(p)/p^M \times K^*/K^{*p^M} \longrightarrow \langle \zeta_{p^M} \rangle.$$

We do not need this structure in a full generality. Let $\widetilde{\Gamma}_K(p)/p^M$ be the quotient of $\Gamma^{\text{ab}}_K(p)/p^M$ by the annihilator of the subgroup $(1 + p\mathcal{O}_K)^\times$ in K^*. Then we have the induced pairing

$$\widetilde{\Gamma}^{\text{ab}}_K(p)/p^M \times (1 + p\mathcal{O}_K)^\times \longrightarrow \langle \zeta_{p^M} \rangle$$

and a basis of open subgroups in $\widetilde{\Gamma}^{\text{ab}}_K(p)/p^M$ consists of the annihilators of the subsets $1 + pD$, where $D \in \mathcal{C}_b(K)$, $D \subset \mathcal{O}_K$ and $\mathcal{C}_b(K)$ is a basis of P-sequentially compact subsets in K from section 1.2.

8.3. Suppose $K_{\centerdot} \in \mathcal{B}^{fa}(N)$ and for any $M \in \mathbb{N}$, there is an $n = n(M)$ such that K_n contains a primitive p^Mth root of unity.

Let $\widetilde{K}$ be the p-adic closure of the union of all K_n. Then for any $M \in \mathbb{N}$, we have a natural identification

$$\Gamma^{\text{ab}}_{\widetilde{K}}(p)/p^M = \varprojlim_n \Gamma^{\text{ab}}_{K_n}(p)/p^M.$$

Notice that if $n_0 \in \mathbb{N}$ and $v \in K^*_{n_0}$, then for $n \gg n_0$, $v \in (1 + p\mathcal{O}_{K_n})^\times \bmod K_n^{*p^M}$ for any given $M \in \mathbb{N}$. Indeed, it will be sufficient to verify this for $M = 1$. Consider the tower $K_{\bar{u}^0} \subset K_{\bar{v}^1} \subset K_{\bar{u}^1} \subset \dots \subset K_{\bar{v}^n} \subset K_{\bar{u}^n} \subset \dots$ from section 4.3.5. Then for any $n \geqslant 1$, there are local parameters $t_{1n}, \dots, t_{Nn}$ in

$K_{\bar{u}^n}$ and $\tilde{t}_{1n},\dots,\tilde{t}_{Nn}$ in $K_{\bar{v}^n}$ such that $t_{1n}^p\equiv\tilde{t}_{1n}\operatorname{mod}\mathrm{m}^1_{K_\bullet}(c_1^*)$, $\dots$, $t_{Nn}^p\equiv\tilde{t}_{Nn}\operatorname{mod}\mathrm{m}^1_{K_\bullet}(c_1^*)$. If $v\equiv\tilde{t}_{11}^{c_1}\dots\tilde{t}_{N1}^{c_N}(1+\tilde{a}_1)\operatorname{mod}K_{\bar{v}^1}^{*p}$, then

$$v\equiv t_{11}^{pc_1}\dots t_{N1}^{pc_N}(1+a_1^p+b_1)\equiv(1+a_1^p+b_1)(1+a_1)^{-p}\equiv 1+b_1+pa_1'\operatorname{mod}K_{\bar{u}^1}^{*p},$$

where $v^1_{K_\bullet}(b_1)\geqslant c_1^*$, $\tilde{a}_1\in\mathrm{m}_{K_{\bar{v}^1}}$, $a_1\in\mathrm{m}_{K_{\bar{u}^1}}$ is such that $\tilde{a}_1\equiv a_1^p\operatorname{mod}\mathrm{m}^1_{K_\bullet}(c_1^*)$ and $a_1'\in\mathrm{m}_{K_{\bar{u}^1}}$. Repeating this procedure m times we obtain that $v\equiv 1+b_m+pa_m'\operatorname{mod}K_{\bar{u}^m}^{*p}$, where $v^1_{K_\bullet}(b_m)\geqslant mc_1^*$ and $a_m'\in\mathrm{m}_{K_{\bar{u}^m}}$. So, if $mc_1^*\geqslant v^1_{K_\bullet}(p)$, then $v\in(1+p\mathcal{O}_{K_{\bar{u}^m}})^\times\operatorname{mod}K_{\bar{u}^m}^{*p}$. It remains to notice that for $n\gg 0$, $K_{\bar{u}^m}\subset K_n$.

Therefore,

$$\Gamma^{\mathrm{ab}}_{\widetilde{K}}(p)/p^M=\varprojlim_n\widetilde{\Gamma}^{\mathrm{ab}}_{K_n}(p)/p^M$$

and the basis of P-open neighborhoods in $\Gamma^{\mathrm{ab}}_{\widetilde{K}}(p)/p^M$ consists of annihilators of all sequentially compact subsets $1+pD\subset(1+p\mathcal{O}_{\widetilde{K}})^\times$, where $D\in\mathcal{C}_b(K_n)$, for some $n\geqslant 0$.

