

Transforms for minimal surfaces in the 5-sphere.

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Abstract

We define two transforms between minimal surfaces with non-circular ellipse of curvature in the 5-sphere, and show how this enables us to construct, from one such surface, a sequence of such surfaces. We also use the transforms to show how to associate to such a surface a corresponding ruled minimal Lagrangian submanifold of complex projective 3-space, which gives the converse of a construction considered in a previous paper, and illustrate this explicitly in the case of bipolar minimal surfaces.

Key words: *Sphere, minimal surface, ellipse of curvature, Lagrangian submanifold, complex projective space.*

Subject class: 53B25, 53B20.

1 Introduction

Let $f : S \rightarrow S^5(1)$ be a minimal immersion of a surface S into the unit 5-sphere. The image of the unit circle in the tangent space under the second fundamental form of f is a central planar ellipse E in the normal space of f called the *ellipse of curvature*. Let $0 \leq \theta \leq \pi/2$ be such that $\cos \theta$ is the ratio of the lengths of the minor and major axes of E . The geometrical significance of θ lies in the fact that if R_θ is the rotation of the normal space through angle θ about the minor axis of E then the orthogonal projection of $R_\theta(E)$ onto the plane containing E is a circle. If N is the unit vector in the normal space orthogonal to the plane containing E , then the transform we will consider is obtained by applying R_θ to N . Of course, there are certain choices of sign and orientation to be made here, and the various choices available give two essentially different transforms. We shall show that these transformed surfaces are also minimal, and that the two transforms are mutual inverses. This enables us to define a sequence $\{f^p : p \in \mathbb{Z}\}$ of minimal immersions into $S^5(1)$ with $f^0 = f$, and we instigate an investigation of this sequence.

The transforms described above are natural generalisations of the polar construction [10] for superconformal minimal surfaces in $S^3(1)$ and $S^5(1)$ (although when the ambient space is $S^3(1)$ the polar is simply the unit normal to the immersion). In the latter situation, the ellipse of curvature is a circle, the angle of rotation θ is zero, and in both situations the

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sequence of minimal immersions is periodic with period two. Our motivation for discovering and studying these transforms comes from [3], where we showed how to associate two minimal surfaces in $S^5(1)$ to a ruled minimal Lagrangian submanifold of complex projective 3-space. We showed that these minimal surfaces were related by the above transforms. In the present paper we show that the construction described in [3] may be reversed, thus showing that all minimal surfaces in $S^5(1)$ whose ellipse of curvature is not a circle may be constructed using the methods of [3]. We then illustrate this explicitly in the case of bipolar minimal surfaces [10] in $S^5(1)$.

2 Minimal surfaces in $S^5(1)$

For the rest of the paper, $f : S \rightarrow S^5(1)$ will denote a minimal immersion of an oriented surface S into $S^5(1)$. We use the orientation and induced metric to give S the structure of a Riemann surface in such a way that f is a conformal immersion. If II denotes the second fundamental form of f in $S^5(1)$ we recall that for each $p \in S$ the subset $E(p)$ of the first normal space of S at p given by

$$E(p) = \{II(X, X) \mid X \text{ is a unit tangent vector to } S \text{ at } p\}$$

is a (possibly degenerate) central ellipse called the *ellipse of curvature* of S at p .

In this section, we assume that S has non-circular non-degenerate ellipse of curvature at every point. We show how to associate a complex moving frame to such an immersion, and obtain the moving frame equations and integrability conditions. The approach we use is based on the theory of harmonic sequences, which is described in [5] for the more general situation of minimal surfaces in $S^m(1)$ or $\mathbb{C}P^m(4)$.

Let $z = x + iy$ be a local complex coordinate on S , and denote $\frac{\partial}{\partial z}$ by ∂ and $\frac{\partial}{\partial \bar{z}}$ by $\bar{\partial}$. We introduce \mathbb{C}^6 -valued functions f_0, f_1, f_2 by

$$f_0 = f, \tag{1}$$

$$f_1 = \partial f, \tag{2}$$

$$f_2 = II(\partial, \partial), \tag{3}$$

where II now denotes the complex bilinear extension of the second fundamental form of f in $S^5(1)$. If $(\ , \)$ is the complex bilinear extension of the standard inner product on \mathbb{R}^6 , it follows that $(f_0, f_1) = 0$ while conformality of f is equivalent to

$$(f_1, f_1) = 0. \tag{4}$$

Thus f_0, f_1, \bar{f}_1 are mutually orthogonal and f_2 is the component of ∂f_1 orthogonal to f_0, f_1, \bar{f}_1 . We note that, by Takahashi's Lemma [12], the minimality of f is equivalent to $\partial \bar{\partial} f_0 = \mu f_0$ for some real-valued function μ .

If $f_2 = a - ib$ where a, b are \mathbb{R}^6 -valued functions, it follows from conformality of f that the ellipse of curvature is homothetic to the image of the map

$$\psi \mapsto II(\cos \psi \frac{\partial}{\partial x} + \sin \psi \frac{\partial}{\partial y}, \cos \psi \frac{\partial}{\partial x} + \sin \psi \frac{\partial}{\partial y}) = 2(a \cos 2\psi + b \sin 2\psi). \tag{5}$$

Since the ellipse of curvature is not a circle we see that $(f_2, f_2) \neq 0$. As noticed by Hopf, the function (f_2, f_2) is holomorphic, so that [4] there exists a complex coordinate z (which we will call an *adapted complex coordinate* for f), defined up to rotations by $\frac{\pi}{2}$, such that $(f_2, f_2) = -1$. In this case

$$-1 = (f_2, f_2) = (a, a) - (b, b) - 2i(a, b),$$

so that

$$(a, a) - (b, b) = -1, \quad (a, b) = 0. \quad (6)$$

It now follows from (5) that a lies along the minor axis and b the major axis of E . It follows from (6) that there is a non-negative function ϕ such that

$$|a| = \sinh \phi, \quad |b| = \cosh \phi, \quad (7)$$

so that the eccentricity e of E is given by

$$e = \sqrt{1 - \frac{|a|^2}{|b|^2}} = \operatorname{sech} \phi. \quad (8)$$

At points where the ellipse of curvature is non-degenerate, a and b are linearly independent, and ϕ is positive. Thus f_2 and \bar{f}_2 are also linearly independent, and we complete our complex moving frame for f by letting N be a real unit vector orthogonal to $\{f_0, f_1, \bar{f}_1, f_2, \bar{f}_2\}$.

It is then straightforward to check that if $\mathcal{F} = \{f_0, f_1, \bar{f}_1, f_2, \bar{f}_2, N\}$ and if $\omega = \log |f_1|^2$, then the matrix A of complex bilinear inner products of the frame vectors of \mathcal{F} is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^\omega & 0 & 0 & 0 \\ 0 & e^\omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & \cosh 2\phi & 0 \\ 0 & 0 & 0 & \cosh 2\phi & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (9)$$

We now write down the moving frame equations for \mathcal{F} . A straightforward computation using (9) shows that if $\alpha = (\partial f_2, N)$, then the moving frame equations for \mathcal{F} may be written in terms of ω, ϕ, α as follows:

$$\begin{aligned} \partial f_0 &= f_1, \\ \partial f_1 &= f_2 + \partial \omega f_1, \\ \partial \bar{f}_1 &= -e^\omega f_0, \\ \partial f_2 &= e^{-\omega} \bar{f}_1 + 2\partial \phi \coth 2\phi f_2 + 2\partial \phi \operatorname{csch} 2\phi \bar{f}_2 + \alpha N, \\ \partial \bar{f}_2 &= -e^{-\omega} \cosh 2\phi \bar{f}_1, \\ \partial N &= -\alpha \operatorname{csch}^2 2\phi (f_2 + \cosh 2\phi \bar{f}_2). \end{aligned}$$

Of course, the corresponding $\bar{\partial}$ equations may be found by taking the conjugates of the above. It follows from uniqueness of solutions of linear differential equations and the integrability conditions $\partial \bar{\partial} \mathcal{F} = \bar{\partial} \partial \mathcal{F}$ that a minimal surface with non-circular non-degenerate

ellipse of curvature in $S^5(1)$ is determined, up to $O(6)$ -congruence, by functions ω , $\phi > 0$, α satisfying the following system of differential equations:

$$\begin{aligned}\bar{\partial}\alpha &= -2\bar{\alpha}\partial\phi \operatorname{csch} 2\phi, \\ \bar{\partial}\bar{\partial}\omega &= -e^\omega + e^{-\omega} \cosh 2\phi, \\ 2\bar{\partial}\bar{\partial}\phi &= \alpha\bar{\alpha} \operatorname{csch} 2\phi - e^{-\omega} \sinh 2\phi.\end{aligned}\tag{10}$$

The functions α , ω and ϕ have geometrical significance: α is analogous to the torsion of a space curve in that it is a measure of the rate at which the surface is pulling away from the great 4-sphere which contains its tangent and first normal space, the metric on the surface is given by $2e^\omega|dz|^2$, and ϕ is a measure of the eccentricity of the ellipse of curvature.

