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# A Regulator Formula for Milnor $K$-groups

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**Abstract.** The classical Abel–Jacobi map is used to geometrically motivate the construction of regulator maps from Milnor $K$-groups $K_n^M(\mathbb{C}(X))$ to Deligne cohomology. These maps are given in terms of some new, explicit $(n-1)$-currents, higher residues of which are defined and related to polylogarithms. We study their behavior in families $X_s$ and prove a rigidity result for the regulator image of the Tame kernel, which leads to a vanishing theorem for very general complete intersections.

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**Key words:** Milnor $K$-theory, regulator, Abel–Jacobi map, polylogarithm, higher Chow group.

## 1. Introduction

This paper concerns some new formulas for the regulator maps

$$K_n^M(\mathbb{C}(X)) \cong CH^n(\eta_X, n) \to H_{\mathcal{D}}^n(\eta_X, \mathbb{Z}(n));$$

the maps themselves, at least in principle, have been around since the publication of [1]. Our aim is to make them more accessible to computation. We will usually be concerned with the case where $X$ is an $(n-1)$-dimensional projective variety, so that (ignoring torsion) the maps take the form

$$R\colon K_n^M(\mathbb{C}(X)) \to H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Q}(n)) \cong \mathrm{Hom}\{H_{n-1}(\eta_X, \mathbb{Z}), \mathbb{C}/\mathbb{Q}(n)\}. \tag{1.1}$$

To motivate our formula, we essentially generalize the approach of Bloch for $n = 2$ in Chapter 8 of [2] (see Section 6.4). We begin in Section 3 with the explicit Hodge-theoretic construction of what amounts to a Deligne cycle-class map for codimension-$n$ relative algebraic cycles on $\eta_X \times (\square^n, \partial\square^n)$, where $(\square^n, \partial\square^n)$ is cubical relative affine $n$-space. We call these cycles graphs, and some nontrivial geometry goes into computing the AJ part of this map. By identifying formal linear combinations $\mathbf{f} \in \otimes^n\mathbb{Z}\{\mathbb{C}(X)^*\}$ with graphs $\gamma_{\mathbf{f}} \in Z^n(\eta_X, n)$, we motivate a formula assigning them to $(n-1)$-currents*

$$\otimes^n\mathbb{Z}\{\mathbb{C}(X)^*\} \to \Gamma({}'\mathcal{D}_X^{n-1}), \qquad \mathbf{f} \mapsto R_{\mathbf{f}}$$

*See Section 4.1 for the definition of ${}'\mathcal{D}_X^{n-1}$.

by means of the $\mathrm{AJ}(\gamma_{\mathbf{f}})$ computation. This essentially descends to a 'Milnor regulator' map of the form (1.1), as described in Section 4.

Here is a concrete example of what a Milnor regulator current looks like, for $n = 3$. If $f, g, h \in \mathbb{C}(S)$ are meromorphic functions on an algebraic surface $S$, let $T_f = f^{-1}(\mathbb{R}^-)$ (where $\mathbb{R}^-$ is considered as the directed path $\overleftarrow{[0,\infty]}$ on $\mathbb{P}^1$) and $\log f$ = the branch with imaginary part $\in (-\pi, \pi]$ and jump along $T_f$, and so on for $g$ and $h$. On the other hand $\mathrm{dlog}\, f$ will mean $\mathrm{d}f/f$; they are related by $\mathrm{d}[\log f] = \mathrm{dlog}\, f - 2\pi \mathrm{i} \delta_{T_f}$. Then if $\mathcal{C}$ is a 'topological' 2-chain ($\dim_{\mathbb{R}} \mathcal{C} = 2$) on $S$ avoiding $|(f)| \cup |(g)| \cup |(h)|$ and $\mathbf{f} := f \otimes g \otimes h$, the period of $R_{\mathbf{f}}$ on $\mathcal{C}$ is by definition

$$\int_{\mathcal{C}} \log f \,\mathrm{dlog}\, g \wedge \mathrm{dlog}\, h + 2\pi \mathrm{i} \int_{\mathcal{C} \cap T_f} \log g \,\mathrm{dlog}\, h - 4\pi^2 \sum_{p \in \mathcal{C} \cap T_f \cap T_g} \log h(p).$$

It is natural to ask whether nontrivial periods $\in \mathbb{C}/\mathbb{Q}(n)$ arise primarily from integrals encircling divisors of the functions in $\mathbf{f}$, i.e. $[\mathcal{C}] \in \ker\{H_{n-1}(\eta_X, \mathbb{Z}) \to H_{n-1}(X, \mathbb{Z})\}$, or from 'nontrivial' topological cycles on $X$ avoiding those divisors. To describe the first kind of periods precisely we employ a local–global spectral sequence to define various 'residues' of the Milnor-regulator currents in Section 5, and relate these to AJ maps (on higher Chow groups) with polylogarithmic properties in Section 6. We then define a subgroup $K_n^M(X) \subset K_n^M(\mathbb{C}(X))$ which produces 'residue-free' currents, taking periods on $\mathrm{coim}\{H_{n-1}(\eta_X) \to H_{n-1}(X)\}$. This is just the Tame kernel for $n = 2, 3$ but is smaller for $n \geqslant 4$.

The resulting restriction

$$R\colon K_n^M(X) \to \mathrm{im}\{H^{n-1}(X, \mathbb{C}/\mathbb{Q}(n)) \to H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Q}(n))\}$$

is called the holomorphic Milnor regulator. It appears to be related to the arithmetic rather than the geometry of $X$, for when $X$ is a very general* complete intersection in $\mathbb{P}^N$, its image is zero: the periods $\int_{\mathcal{C}} R_{\mathbf{f}}$ lie in $\mathbb{Q}(n)$; there is only one exception – for $X$ a general elliptic curve (and $n = 2$). This is the vanishing theorem of Section 8, which in principle is a consequence of Nori connectedness. It is a generalization to higher dimension of results in [6] (also see [14]) for $n = 2$, $X$ a curve.

According to Beilinson rigidity, which is proved in [1] (using rigidity of de Rham classes), $R$ is constant on continuous families in $K_n^M(X)$ for $X$ fixed. What we consider in Section 7 is the situation when $X = X_s$ is allowed to very in a (complete) family of complete intersections in $\mathbb{P}^N$. Let $\{\mathbf{f}_s\} \in \ker(\mathrm{Tame}) \subset K_n^M(\mathbb{C}(X_s))$ be an analytic family (in the sense described in Section 7.1); the associated $R_{\mathbf{f}_s}$ have trivial codimension-1 residues. Then (again excepting the case of elliptic curves) $[R_{\mathbf{f}_s}]$ is a flat section of $\mathcal{H}^{n-1}_{\eta_{X_s}}$ under the Gauss–Manin connection; that

*This can be taken to mean that the coordinates of $X$ (in the moduli space of complete intersections) are algebraically independent.

is, the periods are constant. While this is the basis for the vanishing result, it holds with the weaker assumption (*viz.*, ker(Tame) instead of $K_n^M(X)$) and seems to be a slightly deeper result.