8.4. Suppose $K_\bullet\in\mathcal{B}^{fa}(N)$, $\mathcal{K}\in X(K_\bullet)$ and $\iota:\Gamma_{\widetilde{K}}\longrightarrow\Gamma_{\mathcal{K}}$ is the identification of Galois groups (where $\widetilde{K}$ is the p-adic closure of the $\bigcup_{n\geqslant 0}K_n$) from Theorem 3. Suppose that for each $M\in\mathbb{N}$, a primitive p^Mth root of unity $\zeta_{p^M}\in K_n$ if $n\gg 0$ and consider the groups $\Gamma^{\mathrm{ab}}_{\widetilde{K}}(p)/p^M=\varprojlim_n\widetilde{\Gamma}^{\mathrm{ab}}_{K_n}(p)/p^M$ and $\Gamma^{\mathrm{ab}}_{\mathcal{K}}(p)/p^M$ with the above P-topological structures.

Theorem 4. *For any $M\in\mathbb{N}$, the identification*

$$\iota\operatorname{mod}p^M:\Gamma^{\mathrm{ab}}_{\widetilde{K}}(p)/p^M\longrightarrow\Gamma^{\mathrm{ab}}_{\mathcal{K}}(p)/p^M$$

is a P-homeomorphism.

Proof.

8.4.1. Choose a primitive p^Mth root of unity ζ_{p^M}. Then we can identify $W_M(\mathbb{F}_p)$ and $\langle\zeta_{p^M}\rangle$ and consider the dual to $\iota\operatorname{mod}p^M$ group morphism

$$\tilde{\iota}_M:W_M(\mathcal{K})/(\sigma-\mathrm{id})W_M(\mathcal{K})\longrightarrow\widetilde{K}^*/\widetilde{K}^{*p^M}.$$

Then $\iota\operatorname{mod}p^M$ is P-continuous if and only if $\tilde{\iota}_M$ transforms each sequentially P-compact subset in $W_M(\mathcal{K})/(\sigma-\mathrm{id})W_M(\mathcal{K})$ onto a sequentially P-compact subset in $\widetilde{K}^*/\widetilde{K}^{p^M}$. In other words, $\iota\operatorname{mod}p^M$ is P-continuous if and only if $\tilde{\iota}_M$ is sequentially P-continuous.

Notice that the map $\tilde{\iota}_M$ can be characterised as follows.

Let $\bar{w}\in W_M(\mathcal{K})/(\sigma-\mathrm{id})W_M(\mathcal{K})$ and let $w\in W_M(\mathcal{K})$ be a lifting of $\bar{w}$. Consider $T\in W_M(R_0(N))$ such that $\sigma T-T=w$; then for any $\tau\in\Gamma_{\mathcal{K}}$, $\tau T-T=a_\tau\in W_M(\mathbb{F}_p)$. Let $\bar{v}\in\widetilde{K}^*/\widetilde{K}^{*p^M}$ and $v\in\widetilde{K}^*$ be a lifting of $\bar{v}$. Consider $Z\in\mathbb{C}(N)_p$ such that $Z^{p^M}=v$. Then for any $\tau\in\Gamma_{\widetilde{K}}$, $\tau Z/Z=\zeta_M^{b_\tau}$, where $b_\tau \in W_M(\mathbb{F}_p)$. With respect to the identification $\Gamma_{\widetilde{K}} = \Gamma_{\mathcal{K}}$ given by the construction of the functor $\mathcal{X}_{K_\bullet}$, we have the following criterion:

$$\tilde{\iota}_M(\bar{w}) = \bar{v} \quad \Leftrightarrow \quad a_\tau = b_\tau \ \ \forall \tau \in \Gamma_{\widetilde{K}} = \Gamma_{\mathcal{K}}.$$

8.4.2. As earlier, let $\mathcal{R}(\mathcal{K})$ be the completion of the radical closure of $\mathcal{K}$ (with respect to first valuation). Denote by $\mathcal{O}_{\mathcal{R}(\mathcal{K})}$ its valuation ring. Notice first that the natural embedding $\mathcal{K} \subset \mathcal{R}(\mathcal{K})$ induces a natural identification of P-topological groups

$$W_M(\mathcal{R}(\mathcal{K}))/(\sigma - \mathrm{id})W_M(\mathcal{R}(\mathcal{K})) = W_M(\mathcal{K})/(\sigma - \mathrm{id})W_M(\mathcal{K}).$$

Let ε be Fontaine's element. Recall, $\varepsilon = (\varepsilon^{(n)})_{n\geqslant 0} \in R = R(1) \subset R(N)$ is such that $\varepsilon^{(0)} = 1$, $\varepsilon^{(1)} \neq 1$ and we can assume that $\varepsilon^{(M)} = \zeta_{p^M}$ — this is the primitive p^Mth root of unity chosen in 8.4.1. From the construction of $\mathcal{K} \in \mathrm{cl}(K_\bullet)$ it follows that $\varepsilon \in \mathcal{O}_{\mathcal{R}(\mathcal{K})}$, and therefore, it is invariant under the action of $\Gamma_{\mathcal{K}}$.

