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Conformal non-relativistic hydrodynamics from gravity

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Abstract

We show that the recently constructed holographic duals of conformal non-relativistic theories behave hydrodynamically at long distances, and construct the gravitational dual of fluid flows in a long-wavelength approximation. We compute the thermal conductivity of the holographic conformal non-relativistic fluid. The corresponding Prandtl number is equal to one.

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Contents

1 Introduction

The AdS/CFT correspondence [1, 2, 3] provides an important theoretical framework for studying a class of strongly coupled quantum field theories, and at the same time provides a non-perturbative approach to quantum gravity. Traditionally, the correspondence has been used to study the dynamics of non-abelian gauge theories, since the prototype examples of the correspondence relate string theory/gravity in AdS spacetimes to supersymmetric Yang-Mills theories. However, in recent times the correspondence has been extended to study various condensed matter systems, especially those in the vicinity of a quantum critical point (see [4] and references therein). An example is the possible application to non-relativistic fermions at unitarity [5, 6], which is the focus of the current paper.

Fermionic systems at unitarity, which are realized in experiments with cold atoms, are described by a non-relativistic CFT. This field theory enjoys a symmetry algebra called the Schrödinger algebra [7, 8, 9]. The authors of [5, 6] proposed a class of spacetimes which could describe field theories with non-relativistic conformal invariance holographically, whose metric is

$$ds^2 = r^2 \left(-2\,dx^+\,dx^- - \beta^2\,r^2\,(dx^+)^2 + d\mathbf{x}_d^2\right) + \frac{dr^2}{r^2}. \tag{1.1}$$

These have the Schrödinger algebra as an isometry algebra, and we will henceforth refer to them as Schr spacetimes. A Galilean CFT in d-spatial dimensions is conjectured to be dual to Schr_{d+3}.

Subsequently, the Schr_5 spacetime was realized in string theory [10, 11, 12].[1] It was obtained by starting from a known type IIB solution, the $\mathrm{AdS}_5 \times \mathcal{X}_5$ spacetime, where $\mathcal{X}_5$ is a Sasaki-Einstein manifold, and applying a solution-generating transformation, either the Null Melvin Twist [33, 34] or the TsT transformation [35], to generate the spacetime $\mathrm{Schr}_5 \times \mathcal{X}_5$. The TsT transformation involves a T-duality, followed by a shift, and then a second T-duality. This can be used in any spacetime with a $U(1) \times U(1)$ isometry and, in general, can be thought of as a Melvin Twist, since the transformation adds a background NS-NS B-field, effectively Melvinizing the spacetime. The Null Melvin Twist can be shown to be equivalent to the TsT transformation in the special case when one of the $U(1)$ isometries is null.

Since the string-theoretic embedding of the Schr_5 spacetimes can be achieved by using dualities, we can see that the dual field theory is a deformed version of $\mathcal{N} = 4$ SYM theory (for the special case of $\mathcal{X}_5 = \mathbf{S}^5$). The transformation in [10, 11, 12] breaks the $SU(4)$ symmetry of $\mathcal{N} = 4$ down to an $SU(3) \times U(1)$ subgroup, and the field theory is then the theory twisted by the R-charge associated with the $U(1)$ isometry. One can also view the field theory as the one obtained by decoupling the open string field theory living on D3-branes in a Null Melvin spacetime. More generally any $\mathcal{N} = 1$ superconformal field theory in 3+1 spacetime dimensions (corresponding to a dual Sasaki-Einstein compactification) can be deformed to a two dimensional (spatial) non-relativistic CFT using the $U(1)_R$ symmetry.

Using the same solution-generating transformation, the authors of [10, 11, 12] constructed a two-parameter family of black hole spacetimes with Schr_5 asymptotics. These black holes can be shown to solve the five-dimensional effective equations of motion obtained by a Kaluza-Klein reduction of the 10-dimensional type IIB theory on the $\mathcal{X}_5$. The simplest five dimensional effective action is composed of gravity coupled to a massive vector field and a single scalar field [10], which is a truncation of a more general five-dimensional Lagrangian involving three scalar fields. The latter, remarkably, is a consistent truncation of type IIB supergravity on $\mathcal{X}_5$ [11].

This black hole solution was used to study the equilibrium thermodynamic properties of the field theory and was shown to be dual to the grand canonical ensemble for the dual field theory. The thermodynamics is, not surprisingly, consistent with non-relativistic scale invariance in two spatial dimensions. In particular, it was found that $\varepsilon = P$, where ε is the

[1] For other recent studies of non-relativistic systems in the context of gauge-gravity duality see [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32].

energy density and P is the pressure, as required in non-relativistic CFTs. Furthermore, non-equilibrium transport properties of the non-relativistic plasma were also explored in [10, 12]. In particular, the shear viscosity η of the non-relativistic fluid was calculated and found to take the universal value $\eta/s = 1/4\pi$ typical of strongly interacting field theories with gravity duals.

The present paper is concerned with constructing the gravitational dual of arbitrary fluid flows in the non-relativistic CFT. The basic premise is to build on the recent ideas in the *fluid-gravity correspondence* [36] to construct inhomogeneous black hole solutions with Schr_5 asymptotics. The hydrodynamic description of a fluid (either relativistic or non-relativistic) is an effective field theory which captures the universal long wavelength physics when the system achieves local thermal equilibrium. Building upon previous discussions of the hydrodynamic description of four-dimensional superconformal field theories in the AdS/CFT correspondence (see [37] and references therein), in [36] it was argued that starting from the most general black hole solution in AdS (a boosted Schwarzschild-AdS black hole) one can promote the temperature and velocity fields to local functions of the boundary coordinates. It is then possible to solve for the bulk metric order by order in a boundary derivative expansion and recover the relativistic Navier-Stokes equations, which encode the conservation of the boundary energy-momentum tensor, entirely as a consequence of Einstein's equations. Importantly, it is possible to show that the resulting bulk solutions are genuine black holes, i.e. that they have a regular event horizon [38]. This correspondence has been extended in many directions, including to forced fluids [39], to conformal field theories in various dimensions [40, 41, 42], to charged fluids [43, 44, 45], to Bjorken flow [46, 47], and incompressible non-relativistic fluids [48].

We are interested in constructing inhomogeneous black hole solutions with Schrödinger asymptotics and extending the fluid-gravity correspondence to relate them to fluid flows in the non-relativistic conformal theory. As a first step, we note that a four parameter family of black hole solutions with Schr_5 asymptotics can be obtained by boosting the solutions considered in [10, 11, 12]. We can then promote these parameters to functions of the field theory coordinates and find the solution order by order in a boundary derivative expansion. However, we will show that we can obtain inhomogeneous black hole solutions with the desired asymptotics in an easier way, by simply applying a TsT transformation to the solutions constructed in [36]! We will argue that this in fact captures all the hydrodynamic properties of the system in the planar limit.

To relate these inhomogeneous black hole solutions to fluid flows, we need to calculate the boundary stress tensor from the bulk solutions. Because of the slow fall-off of the metric perturbation in the asymptotically Schrödinger black holes, the usual technique of obtaining the stress tensor by functionally differentiating the action with respect to the boundary metric cannot be straightforwardly applied to these cases (see Appendix A for a discussion of this issue). However, [11] proposed that the stress tensor of the asymptotically AdS space before the TsT transformation can be re-interpreted as the stress tensor complex of the non-relativistic theory. We will adopt this approach. We therefore describe the general reduction of a relativistic stress tensor on the light cone to obtain a non-relativistic stress tensor complex. The structure of the relativistic conformal stress tensor implies that for any non-relativistic conformal theory obtained in this way, the thermal conductivity κ of the non-relativistic fluid is

$$\kappa = 2\,\eta\,\frac{\varepsilon + P}{\rho\,T} \tag{1.2}$$

where ρ is the mass density, or, even more succinctly,

$$\mathrm{Pr} = 1 \tag{1.3}$$

where Pr is the Prandtl number.

The outline of this paper is as follows: we will begin in §2 by discussing how the relativistic fluid equations are reduced on the light-cone to the non-relativistic Navier-Stokes equations. This will allow us to explore the general properties of the non-relativistic stress-tensor complex. We will review some aspects of the Schr_5 solutions in §3 and then describe how to construct inhomogeneous black hole solutions dual to arbitrary fluid flow and give the dual stress tensor to first order in derivatives in §4. We end with a discussion in §5.

2 Light-cone reduction of relativistic fluids

Consider a relativistic fluid in Minkowski space in $d+2$ spacetime dimensions; we will use light-cone coordinates $\{x^+, x^-, \mathbf{x}\}$ and take the metric to be

$$ds^2_{\text{flat}} = \eta_{\mu\nu}\,dx^\mu\,dx^\nu = -2\,dx^+\,dx^- + d\mathbf{x}^2. \tag{2.1}$$

Suppose we view this fluid in the light-cone frame and evolve it in light-cone time x^+. Then, for fixed light-cone momentum P_-, we obtain a system in $d+1$ dimensions with non-relativistic invariance. This is of course familiar from the discrete light-cone quantization (DLCQ) of quantum field theories. In fact, one of the models for studying non-relativistic conformal field theories holographically, suggested in refs. [13, 14], was that one could consider pure AdS, with the relativistic conformal symmetry broken to Galilean symmetry simply by compactification of the x^- coordinate, which singles out a preferred light-cone direction. Note that in this case we are not only compactifying the light-cone direction in the boundary where gravity is non-dynamical (and the metric flat, (2.1)), we are also required to compactify the coordinate in the bulk AdS spacetime. This involves introducing closed null curves in the geometry and the validity of supergravity becomes questionable [11]. We will return to a different gravitational background, viz. (1.1), where x^- does not need to be compactified to achieve Galilean symmetry. Note however that it is still useful to take x^-

compact, so that the momentum P_- is integer quantized, since P_- is interpreted as particle number in the dual theory.