Remark 1 The above equation for ϕ may be used to show that every compact minimal surface S in $S^4(1)$ contains at least one point at which the ellipse of curvature is a point, a line segment or a circle. Otherwise, ϕ would be a smooth globally defined positive function on S satisfying the two-dimensional sinh-Gordon equation.

3 The transforms

In this section we assume that $f : S \rightarrow S^5(1)$ is a minimal immersion with non-circular ellipse of curvature at every point. We show how to associate to f two other minimal immersions of the Riemann surface S into $S^5(1)$ which both induce the same conformal structure on S as that induced by f , and also have non-circular ellipse of curvature at every point. We further show that an adapted complex coordinate z for f is also an adapted complex coordinate for our new minimal immersions. We first consider the open subset of S on which the ellipse of curvature is non-degenerate. Then, as mentioned in the introduction, our new minimal immersions are obtained by rotating N in the normal space through a geometrically significant angle θ about the minor axis of the ellipse of curvature. So, let z be an adapted complex coordinate for f and let $\cos \theta = |a|/|b| = \tanh \phi$. Then, applying rotations of $\pm\theta$ to $\pm N$ gives the four possibilities

$$\pm \frac{1}{\cosh \phi} \frac{b}{|b|} \pm \tanh \phi N.$$

For definiteness, we take N to be such that $\{f_0, \frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}, II(\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial x}), II(\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}), N\}$ is a positively oriented orthogonal moving frame of \mathbb{R}^6 , and the two transforms we will consider are those given by

$$f^+ = -\frac{1}{\cosh \phi} \frac{b}{|b|} + \tanh \phi N,\tag{11}$$

$$f^- = -\frac{1}{\cosh \phi} \frac{b}{|b|} - \tanh \phi N.\tag{12}$$

Thus if we orient the normal space by taking $\{II(\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial x}), II(\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}), N\}$ to be positively oriented, then the (+)transform f^+ is obtained from f by the (+)construction which

is given by (11) and consists of rotating N about the minor axis of the ellipse of curvature through the angle θ anticlockwise, while the $(-)$ transform f^- is obtained from f by the $(-)$ construction which is given by (12) and consists of rotating $-N$ about the minor axis through the angle θ clockwise.

We note that $b = \frac{1}{2}II(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ is a nowhere zero vector, so the orientation of S induces an orthogonal complex structure J on the orthogonal complement of $\{f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, b\}$ in \mathbb{R}^6 . An alternative description of f^ϵ is then given by

$$f^\epsilon = -\frac{1}{|b|} \left(\frac{b}{|b|} + \epsilon J II(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \right).$$

This description is valid and differentiable at points where the ellipse of curvature degenerates to a line segment, and hence we may define our transforms at these points also. We note that $f(S)$ is contained in a totally geodesic $S^3(1)$ in $S^5(1)$ if and only if it is totally geodesic or the ellipse of curvature degenerates to a line segment on the complement of an isolated set of points (where the second fundamental form may vanish), and in this latter case the transforms simply give the unit normal of $f(S)$ in $S^3(1)$. From now on, we assume that we are not in this situation, so that the ellipse of curvature is non-degenerate on an open dense subset of S .

We note the following for later use. Let vol denote the complexification of the standard volume form of \mathbb{R}^6 . Since $\det A = e^{2\omega} \sinh^2 2\phi$, we see that $\text{vol}(f_0, f_1, \bar{f}_1, f_2, \bar{f}_2, N) = \pm e^\omega \sinh 2\phi$. However,

$$\text{vol}(f_0, f_1, \bar{f}_1, f_2, \bar{f}_2, N) = -\frac{1}{4} \text{vol} \left(f_0, \frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}, II(\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial x}), II(\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}), N \right),$$

so that

$$\text{vol}(f_0, f_1, \bar{f}_1, f_2, \bar{f}_2, N) = -e^\omega \sinh 2\phi. \quad (13)$$

We now show that f^+ and f^- both induce the same conformal structure on S as that induced by f . In order to do this, we first define

$$f_1^\epsilon = \partial f^\epsilon.$$

Here and subsequently, we use ϵ as a superscript taking value $+$ or $-$, and use $\epsilon = \pm 1$ in the corresponding equations as appropriate. Then, using (11), (12) and the moving frame equations for \mathcal{F} , we find that

$$f_1^\epsilon = -ie^{-\omega} \bar{f}_1 - \frac{1}{2}(\alpha\epsilon + 2i\partial\phi) \text{sech}^2\phi \left(\text{csch} 2\phi f_2 + \text{coth} 2\phi \bar{f}_2 + \epsilon iN \right). \quad (14)$$

From this, a computation using (9) shows that

$$(f_1^\epsilon, f_1^\epsilon) = 0, \quad (15)$$

$$|f_1^\epsilon|^2 = e^{-\omega} + \frac{1}{2}|\alpha\epsilon + 2i\partial\phi|^2 \text{sech}^2\phi. \quad (16)$$

Thus the maps f^ϵ define conformal immersions of S into $S^5(1)$, so that if $z = x + iy$ then (x, y) are isothermal coordinates not only for the original immersion f but also for the two

newly constructed immersions f^ϵ . We note that the metric induced on S by f^ϵ is given by $2e^{\omega^\epsilon} |dz^2|$, where $\omega^\epsilon = \log |f_1^\epsilon|^2$.

We now show that each f^ϵ is minimal. By Takahashi's lemma, it is sufficient to check that $\partial\bar{\partial}f^\epsilon$ is a multiple of f^ϵ . We first note that it follows quickly from (14) that

$$\begin{aligned}(\partial\bar{\partial}f^\epsilon, f_0) &= 0, \\(\partial\bar{\partial}f^\epsilon, f_1) &= 0,\end{aligned}$$

while a straightforward computer calculation shows that

$$\begin{aligned}(\partial\bar{\partial}f^\epsilon, f_2) &= -i(e^{-\omega} + \frac{1}{2}|\alpha\epsilon + 2i\partial\phi|^2 \operatorname{sech}^2\phi), \\(\partial\bar{\partial}f^\epsilon, N) &= -\epsilon \tanh\phi (e^{-\omega} + \frac{1}{2}|\alpha\epsilon + 2i\partial\phi|^2 \operatorname{sech}^2\phi).\end{aligned}$$

However, it follows from (7), (9), (11) and (12) that

$$(f^\epsilon, f_0) = 0, \tag{17}$$

$$(f^\epsilon, f_1) = 0, \tag{18}$$

$$(f^\epsilon, f_2) = i, \tag{19}$$

$$(f^\epsilon, N) = \epsilon \tanh\phi, \tag{20}$$

and hence, using (16),

$$\partial\bar{\partial}f^\epsilon = -|f_1^\epsilon|^2 f^\epsilon,$$

showing that $\partial\bar{\partial}f^\epsilon$ is indeed a multiple of f^ϵ , so that each f^ϵ is a minimal immersion of S into $S^5(1)$.