Lemma 8.1. *Suppose $\bar{w}$ comes from $w = \dfrac{f}{[\varepsilon] - 1} \bmod p^M \in W_M(\mathcal{R}(\mathcal{K}))$, where $f \in W(\mathcal{O}_{\mathcal{R}(\mathcal{K})})$. Then $\tilde{\iota}_M(\bar{w}) = \bar{v}$, where*

$$\bar{v} = \exp(-p\gamma(\sigma^{-1}f) - \dots - p^M\gamma(\sigma^{-M}f)) \bmod \widetilde{K}^{*p^M}$$

and γ is Fontaine's map (cf. remark in section 7.3).

Proof. Let $U \in W(R_0(N))$ be such that $\sigma U - U = f/([\varepsilon] - 1)$, then for any $\tau \in \Gamma_{\mathcal{K}}$, $\tau U - U = \tilde{a}_\tau \in W(\mathbb{F}_p)$, where $\tilde{a}_\tau \bmod p^M = a_\tau$.

Let $\varepsilon_1 = \sigma^{-1}\varepsilon$, then

$$s = ([\varepsilon] - 1)/([\varepsilon_1] - 1) \in W^1(R(1)) \subset W(R(1)) \subset W(R(N)),$$

where $W^1(R(1)) = \mathrm{Ker}\,\gamma$ and $\gamma : W(R(1)) \longrightarrow \mathcal{O}_{\mathbb{C}_p}$ is Fontaine's map. It is well known (cf. [Fo]), that s generates the ideal $W^1(R(1))$. Notice that similar arguments show that s also generates the kernel $W^1(R(N))$ of the analogue of Fontaine's map from $W(R(N))$ to $\mathcal{O}_{\mathbb{C}(N)_p}$.

Let $T_1 = U([\varepsilon_1] - 1)$. Then $T_1 \in W(R(N))$ and $\sigma T_1 - sT_1 = f$. Let $X = U([\varepsilon] - 1) = sT_1 \in W^1(R(N))$, then

$$\frac{\sigma X}{\sigma s} - X = f$$

and for any $\tau \in \Gamma_{\mathcal{K}}$, $\tau X - X = \tilde{a}_\tau([\varepsilon] - 1)$.

Let $A(N)_{cris}$ be an analogue of Fontaine's A_{cris} constructed by the use of $R(N)$ instead of R. This is the divided power envelope of the $W(R(N))$ with respect to the ideal $W^1(R(N))$, which is generated by s. Proceeding as in

[Ab2] we obtain that if

$$\frac{\sigma m}{p} - m = f \tag{5}$$

where $m \in \mathrm{Fil}^1 A(N)_{cris}$, then for any $\tau \in \Gamma_{\widetilde{K}}$, $\tau m - m = \tilde{a}_\tau \log[\varepsilon]$.

Multiplying both parts of the equality (5) by p and taking exponentials we obtain the equality

$$\sigma Y = Y^p \exp(pf) \tag{6}$$

where $Y \in 1 + \mathrm{Fil}^1 A(N)_{cris}$ and for any $\tau \in \Gamma_{\widetilde{K}}$, $\tau Y/Y = [\varepsilon]^{\tilde{a}_\tau}$. Proceeding again as in [Ab2] we can prove that $Y \in 1 + W^1(R(N))$ (and therefore can forget about the crystalline ring $A(N)_{cris}$; cf. Remark (2) below).

The equation (6) implies that

$$\sigma^M Y = Y^{p^M} \exp(p\sigma^{M-1} f + \cdots + p^M f)$$

and, because σ is bijective on $W(R(N))$, this gives

$$Y = (\sigma^{-M} Y)^{p^M} \exp(p\sigma^{-1} f + \cdots + p^M \sigma^{-M} f). \tag{7}$$

Notice that for any $\tau \in \Gamma_{\widetilde{K}}$, $\tau(\sigma^{-M} Y) = (\sigma^{-M} Y)[\sigma^{-M}\varepsilon]^{\tilde{a}_\tau}$.

Apply Fontaine's map $\gamma : W(R(N)) \longrightarrow \mathcal{O}_{\mathbb{C}(N)_p}$ to both parts of (7). Notice that $\gamma(Y) = 1$, $\gamma(\sigma^{-M} Y) = Z \in 1 + p\mathcal{O}_{\mathbb{C}(N)_p}$, $\gamma([\sigma^{-M}\varepsilon]) = \zeta_{p^M}$ and $\gamma(\sigma^{-s} f) \in \mathcal{O}_{\widetilde{K}}$ for any $s \in \mathbb{Z}$. This gives

$$Z^{p^M} = \exp(-p\gamma(\sigma^{-1} f) - \cdots - p^M \gamma(\sigma^{-M} f)) \in 1 + p\mathcal{O}_{\widetilde{K}}$$

and for any $\tau \in \Gamma_{\widetilde{K}}$, $\tau Z/Z = \zeta_{p^M}^{a_\tau}$.