We remark that we have taken $\dim X = n-1$ throughout Part 3, because the nontrivial (primitive) cohomology of a complete intersection is found in the middle dimension, and $K_n^M$ maps to $H^{n-1}$.

We conclude this paper with an easy, concrete regulator computation on a 'degenerate elliptic curve'; it is a 'toy model' for the harder computations on elliptic curves in [2, 3, 6].

## PART 1. GEOMETRIC CONSTRUCTION OF $R_{\mathbf{f}}$

### 2. Preliminaries

#### 2.1. HIGHER CHOW GROUPS

Define the algebraic $n$-cube

$$\square^n := (\mathbb{P}^1_{\mathbb{C}} \setminus \{1\})^n =: (\mathbb{P}^1)^n \setminus \mathbb{I}^n$$

with faces $\partial\square^n = \bigcup_i \partial_i\square^n = \bigcup_{i,e} \rho^e_{i*}\square^{n-1}$ and more generally codimension-$r$ subfaces $\partial^r\square^n = \bigcup_{\mathbf{i},\mathbf{e}} \rho^{e_1,\ldots,e_r}_{i_1,\ldots,i_r*}\square^{n-r}$, where for $e = 0, \infty$ the face inclusions send $(z_1,\ldots,z_{n-1}) \mapsto (z_1,\ldots,\overset{i}{e},\ldots,z_{n-1})$, and so on. The $n$-cube is also equipped with projections $\pi_{i_1,\ldots,i_r}\colon \square^n \to \square^{n-r}$, where, e.g., for $r = 1$, $\pi_i$ sends $(z_1,\ldots,z_n) \mapsto (z_1,\ldots,\widehat{z_i},\ldots,z_n)$.

Let $Y/\mathbb{C}$ be a (possibly singular) quasiprojective variety and define $c^p(Y,n) :=$ subgroup of $Z^p(Y \times \square^n)$ generated by subvarieties intersecting all subfaces $Y \times (\rho^{\mathbf{e}}_{\mathbf{i}*}\square^{n-r})$ properly, i.e. in the right codimension. (Note that anything can happen at $Y \times \mathbb{I}^n$ if one looks at the closure of such a cycle on $Y \times (\mathbb{P}^1)^n$.) Let $d^p(Y,n) :=$ subgroup of $c^p(Y,n)$ generated by subvarieties pulled back from $Y \times \square^{n-1}$ by some $\pi_i$.

We neglect these latter cycles and write

$$Z^p(Y,n) := c^p(Y,n)/d^p(Y,n),$$

which forms a complex with differential

$$\partial_{\mathcal{B}} := \sum_{i=1}^{n} (-1)^i (\rho_i^{\infty^*} - \rho_i^{0^*})\colon Z^p(Y,n) \to Z^p(Y,n-1);$$

in particular note that $\partial_{\mathcal{B}} \circ \partial_{\mathcal{B}} = 0$ so we have a complex. Define the higher Chow groups as its homology:

$$\mathrm{CH}^p(Y,n) := H_n\{Z^p(Y,\cdot)\}.$$

Note: we identify $\mathrm{CH}^p(\mathbb{C}(X),n) = \mathrm{CH}^p(\operatorname{Spec}\mathbb{C}(X),n) = \mathrm{CH}^p(\eta_X,n)$.

We shall think of $(\square^n, \partial\square^n)$ and $((\mathbb{C}^*)^n, \mathbb{I}^n)$ as dual relative varieties.

## 2.2. MILNOR $K$-GROUPS

We shall write $\mathbb{Z}\{\mathsf{S}\}$ for the free Abelian group on a set $\mathsf{S}$. For any field $\mathsf{F} \supseteq \mathbb{Q}$ and $n \geqslant 2$, let $K_n^M(F)$ denote the quotient of the Abelian group $\otimes^n \mathbb{Z}\{\mathsf{F}^*\}$ by the *Steinberg relations*: the subgroup generated by all permutations of

$$f_1 \otimes f_2 \otimes \cdots \otimes f_n + g_1 \otimes f_2 \otimes \cdots \otimes f_n - f_1 g_1 \otimes f_2 \otimes \cdots \otimes f_n,$$
$$f_1 \otimes f_2 \otimes \cdots \otimes f_n + f_2 \otimes f_1 \otimes \cdots \otimes f_n,$$
$$f_1 \otimes (1 - f_1) \otimes f_3 \otimes \cdots \otimes f_n.$$

(For $n < 2$ just set $K_1^M(\mathsf{F}) := \mathsf{F}^*$ and $K_0^M(\mathsf{F}) := \mathbb{Z}$.) Boldface $\mathbf{f}$ or $\mathbf{g}$ denotes an element of $\otimes^n \mathbb{Z}\{\mathsf{F}^*\}$ while $\{\mathbf{f}\}$ or $\{\mathbf{g}\}$ is the corresponding element of $K_n^M(\mathsf{F})$.

Let $X/\mathbb{C}$ be a smooth projective variety of dimension $d$; to a 'multifunction' $\mathbf{f} = \sum m_j f_{1j} \otimes \cdots \otimes f_{nj} \in \otimes^n \mathbb{Z}\{\mathbb{C}(X)^*\}$ we associate the subvariety $V_{\mathbf{f}} := \bigcup_{i,j} |(f_{ij})|$ and the 'graph cycle'

$$\gamma_{\mathbf{f}} = \Big[\sum m_j (\mathrm{id}_X; f_{1j}, \ldots, f_{nj})_*(X \setminus V_{\mathbf{f}})\Big] \cap ((X \setminus V_{\mathbf{f}}) \times \square^n).$$

Also denote by $V_{\mathbf{f}}^2$ the union of all intersections (and self-intersections) of all components of $V_{\mathbf{f}}$, by $V_{\mathbf{f}}^3$ all intersections of components of $V_{\mathbf{f}}$ and $V_{\mathbf{f}}^2$, and so on

A fundamental result of Totaro [28] says that the graph homomorphism

$$\gamma\colon \otimes^n \mathbb{Z}\{\mathbb{C}(X)^*\} \twoheadrightarrow Z^n(\eta_X, n)$$

so defined, induces an isomorphism

$$K_n^M(\mathbb{C}(X)) \stackrel{\cong}{\to} \mathrm{CH}^n(\eta_X, n).$$

We shall call $\mathbf{f}$ 'good' if the closure $\overline{\gamma_{\mathbf{f}}}$ of its graph to $X \times \square^n$ intersects all subfaces $X \times \partial^r \square^n$ properly, i.e. if $\overline{\gamma_{\mathbf{f}}} \in Z^n(X, n)$. From Bloch's moving lemma [5] we have immediately the following proposition.