Finally in this section, we show that z is also an adapted complex coordinate for f^ϵ . If II^ϵ denotes the second fundamental form of f^ϵ , we put

$$f_2^\epsilon = II^\epsilon(\partial, \partial) = a^\epsilon - ib^\epsilon.$$

As in the moving frame equations for \mathcal{F} , if $\omega^\epsilon = \log |f_1^\epsilon|^2$, then

$$f_2^\epsilon = \partial f_1^\epsilon - \partial\omega^\epsilon f_1^\epsilon. \tag{21}$$

It follows from (11), (12) and the moving frame equations for \mathcal{F} that

$$\begin{aligned}\partial f_1^\epsilon &= if_0 + e^{-\omega} \left(i\partial\omega + \tanh\phi (\alpha\epsilon + 2i\partial\phi) \right) \bar{f}_1 + 2\nu \operatorname{csch}^4 2\phi \sinh^2\phi f_2 \\ &+ (1/2)\nu \coth 2\phi \operatorname{csch} 2\phi \operatorname{sech}^2\phi \bar{f}_2 + i\epsilon\nu \operatorname{csch}^2 2\phi \tanh\phi N,\end{aligned} \tag{22}$$

where

$$\begin{aligned}\nu &= 2\alpha\epsilon\partial\phi(-2 + \cosh 2\phi) + 8i \sinh^2\phi (\partial\phi)^2 \\ &- \epsilon\partial\alpha \sinh 2\phi + i\alpha^2 - 2i \sinh 2\phi \partial\bar{\partial}\phi,\end{aligned}$$

so that, using (9),

$$(\partial f_1^\epsilon, \partial f_1^\epsilon) = -1.$$

Equations (15) and (21) now imply that $(f_2^\epsilon, f_2^\epsilon) = -1$, so that z is also an adapted complex coordinate for each f^ϵ . In particular, each f^ϵ has non-circular ellipse of curvature.

Summarizing the above, we have the following theorem:

Theorem 1 *Let $f : S \rightarrow S^5(1)$ be a minimal immersion with non-circular ellipse of curvature at every point. Then the (+)transform f^+ and the (-)transform f^- of f are both minimal immersions of S into $S^5(1)$ which induce the same conformal structure on S as that induced by f . Moreover, both f^+ and f^- have non-circular ellipse of curvature at every point and an adapted complex coordinate for f is also an adapted complex coordinate for f^+ and f^- .*

Remark 2 The final statement of the above theorem is equivalent to saying that f , f^+ and f^- all have the same $U_{2,-2}$ invariant (see [5] for the definition of this and related invariants).

4 A symmetric adapted moving frame

In this section we assume that f has non-circular ellipse of curvature at each point and is not contained in a totally geodesic $S^3(1)$. We begin the study of f and its transforms by constructing a moving frame \mathcal{B} which gives equal prominence to f and f^ϵ . We also obtain the moving frame equations and integrability conditions for \mathcal{B} .

So, let $\mathcal{B} = \{f_0, f_1, \bar{f}_1, \bar{f}_1^\epsilon, f_1^\epsilon, f_0^\epsilon\}$. It follows quickly from (9) and the moving frame equations for \mathcal{F} that the matrix B of complex bilinear inner products of elements of \mathcal{B} is given by

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^\omega & 0 & -i & 0 \\ 0 & e^\omega & 0 & i & 0 & 0 \\ 0 & 0 & i & 0 & e^{\omega^\epsilon} & 0 \\ 0 & -i & 0 & e^{\omega^\epsilon} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (23)$$

from which we see that

$$\det B = (e^{\omega+\omega^\epsilon} - 1)^2, \quad (24)$$

implying that the vectors in \mathcal{B} are linearly independent as long as $\omega + \omega^\epsilon \neq 0$.

Lemma 1 *We have that $\omega + \omega^\epsilon > 0$ on an open dense subset U of S .*

Proof: It follows from (16) that $\omega + \omega^\epsilon \geq 0$, and that $\omega + \omega^\epsilon = 0$ on an open set if and only if $\alpha = -2i\epsilon\partial\phi$. Taking the derivative of this expression with respect to $\bar{\partial}$ and using the integrability conditions (10) for the moving frame equations for \mathcal{F} it then follows that

$$-4i\epsilon \operatorname{csch} 2\phi \partial\phi\bar{\partial}\phi = e^{-\omega}i\epsilon \operatorname{csch} 2\phi (\sinh^2 2\phi - 4e^\omega \partial\phi\bar{\partial}\phi).$$

Simplifying the above equation then yields the contradiction $0 = e^{-\omega} \sinh 2\phi$. ■

The advantage of the above condition is that on the open dense subset U we can investigate the original immersion f and the new immersion f^ϵ with respect to the frame $\mathcal{B} = \{f_0, f_1, \bar{f}_1, \bar{f}_1^\epsilon, f_1^\epsilon, f_0^\epsilon\}$.

It follows from (24) that $\text{vol}(f_0, f_1, \bar{f}_1, \bar{f}_1^\epsilon, f_1^\epsilon, f_0^\epsilon) = \pm(e^{\omega+\omega^\epsilon} - 1)$. In order to determine the sign we compute the volume explicitly. A straightforward calculation using (13) and (14) yields that

$$\begin{aligned}\text{vol}(f_0, f_1, \bar{f}_1, \bar{f}_1^\epsilon, f_1^\epsilon, f_0^\epsilon) &= \epsilon \operatorname{csch}^2 2\phi \tanh \phi |\alpha\epsilon + 2i\partial\phi|^2 \text{vol}(f_0, f_1, \bar{f}_1, f_2, \bar{f}_2, N) \\ &= -\epsilon e^\omega \operatorname{csch} 2\phi \tanh \phi |\alpha\epsilon + 2i\partial\phi|^2 \\ &= -\epsilon \frac{1}{2} \operatorname{sech}^2 \phi e^\omega |\alpha\epsilon + 2i\partial\phi|^2.\end{aligned}$$

Thus, using (16),

$$\text{vol}(f_0, f_1, \bar{f}_1, \bar{f}_1^\epsilon, f_1^\epsilon, f_0^\epsilon) = -\epsilon(e^{\omega+\omega^\epsilon} - 1). \quad (25)$$

We now introduce the function γ^ϵ by

$$\gamma^\epsilon = (\partial f_1, f_1^\epsilon), \quad (26)$$

and use (23) to write down the moving frame equations for \mathcal{B} in terms of ω , ω^ϵ , γ^ϵ as follows.

$$\begin{aligned}\partial f_0 &= f_1, \\ \partial f_1 &= \frac{-i\gamma^\epsilon + \partial\omega e^{\omega+\omega^\epsilon}}{e^{\omega+\omega^\epsilon} - 1} f_1 + \frac{e^\omega(\gamma^\epsilon + i\partial\omega)}{e^{\omega+\omega^\epsilon} - 1} \bar{f}_1 + i f_0^\epsilon, \\ \partial \bar{f}_1 &= -e^\omega f_0, \\ \partial \bar{f}_1^\epsilon &= -e^{\omega^\epsilon} f_0^\epsilon, \\ \partial f_1^\epsilon &= i f_0 - \frac{e^{\omega^\epsilon}(\gamma^\epsilon - i\partial\omega^\epsilon)}{e^{\omega+\omega^\epsilon} - 1} \bar{f}_1 + \frac{i\gamma^\epsilon + \partial\omega^\epsilon e^{\omega+\omega^\epsilon}}{e^{\omega+\omega^\epsilon} - 1} f_1^\epsilon, \\ \partial f_0^\epsilon &= f_1^\epsilon.\end{aligned}$$

As before, the corresponding $\bar{\partial}$ equations may be found by taking the conjugates of the above. The integrability conditions of the above system of equations are

$$\begin{aligned}\bar{\partial}\gamma^\epsilon &= i(e^\omega - e^{\omega^\epsilon}), \\ \partial\bar{\partial}\omega &= -2 \sinh \omega + \frac{1}{e^{\omega+\omega^\epsilon} - 1} |\gamma^\epsilon + i\partial\omega|^2, \\ \partial\bar{\partial}\omega^\epsilon &= -2 \sinh \omega^\epsilon + \frac{1}{e^{\omega+\omega^\epsilon} - 1} |\gamma^\epsilon - i\partial\omega^\epsilon|^2.\end{aligned} \quad (27)$$

As before, solutions of (27) correspond up to $O(6)$ congruence to a minimal surface and its ϵ -transform.