The above computations imply that $\tilde{\iota}_M(\bar{w}) = \bar{v}$, where

$$\bar{v} = \exp(-p\gamma(\sigma^{-1} f) - \cdots - p^M \gamma(\sigma^{-M} f)) \bmod \widetilde{K}^{*p^M}.$$

The lemma is proved. □

Remarks. (1) The above computations can be used to deduce (in the similar way as in [Ab2]) the explicit formula for the Hilbert symbol for higher-dimensional fields from [Vo].

(2) The use of Fontaine's crystalline ring in the above proof provides a very natural way to pass from the Witt-Artin-Schreier theory to the Kummer theory through the "Bloch-Kato" theory: equation (5) plays a very important role in the paper [BK].

8.4.3. Continue the proof of Theorem 4.

Suppose $t_1, \dots, t_N$ is a system of local parameters in $\mathcal{K}$ (note that $\mathcal{K}$ has a standard F-structure). Denote by k the last residue field of $\mathcal{K}$. Let $\tilde{t}_1 = [t_1], \dots, \tilde{t}_N = [t_N]$ be the Teichmüller representatives of $t_1, \dots, t_N$ in $W_M(\mathcal{K})$. Denote by $O_M(\mathcal{K})$ the set of all power series

$$\sum_{\bar{a}} w_{\bar{a}} \tilde{t}_1^{a_1} \dots \tilde{t}_N^{a_N},$$

where $\bar{a} = (a_1, \dots, a_N) \in \mathbb{Z}^N$ and the coefficients $w_{\bar{a}}$ satisfy the restrictions similar to the restrictions on $\alpha_{\bar{a}}$ from section 1.1. Then $O_M(\mathcal{K})$ is a sequentially P-closed subring in $W_M(\mathcal{K})$.

Let $O_M^0(\mathcal{K})$ be a minimal sequentially P-closed additive subgroup in $O_M(\mathcal{K})$ containing all $w_{\bar{a}} \tilde{t}_1^{a_1} \dots \tilde{t}_N^{a_N}$, such that $w_{\bar{a}} \in W_M(k)$, $(a_1, \dots, a_N) < \bar{0}_N$ and $\gcd(a_1, \dots, a_N, p) = 1$, and the element $\alpha_0 \bmod p^M \in W_M(k)$, such that α_0 has trace 1 in the extension $W(k) \otimes \mathbb{Q}_p / \mathbb{Q}_p$. With the above notation there is the following proposition.

Proposition 8.2. (a) $O_M(\mathcal{K}) = O_M^0(\mathcal{K}) \oplus (\sigma - \mathrm{id}) O_M(\mathcal{K})$.

(b) *A natural embedding $O_M(\mathcal{K}) \subset W_M(\mathcal{K})$ induces a sequentially P-continuous identification of $O_M^0(\mathcal{K})$ and $W_M(\mathcal{K})/(\sigma - \mathrm{id})(W_M(\mathcal{K}))$.*

Proof. For part (a) one can proceed on the level of formal power series. For (b) we must have the following two properties:

$$O_M^0(\mathcal{K}) \cap (\sigma - \mathrm{id}) W_M(\mathcal{K}) = 0$$

and

$$O_M^0(\mathcal{K}) + (\sigma - \mathrm{id}) W_M(\mathcal{K}) = W_M(\mathcal{K}).$$

Both follow easily from (a) and the existence of the embedding $\sigma^{M-1}(W_M(\mathcal{K})) \subset O_M(\mathcal{K})$, which can be proved by induction on M. □

Corollary 8.3. *In $W_M(\mathcal{K})/(\sigma - \mathrm{id}) W_M(\mathcal{K})$, any convergent in the P-topology sequence can be lifted to a convergent sequence in $O_M^0(\mathcal{K}) \subset W_M(\mathcal{K})$.*

Now we can finish the proof of Theorem 4.

Suppose $\{\bar{w}_i\}_{i \geqslant 1} \in W_M(\mathcal{K})/(\sigma - \mathrm{id}) W_M(\mathcal{K})$ is a P-convergent sequence. Lift it to a convergent sequence $\{w_i'\}_{i \geqslant 1}$ in $W_M(\mathcal{K})$. Consider 1-dimensional valuation $v_{\mathcal{K}}^1$ on $R_0(N)$. Then the convergence of $\{w_i'\}_{i \geqslant 1}$ implies that $v_{\mathcal{K}}^1$-valuations of coordinates of all w_i' have a lower bound. Therefore, there is an $s_0 \in \mathbb{N}$ such that for all $i \geqslant 1$, $w_i' = \dfrac{f_i'}{\sigma^{s_0}([\varepsilon] - 1)} \bmod p^M$, where all $f_i' \in W(\mathcal{O}_{\mathcal{R}(\mathcal{K})})$. Clearly, the sequence $\{f_i' \bmod p^M\}_{i \geqslant 1}$ converges in $W_M(\mathcal{O}_{\mathcal{R}(\mathcal{K})})$.