PROPOSITION 2.1. *For any* $\mathbf{f}_0 \in \otimes^n \mathbb{Z}\{\mathbb{C}(X)^*\}$ *there is a Steinberg relation* $\mathbf{g}$ (i.e. $\{\mathbf{g}\} = 0$) *such that* $\mathbf{f} := \mathbf{f}_0 - \mathbf{g}$ *is good.*

The absence of 'corners' from good $\mathbf{f}$ allows us to define elements

$$\partial_i \mathbf{f} := \sum_{x \in X^1} \nu_x(f_i) \cdot f_1|_x \otimes \cdots \otimes \widehat{f_i} \otimes \cdots \otimes f_n|_x, \qquad \partial \mathbf{f} := \sum_i (-1)^i \partial_i \mathbf{f}$$

of $\coprod_{x \in X^1} \otimes^{n-1} \mathbb{Z}\{\mathbb{C}(x)^*\}$; and by the proposition, $\mathbf{f} \mapsto \partial \mathbf{f}$ induces a map

$$\mathrm{Tame}\colon K_n^M(\mathbb{C}(X)) \to \coprod_{x \in X^1} K_n^M(\mathbb{C}(x)).$$

## 2.3. THE DLOG MAP

For us ‘log’ will be a (discontinuous) function from $\mathbb{C}^* \to \{z \in \mathbb{C} | \Im(z) \in (-\pi, \pi]\}$ with branch cut $T = \mathbb{R}^- \setminus \{0\}$ (or $\mathbb{R}^- \cup \{\infty\}$ depending on context); given $f \in \mathbb{C}(X)^*$, dlog $f$ means $\mathrm{d}f/f$ and $T_f = f^{-1}(T)$. On $\square^n$ one has a topological relative $n$-cycle $T_n := T_{z_1} \cap \cdots \cap T_{z_n} \in C_n(\square^n, \partial\square^n)$ and a holomorphic $n$-form $\Omega_n := \mathrm{dlog}\, z_1 \wedge \cdots \wedge \mathrm{dlog}\, z_n \in H^0(\Omega^n_{\square^n}(\log \partial\square^n))$. To a multifunction $\mathbf{f}$ on $X$ one may associate

$$T_{\mathbf{f}} := \sum m_j T_{f_{1j}} \cap \cdots \cap T_{f_{nj}} = \pi_{X_*}\{\overline{\gamma_{\mathbf{f}}} \cap (X \times T_n)\} \in C_{2d-n}(X, V_{\mathbf{f}}),$$

and

$$\Omega_{\mathbf{f}} := \bigwedge^n \mathrm{dlog}\, \mathbf{f} = \sum m_j \frac{\mathrm{d}f_{1j}}{f_{1j}} \wedge \cdots \wedge \frac{\mathrm{d}f_{nj}}{f_{nj}} \in H^0(\Omega^n_X(\log V_{\mathbf{f}})).$$

(For the definition of log-differentials when $V_{\mathbf{f}}$ is not a NCD, see for example [20].) Writing $\varinjlim_{V \subset X} H^0(\Omega^n_X(\log V)) =: H^0(\Omega^n_X(\log))$, we have the following lemma.

LEMMA 2.2. (i) *The assignment* $\mathbf{f} \mapsto \Omega_{\mathbf{f}}$ *descends to a map* ‘dlog’ *on Milnor K-theory, so that the lower-left-hand square of the following diagram commutes*:

$$\begin{array}{ccccc}
 & & & & \varinjlim_{V \subset X} H^n_V(X) \\
 & & & & \downarrow Gy \\
 & & H^0(\Omega^n_X) & \hookrightarrow & H^n(X) \\
 & & \cap\downarrow & & \downarrow \jmath^*_{\eta_X} \\
K^M_n(\mathbb{C}(X)) & \xrightarrow{\mathrm{dlog}} & H^0(\Omega^n_X(\log)) & \hookrightarrow & H^n(\eta_X) \\
\downarrow \mathrm{Tame} & & \downarrow Res & & \\
\coprod_{x \in X^1} K^M_{n-1}(\mathbb{C}(x)) & \xrightarrow{\coprod \mathrm{dlog}} & \coprod_{x \in X^1} H^0(\Omega^{n-1}_{\tilde{x}}(\log)) & &
\end{array}$$

(ii) *Assuming that always* $[\Omega_{\mathbf{f}}] \in \mathrm{im}\{H^n(\eta_X, \mathbb{Z}(n)) \to H^n(\eta_X, \mathbb{C})\}$ [*which will follow from the construction below*], $\{\mathbf{f}\} \in \ker(\mathrm{Tame}) \Longrightarrow \Omega_{\mathbf{f}} = 0$.

*Proof.* (i) dlog is well-defined because $\Omega_{\mathbf{g}} = 0$ if $\{\mathbf{g}\} = 0$; use good $\mathbf{f}$ to check the square commutes (easy).

(ii) Note first the right-hand column is exact: this is from the localization exact sequence in cohomology. Now suppose $\{\mathbf{f}\} \in \ker(\text{Tame})$. Then $\Omega_{\mathbf{f}} \in \ker(\text{Res})$ and we defined a functional on $Z^0_{\mathrm{d}}(\Omega^{2d-n}_{X^\infty})$ by duality:⋆ $\alpha \mapsto \lim_{\epsilon\to 0}\int_{X\setminus N_\epsilon(V_{\mathbf{f}})} \Omega_{\mathbf{f}} \wedge \alpha$. Noting that

$$\lim_{\epsilon\to 0}\int_{X\setminus N_\epsilon(V_{\mathbf{f}})} \Omega_{\mathbf{f}} \wedge \mathrm{d}\beta = \pm \lim_{\epsilon\to 0}\int_{\partial\{N_\epsilon(V_{\mathbf{f}})\}} \Omega_{\mathbf{f}} \wedge \beta$$
$$= \pm \lim_{\epsilon\to 0}\int_{V_{\mathbf{f}}\setminus N_\epsilon(V^2_{\mathbf{f}})} \operatorname{Res}(\Omega_{\mathbf{f}}) \wedge \beta = 0$$

for $\beta \in \Gamma(\Omega^{2d-n-1}_{X^\infty})$, the functional descends to a class

$$[\widetilde{\Omega_{\mathbf{f}}}] \in \left\{\frac{H^{2d-n}(X,\mathbb{C})}{F^{d-n+1}H^{2d-n}}\right\}^\vee \cong F^n H^n(X,\mathbb{C}).$$

If $\omega \in H^0(\Omega^n_X)$ represents this class then $[\jmath^*_{\eta_X}\omega] \equiv [\Omega_{\mathbf{f}}] \in H^n(\eta_X)$ and so (by the diagram) $\Omega_{\mathbf{f}} = \jmath^*_{\eta_X}\omega \in H^0(\Omega^n_X(\log))$. Since $\operatorname{im}(\text{Gy}) \subseteq H^{n-1,1} \oplus \cdots \oplus H^{1,n-1}$, it follows that $H^{n,0}(X) \oplus H^{0,n}(X) \overset{\jmath^*_{\eta_X}}{\hookrightarrow} H^n(\eta_X)$ and thus $[\jmath^*_{\eta_X}\omega] \neq \pm[\jmath^*_{\eta_X}\bar{\omega}]$ unless $\omega = 0$. But $[\Omega_{\mathbf{f}}] \in \operatorname{im}\{H^n(\eta_X,\mathbb{Z}(n)) \to H^n(\eta_X,\mathbb{C})\} \implies [\Omega_{\mathbf{f}}] = \pm\overline{[\Omega_{\mathbf{f}}]}$. □