5 The (+) and (-) constructions are mutual inverses

In this section we use the frame \mathcal{B} introduced in the previous section to prove the following.

Theorem 2 *Let $f : S \rightarrow S^5(1)$ be a minimal immersion with non-circular ellipse of curvature at every point. Then the (+)construction and the (-)construction are mutual inverses in the sense that both f^+ and f^- have non-circular ellipse of curvature and $(f^+)^- = (f^-)^+ = f$.*

Proof: The theorem is clearly true if $f(S)$ is contained in a totally geodesic $S^3(1)$, so we assume otherwise and work on the open dense subset U of S on which $\omega + \omega^+ > 0$. We denote by $p_{\tilde{\epsilon}}(f_0^\epsilon)$, where $\tilde{\epsilon} = -1$, the image of f_0^ϵ by the $\tilde{\epsilon}$ construction. So, by (11) and (12), we have that

$$p_{\tilde{\epsilon}}(f_0^\epsilon) = -\frac{1}{\cosh \phi^\epsilon} \frac{b^\epsilon}{|b^\epsilon|} + \tilde{\epsilon} \tanh \phi^\epsilon N^\epsilon = -\frac{b^\epsilon}{\cosh^2 \phi^\epsilon} + \tilde{\epsilon} \tanh \phi^\epsilon N^\epsilon. \quad (28)$$

Now let

$$v = \frac{b^\epsilon}{\cosh^2 \phi^\epsilon} + f_0.$$

It follows from the ϵ -analogue of (9) and from (23) that v is orthogonal to f_0^ϵ and f_1^ϵ , while, using the ϵ -analogues of (6) and (7), we see that

$$\begin{aligned} (v, f_2^\epsilon) &= \left(\frac{b^\epsilon}{\cosh^2 \phi^\epsilon}, f_2^\epsilon \right) + (f_0, f_2^\epsilon) \\ &= -i \frac{|b^\epsilon|^2}{\cosh^2 \phi^\epsilon} + (f_0, \partial f_1^\epsilon) \\ &= -i - (\partial f_0, f_1^\epsilon) \\ &= -i - (f_1, f_1^\epsilon) \\ &= -i + i = 0, \end{aligned}$$

so that v is a scalar multiple of N^ϵ . We now use the moving frame equations for \mathcal{B} to determine this scalar multiple as follows. First note that

$$\begin{aligned} \text{vol}(f_0^\epsilon, f_1^\epsilon, \bar{f}_1^\epsilon, f_2^\epsilon, \bar{f}_2^\epsilon, v) &= \text{vol}(f_0^\epsilon, f_1^\epsilon, \bar{f}_1^\epsilon, f_2^\epsilon, \bar{f}_2^\epsilon, f_0) \\ &= \text{vol}(f_0^\epsilon, f_1^\epsilon, \bar{f}_1^\epsilon, \partial f_1^\epsilon, \bar{\partial} \bar{f}_1^\epsilon, f_0) \\ &= -\frac{e^{2\omega^\epsilon}}{(e^{\omega+\omega^\epsilon}-1)^2} |\gamma^\epsilon - i\partial\omega^\epsilon|^2 \text{vol}(f_0, f_1, \bar{f}_1, \bar{f}_1^\epsilon, f_1^\epsilon, f^\epsilon) \\ &= \epsilon \frac{e^{2\omega^\epsilon}}{(e^{\omega+\omega^\epsilon}-1)} |\gamma^\epsilon - i\partial\omega^\epsilon|^2, \end{aligned}$$

where the final equality above is obtained using (25).

However, it follows from (27) and the ϵ -analogue of (10) that

$$\frac{e^{\omega^\epsilon}}{(e^{\omega+\omega^\epsilon}-1)} |\gamma^\epsilon - i\partial\omega^\epsilon|^2 = 2 \sinh^2 \phi^\epsilon, \quad (29)$$

so that

$$\text{vol}(f_0^\epsilon, f_1^\epsilon, \bar{f}_1^\epsilon, f_2^\epsilon, \bar{f}_2^\epsilon, v) = 2\epsilon e^{\omega^\epsilon} \sinh^2 \phi^\epsilon.$$

We next note that the ϵ -analogue of (13) gives

$$\text{vol}(f_0^\epsilon, f_1^\epsilon, \bar{f}_1^\epsilon, f_2^\epsilon, \bar{f}_2^\epsilon, N^\epsilon) = -e^{\omega^\epsilon} \sinh 2\phi^\epsilon,$$

so it follows that $v = -\epsilon \tanh \phi^\epsilon N^\epsilon$.

Thus, from (24) we have that

$$\begin{aligned} p_{\tilde{\epsilon}}(f_0^\epsilon) &= -v + f_0 + \tilde{\epsilon} \tanh \phi^\epsilon N^\epsilon \\ &= (\tilde{\epsilon} + \epsilon) \tanh \phi^\epsilon N^\epsilon + f_0 = f_0, \end{aligned}$$

implying that the (+)construction and the (-)construction are mutual inverses. \blacksquare

The above theorem shows that we may associate to a minimal immersion $f : S \rightarrow S^5(1)$ with non-circular ellipse of curvature a sequence $\{f^p : p \in \mathbb{Z}\}$ of such minimal immersions with $f^0 = f$ and, for each p , $f^{p+1} = (f^p)^+$ and $f^{p-1} = (f^p)^-$. Moreover, each element of the sequence induces the same conformal structure on S , and an adapted complex coordinate z for any f^p is an adapted complex coordinate for each element of the sequence.

6 The geometry of the invariants

In the previous section we showed that we may associate to a minimal immersion $f : S \rightarrow S^5(1)$ with non-circular ellipse of curvature a sequence $\{f^p : p \in \mathbb{Z}\}$ of such minimal immersions with $f^0 = f$. For the remainder of the paper we will assume that $f(S)$ is not contained in a totally geodesic $S^3(1)$. We use the superfix p to denote objects connected with f^p . For instance, with each immersion f^p we associate as before the invariants ω^p , ϕ^p and α^p . Moreover with each (+)construction, $f^p \rightarrow f^{p+1}$, we associate the invariants ω^p , ω^{p+1} and $\gamma^{p+1} = (\partial f_1^p, f_1^{p+1})$, while with each (-)construction we associate the invariants ω^p , ω^{p-1} and $\delta^{p-1} = (\partial f_1^p, f_1^{p-1})$. Thus, γ^1 is equal to the invariant γ^+ used in previous sections, while δ^{-1} is equal to γ^- . Since, from (23), we have that $(f_1^p, f_1^{p+1}) = -i$, it is clear that $\delta^p = -\gamma^{p+1}$.

As already mentioned, the geometrical meaning of the invariants ω^p is clear, since they give the metric induced on S by f^p . Also, the final moving frame equation for \mathcal{F} implies that $\alpha^p = 0$ if and only if f^p is not linearly full. In this section we look more closely at this situation, and also obtain a geometrical characterisation of the condition $\gamma^{p+1} = 0$.

We begin with a useful lemma.

Lemma 2 *Let A be an orientation reversing isometry of \mathbb{R}^6 . Then*

$$(Af)^- = A(f^+) , \quad (Af)^+ = A(f^-) .$$

In fact, more generally, for each integer p we have that $(Af)^p = A(f^{-p})$.