Relativistic hydrodynamics in $d+2$ dimensions is formulated in terms of pressure (or, equivalently, the temperature) and the four velocity u^μ, subject to the condition that $\eta_{\mu\nu}\, u^\mu\, u^\nu = -1$. This gives $d+2$ degrees of freedom. At the same time, non-relativistic hydrodynamics in d spatial, one temporal dimensions can be formulated in terms of the mass density ρ, the pressure P, and the spatial velocities v^i, also giving $d+2$ degrees of freedom.

We would like to find a mapping between the degrees of freedom of the $(d+2)$-dimensional theory to the degrees of freedom of the $d+1$ dimensional theory such that the relativistic hydrodynamic equations imply the non-relativistic hydrodynamic equations. We would also like to find how the thermodynamic quantities of the two formulations are related. Finally, we plan to use the map to find out the thermal conductivity of the non-relativistic theory. We will first begin with ideal hydrodynamics and then discuss dissipative terms.

2.1 Ideal fluids

The relativistic hydrodynamics equations are just the conservation of energy and momentum

$$\nabla_\mu T^{\mu\nu} = 0. \tag{2.2}$$

An ideal relativistic fluid has a stress tensor given by[2]

$$T^{\mu\nu} = (\epsilon_{\rm rel} + P_{\rm rel})\, u^\mu\, u^\nu + P_{\rm rel}\, \eta^{\mu\nu}\ , \tag{2.3}$$

where the energy density $\epsilon_{\rm rel}$ is related to the pressure $P_{\rm rel}$ by a thermodynamic equation of state. Equations (2.2) and (2.3) define a system of $d+2$ equations for the $d+2$ unknowns.

Non-relativistic ideal hydrodynamics is described by the continuity equation,

$$\partial_t \rho + \partial_i \left(\rho\, v^i\right) = 0, \tag{2.4}$$

together with the equation of momentum conservation, (here $i = 1, \ldots, d$)

$$\partial_t(\rho\, v^i) + \partial_j \Pi^{ij} = 0, \qquad \Pi^{ij} = \rho\, v^i\, v^j + \delta^{ij} P\,, \tag{2.5}$$

and the equation of energy conservation,

$$\partial_t \left(\varepsilon + \frac{1}{2}\rho\, v^2\right) + \partial_i\, j^i_\varepsilon = 0, \qquad j^i_\varepsilon = \frac{1}{2}\,(\varepsilon + P)\, v^2\, v^i\ . \tag{2.6}$$

where $v^2 = v^i\, v^i$.

[2] We use the subscript "rel" for quantities in relativistic hydrodynamics and indicate quantities in non-relativistic hydrodynamics without subscripts.

Consider the relativistic equations (2.2) on the light-cone. We will consider only solutions to the relativistic equations that do not depend on x^-; that is, all derivatives ∂_- vanish. The coordinate x^+ corresponds to the non-relativistic time t. The equations of energy-momentum conservation are,

$$\partial_+ T^{++} + \partial_i T^{+i} = 0 \ , \qquad \partial_+ T^{+i} + \partial_j T^{ij} = 0 \ , \qquad \partial_+ T^{+-} + \partial_i T^{-i} = 0, \tag{2.7}$$

which reduce to the non-relativistic equations under the following identification: identify T^{++} with the mass density, T^{+i} with the mass flux (which is equal to the momentum density), T^{ij} with the stress tensor, T^{+-} with the energy density, and T^{-i} with the energy flux,

$$\begin{aligned} &T^{++} = \rho, \quad T^{+i} = \rho\, v^i, \quad T^{ij} = \Pi^{ij}, \\ &T^{+-} = \varepsilon + \frac{1}{2}\rho\, v^2, \qquad T^{-i} = j^i_\varepsilon. \end{aligned} \tag{2.8}$$

It is now easy to convince oneself based on (2.8) that the precise mapping between relativistic and non-relativistic hydrodynamic variables is

$$\begin{aligned} u^+ &= \sqrt{\frac{1}{2}\frac{\rho}{\varepsilon + P}} \ , \qquad u^i = u^+ v^i, \\ P_{\rm rel} &= P \, , \qquad\qquad\quad \epsilon_{\rm rel} = 2\,\varepsilon + P. \end{aligned} \tag{2.9}$$

The component of the relativistic velocity u^- can be determined using the normalization condition $u_\mu \, u^\mu = -1$ to be

$$u^- = \frac{1}{2}\left(\frac{1}{u^+} + u^+ \, v^2\right). \tag{2.10}$$

While the analysis has been for a general relativistic fluid with an equation of state $\epsilon_{\rm rel}(P_{\rm rel})$, we will soon focus on conformal fluids. Conformal invariance requires that the stress tensor for the relativistic theory be traceless, $T^\mu_\mu = 0$, which gives us the equation of state $\epsilon_{\rm rel} = (d+1)\, P_{\rm rel}$. In the non-relativistic theory we can once again use the conformal invariance to learn that $2\,\varepsilon = d\, P$.

2.2 Viscous fluids

We now wish to extend our mapping of relativistic hydrodynamics into non-relativistic hydrodynamics to first order in derivatives on both sides. The ideal stress energy tensor (2.3) can be supplemented with dissipative terms, which can be expanded systematically in terms of derivatives of the velocity field and pressure. Specifically, we have

$$T^{\mu\nu} = (\epsilon_{\rm rel} + P_{\rm rel})\, u^\mu\, u^\nu + \eta^{\mu\nu} P_{\rm rel} + \pi^{\mu\nu}, \tag{2.11}$$

where $\pi^{\mu\nu}$ incorporates all the dissipative contributions. For first order viscous hydrodynamics we have

$$\pi^{\mu\nu} = -2\, \eta_{\rm rel}\, \tau^{\mu\nu}, \tag{2.12}$$

where

$$\tau^{\mu\nu} = \frac{1}{2} P^{\mu\alpha} P^{\nu\beta} \left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{d+1} \eta_{\alpha\beta} \nabla_\gamma u^\gamma \right) \tag{2.13}$$

is the shear tensor and we have introduced the spatial projector $P^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$.

We will use the zeroth-order equations of motion to simplify the viscosity term. By using zeroth-order equations, we make an error of second order in derivatives, which can be neglected. Namely, we use the ideal hydrodynamic equations in the following form,

$$\begin{aligned} &u_\mu \nabla^\mu \epsilon_{\rm rel} + (\epsilon_{\rm rel} + P_{\rm rel}) \nabla_\mu u^\mu = 0, \\ &u_\nu \nabla^\nu u^\mu + \frac{\nabla_\perp^\mu P_{\rm rel}}{\epsilon_{\rm rel} + P_{\rm rel}} = 0, \qquad \text{where} \quad \nabla_\perp^\mu \equiv P^{\mu\alpha} \nabla_\alpha \ , \end{aligned} \tag{2.14}$$

to rewrite the stress-energy tensor as

$$\begin{aligned} T^{\mu\nu} &= (\epsilon_{\rm rel} + P_{\rm rel}) u^\mu u^\nu + P_{\rm rel} \eta^{\mu\nu} \\ &\quad - \eta_{\rm rel} \left(\nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{d+1} P^{\mu\nu} \nabla_\alpha u^\alpha - \frac{(u^\mu \nabla_\perp^\nu + u^\nu \nabla_\perp^\mu) P_{\rm rel}}{\epsilon_{\rm rel} + P_{\rm rel}} \right). \end{aligned} \tag{2.15}$$

On the non-relativistic side, we use the ideal hydrodynamic equations in the form

$$\begin{aligned} &\partial_t \rho + \partial_i (\rho v^i) = 0, \\ &\partial_t v^i + v^j \partial_j v^i + \frac{1}{\rho} \partial_i P = 0, \\ &\partial_t \varepsilon + \partial_i (\varepsilon v^i) + P \partial_j v^j = 0. \end{aligned} \tag{2.16}$$

The first-order contributions to the (spatial) stress tensor and the energy flux are

$$\begin{aligned} \Pi^{ij} = \rho v^i v^j + P \delta^{ij} - \eta \sigma^{ij}, \qquad \sigma^{ij} = \partial_i v_j + \partial_j v_i - \tfrac{2}{d} \delta^{ij} \partial_k v^k, \\ j_\varepsilon^i = \left(\varepsilon + P + \tfrac{1}{2} \rho v^2 \right) v^i + \eta \sigma^{ij} v^j - \kappa \partial_i T, \end{aligned} \tag{2.17}$$

where κ is the thermal conductivity and T is the temperature.

By using (2.15) and (2.17), we now establish the mapping between relativistic and non-relativistic viscous hydrodynamics. First, we find $\tau^{++} = 0$, and therefore

$$T^{++} = (\epsilon_{\rm rel} + P_{\rm rel}) (u^+)^2. \tag{2.18}$$

The identification $T^{++} = \rho$ implies then that

$$u^+ = \sqrt{\frac{\rho}{\epsilon_{\rm rel} + P_{\rm rel}}}, \tag{2.19}$$

unchanged from the ideal hydrodynamic level (2.9).