## 3. Abel–Jacobi for Graphs

### 3.1. GEOMETRIC MOTIVATION

Let $\overline{\partial\square^n} = (\mathbb{P}^1)^n \setminus (\mathbb{C}^*)^n$,

$$N_\epsilon(\overline{\partial\square^n}) = \left\{ z_1,\ldots,z_n \in (\mathbb{P}^1)^n \;\middle|\; |z_i| < \epsilon \text{ or } > \frac{1}{\epsilon} \text{ for some } i \right\},$$

$\bar{\square}^n_\epsilon = (\mathbb{P}^1)^n \setminus N_\epsilon(\overline{\partial\square^n})$; and put $\gamma^\epsilon_{\mathbf{f}} := \overline{\gamma_{\mathbf{f}}} \cap (X \times \bar{\square}^n_\epsilon)$. Note that $\partial\gamma^\epsilon_{\mathbf{f}} \subseteq X \times \partial\bar{\square}^n_\epsilon$. Writing $\mathbf{f}^{-1}(\mathcal{S}) := \bigcup_j (\mathrm{id}_X; f_{1j},\ldots,f_{nj})^{-1}(X \times \mathcal{S})$ for a set $\mathcal{S} \subseteq (\mathbb{P}^1)^n$, define $N_\epsilon(V_{\mathbf{f}}) := \mathbf{f}^{-1}(N_\epsilon(\overline{\partial\square^n}))$.

We want to construct cycle-class and Abel–Jacobi maps for our graph cycle $\gamma_{\mathbf{f}}$. To do this, we will view $\gamma^\epsilon_{\mathbf{f}}$ as giving a homology class in

$$H_{2d}((X, \overline{N_\epsilon(V_{\mathbf{f}})}) \times ((\mathbb{C}^*)^n, \mathbb{I}^n)) \cong H_{2d-n}(X, V_{\mathbf{f}}) \otimes H_n((\mathbb{C}^*)^n, \mathbb{I}^n),$$

⋆One can show this integral is convergent for any $C^\infty$ form $\alpha$ (see [18]). Notation: $Z^0_{\mathrm{d}}(\cdots)$ means d-closed sections and $\Omega^{2d-n}_{X^\infty}$ means $C^\infty(2d-n)$-forms.

and integrate $\lim_{\epsilon\to 0}\int_{\gamma_{\mathbf{f}}^{\epsilon}} \Omega_n \wedge \pi_X^*\alpha$ to determine this class (where $\alpha \in Z^0_{\mathrm{d}}(\jmath_!\Omega^{2d-n}_{(X\setminus V_{\mathbf{f}})^\infty})$ has compact support on $X \setminus V_{\mathbf{f}}$). Moreover, $H_n((\mathbb{C}^*)^n, \mathbb{I}^n)$ is generated by the $n$-torus $(S^1)^n$; so there should be a topological $(2d+1)$-chain on $X\times(\mathbb{C}^*)^n$ bounding on (i) $\gamma_{\mathbf{f}}^{\epsilon}$, (ii) {a relative cycle $\mathcal{C}\}\times(S^1)^n$, and (iii) components supported on $X\times\mathbb{I}^n$ and $\overline{N_\epsilon(V_{\mathbf{f}})}\times(\mathbb{C}^*)^n$. That chain will be produced in Section 3.3 by means of a 'homotopy' $\theta$ [formula (3.1)] which pulls $\gamma_{\mathbf{f}}^{\epsilon}$ into the 'topological trashcan' $X\times\mathbb{I}^n$, leaving 'residues' along* $\overline{N_\epsilon(V_{\mathbf{f}})}\times(\mathbb{C}^*)^n$; and it will turn out that $\mathcal{C}$ is just $T_{\mathbf{f}}$. Towards the goal of constructing this chain, we now make some preliminary definitions.

## 3.2. FUNDAMENTAL DOMAIN FOR $(\mathbb{C}^*)^n$

Let $\mathbb{D}$ = closure of $\mathbb{C}^*\setminus T$ with left and right-hand limits *not* identified; there is an obvious map $\mathcal{N}\colon \mathbb{D}\to\mathbb{C}^*$. Write $\tilde{z} := \mathcal{N}^{-1}(z\notin T)$ [one point], $\mathcal{N}^{-1}(z\in T) =: \{z^+, z^-\}$, $\mathcal{N}^{-1}(T) = \{T^+, T^-\}$, $\tilde{T} = T^+ - T^-$. Choose any $C^\infty$ map

$$\theta\colon [0,1]\times\mathbb{D}\to\mathbb{C}^*$$

with $\theta(0,\tilde{z}) = z$, $\theta(1,\tilde{z}) = 1$, so that $\theta(z) := \theta(t,z)$ gives a path from $z$ to 1 in $\mathbb{C}^*$. Also observe that $\theta(z^+) - \theta(z^-) =: \theta^+(z) - \theta^-(z)$ is a circle $(S^1)$, where $\theta^\pm\colon [0,1]\times T\to\mathbb{C}^*$. If $\Gamma$ is a topological cycle compactly supported on $\mathbb{C}^*$, and $\tilde{\Gamma}$ a lift to $\mathbb{D}$, $\theta(\tilde{\Gamma})$ denoted the image of $[0,1]\times\tilde{\Gamma}$ under $\theta$, and $\partial\theta(\tilde{\Gamma}) = \Gamma + \theta^+(\Gamma\cap T) - \theta^-(\Gamma\cup T)$.

More generally, let $\mathbb{D}^n$ = closure (as above) of $(\mathbb{C}^*)^n\setminus\cup T_{z_i}$ and $\mathcal{N}\colon \mathbb{D}^n\to(\mathbb{C}^*)^n$ take $T^+_{z_i}, T^-_{z_i}\mapsto T_{z_i}$, etc. We define various maps

$$\theta_{i_1\ldots i_k}\colon [0,1]^k\times\mathbb{D}^n\to(\mathbb{C}^*)^n$$

by formulas

$$\theta_1(t_1;\tilde{z}_1,\ldots,\tilde{z}_n) = (\theta(t_1,\tilde{z}_1), z_2,\ldots,z_n),$$
$$\theta_{12}(t_1,t_2;\tilde{z}_1,\ldots,\tilde{z}_n) = (\theta(t_1,\tilde{z}_1),\theta(t_2,\tilde{z}_2), z_3,\ldots,z_n),$$