Proof: As A is an orientation reversing isometry, if N is the normal vector associated to f as in equations (11) and (12), then the corresponding normal associated to Af is $-AN$. The first result is now clear from the definitions of the (+) and (-)constructions given in (11) and (12). The second may be proved in a similar manner, and the final statement follows by induction. \blacksquare

Theorem 3 *Let $f : S \rightarrow S^5(1)$ be a minimal immersion, not contained in a totally geodesic $S^3(1)$, with non-circular ellipse of curvature at every point. Let $\{f^p\}$ be the sequence of minimal immersions into $S^5(1)$ determined by f . For each integer q , the following three statements are equivalent:*

1. $\alpha^q = 0$ on an open subset of S ,

2. f^q is not linearly full,

3. there exists an orientation reversing isometry $A \in O(6)$ such that $f^{q+1} = A(f^{q-1})$.

Moreover, in this case, for every integer r we have that f^{q-r} and f^{q+r} are congruent via reflection in the great 4-sphere containing f^q .

Proof: We have already noted the equivalence of the first two statements. Now suppose that condition 3 holds. Taking the $(-)$ -construction of this, we may use Lemma 2 to see that $f^q = A(f^q)$. Since A has at least one eigenvalue equal to -1 , it now follows that f^q is not linearly full and that A is reflection in the great 4-sphere containing f^q .

Conversely suppose that f^q is contained in a totally geodesic $S^4(1)$. In this case N^q is a constant vector, so it is clear from (11) and (12) that f^{q-1} and f^{q+1} are congruent via reflection in the totally geodesic $S^4(1)$ containing f^q . The final statement of the theorem now follows from Lemma 2. \blacksquare

A similar characterisation also exists for $\gamma^{q+1} = 0$. We show in Section 9 that this situation can actually arise; in this case f^q is a bipolar surface in the sense of Lawson [10].

Theorem 4 *Let $f : S \rightarrow S^5(1)$ be a minimal immersion, not contained in a totally geodesic $S^3(1)$, with non-circular ellipse of curvature at every point. Let $\{f^p\}$ be the sequence of minimal immersions into $S^5(1)$ determined by f . For each integer q , $\gamma^{q+1} = 0$ if and only if there exists an orientation reversing isometry $A \in O(6)$ such that $f^{q+1} = A(f^q)$. Moreover, in this case, A is reflection in a great subsphere of $S^5(1)$ and for every integer r we have that $f^{q+1+r} = A(f^{q-r})$.*

Proof: Assume that $f^{q+1} = A(f^q)$. Then,

$$\begin{aligned}\gamma^{q+1} &= (\partial f_1^q, f_1^{q+1}) \\ &= (\partial A(f_1^q), A(f_1^{q+1})).\end{aligned}$$

However,

$$A(f_1^q) = A\partial f^q = \partial A(f^q) = \partial f^{q+1} = f_1^{q+1},$$

while, using Lemma 2,

$$A(f_1^{q+1}) = A(\partial f^{q+1}) = \partial A(f^{q+1}) = \partial((A(f^q))^-) = \partial((f^{q+1})^-) = \partial f^q = f_1^q.$$

Thus

$$\gamma^{q+1} = (\partial f_1^{q+1}, f_1^q) = -(f_1^{q+1}, \partial f_1^q) = -\gamma^{q+1},$$

so that $\gamma^{q+1} = 0$.

Conversely suppose that $\gamma^{q+1} = 0$. It then follows from the integrability conditions (27) that $\omega^q = \omega^{q+1}$. Since the set \mathcal{B} is a basis for \mathbb{C}^6 we may define, for each z , a unique linear map $A(z)$ by

$$\begin{aligned}Af_0^q &= f_0^{q+1}, & Af_1^q &= f_1^{q+1}, & A\bar{f}_1^q &= \bar{f}_1^{q+1}, \\ Af_0^{q+1} &= f_0^q, & Af_1^{q+1} &= f_1^q, & A\bar{f}_1^{q+1} &= \bar{f}_1^q.\end{aligned}$$

However, it follows from the moving frame equations for \mathcal{B} that A does not depend on z , while (23) may be used to show that A is an isometry. It is clear from the definition that A has determinant -1 and A^2 is the identity, so it follows that A is a reflection. As in the previous theorem, the final statement follows from Lemma 2. \blacksquare

In the previous two theorems we have considered two situations which led to the conclusion that two elements in a sequence $\{f^p\}$ are congruent via an orientation reversing isometry. An easy application of Lemma 2 quickly leads to the following theorem, which shows that the above situations are the only ones for which this can happen.

Theorem 5 *Let $f : S \rightarrow S^5(1)$ be a minimal immersion, not contained in a totally geodesic $S^3(1)$, with non-circular ellipse of curvature at every point. Let $\{f^p\}$ be the sequence of minimal immersions into $S^5(1)$ determined by f . Suppose that two elements f^q and f^r of the sequence are congruent via an orientation reversing isometry. Then there are two possibilities, depending on the parity of $q - r$. Either*

1. *there exists an integer s for which $\alpha^s = 0$ on an open subset of S , or*
2. *there exists an integer s for which $\gamma^{s+1} = 0$.*

Finally in this section, we note that if two elements of a sequence $\{f^p\}$ are related by an orientation preserving isometry then the sequence is periodic in a natural sense. This situation can actually arise, and we plan to investigate this further at a later date.

7 Minimal surfaces and ruled Lagrangian submanifolds

In a previous paper [3], we studied minimal Lagrangian submanifolds of complex projective 3-space $\mathbb{C}P^3(4)$ which admit a foliation by asymptotic curves. Such submanifolds can be divided into three types.

1. Those which additionally satisfy Chen's equality [8]. These were studied and characterized in [1], [2], and are closely related to minimal immersions of surfaces in $S^5(1)$ with ellipse of curvature a circle.
2. Those for which the unit tangent vectors to the asymptotic curves form a Killing vector field. It is shown in [3] that these are related to minimal surfaces in $S^3(1)$, and classification theorems are obtained in [7].
3. All the rest. In [3] we showed how to construct, starting from such a submanifold, a pair of minimal immersions of a surface S into $S^5(1)$ with non-circular ellipse of curvature which are related by the (+) and (-) constructions.

In this section we deal with the converse of the construction described in [3]. We show how to associate to a minimal immersion f of a surface in $S^5(1)$ with non-circular ellipse of curvature a Lagrangian submanifold of $CP^3(4)$ belonging to the third type. We will then show that the construction given in [3] associates to this Lagrangian submanifold the immersion f and its (+)transform. We first briefly describe the construction given in [3]. Let M be a minimal Lagrangian submanifolds of $CP^3(4)$ which admits a foliation by asymptotic curves. We construct an orthonormal moving frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ along M such that \mathbf{e}_1 is tangential to the asymptotic curves and $\mathbf{e}_2, \mathbf{e}_3$ are eigenvectors of the second fundamental form $A_{J\mathbf{e}_1}$ of M with respect to the normal $J\mathbf{e}_1$ with corresponding eigenvalues $\pm\lambda$ ($\lambda > 0$). Since M is Lagrangian in $CP^3(4)$, there is a horizontal lift $\mathbf{E}_0 : M \rightarrow S^7(1)$ of M to the unit 7-sphere of the Hopf fibration $\pi : S^7(1) \rightarrow CP^3(4)$ [11]. By setting $\mathbf{E}_j = d\mathbf{E}_0(\mathbf{e}_j)$ for $j = 1, 2, 3$, we then define a map $\mathcal{E} = (\mathbf{E}_0, \dots, \mathbf{E}_3) : M \rightarrow U(4)$, where $U(4)$ denotes the unitary group, but, by choosing a suitable horizontal lift, we may assume that the image of \mathcal{E} is contained in the special unitary group $SU(4)$. Composing \mathcal{E} with a suitable standard double cover of the orthogonal group $SO(6)$ by $SU(4)$ gives a map $\mathcal{U} = (\mathbf{U}_1, \dots, \mathbf{U}_6) : M \rightarrow SO(6)$. It turns out that \mathbf{U}_2 and \mathbf{U}_4 are constant along the asymptotic curves of M , and their images are minimal surfaces in $S^5(1)$. If M is a Lagrangian submanifold of the third type mentioned at the beginning of this section, then it is shown in [3] that the ellipse of curvature of these surfaces is not a circle, and the surfaces are related by the transforms discussed in the current paper.