Next, we find

$$\tau^{i+} = -\eta_{\rm rel} \left(\partial_i u^+ - \frac{u^+ \partial_i P_{\rm rel}}{2 (\epsilon_{\rm rel} + P_{\rm rel})} \right). \tag{2.20}$$

On the other hand, we still want to map $T^{+i} = \rho\, v^i$. This means that there is now a correction to the relation between u^i and v^i:

$$u^i = u^+ \left[v^i + \frac{\eta_{\rm rel}}{\rho} \left(\partial_i u^+ - \frac{u^+}{2\,(\epsilon_{\rm rel} + P_{\rm rel})}\, \partial_i P_{\rm rel} \right) \right]. \tag{2.21}$$

For T^{ij}, after some algebra, we find

$$T^{ij} = \rho\, v^i\, v^j + P_{\rm rel}\, \delta^{ij} - \eta_{\rm rel}\, u^+ \left(\partial_i v_j + \partial_j v_i - \frac{2}{d}\, \delta^{ij}\, \partial_k v^k \right), \tag{2.22}$$

which implies that the pressures on the two sides still coincide,

$$P_{\rm rel} = P, \tag{2.23}$$

and the relationship between the viscosities is

$$\eta_{\rm rel} = \frac{\eta}{u^+}. \tag{2.24}$$

Note that our identifications automatically give a first-order correction σ_{ij} in the non-relativistic theory with the correct tensor structure. That is, the trace-free relativistic shear tensor gives a trace free spatial stress tensor in the non-relativistic theory.

Regarding the other components of the stress tensor, after some calculations involving many cancellations, one discovers that T^{+-} is

$$T^{+-} = \frac{1}{2}\,(\epsilon_{\rm rel} - P_{\rm rel}) + \frac{1}{2}\,\rho\, v^2, \tag{2.25}$$

which means that the relationship between relativistic and non-relativistic energy densities remain unchanged,

$$\epsilon_{\rm rel} = 2\,\varepsilon + P. \tag{2.26}$$

Finally, for T^{-i} we find

$$T^{-i} = \left(\varepsilon + P + \frac{1}{2}\,\rho\, v^2 \right)\, v^i - \eta_{\rm rel}\, u^+ \sigma^{ij}\, v_j + \eta_{\rm rel}\, \delta^{ij}\, \frac{\partial_j u^+}{(u^+)^2} - \eta_{\rm rel}\, \frac{u^+}{\rho}\, \delta^{ij}\, \partial_j P. \tag{2.27}$$

Thus we have to require that

$$\eta_{\rm rel}\, \frac{\partial_i u^+}{(u^+)^2} - \eta_{\rm rel}\, \frac{u^+}{\rho}\, \partial_i P = -\kappa\, \partial_i T. \tag{2.28}$$

In order to see that the left hand side is indeed proportional to $\partial_i T$, we need to use the mapping (2.19) and the equation of state for a non-relativistic theory.

Focusing specifically now on conformally invariant fluids, using (2.19) and $\varepsilon = \frac{d}{2}\, P$, we find

$$\eta_{\rm rel}\, \frac{\partial_i u^+}{(u^+)^2} - \eta_{\rm rel}\, \frac{u^+}{\rho}\, \partial_i P = -\eta_{\rm rel} \sqrt{\frac{\varepsilon + P}{2\,\rho}}\; \partial_i \ln \left(\frac{P^{(d+4)/(d+2)}}{\rho} \right). \tag{2.29}$$

Recalling that the equation of state of the holographic non-relativistic liquid is [25]

$$P = \alpha \left(\frac{T^2}{\mu} \right)^{(d+2)/2} , \tag{2.30}$$

the argument of the logarithm in (2.29) is T^2 up to a constant. Therefore, the left hand side of (2.28) is indeed proportional to $\partial_i T$, and one reads out the value for the thermal conductivity:

$$\kappa = 2\,\eta\,\frac{\varepsilon + P}{\rho\,T}. \tag{2.31}$$

Let us now compute the Prandtl number. The Prandtl number is defined as the ratio of the kinematic viscosity ν and the thermal diffusivity χ,

$$\mathrm{Pr} = \frac{\nu}{\chi}, \tag{2.32}$$

where

$$\nu = \frac{\eta}{\rho}\,, \qquad \chi = \frac{\kappa}{\rho\,c_p}, \tag{2.33}$$

where c_p is the specific heat at constant pressure. We note the definition of the heat capacity at constant volume:

$$C_v = \left(\frac{\partial H}{\partial T} \right)_{P,N} , \tag{2.34}$$

where $H = E + PV$ is the enthalpy. Write $H = w\,V = w\,N/n$. We then find

$$C_p = N\,\frac{\partial}{\partial T}\left(\frac{w}{n} \right)_P = -\frac{N\,w}{n^2}\left(\frac{\partial n}{\partial T} \right)_P \tag{2.35}$$

where we have used the fact that $w = (d/2+1)\,P$ and is fixed at fixed P. At fixed P, $\mu \sim T^2$, and $n = \partial P/\partial T \sim 1/T^2$, and $\partial n/dT = -2n/T$. Therefore

$$C_p = \frac{2\,w\,N}{T\,n} \tag{2.36}$$

and $c_p = C_p/M$ (M being the total mass) is equal to $2w/\rho T$. Thus we find:

$$\mathrm{Pr} = \frac{2\,w\,\eta}{\rho\,T\,\kappa} = 1. \tag{2.37}$$

Note that this result is valid for any non-relativistic conformal fluid obtained from the DLCQ of a relativistic conformal fluid.

3 Thermal description of non-relativistic CFTs

Non-relativistic conformal field theories with Schrödinger symmetry in d spatial dimensions are dual to Schr_{d+3} spacetime [5, 6]. The Schr_5 spacetimes can be realized in string theory as the near-horizon geometry of D3-branes probing a Null Melvin universe [10].[3] Furthermore, this background can be obtained as a solution to a 5-dimensional Lagrangian which is a consistent truncation of IIB supergravity [11], involving gravity coupled to a massive vector field and three scalars:

$$\begin{aligned}
\mathcal{S}_{\rm bulk} &= \frac{1}{16\,\pi G_5}\int d^5x\,\sqrt{-g}\,\Big(R+V(\phi_i)-5\,(\partial\phi_1)^2-\frac{15}{2}\,(\partial\phi_2)^2-\frac{1}{2}\,(\partial\phi_3)^2\\
&\qquad -\frac{1}{4}\,g(\phi_i)\,F_{\mu\nu}F^{\mu\nu}-4\,e^{-2\phi_1-3\phi_2-\phi_3}\,A_\mu A^\mu\Big)\,,\\
V(\phi_i) &= 24\,e^{-\phi_1-4\,\phi_2}-4\,e^{-6\,\phi_1-4\,\phi_2}-8\,e^{-10\,\phi_2}\,,\\
g(\phi_i) &= e^{4\phi_1+\phi_2-\phi_3}.
\end{aligned}\tag{3.1}$$

In fact, all known solutions are solutions to a slightly simpler theory outlined in [10], where the three scalars are linearly related as

$$\{\phi_1,\phi_2,\phi_3\}=\{-\frac{2}{5},-\frac{1}{15},1\}\,\phi. \tag{3.2}$$

This allows one to consider the five-dimensional effective action

$$S=\frac{1}{16\,\pi G_5}\int d^5x\sqrt{-g}\left(R-\frac{4}{3}(\partial_\mu\phi)(\partial^\mu\phi)-\frac{1}{4}e^{-8\phi/3}F_{\mu\nu}F^{\mu\nu}-4A_\mu A^\mu-V(\phi)\right), \tag{3.3}$$

where

$$V(\phi)=4\,e^{2\phi/3}(e^{2\phi}-4). \tag{3.4}$$

The field theory on the D-branes in the appropriate decoupling limit is $\mathcal{N}=4$ Super Yang-Mills plus a non-commutative deformation, giving rise to a version of the dipole field theories discussed in [33, 49]. Consider the fields in $\mathcal{N}=4$ SYM (with gauge group $SU(N)$) which transform under the global symmetry $SO(4,2)\times SO(6)$. Picking a $U(1)\subset SO(6)$ we deform the field theory by replacing ordinary products appearing in the Lagrangian by a star-product [11]

$$f\star g=e^{i\,\beta\left(P_-^f\,R^g-P_-^g\,R^f\right)}\,fg \tag{3.5}$$

where P_- is the momentum along the null x^- direction and R is the $U(1)$ charge. This product is a hybrid between the usual non-commutative star-product which only involves spatial momenta and the β-deformation of $\mathcal{N}=4$ which involves only the R-charges. Of importance later will be the fact that this deformed field theory inherits some of the properties from $\mathcal{N}=4$ SYM. In the large N limit we in fact expect that the planar sector of the deformed theory to be identical to that of $\mathcal{N}=4$ SYM [11].[4]

[3] The near-horizon limit in this case needs to be taken keeping in mind that we want to realize the scaling symmetry consistent with the Schrödinger algebra. We thank James Lucietti for a useful discussion about this issue.