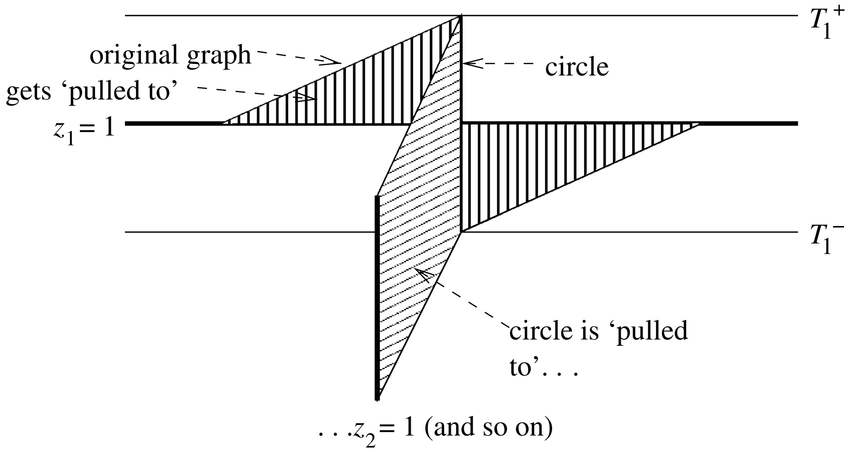
and so on, often omitting parameters $t_i$ in the argument to indicate corresponding chains. Again certain restrictions descend to subsets of $(\mathbb{C}^*)^n$; e.g. by restricting $\theta_{12}$ to $T^+_{z_1}\cap T^-_{z_2}$ we have $\theta^{+-}_{12}\colon [0,1]^2\times(T_{z_1}\cap T_{z_2})\to(\mathbb{C}^*)^n$ and more generally $\theta^{s_1\ldots s_k}_{i_1\ldots i_k} =: \theta^s_i$. For $z\in T_{z_{i_1}}\cap\cdots\cap T_{z_{i_k}}$, the formal sum (considered as a chain) $\sum_{s_1,\ldots,s_k=+,-}(-1)^{\prod s_j}\theta^{\mathbf{s}}_{\mathbf{i}}(z)$ yields a topological $k$-torus $(S^1)^k$.

All of these definitions make sense crossed with $X$. Below we shall use the shorthand $T_{z_i}$ for $X\times T_{z_i}$, and so on.

*We cannot say $\partial\overline{N_\epsilon(V_{\mathbf{f}})}$ here because in general $\mathbf{f}^{-1}(\partial\bar{\square}^n_\epsilon)$ only $\subset\overline{N_\epsilon(V_{\mathbf{f}})}$.

### 3.3. BOUNDING CHAIN FOR $\gamma_{\mathbf{f}}^{\epsilon}$

The idea is to use $\theta_1$ to push $\gamma_{\mathbf{f}}^{\epsilon}$ down (along the first coordinate) to $z_1 = 1$, with discrepancy arising from $\gamma_{\mathbf{f}}^{\epsilon} \cap T_{z_1}$ which we then push (along the second coordinate, using $\theta_{12}$) down to $z_2 = 1$, and so on.




If $\widetilde{\gamma_{\mathbf{f}}^{\epsilon}}$ is the preimage of $\gamma_{\mathbf{f}}^{\epsilon}$ under $\mathcal{N}\colon X \times \mathbb{D}^n \to X \times (\mathbb{C}^*)^n$, one writes the chain

$$\theta(\widetilde{\gamma_{\mathbf{f}}^{\epsilon}}) := \theta_1(\widetilde{\gamma_{\mathbf{f}}^{\epsilon}}) - \theta_{12}(\widetilde{\gamma_{\mathbf{f}}^{\epsilon}} \cap \tilde{T}_{z_1}) + \cdots + \pm\theta_{1\ldots n}(\widetilde{\gamma_{\mathbf{f}}^{\epsilon}} \cap \tilde{T}_{z_1} \cap \cdots \cap \tilde{T}_{z_n}) \tag{3.1}$$

whose boundary is a sort of 'geometric collapsing sum' yielding

$$\partial\theta(\widetilde{\gamma_{\mathbf{f}}^{\epsilon}}) = \gamma_{\mathbf{f}}^{\epsilon} \pm \sum_{\mathbf{s}} (-1)^{\mathbf{s}} \theta_{1\ldots n}^{\mathbf{s}}(\gamma_{\mathbf{f}}^{\epsilon} \cap T_n) + \theta(\widetilde{\partial\gamma_{\mathbf{f}}^{\epsilon}}) \tag{3.2}$$

plus terms supported on $X \times \mathbb{I}^n$. The second term is supported *over* $T_{\mathbf{f}}$ with fibers $\sim (S^1)^n$; we may tacitly modify $\theta(\widetilde{\gamma_{\mathbf{f}}^{\epsilon}})$ by a 'trivial' chain (which would not affect integrals) uniformizing these tori, and replace the second term by $T_{\mathbf{f}} \times (S^1)^n$. The third term is in $\overline{N_\epsilon(V_{\mathbf{f}})} \times (\mathbb{C}^*)^n$ (trivial for our immediate purposes) but gives some insight into *residues* of the AJ (and cycle-class) constructions below. Ignoring codimension 2, it turns out that $\theta(\widetilde{\gamma_{\mathbf{f}}^{\epsilon}}) \approx \sum_i S_{z_i}^1 \times \theta(\widetilde{\gamma_{\partial_i \mathbf{f}}^{\epsilon}})$ is a sufficiently accurate approximation (for good $\mathbf{f}$), making the construction 'telescopic' in codimension 1 (only!). Pursuing this leads to a geometric proof of Proposition 5.3.

For $\alpha \in \Omega^{2d-n}_{X^\infty,c}(X \setminus V_{\mathbf{f}})$ d-closed and compactly supported on [a closed subset of] $X \setminus N_\epsilon(V_{\mathbf{f}})$, we have that

$$\begin{aligned} 0 &= \pm \int_{\theta(\widetilde{\gamma^\epsilon_{\mathbf{f}}})} \Omega_n \wedge \pi^* \mathrm{d}\alpha = \int_{\theta(\widetilde{\gamma^\epsilon_{\mathbf{f}}})} \mathrm{d}\{\Omega_n \wedge \pi^*\alpha\} = \int_{\partial\theta(\widetilde{\gamma^\epsilon_{\mathbf{f}}})} \Omega_n \wedge \pi^*\alpha \\ &= \int_{\gamma^\epsilon_{\mathbf{f}}} \Omega_n \wedge \pi^*\alpha \pm \int_{T_{\mathbf{f}}\times(S^1)^n} \Omega_n \wedge \pi^*\alpha + \int_{\theta(\partial\widetilde{\gamma^\epsilon_{\mathbf{f}}})} \Omega_n \wedge \pi^*\alpha \\ &= \int_X \Omega_{\mathbf{f}} \wedge \alpha \pm (2\pi\mathrm{i})^n \int_{T_{\mathbf{f}}} \alpha + 0; \end{aligned}$$

that is,

$$[\Omega_{\mathbf{f}}] \equiv \left[(2\pi\mathrm{i})^n \int_{T_{\mathbf{f}}}\right] \in \{H^{2d-n}(X, V_{\mathbf{f}})\}^\vee \cong H^n(X \setminus V_{\mathbf{f}}, \mathbb{C}).$$