For any $i \geqslant 1$, set $w_i = \sigma^{-s_0} w_i' = \dfrac{f_i}{[\varepsilon] - 1}$, where $f_i = \sigma^{-s_0} f_i'$. Then all w_i are still liftings of $\bar{w}_i$ to $W_M(\mathcal{R}(\mathcal{K}))$ and $\{f_i \bmod p^M\}_{i \geqslant 1}$ is a converging sequence of elements in $W_M(\mathcal{O}_{\mathcal{R}(\mathcal{K}))})$.

Applying Lemma 8.1 we obtain for all $i \geqslant 1$, that $\tilde{\iota}_M(\bar{w}_i) = \bar{v}_i$, where

$$v_i \equiv exp(-p\gamma(\sigma^{-1}f_i) - \cdots - p^M\gamma(\sigma^{-M}f_i)) \bmod \widetilde{K}^{*p^M}.$$

Clearly, such a sequence $\{\bar{v}_i\}_{i\geqslant 1}$ is P-convergent and its limit is the image of the limit of $\bar{w}_i$ under $\tilde{\iota}_M$. Therefore, $\tilde{\iota}_M$ is sequentially P-continuous. We omit the verification that the inverse map is also sequentially P-continuous.

The theorem is proved.

9. The Grothendieck conjecture for higher-dimensional local fields

9.1. Suppose K, K' are 1-dimensional local fields from the category $\mathrm{LF}(1) = \mathrm{LF}_0(1)\coprod \mathrm{LF}_p(1)$. Then any isomorphism $f \in \mathrm{Hom}_{\mathrm{LF}(1)}(K, K')$ is given by an automorphism of $\mathbb{C}(1)_p$ or, respectively, $\mathcal{C}(1)_p$ such that $f(K) = K'$. Therefore, f induces the isomorphism of profinite groups

$$f^* : \Gamma_{K'} \longrightarrow \Gamma_K$$

such that for any $v \geqslant 0$, $f^*(\Gamma_{K'}^{(v)}) = \Gamma_K^{(v)}$, where $\Gamma_K^{(v)}$ is the ramification subgroup with the upper number $v \geqslant 0$.

The inverse statement was proved in [Mo] in the mixed characteristic case and in [Ab4] if the characteristic of the residue fields of K and K' is $\geqslant 3$. It is known as a local (1-dimensional) analogue of the Grothendieck conjecture and can be stated in the following form:

If $\iota : \Gamma_{K'} \longrightarrow \Gamma_K$ is an isomorphism of profinite groups such that for any $v \geqslant 0$, $\iota(\Gamma_{K'}^{(v)}) = \Gamma_K^{(v)}$, then there is an $f \in \mathrm{Hom}_{\mathrm{LF}(1)}(K, K')$ such that $\iota = f^*$.

9.2. Suppose $N \geqslant 1$ and $\mathcal{K}, \mathcal{K}' \in \widetilde{\mathrm{LF}}_R(N)$. Suppose $f \in \mathrm{Hom}_{\widetilde{\mathrm{LF}}_R(N)}(\mathcal{K}, \mathcal{K}')$ is an isomorphism. In other words, $f : R_0(N) \longrightarrow R_0(N)$ is sequentially P-continuous and compatible with F-structures field automorphism such that for all $1 \leqslant i \leqslant N$, $f(\mathcal{K}(i)R(\mathcal{K}(i-1))) = \mathcal{K}'(i)R(\mathcal{K}'(i-1))$. Then $f^* : \Gamma_{\mathcal{K}'} \longrightarrow \Gamma_{\mathcal{K}}$ is an isomorphism of profinite groups such that for any $j \in J(N)$, $f^*(\Gamma_{\mathcal{K}'}^{(j)}) = \Gamma_{\mathcal{K}}^{(j)}$ (cf. [Ab5]). We point out that in the case of higher-dimensional local fields $\mathcal{K}$ of positive characteristic, the knowledge of their Galois group together with its ramification filtration is sufficient to recover the isomorphism class of $\mathcal{K}$ only in the category $\widetilde{\mathrm{LF}}_R(N)$.