[4] As discussed in §1 these statements extend to generic $\mathcal{N}=1$ superconformal field theories which we deform by a $U(1)_R$ symmetry.

The Lagrangian (3.3) has a black hole solution which was obtained by the Null Melvin Twist [10, 12] or TsT solution-generating transformation [11] in 10 dimensions. The black hole geometry is given as

$$\begin{aligned} ds_E^2 &= r^2\,k(r)^{-\frac{2}{3}}\left(\left[\frac{1-f(r)}{4\,\beta^2}-r^2\,f(r)\right](dx^+)^2+\frac{\gamma^2}{r^4}\,(dx^-)^2-[1+f(r)]\;dx^+\,dx^-\right)\\ &\quad+\;k(r)^{\frac{1}{3}}\left(r^2 d\mathbf{x}^2+\frac{dr^2}{r^2\,f(r)}\right), \end{aligned} \tag{3.6}$$

with the massive vector and scalars taking the form

$$\begin{aligned} A &= \frac{r^2}{k(r)}\left(\frac{1+f(r)}{2}\,dx^+-\frac{\gamma^2}{r^4}\,dx^-\right),\\ e^{\phi} &= \frac{1}{\sqrt{k(r)}}\,, \end{aligned} \tag{3.7}$$

where $f(r)$ and $k(r)$ are

$$f(r)=1-\frac{r_+^4}{r^4}\,,\qquad k(r)=1+\frac{\gamma^2}{r^2}\,, \tag{3.8}$$

with $\gamma^2\equiv\beta^2\,r_+^4$. Note that in these light-cone coordinates, the solution asymptotically approaches the vacuum solution (1.1) at large r, and also reduces to (1.1) when we set $r_+=0$. It has been argued in [10] that the black hole spacetime (3.6) corresponds in the field theory to a grand canonical ensemble at temperature

$$T=\frac{r_+}{\pi\,\beta} \tag{3.9}$$

and a chemical potential for particle number

$$\mu=\frac{1}{2\,\beta^2}\,. \tag{3.10}$$

Note that the Schrödinger algebra involves a conserved charge associated with particle number, which geometrically is realized via the Killing field ∂_-. These results were obtained with minor differences in the derivation in [11, 12].

Using a Euclidean action calculation, the authors of [10] derived the conserved charges of the black hole. This calculation required a careful analysis of the boundary counterterms, since the metric (3.6) has rather complicated asymptotics. In Appendix A we discuss the calculation of the conserved charges in a Hamiltonian formulation, supplementing the analysis of [10]. The conserved charges associated with the Killing symmetries ∂_- and ∂_+ in (3.6) translate in the field theory to particle number N and total energy E. To obtain finite values we assume that the x^--direction is compactified with period Δx^-. The results obtained in [10] are

$$\langle N\rangle=\langle P_-\rangle\frac{\Delta x^-}{2\pi}=\frac{\pi^2\,T^4}{64\,G_5\,\mu^3}\,V\,(\Delta x^-)^2\;=\frac{\gamma^2}{8\pi^2\,G_5}\,V\,(\Delta x^-)^2, \tag{3.11}$$

and

$$\langle E\rangle = \frac{\pi^3\,T^4}{64\,G_5\,\mu^2}\,V\,\Delta x^- \;=\frac{r_+^4}{16\pi\,G_5}\,V\,\Delta x^-, \tag{3.12}$$

where we have given the results in terms of the physical temperature and chemical potential, and also in terms of the parameters in the bulk solution to emphasize the physical interpretation of the parameters γ and r_+. Furthermore, the pressure is given in the grand canonical ensemble directly in terms of the Gibbs potential $\mathcal{Q}(T,\mu,V)$:

$$P\,V = -\mathcal{Q}(T,\mu,V) = \frac{\pi^3\,T^4}{64\,G_5\,\mu^2}\,V\,\Delta x^- \;, \tag{3.13}$$

leading thus to an equation of state

$$P\,V = E \quad\Longrightarrow\quad P = \varepsilon. \tag{3.14}$$

In addition, we note for future reference that the entropy of the black hole is given by

$$S = \frac{\pi^3\,T^3}{16\,G_5\,\mu^2}\,V\,\Delta x^- \;. \tag{3.15}$$

4 Hydrodynamic description of non-relativistic CFTs

The black hole solution (3.6) corresponds to an equilibrium configuration of the non-relativistic field theory. We would like to study departures from equilibrium in the continuum limit by considering local patches of the field theory in local equilibrium. This is the hydrodynamic limit, where one has local variations of energy density and particle number; these local domains evolve according to the laws of hydrodynamics which were described in §2.

To obtain a gravitational description of the fluid dynamical regime, we need to patch together local domains of equilibrated fluid – the precise manner in which this can be achieved was outlined in [36]. The authors presented an algorithmic procedure to construct gravitational solutions starting from the equilibrium black hole solution. The idea is to consider the most general stationary solution for the equilibrium – for the case of relativistic superconformal theories in $d+2$ dimensions, this is given by a boosted Schwarzschild-AdS$_{d+3}$ black hole, which is specified by $d+1$ parameters: a horizon size r_+ and a unit normalized velocity field u_μ. Choosing Eddington-Finkelstein like ingoing coordinates which are regular on the horizon, the metric takes the form:

$$ds^2 = 2\,u_\mu\,dx^\mu dr - r^2\,f(r)\,u_\mu u_\nu\;dx^\mu dx^\nu + r^2\,P_{\mu\nu}\,dx^\mu dx^\nu \tag{4.1}$$

where $P_{\mu\nu} = \eta_{\mu\nu} + u_\mu\,u_\nu$ is the spatial projector introduced earlier and $f(r) = 1-(r_+/r)^{d+2}$.

To study the hydrodynamic configurations one promotes the parameters r_+ and u_μ to functions of the boundary coordinates x^μ. Then one recursively solves for the bulk metric order by order in a boundary derivative approximation. This procedure was systematically carried out to second order in [36]. To ensure regularity of the bulk solution, and in particular to guarantee the existence of a regular future event horizon, it was necessary to adapt coordinates wherein lines of constant boundary x^μ corresponded to ingoing null geodesics in the bulk. This coordinate chart was utilized in identifying how the locally equilibrated boundary domains evolve in the bulk radial direction – the fluid in such a domain was evolved along a tube in the bulk centered about the ingoing null geodesic. The size of these tubular domains is set by the local energy density and they provide the bulk analog of patching together pieces of equilibrated fluid.

4.1 Gravitational dual of a non-relativistic fluid: Direct construction

We now turn to the analogous calculation for the non-relativistic CFT discussed in §3. We should first write the metric (3.6) in a form that is regular through the future horizon. To do so, perform a coordinate transformation

$$x^+ \to x^+ + \beta\, p(r) \;, \qquad x^- \to x^- + \frac{1}{2\,\beta}\, p(r) \tag{4.2}$$

with $p(r)$ chosen such that the metric is regular on the horizon $r = r_+$. A convenient choice is[5]

$$p'(r) = \frac{1}{r^2\, f(r)} \;, \tag{4.3}$$

leading to a metric

$$\begin{aligned} ds^2 \;\; &= \;\; r^2\, k(r)^{-\frac{2}{3}} \left(\left[\frac{1-f(r)}{4\beta^2} - r^2\, f(r) \right] (dx^+)^2 + \frac{\gamma^2}{r^4}\, (dx^-)^2 - [1+f(r)]\; dx^+\, dx^- \right) \\ & \quad - k^{-\frac{2}{3}} \left[\left(\frac{1}{\beta} + 2\,\beta\, r^2 \right) dx^+ + 2\,\beta\, dx^- \right] dr - k^{-\frac{2}{3}}\, \beta^2\, dr^2 + k(r)^{\frac{1}{3}}\, r^2 d\mathbf{x}^2 \;. \end{aligned} \tag{4.4}$$

This gives a two-parameter family of solutions of the 5d effective Lagrangian (3.3). We can construct a four parameter family of solutions trivially by performing a boost via the coordinate transformation

$$\mathbf{x} \to \mathbf{x} + \mathbf{v}\, x^+ \;, \qquad x^- \to x^- + \mathbf{v}\cdot\mathbf{x} + \frac{1}{2}\, v^2\, x^+. \tag{4.5}$$

Since the background metric (1.1) has Galilean invariance, this boost does not change the asymptotic form of the metric.

This boosted black hole, characterized by the parameters $\{r_+, \beta, \mathbf{v}\}$, is the starting point for a systematic hydrodynamic analysis. Following [38] we should promote the parameters

[5] If one wanted further ∂_r to be null then one could instead choose $p'(r) = \pm\sqrt{\frac{k(r)}{1+\beta^2\, r^2}}\, \frac{1}{r^2\, f(r)}$.

$\{r_+, \beta, \mathbf{v}\}$ to fields depending on $(x^+, \mathbf{x})$ and consistently solve the equations of motion order by order in derivatives in the x^+ and $\mathbf{x}$ directions. This process, while straightforward, is rendered cumbersome by the presence of additional fields in the action (3.1). We will therefore resort to a trick to recover the dual geometry.