Since $[(2\pi\mathrm{i})^n \int_{T_{\mathbf{f}}}] \in \operatorname{im}\{H^n(X \setminus V_{\mathbf{f}}, \mathbb{Z}(n)) \to H^n(X \setminus V_{\mathbf{f}}, \mathbb{C})\}$ this completes the proof (see Lemma 2.2) that $\mathrm{Tame}\{\mathbf{f}\} = 0 \Longrightarrow \Omega_{\mathbf{f}} = 0$. We shall henceforth work under the assumption that $\mathrm{Tame}\{\mathbf{f}\}$ is trivial or $n > d[\Longrightarrow \Omega_{\mathbf{f}} = 0$ by type considerations]. Either condition clearly $\Longrightarrow [T_{\mathbf{f}}] \sim 0$ in $H_{2d-n}(X, V_{\mathbf{f}}; \mathbb{Q})$,⋆ so that there is a membrane $\zeta$ with $\partial\zeta = T_{\mathbf{f}}+$ chains supported on $V_{\mathbf{f}}$; we often write $\partial^{-1}_{(X,V_{\mathbf{f}})} T_{\mathbf{f}}$ for $\zeta$. Define

$$\partial^{-1}_\epsilon \gamma_{\mathbf{f}} := \theta(\widetilde{\gamma^\epsilon_{\mathbf{f}}}) \mp \zeta \times (S^1)^n; \tag{3.3}$$

modulo $X \times \mathbb{I}^n$ and $\overline{N_\epsilon(V_{\mathbf{f}})} \times (\mathbb{C}^*)^n$ this now bounds exactly on $\gamma^\epsilon_{\mathbf{f}}$. We summarize in the following proposition.

PROPOSITION 3.1. *If* $\mathrm{Tame}\{\mathbf{f}\} = 0$ *or* $n > d$ *then the relative cycle-class of* $\gamma_{\mathbf{f}}$ *in* $H_{2d}((X, N_\epsilon(V_{\mathbf{f}})) \times ((\mathbb{C}^*)^n, \mathbb{I}^n))$ *is trivial.*

### 3.4. COMPUTATION OF AJ

We now propose to define $\mathrm{AJ}(\gamma_{\mathbf{f}})$ by integrating 'test forms' over $\partial^{-1}_\epsilon \gamma_{\mathbf{f}}$; intuitively this should map to

$$\frac{H^{2n-1}((X \setminus V_{\mathbf{f}}) \times (\square^n, \partial\square^n))}{F^n\{\mathrm{num}\} + \{\mathrm{num}\}_{\mathbb{Z}}} \cong \frac{\{F^{d+1}H^{2d+1}((X, V_{\mathbf{f}}) \times ((\mathbb{C}^*)^n, \mathbb{I}^n))\}^\vee}{\mathrm{periods}}$$

the expression in braces is

$$F^{d-n+1}H^{2d-n+1}(X, V_{\mathbf{f}}) \otimes \langle \Omega^n \rangle = H^{2d-n+1}(X, V_{\mathbf{f}}) \otimes \langle \Omega^n \rangle,$$

which is to say the $((\mathbb{C}^*)^n, \mathbb{I}^n)$ part absorbs all Hodge-theoretic restrictions on the test forms.

⋆$[T_{\mathbf{f}}]$ may still be torsion in $H_{2d-n}(X, V_{\mathbf{f}}; \mathbb{Z})$, so in particular the membrane may require $\mathbb{Q}$-coefficients.

*Remark.* Since there is actually nothing preventing us, then, from taking an *integral* basis for these test forms, we might as well substitute [Poincaré duals of] topological $(2d-n+1)$-cycles. This is the point of view we take in subsequent sections.

DEFINITION. $\mathrm{AJ}(\gamma_{\mathbf{f}}) \in H^{n-1}(X \setminus V_{\mathbf{f}}, \mathbb{C}/\mathbb{Q}(n)) \cong \{H^{2d-n+1}(X, V_{\mathbf{f}})\}^{\vee}/\mathrm{im}\{H_{2d-n+1}(X, V_{\mathbf{f}}; \mathbb{Q}(n))\}$ is the functional on forms $\omega \in \Gamma(\Omega^{2d-n+1}_{X^{\infty},c})$ supported on a compact subset of $X \setminus \overline{N_\epsilon(V_{\mathbf{f}})}$, given by the integral $\int_{\partial_\epsilon^{-1}\gamma_{\mathbf{f}}} \Omega_n \wedge \pi_X^* \omega$. The ambiguity of $\mathbb{Q}(n)\cdot\{\text{periods}\}$ is generated by the choice of $\zeta = \partial^{-1}_{(X,V_{\mathbf{f}})} T_{\mathbf{f}}$.

*Remark.* We usually take the image of $\mathrm{AJ}(\gamma_{\mathbf{f}}) \in H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Z}(n))$ by the direct limit.

Now recall that

$$\partial_\epsilon^{-1}\gamma_{\mathbf{f}} = \theta_1(\widetilde{\gamma_{\mathbf{f}}^\epsilon}) + \sum_{k=2}^{n} (-1)^{k-1}\theta_{1\dots k}(\widetilde{\gamma_{\mathbf{f}}^\epsilon} \cap \tilde{T}_{z_1} \cap \cdots \cap \tilde{T}_{z_{k-1}}) + \partial^{-1}_{(X,V_{\mathbf{f}})} T_{\mathbf{f}} \times (S^1)^n.$$

The first term of the integral defining $\mathrm{AJ}(\gamma_{\mathbf{f}})$ is (partially) computed by integrating $\mathrm{dlog}\, z_1$ along the fibers of $\theta_1(\widetilde{\gamma_{\mathbf{f}}^\epsilon})$:

$$\begin{aligned}
&\int_{\theta_1(\widetilde{\gamma_{\mathbf{f}}^\epsilon})} \mathrm{dlog}\, z_1 \wedge \cdots \wedge \mathrm{dlog}\, z_n \wedge \pi^*\omega \\
&\quad = \int_{\gamma_{\mathbf{f}}^\epsilon} \log z_1 \, \mathrm{dlog}\, z_2 \wedge \cdots \wedge \mathrm{dlog}\, z_n \wedge \pi^*\omega \\
&\quad = \sum m_j \int_X \log f_{1j} \, \mathrm{dlog}\, f_{2j} \wedge \cdots \wedge \mathrm{dlog}\, f_{nj} \wedge \pi^*\omega.
\end{aligned}$$

