THE ROLE OF DEFINITIONS IN EXAMPLE CLASSIFICATION

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This paper reports an empirical study of students' classification of sequences before and after meeting explicit definitions of 'increasing' and 'decreasing'. In doing so, it explores 1) students' interpretations of the definitions and 2) the appropriateness of this apparently straightforward context for teaching students about the status of mathematical definitions. In particular, it demonstrates that students' spontaneous conceptions in this context can be inconsistent with definitions, and it explores the extent to which exposure to formal definitions influences these conceptions. The results show an interesting pattern of modified classifications, which demonstrates increased consistency with the definitions but shows problems with some pivotal examples.

INTRODUCTION AND THEORETICAL BACKGROUND

Undergraduate students are often unaware of the status of definitions in mathematical theory. They may be unable to state important definitions, even after a substantial period of study, and many appear to reason about mathematical concepts using concept images instead of definitions (Tall & Vinner, 1981). This can be particularly problematic in Analysis, in which spontaneous conceptions, based on everyday use of terms or informal experience with concepts such as *limit*, can be at odds with the formal definitions (Williams, 1991; Cornu, 1991).

For success in undergraduate pure mathematics, it is vital to learn to use definitions correctly in making classifications and in constructing general proofs. It is therefore important for mathematics educators to study ways to help students achieve this, and this paper approaches the classification issue by analysing students' responses to a task that required them to classify examples spontaneously and then using (previously unseen) definitions.

To design the task, we first identified a context in which there is likely to be disparity between spontaneous conceptions (Cornu, 1991) and the extension of the formal definition. In Analysis, the obvious place to start is with the limit concept, since this is central in the subject and much work has been done in establishing common misconceptions (Williams, 2001). However, limit definitions are logically complex (involving three nested quantifiers) so any investigation of their use in classification is likely be confounded by difficulties in understanding their logical structure (see, for example, Dubinsky, Elterman & Gong, 1988). Thus, the research reported here used the concepts of *increasing* and *decreasing* for infinite sequences of real numbers. The definitions for these concepts are logically simple (they involve only

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one quantifier) and some classifications based on them are counterintuitive: for example, constant sequences are classified as both increasing and decreasing and sequences such as 0,1,0,1,0,1,0,1,… is classified as neither increasing nor decreasing. Specifically, the research addressed the following questions:

RQ1: To what extent are students' spontaneous conceptions about increasing and decreasing sequences inconsistent with definition-based judgments?

RQ2: When given a basic introduction to the definitions, can students work with these and correctly revise their judgments?

If there are sufficient inconsistent spontaneous conceptions and evidence that exposure to the definitions led to revisions, this simple context would arguably be appropriate for raising students' awareness of the way mathematical definitions are used to resolve ambiguity or disagreement by precisely specifying a concept.

In theoretical terms one might say that this research sets out to investigate the participants' *example spaces* and their ability to modify the structure of these to better mirror the *conventional example space* (Watson & Mason, 2005). Previous research has tended to use example *generation* tasks: Zazkis & Leikin (2007), for instance, considered what such tasks can reveal about the accessibility, richness and generality of individual's example spaces. However, in this case example generation was considered unlikely to lead to interesting results because new undergraduates' experience with sequences is likely to be limited to work with arithmetic and geometric sequences. Since we wished to gain insight into students' responses in counterintuitive as well as 'obvious' cases, we used an example *classification* task with the deliberate inclusion of examples such as those above. These examples were expected to be *pivotal* for at least some of the participants, in the sense that they might cause students to experience uncertainty and/or to recognise and question initial assumptions (Zaskis & Chernoff, 2008).

METHOD

187 students completed the task as part of a regularly timetabled Analysis lecture (within a standard lecture course not given by the researcher). All participants were in the first term of their first year of a mathematics degree at a high-ranking UK university. The entry requirements for the degree included grade A for both mathematics and further mathematics A-levels (or equivalent), effectively the highest possible pre-university mathematics requirement in the UK.