4.2 Inhomogeneous black holes via TsT transform

The solution we want to obtain is characterized by four parameters depending on the coordinates $(x^+, \mathbf{x})$. The general inhomogeneous black hole of [36] also depends on four parameters, r_+ and u_μ, and if we specialize to solutions independent of x^-, they also depend on the coordinates $(x^+, \mathbf{x})$. For such solutions, we can then apply the same TsT transformation used to obtain the black hole solution (3.6) to obtain inhomogeneous black holes with Schrödinger asymptotics. Since these solutions will by construction reduce to (3.6) when the parameters are constants, we argue that they are precisely the required inhomogeneous solutions. Below, we discuss the action of the TsT transformation on the solution of [36].

The TsT transformation is a solution-generating technique in string theory wherein one uses a twisted T-duality to add NS-NS flux to a given solution [35]. Consider a solution to type IIB supergravity of the form $\mathcal{M}_5 \times \mathcal{X}_5$ where $\mathcal{X}_5$ is a Sasaki-Einstein space, which we view as a $U(1)$ fibration over a base $\mathcal{B}$, and $\mathcal{M}_5$ is a constant negatively curved spacetime, which we will take to be asymptotically AdS$_5$. The simplest case is when $\mathcal{X}_5 = \mathbf{S}^5$ and the base is thus a $\mathbf{CP}^2$. We assume that $\mathcal{M}_5$ admits a Killing vector field ∂_-, so that $\mathcal{M}_5 \times \mathcal{X}_5$ has an isometry group $U(1)_{x^-} \times U(1)_\psi$. Viewed as an eight dimensional solution upon Kaluza-Klein reduction, we have an $SL(2, \mathbf{R})$ symmetry group, which can be used to generate new solutions. This is the TsT transformation.

We start from the solutions dual to arbitrary fluid flow for hydrodynamics of relativistic superconformal theories of [36],

$$ds^2 = g_{AB}\, dX^A\, dX^B = -2\, u_\mu\, \mathcal{S}(r,x)\, dx^\mu\, dr + \chi_{\mu\nu}(r,x)\, dx^\nu\, dx^\nu, \tag{4.6}$$

where $X^A = \{x^\mu, r\}$. The metric dual to viscous fluid dynamics for the relativistic conformal fluids is given by (4.6) with $\mathcal{S}(r,x) = 1$ and

$$\chi_{\mu\nu} = r^2\, \left(P_{\mu\nu} - f(r,x)\, u_\mu\, u_\nu\right) + \frac{2}{r_+}\, r^2\, F(r,x)\, \tau_{\mu\nu} + \frac{2}{3}\, r\, u_\mu\, u_\nu\, \nabla_\lambda u^\lambda - r\, u^\lambda \nabla_\lambda \left(u_\nu u_\mu\right), \tag{4.7}$$

where $P_{\mu\nu}$ is the spatial projector defined after (2.13), $f(r,x)$ is the function $f(r)$ appearing in (3.8) with $r_+ \to r_+(x)$, and

$$F(r,x) = \frac{1}{4}\left[\ln\left(\frac{(r + r_+(x))^2 (r^2 + r_+(x)^2)}{r^4}\right) - 2\, \arctan(r/r_+(x)) + \pi\right]. \tag{4.8}$$

It is important to note that u_μ and r_+ are no longer parameters, but regarded as functions of x^μ. To apply the TsT transformation, we need to choose a direction x^-, and assume r_+, u_μ are independent of x^-.

The general map between the parameters of a relativistic fluid and the parameters of a non-relativistic fluid to first order in derivatives was written in (2.19), (2.21) and (2.23). When we write the relativistic fluid in terms of the gravitational dual spacetime, we can rewrite this mapping as a mapping between the functions r_+, u_μ characterizing the asymptotically AdS spacetime and the functions r_+, β, v_i appropriate in the non-relativistic case. The function r_+ is the same in both descriptions, and the map from u^μ to β, v^i is

$$u^+ = \beta, \quad u^i = \beta \left[v^i + \frac{1}{4\beta^2} \partial_i \frac{\beta}{r_+} \right]. \tag{4.9}$$

The full 10 dimensional metric is a direct sum of the metric g_{AB} on $\mathcal{M}_5$ given in (4.6) and a Sasaki-Einstein space $\mathcal{X}_5$,

$$\begin{aligned} ds^2_E &= g_{AB}\, dX^A\, dX^B + h^2\, (d\psi + A)^2 + ds^2(\mathcal{B}), \\ F_{(5)} &= 4\, (\mathrm{Vol}(\mathcal{M}_5) + h\, \mathrm{Vol}(\mathcal{B}) \wedge (d\psi + A)) \,. \end{aligned} \tag{4.10}$$

Under the TsT transformation the metric (4.10) gets mapped to a new solution of Type IIB supergravity [11][6]

$$\begin{aligned} ds^2_E &= e^{-\frac{\varphi}{2}} \left(g_{AB}\, dX^A\, dX^B - e^{2\varphi}\, h^2\, g_{A-}\, g_{B-}\, dX^A dX^B + e^{2\varphi}\, h^2\, (d\psi + A)^2 + ds^2(\mathcal{B}) \right), \\ F_{(5)} &= 4\, (\mathrm{Vol}(\mathcal{M}_5) + \mathrm{Vol}(\mathcal{B}) \wedge h\, (d\psi + A)) \,, \\ B_{(2)} &= e^{2\varphi}\, h^2\, g_{B-}\, dX^B \wedge (d\psi + A), \\ e^{-2\varphi} &= 1 + h^2\, g_{--} \,. \end{aligned} \tag{4.11}$$

This solution can be Kaluza-Klein reduced back to five dimensions to give a metric (restricting to situations where the norm of the Reeb vector ∂_ψ is fixed to $h^2 = 1$),

$$ds^2_5 = e^{-\frac{2}{3}\phi} \left(g_{AB} - e^{2\phi}\, g_{A-}\, g_{B-} \right) dX^A\, dX^B, \tag{4.12}$$

supported by

$$\begin{aligned} A &= e^{2\phi}\, g_{A-}\, dX^A, \\ e^{2\phi} &= \frac{1}{1 + g_{--}}. \end{aligned} \tag{4.13}$$

As before this five dimensional solution solves the equations of motion arising from (3.1) with the scalars being related as in (3.2).

[6] In [11], the TsT transformation involved an arbitrary parameter σ; however, this can be absorbed into a redefinition of the coordinate x^- by a boost in the $x^\pm$ plane. We have found it more transparent to fix the parameter in the TsT transformation and keep instead the velocity β in (4.9) as the free parameter corresponding to the choice of x^-.

Restricting to configurations which have ∂_- as an isometry, after a TsT transformation on the metric (4.6) we get a new metric of the form

$$
\begin{aligned}
ds^2_E &= e^{-\frac{2}{3}\phi}\left(-2\,u_\mu\,\mathcal{S}\,dx^\mu\,dr + \left[\chi_{AB} - \frac{\widetilde{\chi}_A\,\widetilde{\chi}_B}{1+\chi_{--}}\right] dX^A\,dX^B\right), \\
A &= e^{2\phi}\,\widetilde{\chi}_A\,dX^A, \\
e^{2\phi} &= \frac{1}{1+\chi_{--}},
\end{aligned}
\tag{4.14}
$$

with

$$
\widetilde{\chi}_A = \chi_{A-} - u_-\,\mathcal{S}\,\delta^r_A\,. \tag{4.15}
$$

The TsT transform converts the asymptotically AdS_5 spacetime (4.6) to an asymptotically Schr_5 spacetime, which depends on the r_+, β, v_i defined in (4.9) which are arbitrary functions of $(x^+, \mathbf{x})$. This provides the required inhomogeneous generalization of the black hole solution (3.6). We will not write the result of applying the transformation more explicitly, as it is quite complicated, and its construction is a straightforward exercise. Also note that since the internal manifold $\mathcal{X}_5$ plays a minimal role in our construction, it is easy to verify that the solution (4.14) also solves the consistent truncation action (3.1) with the scalars still related according to (3.2).

4.3 Properties of black holes dual to non-relativistic fluids

We will now discuss some of the physical properties of the solution (4.14). The solutions (4.14) are the most general long-wavelength regular solutions dual to configurations of the dual non-relativisitic conformal field theory and are valid to leading order in the boundary derivative expansion. These geometries solve the field equations arising from (3.3) provided the boundary stress tensor complex satisfies the non-relativistic Navier-Stokes equations. This follows from the fact that (4.6) are the most general regular solutions dual to the relativistic field theory and that the regularity properties of the black hole solutions are unaffected by the TsT transformation. The regularity of the solutions (4.6), in particular, the fact that they have a regular event horizon, was demonstrated in [38]. This result required that the variations in the boundary directions parameterized by x^μ are slow. In employing the TsT transformation, all we required was that ∂_- be a Killing vector; so there is no variation of the relativistic fluid in the light cone direction and one can of course take the variations in all the other directions to be appropriately slow. To explicitly demonstrate that the solution (4.14) has a regular event horizon, one can follow the perturbative construction of [38]. From this analysis it is easy to infer that for viscous non-relativistic fluids, the location of the horizon will remain at $r = r_+(x^+, \mathbf{x})$.