More generally,

$$\begin{aligned}
&\int_{\theta_{1\dots k}(\widetilde{\gamma_{\mathbf{f}}^\epsilon} \cap \tilde{T}_{z_1} \cap \cdots \cap \tilde{T}_{z_{k-1}})} \Omega_n \wedge \pi^*\omega \\
&\quad = \int_{(\widetilde{\gamma_{\mathbf{f}}^\epsilon} \cap \tilde{T}_{z_1} \cap \cdots \cap \tilde{T}_{z_{k-1}}) \times [0,1]^k} \theta^*_{1\dots k}(\mathrm{dlog}\, z_1 \wedge \cdots \wedge \mathrm{dlog}\, z_n \wedge \pi^*\omega) \\
&\quad = \int_{(\widetilde{\gamma_{\mathbf{f}}^\epsilon} \cap \tilde{T}_{z_1} \cap \cdots \cap \tilde{T}_{z_{k-1}}) \times [0,1]^k} \theta^*(\mathrm{dlog}\, z_1) \wedge \cdots \wedge \theta^*(\mathrm{dlog}\, z_k) \wedge \\
&\qquad \wedge \mathrm{dlog}\, z_{k+1} \wedge \mathrm{dlog}\, z_n \wedge \pi^*\omega \\
&\quad = (2\pi \mathrm{i})^{k-1} \int_{\gamma_{\mathbf{f}}^\epsilon \cap T_{z_1} \cap \cdots \cap T_{z_{k-1}}} \log z_k \, \mathrm{dlog}\, z_{k+1} \wedge \cdots \wedge \mathrm{dlog}\, z_n \wedge \pi^*\omega \\
&\quad = (2\pi \mathrm{i})^{k-1} \sum m_j \int_{T_{f_{1j}} \cap \cdots \cap T_{f_{k-1}}} \log f_{kj} \, \mathrm{dlog}\, f_{(k+1)j} \wedge \cdots \wedge \mathrm{dlog}\, f_{nj} \wedge \omega,
\end{aligned}$$

where $\log f_{kj}$ is understood as having argument $\in (-\pi, \pi]$. Therefore the whole integral $[\mathrm{AJ}(\gamma_{\mathbf{f}})](\omega) =$

$$\begin{aligned}\int_{\partial_\epsilon^{-1}\gamma_{\mathbf{f}}} \Omega_n \wedge \pi^*\omega &= \sum m_j \Bigg\{ \int_X \log f_{1j}\, \mathrm{dlog} f_{2j} \wedge \cdots \wedge \mathrm{dlog} f_{nj} \wedge \omega - \\ &\quad - 2\pi\mathrm{i} \int_{T_{f_{1j}}} \log f_{2j}\, \mathrm{dlog} f_{3j} \wedge \cdots \wedge \mathrm{dlog} f_{nj} \wedge \omega + \\ &\quad + \cdots \pm (2\pi\mathrm{i})^{n-1} \int_{T_{f_{1j}} \cap \cdots \cap T_{f_{(n-1)j}}} \log f_{nj} \wedge \omega \Bigg\} \mp \\ &\quad \mp (2\pi\mathrm{i})^n \int_{\partial^{-1}_{(X,V_{\mathbf{f}})} T_{\mathbf{f}}} \omega \\ &=: \int_X R_{\mathbf{f}} \wedge \omega \mp (2\pi\mathrm{i})^n \int_{\partial^{-1}_{(X,V_{\mathbf{f}})} T_{\mathbf{f}}} \omega \\ &=: \int_X R'_{\mathbf{f}} \wedge \omega.\end{aligned}$$

PROPOSITION 3.2. $\mathrm{AJ}(\gamma_{\mathbf{f}}) \in H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Q}(n))$ *is computed by the class of the current* $R'_{\mathbf{f}}$.

Any given topological $(n-1)$-cycle $\mathcal{C}$ on $X \setminus V_{\mathbf{f}}$ has a $C^\infty$ Poincaré-dual $(2d - n - 1)$-form $\omega_{\mathcal{C}}$ compactly supported on $X \setminus V_{\mathbf{f}}$; so one defines periods $\int_{\mathcal{C}} R'_{\mathbf{f}} := \int_X R'_{\mathbf{f}} \wedge \omega_{\mathcal{C}}$ and observes that their values $\in \mathbb{C}/\mathbb{Q}(n)$ determine $[R'_{\mathbf{f}}] = \mathrm{AJ}(\gamma_{\mathbf{f}})$. Since $(2\pi\mathrm{i})^n \int_\zeta \omega_{\mathcal{C}}$ is an intersection number $\in \mathbb{Q}(n)$, from this point of view $[R_{\mathbf{f}}]$ will suffice.

*Remark.* One can also check geometrically that $\mathbf{f} \mapsto \mathrm{AJ}(\gamma_{\mathbf{f}})$ descends to Milnor $K$-theory (see [18, Section 2.1]), but it is much easier to do algebraically with the bit of formalism we develop in the next section.

## 4. Milnor-regulator Currents

### 4.1. CURRENTS ON $X$ AND $\eta_X$

An *m-current* on $X$ is a section of the sheaf ${}'\mathcal{D}_X^m = \mathcal{D}(\Omega_{X^\infty}^{2d-m})$ of distributions on $C^\infty$- forms.

EXAMPLE 4.1. Let $Y \subseteq X$ be a real-codimension-$k$ analytic subset, $\eta$ an $\ell$-form with singularities along $D \subset Y$. Then $\eta \cdot \delta_Y$ defines a current $\in \Gamma({}'\mathcal{D}_X^{k+\ell})$ by the formula $\int_X (\eta \cdot \delta_Y) \wedge \omega := \lim_{\epsilon \to 0} \int_{Y \setminus N_\epsilon(D)} \eta \wedge \iota_Y^* \omega$, provided the limit converges for all $\omega \in \Gamma(\Omega_{X^\infty}^{2d-k-\ell})$.

Like $C^\infty$-forms, currents form a complex of acyclic sheaves with

$$\mathbb{H}^*(X, {}'\mathcal{D}_X^\bullet) \cong H^*(\Gamma(X, {}'\mathcal{D}_X^\bullet)) \cong H^*_{\mathrm{DR}}(X, \mathbb{C}),$$

under the differential defined by [15]

$$\int_X \mathrm{d}[S] \wedge \omega = (-1)^{1+\deg S} \int_X S \wedge \mathrm{d}\omega.$$

EXAMPLE 4.2. $d[d\log f] = 2\pi i \delta_{(f)}$ on $X$.