The task presented here was part of an intervention lasting approximately 25 minutes. The students were informed that the task would help the researcher understand their thinking, that they should work alone, that the responses would be treated as anonymous and that their lecturer would be given a summary but that the tasks would not influence their grade. They were also told that they could opt out by choosing not to hand in their paper. In the *spontaneous classification* phase, the students were asked to fill in a table to indicate whether they would classify each of the sequences in Table 1 as increasing, decreasing, both or neither, or whether they were not sure.

A:0,1,0,1,0,1,0,1,…	F: 1,3,2,4,3,5,4,6,…
B: 1,4,9,16,25,36,49,64,…	G:6,6,7,7,8,8,9,9,…
C: $1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6},\frac{1}{7},\frac{1}{8},K$	H:0,1,0,2,0,3,0,4,…
D: $1,-1,2,-2,3,-3,4,-4,K$	J: $10\frac{1}{2},10\frac{3}{4},10\frac{7}{8},10\frac{15}{16},10\frac{31}{32},K$
E: 3,3,3,3,3,3,3,3,…	K: $-2,-4,-6,-8,-10,K$

Table 1: Classification task examples

The students were then shown definitions of 'increasing' and of 'decreasing', stated in notation consistent with that used in their course:

A sequence $\{x_n\}_{n=1}^{\infty}$ is <u>increasing</u> if and only if $\forall n \in \mathbf{N}$, $x_{n+1} \geq x_n$.

A sequence $\{x_n\}_{n=1}^{\infty}$ is <u>decreasing</u> if and only if $\forall n \in \mathbf{N}$, $x_{n+1} \leq x_n$.

These definitions were accompanied by a brief verbal explanation. In the following *definition-based classification* phase, the students were asked to fill in another table to show, *according to the definitions*, whether each of the sequences was increasing, decreasing, both or neither (without a 'not sure' option).

RESULTS

Spontaneous classifications

Table 2 shows the responses to the spontaneous classification task. The shaded cell in each row indicates the response consistent with the definitions (of course, at this stage it makes no sense to consider any responses 'incorrect').

	Inc	Dec	Both	Neither	Not Sure
A			86	93	4
B	187				
C		185	2		
D	1	4	154	23	5
E			0	187	
F	83		84	12	8
G	162	1		12	11
H	60		86	21	20
J	178	5	2	1	1
K	2	182	1	1	1

Table 2: Spontaneous classifications

This shows that, in many cases, spontaneous conceptions were not consistent with the definitions. Although it was the case that for sequences that are either increasing or decreasing but not both (B, C, G, J and K) a substantial majority gave a response consistent with the definitions, in all other cases, substantial numbers (often the vast majority) gave a response inconsistent with the definitions. In particular:

- Every participant classified the constant sequence as *neither*.
- Approximately half classified the sequence A: 0,1,0,1,0,1,0,1,… as *neither* (consistent with the definitions) and half as *both*.
- Only 23 participants classified the sequence D: $1,-1,2,-2,3,-3,4,-4,\ldots$ as *neither* (consistent), with a substantial majority classifying it as *both*.
- There was considerable difference of opinion regarding F: 1,3,2,4,3,5,4,6,… and H: 0,1,0,2,0,3,0,4,…. Only 12 and 21 respectively gave the response *neither* (consistent), with many more in each case selecting *increasing* or *both*.

It is worth noting that few students made use of the *not sure* option. It had been anticipated that more would do so, but with hindsight it seems reasonable that students who have not done much study of definitions and counterexamples would be comfortable making classifications in the absence of precise criteria.

Definition-based classifications

Table 3 shows the responses to the definition-based classification task. Again, responses consistent with the definitions are indicated by a shaded cell.