We will use the notation and terminology of [3], and work throughout on the open subset U of S on which $\omega + \omega^+ > 0$. In order to simplify the notation, we denote the immersion of S into $S^5(1)$ by f and let g be the (+)transform of f . As usual, we let z be an adapted complex coordinate, put $f_0 = f$, $f_1 = \partial f$, $g_0 = g$ and $g_1 = \partial g$.

We will show that, for a suitable interval I of real numbers, $M = I \times U$ may be realised as a minimal Lagrangian submanifold of $CP^3(4)$ of type 3 such that if $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal frame along M of the type described above and in [3], then the corresponding map $\mathcal{U} = (\mathbf{U}_1, \dots, \mathbf{U}_6) : M \rightarrow SO(6)$ has

$$\mathbf{U}_2(t, z) = g_0(z), \quad \mathbf{U}_4(t, z) = f_0(z), \quad (30)$$

where t is the standard real coordinate on I .

In fact, we use the invariants ω , ω^+ and γ^+ to construct a map $\mathcal{U} = (\mathbf{U}_1, \dots, \mathbf{U}_6) : M \rightarrow SO(6)$ satisfying (30), with the property that if $\Omega = \mathcal{U}^{-1}d\mathcal{U}$ then Ω has the form of (33) of [3] for suitable functions $z_{21}^2, z_{12}^3, z_{22}^3, z_{32}^3, \lambda, a$ and b on M , and linearly independent 1-forms $\omega_1, \omega_2, \omega_3$ on M . Having done this, it is straightforward to deduce that we may reverse the construction given in [3] in order to construct from such a map \mathcal{U} our required Lagrangian submanifold of $CP^3(4)$, with the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ being the basis of vectors dual to $\{\omega_1, \omega_2, \omega_3\}$.

We begin by noting from (42) of [3] that in order for (30) to hold we require that

$$d\mathbf{U}_2(\mathbf{e}_2 - i\mathbf{e}_3) = 2\theta_1\sqrt{\lambda}g_1, \quad (31)$$

where θ_1 is a fourth root of unity. In fact, we may assume that $\theta_1 = 1$ by rotating our adapted complex coordinate z through a suitable multiple of $\pi/2$. In a similar manner,

$$d\mathbf{U}_4(\mathbf{e}_2 - i\mathbf{e}_3) = 2\theta_2\sqrt{\lambda}f_1, \quad (32)$$

for some fourth root of unity θ_2 .

We next note that for (34) and (35) of [3] to hold we must have that

$$d\mathbf{U}_2(\mathbf{e}_2 - i\mathbf{e}_3) = (z_{12}^3 - 1 - iz_{21}^2)(\mathbf{U}_1 + i\mathbf{U}_3) - \lambda(\mathbf{U}_5 + i\mathbf{U}_6),$$

and

$$d\mathbf{U}_4(\mathbf{e}_2 - i\mathbf{e}_3) = i\lambda(\mathbf{U}_1 - i\mathbf{U}_3) + i(1 + z_{12}^3 - iz_{21}^2)(\mathbf{U}_5 - i\mathbf{U}_6),$$

so that, from (31) and (32),

$$2\sqrt{\lambda}g_1 = (z_{12}^3 - 1 - iz_{21}^2)(\mathbf{U}_1 + i\mathbf{U}_3) - \lambda(\mathbf{U}_5 + i\mathbf{U}_6), \quad (33)$$

$$2\theta_2\sqrt{\lambda}f_1 = i\lambda(\mathbf{U}_1 - i\mathbf{U}_3) + i(1 + z_{12}^3 - iz_{21}^2)(\mathbf{U}_5 - i\mathbf{U}_6). \quad (34)$$

However, it follows from (23) that $-i = (f_1, g_1)$, so orthonormality of \mathcal{U} requires that

$$\begin{aligned} -4\theta_2\lambda i &= 2i\lambda(z_{12}^3 - 1 - iz_{21}^2) - 2i\lambda(1 + z_{12}^3 - iz_{21}^2) \\ &= -4i\lambda, \end{aligned}$$

so that $\theta_2 = 1$. Therefore, in order to construct \mathcal{U} , it is necessary to determine real-valued functions $z_{12}^3(t, z)$, $z_{21}^2(t, z)$, $\lambda(t, z)$, and orthonormal vector fields $\mathbf{U}_1(t, z)$, $\mathbf{U}_3(t, z)$, $\mathbf{U}_5(t, z)$, $\mathbf{U}_6(t, z)$ in \mathbb{R}^6 satisfying (33) and (34) (with $\theta_2 = 1$ in this latter equation). However, as $(f_1, \bar{f}_1) = e^\omega$ and $(g_1, \bar{g}_1) = e^{\omega^+}$ we see that

$$\begin{aligned} 2\lambda e^{\omega^+} &= (z_{12}^3 - 1)^2 + (z_{21}^2)^2 + \lambda^2, \\ 2\lambda e^\omega &= (1 + z_{12}^3)^2 + (z_{21}^2)^2 + \lambda^2. \end{aligned}$$

Thus

$$z_{12}^3 = \frac{1}{2}\lambda(e^\omega - e^{\omega^+}), \quad (35)$$

and

$$(z_{21}^2)^2 + \left(1 - \frac{1}{2}\lambda(e^\omega + e^{\omega^+})\right)^2 = \lambda^2(e^{\omega+\omega^+} - 1).$$

As λ and $(e^{\omega+\omega^+} - 1)$ are both positive, we may define z_{21}^2 and λ by taking

$$\lambda = \frac{2}{e^\omega + e^{\omega^+} + 2\cos t\sqrt{e^{\omega+\omega^+} - 1}}, \quad (36)$$

$$z_{21}^2 = \lambda \sin t \sqrt{e^{\omega+\omega^+} - 1}. \quad (37)$$

We restrict t to lie on a suitable subinterval I of $(0, \pi)$, in order to ensure that λ is well defined and $z_{21}^2 > 0$.

We have now obtained, through (35), (36) and (37), formulae for z_{12}^3 , λ and z_{21}^2 in terms of ω , ω^+ and γ^+ . We next obtain \mathbf{U}_1 , \mathbf{U}_3 , \mathbf{U}_5 and \mathbf{U}_6 as the solutions to (33) and (34). In the next section we will discuss a particular special case of the construction detailed in this section, so we will write down the solution to (33) and (34) and verify their properties explicitly (for which we used Mathematica), although the properties we obtain may be deduced directly from (33) and (34).

So, if

$$C = \sqrt{\frac{\lambda}{e^{\omega+\omega^+}-1}},$$

we find from (33) and (34) that

$$\mathbf{U}_1 + i\mathbf{U}_3 = -C \left((\sqrt{e^{\omega+\omega^+}-1} + e^{-it+\omega})g_1 + ie^{-it}\bar{f}_1 \right), \quad (38)$$

$$\mathbf{U}_5 + i\mathbf{U}_6 = C \left(e^{-it}g_1 + i(\sqrt{e^{\omega+\omega^+}-1} + e^{-it+\omega^+})\bar{f}_1 \right). \quad (39)$$

It is now a straightforward computation using (23) to verify that \mathbf{U}_1 up to \mathbf{U}_6 defined by (30),(38) and (39) are orthonormal vectors and that, using (25),

$$\begin{aligned} \text{vol}(\mathbf{U}_1, \dots, \mathbf{U}_6) &= -\frac{1}{e^{\omega+\omega^+}-1} \text{vol}(f_0, f_1, \bar{f}_1, \bar{g}_1, g_1, g_0) \\ &= 1. \end{aligned}$$

Thus $\mathcal{U} = (\mathbf{U}_1, \dots, \mathbf{U}_6) : M \rightarrow SO(6)$, so that $\Omega = \mathcal{U}^{-1}d\mathcal{U}$ is a skew symmetric matrix whose second and fourth columns have the correct form.

It remains to find the linearly independent 1-forms $\omega_1, \omega_2, \omega_3$ on M , and real valued functions z_{22}^3, z_{32}^3, a and b on M such that the entries of $\Omega = \mathcal{U}^{-1}d\mathcal{U}$ are as given in (33) of [3].