In analyzing the regularity of the geometries constructed in [36], it was important to work in a well behaved coordinate chart. As explained there and subsequently elaborated in

[38, 42] the solutions can be thought of as being tubewise approximated by a homogeneous black hole solution, with the tubes being domains in the bulk centered around radially ingoing null geodesics with width set by the scale of variation in the boundary directions x^μ. In the non-relativistic case (4.14), the metric is not *a priori* written in coordinates adapted to the radially ingoing geodesics. To see this note that the gauge choice employed in [36, 38] was to set $g_{r\mu} \propto u_\mu$.[7] On the other hand, in the non-relativistic case we have non-trivial g_{rr} arising after the TsT transformation coming from the non-vanishing $\widetilde{\chi}_r$. This can of course be removed by a coordinate transformation, and then the issue of regularity boils down to the analysis presented in [38].

To complete the fluid-gravity correspondence, we need to identify the stress tensor complex associated with the inhomogeneous solutions constructed above. The boundary field theory dual to the Schr_5 background was argued to be a non-commutative deformation of $\mathcal{N} = 4$ SYM. This fact allows us to argue that the planar sector of the deformed theory with non-relativistic invariance is in fact identical to the parent $\mathcal{N} = 4$ theory [11], and the direct computations of the thermodynamics in [10] are consistent with this picture. We therefore argue that, as advocated in [11], we can identify the non-relativistic stress tensor complex corresponding to the geometry after the TsT transformation with the one we have prior to the transformation. That is, the dual stress tensor complex is obtained simply by performing the DLCQ reduction described in §2 for the relativistic stress tensor corresponding to the geometry (4.6) before the TsT transformation.

More explicitly, the relativistic stress tensor corresponding to (4.6) (to first order in derivatives) is of the form (2.11) with[8]

$$\epsilon_{\rm rel} = 3\, P_{\rm rel}, \quad P_{\rm rel} = \frac{r_+^4}{16\pi\, G_5}, \quad \eta_{\rm rel} = \frac{r_+^3}{16\pi\, G_5}. \tag{4.16}$$

Thus, using the identifications in section 2, the non-relativistic stress tensor complex dual to the geometry (4.14) will be of the form (2.17) with[9]

$$\varepsilon = P, \quad P = \frac{r_+^4}{16\pi\, G_5}, \quad \rho = \frac{\beta^2 r_+^4}{4\pi\, G_5}, \quad \eta = \frac{r_+^3 \beta}{16\pi\, G_5}, \quad \kappa = \frac{r_+^2}{16\, G_5}. \tag{4.17}$$

The boundary stress tensor complex for the solution (4.14) comprises a spatial stress tensor Π_{ij}, particle density ρ, an energy flux j^i_ε, momentum flux $\rho\, v^i$ and energy ε. While we have argued that in the large N planar limit these quantities can be derived by reducing

[7] In fact, as discussed in [42], one can simplify the metric further, by demanding that the radially ingoing null geodesics are affinely parameterized by r. This is equivalent to setting $\mathcal{S}(r, x) = 1$ in (4.6).

[8] Note that $\frac{1}{16\pi\, G_5}$ gives the effective central charge of the dual field theory. For the case of deformed $\mathcal{N} = 4$ SYM to the non-relativistic dipole theory, this evaluates to $\frac{N^2}{8\,\pi^2}$ where N is the rank of the gauge group.

[9] Note that the stress tensor complex obtained from the relativistic stress tensor in §2 is a local density in the x^- direction. From the non-relativistic field theory point of view, it is more natural to multiply by Δx^- to obtain an object which is a local density only in the spatial directions.

the relativistic stress tensor on the light-cone (using thus the properties of the TsT transformation), it should in principle also be possible to obtain these by direct computation in the geometry (4.14). It turns out to be easy to use the counter-term construction proposed in [10] to extract the spatial stress tensor without trouble. However, it is not clear how to calculate the energy ϵ and the particle density ρ in an analogous fashion. In [10], the energy ϵ and the particle density ρ for the system in thermal equilibrium were obtained by a Euclidean action calculation. In Appendix A, we discuss the calculation of these quantities using canonical methods.

5 Discussion

We have discussed the hydrodynamic limit of d spatial dimensional non-relativistic conformal field theories and their dual gravitational solutions, which are inhomogeneous black holes with Schr_{d+3} asymptotics. We employed the fact that starting with a relativistic hydrodynamical system one can obtain non-relativistic fluid dynamical equations upon an appropriate light-cone reduction. Relativistic hydrodynamics in $d+2$ spacetime dimensions descends to non-relativistic hydrodynamics in d spatial dimensions. We demonstrated this both for ideal fluids and also for dissipative fluids where we restricted attention to first order in the gradient expansion. The latter allowed us to recover the heat conductivity of the non-relativistic system in terms of the shear viscosity and state parameters of the parent relativistic fluid. In particular, for non-relativistic CFTs we have shown that the Prandtl number is unity.[10] Since the first order dissipative coefficients for the non-relativistic CFTs arise from the shear viscosity of the relativistic system, it is not surprising that the rates of momentum diffusion and thermal conduction are correlated.

Our construction of the geometries dual to the non-relativistic fluids uses as its starting point the asymptotically AdS spacetimes dual to relativistic hydrodynamics. Starting with the solutions constructed in [36] we were able to construct inhomogeneous black holes with Schr asymptotics using the TsT transformation described in [11]. The TsT transform deforms the boundary field theory to a non-local quantum field theory through the introduction of a star-product (3.5) – we are therefore looking at the hydrodynamic limit of these deformed superconformal field theories.

Given these black hole solutions we can in principle try to extract the non-relativistic stress tensor complex of the dual field theory using an appropriate boundary stress tensor construction. Instead of doing so, we adopted the philosophy advocated in [11] – the hydrodynamics of the non-relativistic theory is given by the light-cone reduction of the undeformed superconformal theory in the planar limit. This can be justified by recalling that for superconformal field theories with gauge group of rank N, the star-product deformation resulting

[10] For comparision, Pr(water) ≈ 7, Pr(mercury) $\sim 10^{-3}$, and Pr(air) ≈ 0.7.

from the TsT transformation leaves the planar sector of the theory unchanged [35]. This allows us to efficiently extract the stress tensor for the non-relativistic viscous fluid. One should in principle explicitly compute the stress-tensor complex using the counter-term subtraction scheme discussed in [10]. It turns out to be easy to check that one can indeed extract the spatial stress tensor. Furthermore, by rewriting the action in a Hamiltonian formulation we have been able to extract the energy and particle density. But we have encountered some difficulties in proving that the Hamiltonian generates the time-translation symmetry of the non-relativistic field theory. This is an interesting open problem which we hope to return to in the future. It is also worth remarking that to compute equilibrium thermodynamics we can take a simpler route proposed in [25] – use a background subtraction scheme where the reference spacetime is not the vacuum Schr_d spacetime, but corresponds to a state with zero temperature at finite particle density.[11]

Our discussion of the light-cone reduction of the relativistic equations was restricted to ideal and viscous fluids, i.e., up to and including the first order in the gradient expansion. Fluid dynamics in general can be viewed as an effective field theory with an infinite number of irrelevant terms obtained as usual in a derivative expansion. It would be interesting to understand the light-cone reduction of the second order relativistic conformal hydrodynamic stress tensor constructed in [36, 50] (see [41, 42] for the result in arbitrary dimensions). In particular, the various relaxation times encountered at second order should descend to interesting transport coefficients for the non-relativistic theory. It must however be mentioned that unlike the relativistic case, where causality issues [50] require us to consider non-linear hydrodyanamics, one is not similarly forced to consider higher-order terms in the non-relativistic setting. It would also be interesting to carry out a systematic analysis of non-relativistic conformal hydrodynamics by looking at the constraints on the allowed tensor structures coming from the Schrödinger symmetry, paralleling the relativistic analyses of [50, 51].

The fluid dynamics we have discussed here has been restricted to conformal fluids with Schrödinger symmetry. One main feature of such fluids is that they are in general compressible; this follows from the fact that the energy density is related to the pressure $2\,\varepsilon = d\,P$ through the equation of state (which in turn follows from scale invariance). To make contact with the usual studies of incompressible Navier-Stokes equations we need to ensure that we can decouple the fluctuations in the density. This can be achieved by looking at low frequency modes which do not excite the propagating sound mode in a hydrodynamic system. In fact, this limit was discussed recently in the context of the fluid-gravity correspondence in [48, 52] where the authors showed that starting from a parent relativistic conformal fluid dynamical system one can recover incompressible Navier-Stokes equations in a suitable scaling limit. Curiously the limit procedure reveals an interesting structure in the fluid equations – they are scale invariant under a new scaling symmetry. This symmetry is different from the

[11] In the notation used in [10] this corresponds to the limit $r_+ \to 0$ and $\beta \to \infty$ with $\gamma = \beta\, r_+^2$ held fixed.

Schrödinger symmetry enjoyed by the fluids under consideration in this paper. We arrived at our hydrodynamic description in d spatial dimensions via a light-cone reduction of a $d+2$ dimensional relativistic theory. In [48, 52] the authors derive non-relativistic incompressible hydrodynamics in $d+1$ spatial dimensions by a suitable limit of the relativistic theory. It would be interesting to the take the incompressible limit of the non-relativistic conformal fluid described here to compare with their results.

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A Hamiltonian calculation in the bulk

Since the field theory dual of the asymptotically Schr geometries is non-relativistic, we might not expect the construction of a spacetime stress tensor for asymptotically AdS geometries to have a natural counterpart for these solutions. We might instead expect the map from bulk to boundary to involve some kind of Hamiltonian framework, which treats spatial and time directions separately. In this appendix, we describe the construction of a Hamiltonian for the bulk geometry from the action constructed in [10]. This is a generalization of the calculation of [53], using the known form of the action to determine the correct boundary terms for the Hamiltonian framework.