	Inc	Dec	Both	Neither
A		1	54	132
B	187			
C	1	186		
D		1	65	121
E	6		86	94
F	9		57	121
G	143		7	37
H	2		57	127
J	181	4	2	0
K		186	0	1

Table 3: Definition-based classifications

This shows that there were noticeable changes towards responses consistent with the definitions. Again, for the sequences that are either increasing or decreasing but not both, a substantial majority gave a response consistent with the definitions. Exposure to the definitions also meant that in all other cases but one, a majority responded consistently with these. However, these majorities were not overwhelming so it is not reasonable to say that the answer to RQ2 is an unqualified *yes*. In particular:

- The number classifying G:6,6,7,7,8,8,9,9,… as *increasing* (consistent) dropped from 162 to 143, with 37 participants now classifying this as *neither*. This could be due to misinterpreting the inequality to mean 'strictly less than' or perhaps becoming confused by the universal quantifier and believing the order relationship has to be the same for each pair of consecutive terms.
- Just under half (86) classified the constant sequence E as *both* (consistent). Almost exactly half (93) once again classified it as *neither*. Interestingly, 6 now classified it as *increasing.* It is possible that the latter found that it did satisfy the definition of increasing and assumed that this precluded being decreasing.
- For each of the four sequences A, D, F and H, approximately two thirds of the participants gave the classification *neither* (consistent), with approximately one third giving the classification *both*. The *both* response is in line with the most common spontaneous classifications, so could occur when participants simply do not change their minds. It is also in line with a misinterpretation of the universal quantifier in the definitions, for example classifying 0,1,0,1,0,1,0,1,… as *both* because it is true that for all n, either $x_{n+1} \geq x_n$ or $x_{n+1} \leq x_n$.

Individual responses

Analysis across all the participants shows that exposure to the definitions in this context did lead to a marked (but far from complete) move towards classifications consistent with the definitions. For each of the apparently counterintuitive cases (except for the constant sequence), about two thirds of the definition-based classifications were correct. Further examination of the individual participants' responses allows us to examine the question of whether this means that about two thirds of the participants 'got the idea' and gave entirely correct classifications. It also allows us to discern some internally consistent interpretations of the definitions that might indicate key misunderstandings. Table 4 summarises all the profiles of the definition-based classifications. These profiles account for over 80% of the participants and all the distinct profiles associated with four or more participants.

Response Profile	*n*	*Profile description*
N I D N B N I N I D	54	Correct
N I D N N N I N I D	30	Correct except *constant* classified as *neither*
N I D N N N N N I D	17	*Neither* applied to all *both* and all *neither*
B I D B N B I B I D	16	*Both* and *neither* switched
B I D B N B N B I D	9	*Both* and *neither* switched; 'steps' classified as *neither*
B I D B B B I B I D	12	*Both* applied to all *both* and all *neither*
B I D B B B B B I D	4	*Both* applied to all *both* and all *neither* and 'steps'

Table 4: Common definition-based response profiles

This data tells us that it is <u>not</u> the case that two thirds of the participants fully "got the idea". In fact, only 54 (29%) gave a correct set of classifications, although a further 30 (16%) were correct for all except the constant sequence and 17 (9%) gave the response *neither* for all both and all neither sequences (correct for all except the constant and 'steps' [G: 6,6,7,7,8,8,9,9,…] sequences). In addition, it tells us that 41 (22%) either switched around *both* and *neither* or applied *both* to the majority of the sequences. These latter profiles could indicate interpretations in which the universal quantifier is misunderstood or ignored, or could indicate that participants were thinking of sequences such as D: $1,-1,2,-2,3,-3,4,-4,K$ as two different sequences ($1,2,3,4,\ldots$ as increasing and $-1, -2, -3, -4,\ldots$ as decreasing). This thinking would be consistent with findings of Tall & Vinner (1981).

Because a relatively small number of participants gave fully correct definition-based classifications, we also examine the profiles for the spontaneous classifications. This allows us to investigate whether the students who gave correct definition-based classifications were already mostly correct in their spontaneous classifications, or whether some did reach a correct profile by making a substantial change in their interpretation. This is more difficult to do, because (unsurprisingly) there was more variation among the spontaneous classifications. Indeed, there were only two profiles given by more than eight students; we discuss each of these here.