Since $d\mathbf{U}_2(\partial/\partial t) = 0$, for (33) of [3] to hold we need that $\omega_2(\partial/\partial t) = \omega_3(\partial/\partial t) = 0$. Also, from (31) with $\theta_1 = 1$, we have

$$\begin{aligned} d\mathbf{U}_2(\mathbf{e}_2) &= \sqrt{\lambda} dg(\partial/\partial x) = \sqrt{\lambda} d\mathbf{U}_2(\partial/\partial x), \\ d\mathbf{U}_2(\mathbf{e}_3) &= \sqrt{\lambda} dg(\partial/\partial y) = \sqrt{\lambda} d\mathbf{U}_2(\partial/\partial y), \end{aligned}$$

from which it follows that $\mathbf{e}_2 - \sqrt{\lambda}\partial/\partial x$ and $\mathbf{e}_3 - \sqrt{\lambda}\partial/\partial y$ are multiples of $\partial/\partial t$. Thus

$$\omega_2 = \frac{1}{\sqrt{\lambda}}dx, \quad \omega_3 = \frac{1}{\sqrt{\lambda}}dy, \quad (40)$$

which also ensures that (34) holds with $\theta_2 = 1$.

We now consider the columns of Ω other than the second and fourth. These have the correct form if and only if we have that, modulo \mathbf{U}_2 and \mathbf{U}_4 ,

$$\begin{aligned} d(\mathbf{U}_1 + i\mathbf{U}_3) &\equiv i \left((1 + z_{12}^3)\omega_1 + z_{22}^3\omega_2 + z_{32}^3\omega_3 \right) (\mathbf{U}_1 + i\mathbf{U}_3) \\ &\quad + (c\lambda^{-\frac{1}{2}}dz - i\lambda\omega_1)(\mathbf{U}_5 + i\mathbf{U}_6), \quad (\text{mod } \mathbf{U}_2, \mathbf{U}_4), \end{aligned} \quad (41)$$

$$\begin{aligned} d(\mathbf{U}_5 + i\mathbf{U}_6) &\equiv -(c\lambda^{-\frac{1}{2}}d\bar{z} + i\lambda\omega_1)(\mathbf{U}_1 + i\mathbf{U}_3) \\ &\quad - i \left((z_{12}^3 - 1)\omega_1 + z_{22}^3\omega_2 + z_{32}^3\omega_3 \right) (\mathbf{U}_5 + i\mathbf{U}_6), \quad (\text{mod } \mathbf{U}_2, \mathbf{U}_4), \end{aligned} \quad (42)$$

where $c = -b - ia$. In particular, ω_1 must satisfy

$$(d(\mathbf{U}_1 + i\mathbf{U}_3), \mathbf{U}_1 - i\mathbf{U}_3) + (d(\mathbf{U}_5 + i\mathbf{U}_6), \mathbf{U}_5 - i\mathbf{U}_6) = 4i\omega_1. \quad (43)$$

Computing the lefthand side explicitly using (38) and (39), we see that (43) holds if and only if

$$\omega_1\left(\frac{\partial}{\partial t}\right) = -\frac{1}{2}, \quad (44)$$

$$\omega_1(\bar{\partial}) = -\frac{1}{4}i\frac{2i\bar{\gamma}^+ + e^{\omega+\omega^+}\bar{\partial}(\omega-\omega^+)}{e^{\omega+\omega^+}-1}. \quad (45)$$

We use these expressions to define ω_1 , in which case (40), implies that ω_1 , ω_2 and ω_3 are linearly independent 1-forms on M .

A straightforward computation using (38) and (39) now shows that

$$\begin{aligned} (d(\mathbf{U}_1 + i\mathbf{U}_3), \mathbf{U}_5 - i\mathbf{U}_6)\left(\frac{\partial}{\partial t}\right) &= i\lambda, \\ (d(\mathbf{U}_1 + i\mathbf{U}_3), \mathbf{U}_1 - i\mathbf{U}_3)\left(\frac{\partial}{\partial t}\right) - (d(\mathbf{U}_5 + i\mathbf{U}_6), \mathbf{U}_5 - i\mathbf{U}_6)\left(\frac{\partial}{\partial t}\right) + 2iz_{12}^3 &= 0, \\ (d(\mathbf{U}_1 + i\mathbf{U}_3), \mathbf{U}_5 - i\mathbf{U}_6)(\bar{\partial}) &= -\frac{1}{2}\lambda\frac{2i\bar{\gamma}^+ + e^{\omega+\omega^+}\bar{\partial}(\omega-\omega^+)}{e^{\omega+\omega^+}-1} = -2i\lambda\omega_1(\bar{\partial}), \end{aligned}$$

so it only remains to define the complex-valued function c on M and the real valued functions z_{22}^3 and z_{32}^3 in such a way that (41) and (42) hold. This may be done explicitly and uniquely by calculating $(d(\mathbf{U}_1 + i\mathbf{U}_3), \mathbf{U}_5 - i\mathbf{U}_6)(\partial)$, $(d(\mathbf{U}_1 + i\mathbf{U}_3), \mathbf{U}_1 - i\mathbf{U}_3)\left(\frac{\partial}{\partial x}\right)$, and $(d(\mathbf{U}_1 + i\mathbf{U}_3), \mathbf{U}_1 - i\mathbf{U}_3)\left(\frac{\partial}{\partial y}\right)$. We have thus proved the following theorem.

Theorem 6 *Let $f : S \rightarrow S^5(1)$ be a minimal immersion, not contained in a totally geodesic $S^3(1)$, with non-circular ellipse of curvature at every point. Then there exists a minimal Lagrangian submanifold of $CP^3(4)$, admitting a foliation by asymptotic curves, for which the construction described in [3] yields f and its (+)transform on an open dense set.*

We remark that, since we have shown in the previous section that such minimal surfaces are part of a sequence, minimal Lagrangian submanifolds of type 3 also form a sequence. However, up to now, we do not know geometrically (without using this detour over minimal surfaces) how to associate one with its successor.

In the next section we will give an example in which we can explicitly describe the reverse construction detailed in this section.

8 Lawson's bipolar surfaces

Let $f : S \rightarrow S^5(1)$ be a minimal immersion with non-circular ellipse of curvature, which is not contained in a totally geodesic $S^3(1)$. As usual, we will work on the open dense subset U of S on which $\omega + \omega^+$ is non-zero, and let z be an adapted complex coordinate for f . Using the notation of the previous section, we will consider the special case in which the invariant $\gamma^+ = (\partial f_1, g_1)$ is identically zero. We will show that, in this case, f is the bipolar surface in the sense of Lawson [10] of a minimal surface in $S^3(1)$.

We begin by noting that if $\gamma^+ = 0$ then (27) implies that $\omega = \omega^+$ while (27) and Lemma 1 imply that ω is a positive solution of the following partial differential equation:

$$\partial\bar{\partial}\omega = -2\sinh\omega + \frac{1}{e^{2\omega}-1}|\partial\omega|^2. \quad (46)$$

Conversely given a positive solution of the above differential equation, there exists a corresponding minimal surface in $S^5(1)$ with non-circular non-degenerate ellipse of curvature and induced metric $2e^\omega|dz|^2$.

It is convenient to rewrite the above differential equation by making the substitution $e^\omega = \cosh\eta$. A short calculation shows that ω satisfies (46) if and only if the function η satisfies the sinh-Gordon equation

$$\partial\bar{\partial}\eta = -\sinh\eta. \quad (47)$$

We recall [9] that a solution η of the sinh-Gordon equation determines an S^1 -family of non-totally geodesic minimal immersions in $S^3(1)$ whose induced metric is $e^\eta|dz|^2$, and we will see that f is the bipolar in $S^5(1)$ of the minimal immersion in this family for which the coordinate curves are the lines of curvature.