The idea of the calculation in [53] is to choose a foliation of the spacetime by constant time slices, and rewrite the covariant action in the form $\mathcal{S} = \int dt\,[p\dot{q} - H]$, keeping careful track of the boundary terms. This will then give us a form for the Hamiltonian including boundary terms at infinity. For simplicity we start from the truncated version of the action used in [10],[12]

$$\begin{aligned} \mathcal{S} &= \frac{1}{16\pi\, G_5}\int d^5x\sqrt{-g}\left(R - \frac{4}{3}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{4}e^{-8\phi/3}F_{\mu\nu}F^{\mu\nu} - 4\,A_\mu A^\mu - V(\phi)\right) \\ &+\frac{1}{16\pi G_5}\int d^4\xi\,\sqrt{-h}\,\left(2\,K - 6 + A_\mu A^\mu + c\,A_\mu A^\mu\phi + b\,(A_\mu A^\mu)^2 + (2\,c - 4\,b + 3)\phi^2\right) \end{aligned} \tag{A.1}$$

for some arbitrary constants c, d. To simplify the formulae below, we will work with $c = b = 0$, but it is straightforward to carry out the analysis in the general case. We take the constant

[12] The analysis can be easily extended to the consistent truncation Lagrangian of [11] modulo the cost of some cumbersome expressions. We find that we have an eight parameter set of boundary terms which lead to the finite on-shell action reported in [10].

time slices to be the surfaces Σ_+ of constant x^+. The coordinates on Σ_+ will be denoted collectively as χ^μ and the induced metric is $\kappa_{\mu\nu}$.[13] The lapse and shift N, N^μ are defined by considering a vector u^μ defined on Σ_+, such that $u^\mu \nabla_\mu x^+ = 1$, and decomposing this vector into the normal n^μ to Σ_+ and the shift,

$$u^\mu = N\, n^\mu + N^\mu. \tag{A.2}$$

We also have a constant r surface which is our cut-off on the spacetime, with coordinates ξ^μ and induced metric $h_{\mu\nu}$. To top off the list of surfaces we have the constant time surfaces on the cut-off surfaces which we will denote as Σ_+^∞ and we reserve ζ^μ for the coordinates and $\sigma_{\mu\nu}$ for the induced metric on Σ_+^∞.

The analysis of the gravitational part of the action (A.2) is very similar to the calculation in [53], and leads to

$$H_g = \int_{\Sigma_+} d^4\chi \left(N\,\mathcal{H}_g + N^\mu\,\mathcal{H}_{\mu,g}\right) - \int_{\Sigma_+^\infty} d^3\zeta \sqrt{\sigma}\left[\frac{N}{16\pi G_5}(2\,{}^{(3)}K - 6) - 2\,N^\mu\,\frac{p_{\mu\nu}}{\sqrt{\kappa}}\,r^\nu\right], \tag{A.3}$$

where $p_{\mu\nu}$ is the gravitational conjugate momentum, $p_{\mu\nu} = \sqrt{\kappa}(K_{\mu\nu} - K g_{\mu\nu})$, and r^μ is the unit normal to Σ^∞ and $\kappa = \det(\kappa_{\mu\nu})$. The contributions to the bulk Hamiltonian and Hamiltonian density constraints are

$$\mathcal{H}_g = \frac{16\pi\, G_5}{\sqrt{\kappa}}\, p_{\mu\nu}\, p^{\mu\nu} - \frac{16\pi\, G_5}{3\,\sqrt{\kappa}}\, p^\mu{}_\mu\, p^\nu{}_\nu - \frac{\sqrt{\kappa}}{16\pi\, G_5}\,\mathcal{R}, \quad \mathcal{H}_{\mu,g} = -2\sqrt{\kappa}\,\mathcal{D}_\nu\left(\frac{p_\mu{}^\nu}{\sqrt{\kappa}}\right). \tag{A.4}$$

Here $\mathcal{D}_\mu$ is the covariant derivative associated with the spatial metric $\kappa_{\mu\nu}$. Only the boundary term will contribute to the on-shell Hamiltonian. We note that this can be rewritten as

$$H_g^{os} = -\int d^3\zeta\,\left[\sqrt{\sigma}\,\frac{N}{16\pi G_5}\left(2\,{}^{(3)}K - 6\,N\right) - 2\,N^\mu\,p_\mu^r\right]. \tag{A.5}$$

For the matter part of the action,

$$\begin{aligned}\mathcal{S}_m \;&=\; \frac{1}{16\pi\, G_5}\int d^5x\,\sqrt{-g}\left(-\frac{4}{3}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{4}e^{-8\phi/3}F_{\mu\nu}F^{\mu\nu} - 4\,A_\mu A^\mu - V(\phi)\right) \\ &\quad + \frac{1}{16\pi\, G_5}\int d^4\xi\sqrt{-h}\,\left(A_\alpha A^\alpha + 3\,\phi^2\right),\end{aligned} \tag{A.6}$$

performing the space-time split and introducing the momenta

$$p_\phi = \frac{1}{6\pi\, G_5}\frac{\sqrt{\kappa}}{N}\,\left(\dot{\phi} - N^\mu\partial_\mu\phi\right), \tag{A.7}$$

$$p_A^\lambda = \frac{e^{-8\phi/3}}{16\pi\, G_5}\,\frac{\sqrt{\kappa}}{N}\,\left(\dot{A}_\mu - \partial_\mu A_u - N^\nu F_{\nu\mu}\right)\kappa^{\lambda\mu}, \tag{A.8}$$

[13] We will not distinguish between the indices used for different hypersurfaces.

where dot denotes differentiation wrt x^+, we can rewrite this action as

$$\begin{aligned}\mathcal{S}_m &= \int dx^+ \int_{\Sigma_+} d^4\chi \left[p_\phi \dot{\phi} + p_A^\lambda \dot{A}_\lambda + \frac{2\sqrt{\kappa}}{8\pi\, G_5\, N}(A_+ - N^\mu A_\mu)^2 - (\partial_\mu A_+)\, p_A^\mu - N\,\mathcal{H}'_m - N^\mu\,\mathcal{H}'_{\mu,m}\right] \\ &\quad + \int dx^+ \int_{\Sigma_+^\infty} d^3\zeta \sqrt{\sigma}\left[-\frac{1}{16\pi G_5 N}(A_+ - N^\alpha A_\alpha)^2 + \frac{N}{16\pi G_5}\left(A_\alpha A^\alpha + 3\,\phi^2\right)\right]. \qquad \text{(A.9)}\end{aligned}$$

We need to integrate the term $(\partial_\mu A_+)\, p_A^\mu$ by parts; A_+ will then be a non-dynamical field, and we can eliminate it using its equation of motion,

$$A_+ = N^\mu A_\mu - 2\pi\, G_5\, N\, \mathcal{D}_\mu \frac{p_A^\mu}{\sqrt{\kappa}}, \qquad \text{(A.10)}$$

This gives us the matter action in its final form,

$$\begin{aligned}\mathcal{S}_m &= \int dx^+ \int_{\Sigma_+} d^4\chi \left[p_\phi\, \dot{\phi} + p_A^\lambda\, \dot{A}_\lambda - N\,\mathcal{H}_m - N^\mu\,\mathcal{H}_{\mu,m}\right] \qquad \text{(A.11)} \\ &\quad + \int dx^+ \int_{\Sigma_+^\infty} d^3\zeta \sqrt{\sigma}\left[-\frac{1}{16\pi\, G_5\, N}(A_+ - N^\alpha A_\alpha)^2 - A_+ \frac{p_A^\mu}{\sqrt{\kappa}}\, r_\mu \right. \\ &\quad \left. + \frac{N}{16\pi\, G_5}\left(A_\alpha A^\alpha + 3\,\phi^2\right)\right],\end{aligned}$$

where

$$\begin{aligned}\mathcal{H}_m &= \frac{8\pi\, G_5 \sqrt{\kappa}}{8}\left(\mathcal{D}_\mu \frac{p_A^\mu}{\sqrt{\kappa}}\right)^2 + \frac{24\pi\, G_5}{8\sqrt{\kappa}} p_\phi^2 + \frac{8\pi\, G_5}{\sqrt{\kappa}}\, e^{8\phi/3}\, p_A^\mu\, p_A^\nu\, \kappa_{\mu\nu} \qquad \text{(A.12)} \\ &\quad + \frac{\sqrt{\kappa}}{16\pi\, G_5}\left[\frac{4}{3}\,\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{4}\, e^{-8\phi/3}\, F_{\mu\nu}F^{\mu\nu} + 4\, A_\mu A^\mu + V(\phi)\right],\end{aligned}$$

and

$$\mathcal{H}_{\mu,m} = \partial_\mu\phi\, p_\phi + F_{\mu\nu}\, p_A^\nu - A_\mu \sqrt{\kappa}\, \mathcal{D}_\nu \frac{p_A^\nu}{\sqrt{\kappa}}. \qquad \text{(A.13)}$$

Thus, the matter contribution to the on-shell Hamiltonian is

$$H_m^{os} = -\int_{\Sigma_+^\infty} d^3\zeta\, \sqrt{\sigma}\, \left[-\frac{1}{16\pi\, G_5\, N}(A_+ - N^\alpha A_\alpha)^2 - A_+ \frac{p_A^\mu}{\sqrt{\kappa}}\, r_\mu + \frac{N}{16\pi\, G_5}\left(A_\alpha A^\alpha + 3\,\phi^2\right)\right]. \qquad \text{(A.14)}$$

On-shell

$$A_+ - N^\alpha A_\alpha \propto A^+ = 0, \qquad \text{(A.15)}$$

so we can drop the first term, which will not contribute to the value or first variation of the Hamiltonian, and write

$$H_m^{os} = -\int_{\Sigma_+^\infty} d^3\zeta\, \left[-A_+\, p_A^r + \frac{N\sqrt{\sigma}}{16\pi\, G_5}\left(A_\alpha A^\alpha + 3\,\phi^2\right)\right], \qquad \text{(A.16)}$$

which is closer in form to the gravitational part.