42 participants (22%) spontaneously gave the 'both and neither switched' profile, suggesting that this is the most natural interpretation of combinations of the concepts *increasing* and *decreasing* for sequences. It is also internally consistent: these students apparently considered a sequence to be *both* if some terms are less than their predecessors and some are greater, and *neither* if all terms are the same. Of these 42 participants, for the definition-based classification:

- 12 changed to CORRECT.
- 9 changed to correct except constant classified as neither.
- 9 remained with both and neither switched.

The second prevalent profile, given by 28 participants (15%), also shows some internal consistency, although in a way that might not be recognised as such by a mathematician accustomed to precisely formulated property specifications. These participants classified sequences A:0,1,0,1,0,1,0,1,… and D:$1,-1,2,-2,3,-3,4,-4,K$ as *neither*, which is consistent with the definitions. They classified F:1,3,2,4,3,5,4,6,…, G:6,6,7,7,8,8,9,9,… and H:0,1,0,2,0,3,0,4,… as *increasing*, perhaps indicating that the presence of a general 'upward trend' was enough to gain this classification, and without apparently experiencing the fact that H has infinitely many zero terms as problematic. Of these 28 participants:

- 12 changed to CORRECT.
- 6 changed to neither applied to all both and all neither.

This analysis of common spontaneous classifications gives some indication that there is not a straightforward relationship between those who gave correct definition-based interpretations and 'almost correct' initial responses. In particular, a sizeable proportion of those who moved to correct classifications did so from substantially different spontaneous classifications. We continue to analyse the data for patterns in the responses to the ten separate sequences.

PEDAGOGICAL IMPLICATIONS

Exposure to a broad range of examples may be appropriate in and of itself, since research indicates that at least some students do not spontaneously generate examples in response to definitions (Dahlberg & Housman, 1997), that one reason for students' difficulties with proof is that they do not have well-developed example spaces (Moore, 1994) and that at least some successful mathematicians use examples extensively to support their reasoning (Alcock & Inglis, 2008). A task such as that used here provides exposure to a deliberately broad range of examples, with the specific aim of including some for which there is likely to be conflict between spontaneous and correct definition-based classifications.

With this in mind, a lecturer might use the outcomes of this study in several ways. First, they might simply give extra attention to the issue of definition-based classification, since even simple definitions may not be applied reliably by students who are unaccustomed to this type of reasoning, and even an apparent move toward correct definition use might mask underlying misconceptions that are resistant to change. Second, this task could be used as an introduction to these concepts, with subsequent discussion focused on the common misinterpretations. Third, one could run a similar intervention in which students were allowed to confer with each other at some stage. In this case, examples A: 0,1,0,1,0,1,0,1,…, F: 1,3,2,4,3,5,4,6,… and H: 0,1,0,2,0,3,0,4,… have particular potential as pivotal examples, since they seem to be those for which there is considerable variation in initial classifications so that there would likely be disagreements to be resolved among the class.

Such suggestions depend, of course, upon the generalisability of the results presented here. It might be thought that because these students attend a high-ranking institution, they would better at all types of mathematical tasks than is typical. However, precisely because they are considered successful and capable, these students study Analysis in the first term of their first year at university. At many other institutions students do not study it until the second term or second year, and thus come to it with more experience of learning university mathematics, and more experience of working with definitions in subjects such as elementary set theory, linear algebra, etc. Overall, we do not see a strong reason to believe that the responses would be substantially different for those studying the same material at other institutions, though obviously more research is needed to establish this.

RESEARCH IMPLICATIONS

The main question of interest here is whether one or more interventions like this can have a positive impact on students' engagement with definitions in general and on their eventual attainment. That is, does repeated exposure to challenges to one's spontaneous conceptions, and subsequent work with definitions, lead to an underlying cognitive shift in the process of making classification judgements and possibly in the use of definitions in mathematics more broadly? Further study would be necessary to establish whether the changes observed in this study have a long-term effect, even in this restricted context. This could be explored in two senses: whether just this one exposure would have a lasting impact over a longer time frame or whether further exposure to similar tasks in the same context would have increased effect, even for those classifications that appear counterintuitive and resistant to change.

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