In addition, suppose $\mathcal{E}$ is a finite extension of $\mathcal{K}$ in $R_0(N)$ and $f(\mathcal{E}) = \mathcal{E}'$. Then $\mathcal{E}'$ is a finite extension of $\mathcal{K}'$ such that $f^*(\Gamma_{\mathcal{E}'}) = \Gamma_{\mathcal{E}}$. Let $M \in \mathbb{N}$. Consider the induced isomorphism of the maximal abelian quotients modulo p^Mth powers

$$f_M^* : \Gamma_{\mathcal{E}'}^{\mathrm{ab}}(p)/p^M \longrightarrow \Gamma_{\mathcal{E}}^{\mathrm{ab}}(p)/p^M.$$

It is dual to the isomorphism of additive groups

$$f_M : W_M(\mathcal{E})/(\sigma - \mathrm{id})W_M(\mathcal{E}) \longrightarrow W_M(\mathcal{E}')/(\sigma - \mathrm{id})W_M(\mathcal{E}').$$

Clearly, f_M is sequentially P-continuous and, therefore, maps sequentially P-compact subsets to sequentially P-compact subsets. This implies that f_M^* is P-continuous for all $M \in \mathbb{N}$.

The inverse statement appears as an analogue of the Grothendieck conjecture for higher-dimensional local fields of characteristic p.

Theorem 5. *With the above notation, suppose that $p \geqslant 3$ and $\iota : \Gamma_{\mathcal{K}'} \longrightarrow \Gamma_{\mathcal{K}}$ is an isomorphism of profinite groups such that*

(a) *for any $j \in J(N)$, $\iota(\Gamma_{\mathcal{K}'}^{(j)}) = \Gamma_{\mathcal{K}}^{(j)}$;*

(b) *if $\mathcal{E}$ and $\mathcal{E}'$ are finite extensions of $\mathcal{K}$ and, resp., $\mathcal{K}'$ in $R_0(N)$ such that both $\mathcal{E}$ and $\mathcal{E}'$ have a standard F-structure and $\iota(\Gamma_{\mathcal{E}'}) = \Gamma_{\mathcal{E}}$, then for all $M \geqslant 1$, the induced isomorphism $\iota_M : \Gamma_{\mathcal{E}'}^{\mathrm{ab}}(p)/p^M \longrightarrow \Gamma_{\mathcal{E}}^{\mathrm{ab}}(p)/p^M$ is P-continuous.*

Then there is an isomorphism $f \in \mathrm{Hom}_{\widetilde{\mathrm{LF}}_R(N)}(\mathcal{K}, \mathcal{K}')$ such that $f^* = \iota$.

This statement was proved in [Ab6] in the case $N = 2$. The case of general N can be done along the same lines.

Remarks. (1) Actually, in the statement of the main theorem in [Ab6] there was no requirement that $\mathcal{E}$ and $\mathcal{E}'$ have a standard F-structure. But in the proof we applied this condition only to fields, which have a standard F-structure. Also, in [Ab6] there was a requirement about the P-continuity of the induced group isomorphism $\iota^{\mathrm{ab}} : \Gamma_{\mathcal{E}'}^{\mathrm{ab}} \longrightarrow \Gamma_{\mathcal{E}}^{\mathrm{ab}}$ but again, in the proof, we applied this property only to the induced isomorphism of the Galois groups $\Gamma_{\mathcal{E}'}^{\mathrm{ab}}(p)$ and $\Gamma_{\mathcal{E}}^{\mathrm{ab}}(p)$ of the maximal abelian p-extensions of $\mathcal{E}'$ and $\mathcal{E}$.

(2) The restriction $p \geqslant 3$ appears because our proof is based on the nilpotent Artin-Schreier theory, which allows us to study the maximal quotient of the Galois group $\Gamma_{\mathcal{K}}(p)$ of nilpotent class $< p$ together with induced ramification filtration. If $p = 2$, this gives us only information about $\Gamma_{\mathcal{K}}(2)^{ab}$ which is not sufficient to establish such a result. For $p \geqslant 3$, the proof uses only the explicit description of the ramification filtration in the group $\Gamma_{\mathcal{K}}(p)/C_3(\Gamma_{\mathcal{K}}(p))$.

9.3. Suppose $N \geqslant 1$ and $K, K' \in \mathrm{LF}_0(N)$. Any sequentially P-continuous and compatible with F-structures field automorphism $f : \mathbb{C}(N)_p \longrightarrow \mathbb{C}(N)_p$ such that $f(K) = K'$ induces an isomorphism of profinite groups $f^* : \Gamma_{K'} \longrightarrow \Gamma_K$ such that $f^*(\Gamma_{K'}^{(j)}) = \Gamma_K^{(j)}$ for any $j \in J(N)$.

Suppose E is a finite extension of K; then $E' = f(E)$ is a finite extension of K'. If both E and E' contain a primitive p^Mth root of unity, then the groups

$\Gamma_E^{\mathrm{ab}}(p)/p^M$ and $\Gamma_{E'}^{\mathrm{ab}}(p)/p^M$ are provided with the P-topological structure (cf. section 8.2) and the induced isomorphism

$$f_M^* : \Gamma_{E'}^{\mathrm{ab}}(p)/p^M \longrightarrow \Gamma_E^{\mathrm{ab}}(p)/p^M$$

is P-continuous.