For our spacetime (3.6), we have (writing for brevity $\ell(r) = \sqrt{f(r)}\,k(r)^{1/6}$)

$$
\begin{aligned}
u^\mu &= \frac{\partial}{\partial x^+}, \\
n^\mu &= -\frac{\gamma}{r^3\,\ell(r)}\frac{\partial}{\partial x^+} - \frac{r\,(1+f(r))}{2\,\gamma\,\ell(r)}\frac{\partial}{\partial x^-},
\end{aligned}
\tag{A.17}
$$

which leads to

$$
\begin{aligned}
N &= -\frac{1}{\gamma}\,r^3\,\ell(r), \\
N^\mu &= -r^4\frac{1+f(r)}{2\,\gamma^2}\frac{\partial}{\partial x^-},
\end{aligned}
\tag{A.18}
$$

and r^μ being the unit-normal to Σ_+^∞ is just given as

$$
r^\mu = \frac{r\,f(r)}{\ell(r)}\frac{\partial}{\partial r}. \tag{A.19}
$$

Direct computation gives

$$
\begin{aligned}
\sqrt{\tilde{h}} &= \gamma\,r, \\
{}^{(3)}K &= \frac{f(r)}{\ell(r)}\ , \\
\frac{p_{-r}}{\sqrt{k}} &= -\frac{N}{8\pi\,G_5}\,\frac{\gamma^2}{r^5\,f(r)\,k(r)}.
\end{aligned}
\tag{A.20}
$$

The gravitational part of the Hamiltonian then involves

$$
\begin{aligned}
H_g^{os} &= -\int_{\Sigma_+^\infty} d^3\zeta\,\sqrt{\sigma}\,\frac{N}{16\pi\,G_5}\left(2\,\frac{f(r)^{1/2}}{k(r)^{1/6}} - 6 + 2\,k(r)^{-7/6}\frac{(1+f(r))}{f(r)^{1/2}}\right) \\
&\approx -\int_{\Sigma_+^\infty} d^3\zeta\,\sqrt{\sigma}\,\frac{N}{16\,\pi G_5}\left(-5\,\frac{\gamma^2}{r^2} + \frac{21}{4}\,\frac{\gamma^4}{r^4} - \frac{r_+^4}{r^4}\right).
\end{aligned}
\tag{A.21}
$$

We see that the r^4 divergence cancels. The r^2 divergence will cancel against a contribution from the matter part, which we turn to next.

On the 3-boundary,

$$
A_\alpha A^\alpha = \frac{\gamma^2}{r^2\,k(r)^{4/3}}, \tag{A.22}
$$

so

$$
A_\alpha A^\alpha + 3\,\phi^2 \approx \frac{\gamma^2}{r^2} - \frac{7}{12}\,\frac{\gamma^4}{r^4}. \tag{A.23}
$$

We also evaluate

$$
-\,A_+\frac{p_A^r}{\sqrt{\kappa}}r_r = \frac{e^{-8\phi/3}}{8\pi\,G_5\,N}\,\frac{r\,f(r)^{1/2}}{k(r)^{1/6}}\,A_+(\partial_r A_+ - N^-\partial_r A_-) \approx \frac{N}{16\pi\,G_5}\left(\frac{4\gamma^2}{r^2} - \frac{14}{3}\,\frac{\gamma^4}{r^4}\right) \tag{A.24}
$$

to obtain

$$H_m^{os} \approx -\int_{\Sigma_+^\infty} d^3\zeta\,\sqrt{\sigma}\,\frac{N}{16\pi\,G_5}\left(5\,\frac{\gamma^2}{r^2} - \frac{21}{4}\,\frac{\gamma^4}{r^4}\right). \tag{A.25}$$

As a result, the total Hamiltonian is

$$H^{os} = \frac{1}{16\pi\,G_5}\int_{\Sigma_+^\infty} d^3\zeta\,\sqrt{\sigma}\,N\,\frac{r_+^4}{r^4} = -\frac{1}{16\pi\,G_5}\int_{\Sigma_+^\infty} d^3\zeta\,r_+^4, \tag{A.26}$$

in agreement with our expectations: this matches the total energy obtained by Euclidean methods in [10].

It is easy to also extract the particle number density from this calculation; we simply consider shifting the vector $t^\mu \to t^\mu + \alpha^\mu$, where $\alpha^\mu \partial_\mu t = 0$, which shifts $N^\mu \to N^\mu + \alpha^\mu$. This will change

$$p_A^\lambda \to p_A^\lambda - \frac{e^{-8\phi/3}\sqrt{\kappa}}{16\pi\,G_5 N}\alpha^\nu\,F_{\nu\mu}\,k^{\lambda\mu}, \tag{A.27}$$

and hence redefines the Hamiltonian by

$$H^{os} \to H^{os} - \int_{\Sigma_+^\infty} d^3\zeta\,\sqrt{\sigma}\,\left[-2\,\alpha^\mu\,\frac{p_{\mu\nu}}{\sqrt{\kappa}}\,r^\nu + \frac{e^{-8\phi/3}}{16\pi\,G_5\,N}\alpha^\nu\,F_{\nu\mu}\,r^\mu - \frac{1}{16\pi\,G_5 N}(\alpha^\nu A_\nu)^2\right]. \tag{A.28}$$

We should interpret this new Hamiltonian associated with $t^\mu + \alpha^\mu$ as a combination of the energy and the momentum in the direction specified by α^μ. We see immediately that if α^ν points in one of the spacelike directions, the change in the Hamiltonian vanishes, which agrees with our expectation that the spatial momentum densities vanish in the state we are considering. If we take $\alpha^\nu = \alpha\,\delta_-^\nu$, then the gravity term is

$$\frac{p_{-r}}{\sqrt{\kappa}}\,r^r = -\frac{N}{8\pi\,G_5}\,\frac{\gamma^2}{r^5\,f(r)\,k(r)}\frac{r\,f(r)}{\ell(r)} \approx -\frac{N}{8\pi\,G_5}\,\frac{\gamma^2}{r^4}. \tag{A.29}$$

The matter terms give

$$\frac{e^{-8\phi/3}}{16\pi\,G_5\,N}\alpha^\nu\,F_{\nu\mu}\,r^\mu \approx -\frac{\alpha N}{8\pi\,G_5}\frac{\gamma^4}{r^6}, \quad \frac{1}{16\pi\,G_5\,N}\,(\alpha^\nu A_\nu)^2 \approx \frac{\alpha^2\,N}{16\pi\,G_5}\,\frac{\gamma^6}{r^{10}}, \tag{A.30}$$

so only the gravitational term contributes to the change in the Hamiltonian, which is

$$H^{os} \to H^{os} - \int_{\Sigma_+^\infty} d^3\zeta\,\sqrt{\sigma}\,N\frac{1}{4\pi\,G_5}\,\frac{\gamma^2}{r^4} = H^{os} - \int_{\Sigma_+^\infty} d^3\zeta\,\frac{1}{4\pi\,G_5}\gamma^2, \tag{A.31}$$

consistent with the value $\langle P_- \rangle = \gamma^2 V \Delta x^- / 4\pi G_5$ obtained in the Euclidean approach.

Thus, this approach gives us a definition of the Hamiltonian whose on-shell value appears correct. We should ask if the Hamiltonian so defined is the generator of the asymptotic time-translation symmetry on this class of spacetimes. To be the generator of the symmetry, our Hamiltonian should satisfy [54]

$$\delta H = \int d^4\chi\,\left[\delta p_{\mu\nu}\,E^{\mu\nu} + \delta p_\psi\,E^\psi + \delta k^{\mu\nu}\,A_{\mu\nu} + \delta\psi\,A_\psi\right], \tag{A.32}$$

with no surface terms, where ψ denotes the collection of matter fields. Now, our Hamiltonian constructed from the covariant action should satisfy this property by construction, since (A.32) is just the requirement for Hamilton's principle $\delta \int dt\,(p\dot{q} - H) = 0$ to have solutions, and the boundary terms in our action $\mathcal{S} = \int dt\,(p\dot{q} - H)$ were chosen precisely so as to ensure that $\delta\mathcal{S} = 0$ on-shell. However, in [10], the action was only shown to satisfy $\delta\mathcal{S} = 0$ for a restricted set of boundary conditions on the variations, relating the leading order variations of different fields. It would therefore be useful to check directly that the Hamiltonian we have proposed satisfies (A.32). We have encountered difficulties in performing this check; we leave their resolution to future work.

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