Specialising the formulae of the previous section to the case $\gamma^+ = 0$, we obtain

$$\begin{aligned} \lambda &= \frac{1}{\cosh\eta + \cos t \sinh\eta}, \\ z_{21}^2 &= \frac{\sin t \sinh\eta}{\cosh\eta + \cos t \sinh\eta}, \\ z_{12}^3 &= 0, \\ \omega_1 &= -\frac{1}{2}dt, \\ \omega_2 &= \frac{1}{\sqrt{\lambda}}dx, \\ \omega_3 &= \frac{1}{\sqrt{\lambda}}dy, \\ b &= \frac{1}{2}\lambda^{3/2}\eta_y \sin t, \\ a &= \frac{1}{2}\lambda^{3/2}\eta_x \sin t, \\ z_{32}^3 &= \frac{1}{2}\lambda^{3/2}\eta_x(\cos t \cosh\eta + \sinh\eta), \\ z_{22}^3 &= -\frac{1}{2}\lambda^{3/2}\eta_y(\cos t \cosh\eta + \sinh\eta). \end{aligned} \quad (48)$$

In particular, $\partial/\partial t = -\frac{1}{2}\mathbf{e}_1$, $\partial/\partial x = \frac{1}{\sqrt{\lambda}}\mathbf{e}_2$, and $\partial/\partial y = \frac{1}{\sqrt{\lambda}}\mathbf{e}_3$.

Substituting these expressions into (25) of [3] we see that the horizontal lift F to $S^7(1)$ of the minimal Lagrangian immersion into $\mathbb{C}P^3(4)$ corresponding to f satisfies the following

system of differential equations:

$$F_{tt} = -F/4, \quad (49)$$

$$F_{tx} = \frac{-(i+\sin t \sinh \eta)}{2(\cosh \eta + \cos t \sinh \eta)} F_x, \quad (50)$$

$$F_{ty} = \frac{i-\sin t \sinh \eta}{2(\cosh \eta + \cos t \sinh \eta)} F_y, \quad (51)$$

$$F_{xx} = -(\cosh \eta + \cos t \sinh \eta)F + 2(\sin t \sinh \eta - i)F_t \\ + \frac{\eta_x (\cos t \cosh \eta + i \sin t + \sinh \eta)}{2(\cosh \eta + \cos t \sinh \eta)} F_x - \frac{\eta_y (\cos t \cosh \eta - i \sin t + \sinh \eta)}{2(\cosh \eta + \cos t \sinh \eta)} F_y, \quad (52)$$

$$F_{xy} = \frac{\eta_y (\cos t \cosh \eta + i \sin t + \sinh \eta)}{2(\cosh \eta + \cos t \sinh \eta)} F_x + \frac{\eta_x (\cos t \cosh \eta - i \sin t + \sinh \eta)}{2(\cosh \eta + \cos t \sinh \eta)} F_y, \quad (53)$$

$$F_{yy} = -(\cosh \eta + \cos t \sinh \eta)F + 2(i + \sin t \sinh \eta)F_t \\ - \frac{\eta_x (\cos t \cosh \eta + i \sin t + \sinh \eta)}{2(\cosh \eta + \cos t \sinh \eta)} F_x + \frac{\eta_y (\cos t \cosh \eta - i \sin t + \sinh \eta)}{2(\cosh \eta + \cos t \sinh \eta)} F_y. \quad (54)$$

It follows from (49) that we may write

$$F(t, u, v) = G_1(x, y) \cos \frac{t}{2} + iG_2(x, y) \sin \frac{t}{2}, \quad (55)$$

for suitable \mathbb{C}^4 -valued functions G_1 and G_2 . Substituting this into (50) and carrying out significant but elementary simplification we find that

$$(G_2)_x = -e^{-\eta}(G_1)_x, \quad (56)$$

while similar reasoning using (51) gives that

$$(G_2)_y = e^{-\eta}(G_1)_y. \quad (57)$$

Using (56) and (57) we find, after some calculation, that (52), (53) and (54) are equivalent to

$$(G_1)_{xx} = \frac{1}{2}\eta_x(G_1)_x - \frac{1}{2}\eta_y(G_1)_y + G_2 - e^\eta G_1, \\ (G_1)_{xy} = \frac{1}{2}\eta_y(G_1)_x + \frac{1}{2}\eta_x(G_1)_y, \\ (G_1)_{yy} = -\frac{1}{2}\eta_x(G_1)_x + \frac{1}{2}\eta_y(G_1)_y - G_2 - e^\eta G_1. \quad (58)$$

We now note that since $|F| = 1$, (55) implies that $|G_1| = |G_2| = 1$ and that G_1 is real orthogonal to G_2 . The horizontality of F further shows that G_1 and G_2 are unitarily orthogonal.

The coefficients in the system (58) are all real, so the real subspace spanned by G_1 , G_2 , $(G_1)_x$ and $(G_1)_y$ is constant. We identify this subspace with \mathbb{R}^4 by picking an orthonormal basis. But the system (58) is exactly that of a minimal surface G_1 in $S^3(1)$ with unit normal G_2 , induced metric $ds^2 = e^\eta |dz|^2$ with the complex coordinate chosen such that the second fundamental form $\tilde{I}I$ of G_1 satisfies $(\tilde{I}I(\partial, \partial), \tilde{I}I(\partial, \partial)) = 1/4$. In particular, since $\tilde{I}I(\partial/\partial x, \partial/\partial y) = 0$, we see that the coordinate curves of G_1 are the lines of curvature.

Applying now the definition of U_4 of [3] and the expressions for ω_1 , ω_2 and ω_3 obtained in (48), we get that

$$f = \frac{1}{i\sqrt{2}}(F \wedge (-2F_t) - \lambda F_x \wedge F_y) \subset \Lambda^2 \mathbb{C}^4.$$

An easy calculation using (55) now shows that

$$f = \frac{1}{\sqrt{2}}(ie^{-\eta}G_{1x} \wedge G_{1y} - G_1 \wedge G_2). \quad (59)$$

According to Lawson in [10], the *bipolar surface* of the minimal surface G_1 in $S^3(1)$ with unit normal G_2 is the surface in $S^5(1)$ given by $G_1 \wedge G_2$ in $\Lambda^2\mathbb{R}^4 = \mathbb{R}^6$. This is a minimal surface in $S^5(1)$. If we include $\Lambda^2\mathbb{R}^4$ in $\Lambda^2\mathbb{C}^4$ via $v \mapsto (1/\sqrt{2})(v - i \star v)$, where \star denotes the Hodge star operator on $\Lambda^2\mathbb{R}^4$, then it follows immediately from (59) that f is the bipolar of G_1 .

Conversely, let $G_1(z)$ be a non-totally geodesic minimal immersion in $S^3(1)$, other than the Clifford torus, with unit normal $G_2(z)$. By restricting to an open dense subset if necessary, we may assume that z is such that the induced metric is $ds^2 = e^\eta|dz|^2$, where η is a positive function satisfying (47), and that the second fundamental form $\tilde{I}I$ of G_1 satisfies $(\tilde{I}I(\partial, \partial), \tilde{I}I(\partial, \partial)) = 1/4$. Then G_1 and G_2 will satisfy the system (58), so if we define F using (55) then F is horizontal in $S^7(1)$ and satisfies the system (49)-(54). In particular, we have that $z_{12}^3 = 0$ and $\omega_1(\partial) = 0$. If we apply the construction of [3] to the projection of F to $\mathbb{C}P^3$, then we will obtain the bipolar $f(z)$ of $G_1(z)$. It then follows from (35) that $\omega = \omega^+$, so that (45) gives that $\gamma^+ = 0$. Hence, using Theorem 4, we obtain the following.

Theorem 7 *Let $f : S \rightarrow S^5(1)$ be a minimal immersion, not contained in a totally geodesic $S^3(1)$, with non-circular ellipse of curvature at every point. Then, on an open dense subset of S , the following three statements are equivalent:*

1. $\gamma^+ = 0$,
2. f is the bipolar surface of a non-totally geodesic minimal surface in $S^3(1)$ which is not the Clifford torus,
3. the (+)transform f^+ is the reflection of f in a great subsphere of $S^5(1)$.

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