Consider the inverse statement.

Theorem 6. *With the above notation, suppose that $p \geqslant 3$ and $\iota : \Gamma_{K'} \longrightarrow \Gamma_K$ is an isomorphism of profinite groups such that*

(a) *for all $j \in J(N)$, $\iota(\Gamma_{K'}^{(j)}) = \Gamma_K^{(j)}$;*

(b) *if E, E' are finite extensions of K and, resp., K' such that both contain a primitive p^Mth root of unity and $\iota(\Gamma_{E'}) = \Gamma_E$, then the induced isomorphism*

$$\iota_M : \Gamma_{E'}^{\mathrm{ab}}(p)/p^M \longrightarrow \Gamma_E^{\mathrm{ab}}(p)/p^M$$

is P-continuous.

Then there is a (unique) $f \in \mathrm{Hom}_{\mathrm{LF}_0(N)}(K, K')$ such that $f(K) = K'$ and $f^* = \iota$.

Remark. Modulo some technical details and notation (cf. Remark (1) in section 9.2) this statement was announced in [Ab5].

Proof.

9.3.1. Notice first that the compatibility of ι with ramification filtrations gives for $1 \leqslant r \leqslant N$, the group isomorphisms $\iota(r) : \Gamma_{K'(r)} \longrightarrow \Gamma_{K(r)}$, which are induced by ι. All these isomorphisms are also compatible with the corresponding ramification filtrations.

In particular, $\iota(1)$ is compatible with ramification filtrations isomorphism of the absolute Galois groups of 1-dimensional local fields $K(1)$ and $K'(1)$. Therefore, by the 1-dimensional case of a local analogue of the Grothendieck conjecture (cf. section 8.1) $\iota(1)$ is induced by a field isomorphism $f(1) : \mathbb{C}_p \longrightarrow \mathbb{C}_p$ such that $f(1)(K(1)) = K'(1)$.

9.3.2. Prove the existence of $F_{\bullet}, F'_{\bullet} \in \mathcal{B}^{fa}(N)$ such that for all $n \geqslant 0$,

(a) $F_0 \supset K$, $F'_0 \supset K'$;

(b) $\iota(\Gamma_{F'_n}) = \Gamma_{F_n}$;

(c) $\zeta_{p^n} \in F_n$ and $\zeta_{p^n} \in F'_n$, where ζ_{p^n} is a primitive p^nth root of unity.

Let $L_0 = \mathbb{Q}_p\{\{t_N\}\}\dots\{\{t_2\}\}$ be a basic N-dimensional local field. Then K and K' are its finite extensions with induced F-structures. Consider $E_{\bullet} \in \mathcal{B}(N)$ such that for all $n \geqslant 1$, $E_n = E_0(\zeta_{p^n}, \sqrt[p^n]{t_2}, \dots, \sqrt[p^n]{t_N})$. Clearly, $E_{\bullet} \in \mathcal{B}^a(N)$ (even more, $E_{\bullet} \in \mathcal{B}^{fa}(N)$): It is easy to see that $n^*(E_{\bullet}) = 1$ and $c^*(1, E_{\bullet}) = 1$, because for all $n \in \mathbb{N}$, $\mathrm{pr}_1(j(E_{n+1}(r)/E_n(r))) = p^n$. Indeed, if $n \geqslant 1$ and $\theta = \sqrt[p^{n+1}]{t_r}$, then for any $\tau \in \mathrm{Gal}(E_{n+1}(r)/E_n(r)E_{n+1}(r-1))$, $\tau \neq \mathrm{id}$, it holds that

$$v_{E_{n+1}(r)}(\tau\theta - \theta) = v_{E_{n+1}(r)}(\theta(\zeta_p - 1)) = v_{E_{n+1}(r)}(\theta) + (p^n, 0, \dots, 0).$$

Let $L_\bullet = KE_\bullet$. Then $L_\bullet \in \mathcal{B}^a(N)$ by Proposition 4.1. Introduce $L'_\bullet = \{L'_n \mid n \geqslant 0\} \in \mathcal{B}(N)$ such that $\iota(\Gamma_{L'_n}) = \Gamma_{L_n}$. Then $L'_\bullet \in \mathcal{B}^a(N)$ because ι is compatible with ramification filtrations.

Suppose $n^* = n^*(L_\bullet)$ is an index parameter for $L_\bullet$ (cf. section 4.2). Clearly, n^* can be taken also as an index parameter for $L'_\bullet$. Choose a finite extension $M(N-1)$ of $L_{n^*}(N-1)$ such that if $M = L_{n^*}M(N-1)$, then $(M, M(N-1)) \in \mathrm{LC}(N)$ is standard (cf. Theorem 1). We can enlarge (if necessary) $M(N-1)$ to satisfy the following property: If $M(N-1)'$ is such that $\iota(N-1)(\Gamma_{M(N-1)'}) = \Gamma_{M(N-1)}$ and $M' = L'_{n^*}M(N-1)'$, then $(M', M(N-1)') \in \mathrm{LC}(N)$ is standard. Therefore, the towers $M_\bullet = L_\bullet M(N-1)$ and $M'_\bullet = L'_\bullet M(N-1)'$ are such that for all $n \geqslant 0$, $\iota(\Gamma_{M'_n}) = \Gamma_{M_n}$ and both $(M_{n^*}, M_{n^*}(N-1))$ and $(M'_{n^*}, M'_{n^*}(N-1)) \in \mathrm{LC}(N)$ are standard.

Apply the above procedure to $(N-1)$-dimensional towers $M_\bullet(N-1)$, $M'_\bullet(N-1) \in \mathcal{B}^a(N-1)$ with a parameter $m^* \geqslant n^*$ and so on. Finally, we obtain finite separable extensions $F_\bullet$ and $F'_\bullet$ of $L_\bullet$ and, resp., $L'_\bullet$, which still satisfy the above requirements (a)–(b) but are already objects of the category $\mathcal{B}^{fa}(N)$. Clearly, for all $n \in \mathbb{N}$, $\zeta_{p^n} \in F_n$. Then $\iota(1)(\Gamma_{F'_n}(1)) = \Gamma_{F_n}(1)$ implies that $f(1)(F_n) = F'_n$ and $\zeta_{p^n} \in F'_n$.

9.3.3. Let $\mathcal{F} \in \mathrm{cl}(F_\bullet)$ and $\mathcal{F}' \in \mathrm{cl}(F'_\bullet)$ (cf. section 5). By Theorem 2, the group isomorphism ι induces the identification

$$\iota_{F_\bullet} : \Gamma_{\mathcal{F}'} \longrightarrow \Gamma_{\mathcal{F}}$$

which is compatible with ramification filtrations on these groups.

Suppose the finite extensions $\mathcal{E}/\mathcal{F}$ and $\mathcal{E}'/\mathcal{F}'$ are such that $\iota_{F_\bullet}(\Gamma_{\mathcal{E}'}) = \Gamma_{\mathcal{E}}$. If $\mathcal{E}$ and $\mathcal{E}'$ have standard F-structures, then $\mathcal{E} \in \mathrm{cl}(E_\bullet)$ and $\mathcal{E}' \in \mathrm{cl}(E'_\bullet)$, where $E_\bullet, E'_\bullet \in \mathcal{B}^{fa}(N)$ are finite separable extensions of $F_\bullet$ and $F'_\bullet$, respectively. Therefore, we can apply Theorem 4 to deduce from the condition (b) of the statement of our theorem that for any $M \in \mathbb{N}$, the induced identification

$$\iota_{F_\bullet, M} : \Gamma^{\mathrm{ab}}_{\mathcal{E}'}(p)/p^M \longrightarrow \Gamma^{\mathrm{ab}}_{\mathcal{E}}(p)/p^M$$

is P-continuous.

Therefore, by the charactersitic p case of the Grothendieck conjecture (cf. Theorem 5 in section 9.2), the isomorphism $\iota_{F_\bullet}$ is induced by a sequentially P-continuous field isomorphism $f_R : R_0(N) \longrightarrow R_0(N)$, such that $f_R|\mathcal{F}$ is an isomorphism between $\mathcal{F}$ and $\mathcal{F}'$ in the category $\widetilde{\mathrm{LF}}_R(N)$.

9.3.4. Clearly, $f_R|_{R_0(1)}$ is induced by the $f(1) : \mathbb{C}(1)_p \longrightarrow \mathbb{C}(1)_p$ from section 9.3.1. Therefore, f_R leaves invariant the subgroup of Fontaine's elements $\langle \varepsilon \rangle$ and by Proposition 7.4, f_R is induced by a field automorphism $f : \mathbb{C}(N)_p \longrightarrow \mathbb{C}(N)_p$.

The characteristic property of the field automorphism f_R is that it transforms the action of any $\tau \in \Gamma_{K'}$ on $R_0(N)$ into the action of $\iota(\tau) \in \Gamma_K$ on $R_0(N)$. Therefore, f satisfies the same property and we have

$$f(K) = f(\mathbb{C}(N)_p^{\Gamma_K}) = \mathbb{C}(N)_p^{\Gamma_{K'}} = K'.$$

So, $f \in \mathrm{Hom}_{\mathrm{LF}_0(N)}(K, K')$ and Theorem 6 is proved.

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Department of Mathematics and Sciences, Durham University, Science Laboratories, South Road, Durham, DH1 3LE, United Kingdom

E-mail address: victor.abrashkin@durham.ac.uk