A different kind of example concerns us here, where we are not interested in 'residues' of this sort, and want to work 'away' from $V_{\mathbf{f}}$. If $\jmath\colon X \setminus V_{\mathbf{f}} \hookrightarrow X$ is the inclusion, $\jmath_! \Omega^{2d-m}_{(X\setminus V_{\mathbf{f}})^\infty}$ is the sheaf of $C^\infty$-forms compactly supported away from $V_{\mathbf{f}}$. Let ${}'\mathcal{D}^m_{V^\infty} \subseteq {}'\mathcal{D}^m_X$ be the subsheaf of currents annihilating these forms; these are the 'currents on $X$ supported on $V$' (*distinct* from currents ${}'\mathcal{D}^m_V$ 'on $V$'). Ignoring these we get currents on $X \setminus V$,

$$ {}'\mathcal{D}^m_{(X\setminus V)^\infty} := {}'\mathcal{D}^m_X / {}'\mathcal{D}^m_{V^\infty} = \mathcal{D}(\jmath_! \Omega^{2d-m}_{(X\setminus V)^\infty}). $$

These form a complex, and $\int_X S \wedge \omega$ induces a perfect pairing between

$$ H^m\{\Gamma(X, {}'\mathcal{D}^\bullet_{(X\setminus V)^\infty})\} \cong H^m(X \setminus V, \mathbb{C}), $$

and

$$ H^{2d-m}\{\Gamma(X, \jmath_! \Omega^\bullet_{(X\setminus V)^\infty})\} \cong H^{2d-m}_c(X \setminus V, \mathbb{C}). $$

EXAMPLE 4.3. $d[\log f] = d\log f - 2\pi i \delta_{T_f}$ on $\eta_X$, where $T_f$ is oriented so that $\partial T_f = (f)_0 - (f)_\infty$.

With the convention that the $\delta_{T_{f_i}}$'s anti-commute with the $d\log f_i$'s and the like, we may differentiate combinations of these exactly like forms (with regard to signs).

## 4.2. FORMALIZATION OF THE MAP $\{\mathbf{f}\} \mapsto R_{\mathbf{f}}$

Define a map

$$ R\colon \otimes^n \mathbb{Z}\{\mathbb{C}(X)^*\} \to \Gamma({}'\mathcal{D}^{n-1}_X) $$

induced by sending

$$ f_1 \otimes \cdots \otimes f_n = \mathbf{f} \mapsto R_{\mathbf{f}} := \sum_{i=1}^n (\pm 2\pi i)^{i-1} \log f_i \, d\log f_{i+1} \wedge \wedge \cdots \wedge d\log f_n \cdot \delta_{T_{f_1} \cap \cdots \cap T_{f_{i-1}}}, $$

where $\pm = (-1)^{n-1}$. Singularities are integrable even for $\mathbf{f}$ 'bad' so this makes sense as a current on all of $X$. Now modulo ${}'\mathcal{D}^n_{V_{\mathbf{f}}^\infty}$, applying d gives a collapsing sum:

$$ \begin{aligned} d[R_{\mathbf{f}}] &= \sum_{i=1}^n (\pm 2\pi i)^{i-1} d\log f_i \wedge \cdots \wedge d\log f_n \cdot \delta_{T_{f_1} \cap \cdots \cap T_{f_{i-1}}} - \\ &\quad - \sum_{i=1}^n (\pm 2\pi i)^i \, d\log f_{i+1} \wedge \cdots \wedge d\log f_n \cdot \delta_{T_{f_1} \cap \cdots \cap T_{f_i}} \\ &= d\log f_1 \wedge \cdots \wedge d\log f_n - (2\pi i)^n \delta_{T_{f_1} \cap \cdots \cap T_{f_n}} = \Omega_{\mathbf{f}} - (2\pi i)^n \delta_{T_{\mathbf{f}}}. \end{aligned} $$

Assuming $n > d$ or Tame$\{\mathbf{f}\} = 0$, $\Omega_{\mathbf{f}} = 0$ and $\partial^{-1}_{(X,V_{\mathbf{f}})} T_{\mathbf{f}} =: \zeta_{\mathbf{f}}$ exists, so that the $(n-1)$-current $R'_{\mathbf{f}} = R_{\mathbf{f}} + (2\pi \mathrm{i})^n \delta_{\zeta_{\mathbf{f}}}$ has $\mathrm{d}[R'_{\mathbf{f}}] \in \Gamma({}'\mathcal{D}^n_{V^\infty})$ and so is *closed* in $\Gamma({}'\mathcal{D}^\bullet_{(X\setminus V)^\infty})$. We get a class in $H^{n-1}(X\setminus V_{\mathbf{f}}, \mathbb{C})$ well defined up to classes generated by currents $\{(2\pi \mathrm{i})^n \delta_{\mathcal{C}} | [\mathcal{C}] \in H_{n-1}(X, V_{\mathbf{f}}; \mathbb{Q})\}$ as before.

PROPOSITION 4.4. *For $n > d$ the assignment $\mathbf{f} \mapsto R'_{\mathbf{f}}$ induces a well-defined map $R'\colon K^M_n(\mathbb{C}(X)) \to H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Q}(n))$.*

*Proof.* For $\mathbf{f}$ any Steinberg we must write $R'_{\mathbf{f}}$ (for some choice of $\zeta_{\mathbf{f}}$) as a coboundary $\mathrm{d}[S]$ in $\Gamma({}'\mathcal{D}^\bullet_{(X\setminus V_{\mathbf{f}})^\infty})$ – that is, modulo components with support in codimension $\geqslant 1$.

*Case 1:* $\mathbf{f} = f \otimes (1-f) \otimes \mathbf{g}$.

As $T_f \cap T_{1-f} \subset V_{\mathbf{f}}$, we may choose $\zeta_{\mathbf{f}} = 0$ and so

$$R'_{\mathbf{f}} = \log f \,\mathrm{dlog}(1-f) \wedge \Omega_{\mathbf{g}} + (-1)^{n-1} 2\pi \mathrm{i} \log(1-f)\Omega_{\mathbf{g}} \cdot \delta_{T_f}.$$

Now one may view all the polylogarithms as single-valued functions with branch cut at $T_{1-z} = [1,\infty] \subset \mathbb{R}^+$, and so as 0-currents on $\mathbb{P}^1$ with

$$\mathrm{d}[\mathrm{Li}_n(z)] = \mathrm{Li}_{n-1}(z)\,\mathrm{dlog}\, z + \frac{2\pi \mathrm{i}}{(n-1)!} \log^{n-1} z \cdot \delta_{T_{1-z}}.$$

(This formula follows from the discussion of monodromy in [16].) Setting $S = -\mathrm{Li}_2(1-f) \wedge \Omega_{\mathbf{g}}$, we have $\mathrm{d}[S] = R'_{\mathbf{f}}$.

*Case 2:* $\mathbf{f} = \mathbf{g} \otimes f_1 f_2 - \mathbf{g} \otimes f_1 - \mathbf{g} \otimes f_2$.

The difficulty here is the branch change: $\log f_1 + \log f_2 - \log f_1 f_2 =: 2\pi \mathrm{i}\delta_{\Delta_f} \neq 0$, for some $(2d-1)$-chain $\Delta_f$ with $\partial \Delta_f = T_{f_1 f_2} - T_{f_1} - T_{f_2}$. Since $\mathrm{dlog}\, f_1 f_2 = \mathrm{dlog}\, f_1 + \mathrm{dlog}\, f_2$, most terms cancel in $R_{\mathbf{f}} = R_{\mathbf{g}\otimes f_1 f_2} - R_{\mathbf{g}\otimes f_1} - R_{\gamma \otimes f_2} = \pm (2\pi \mathrm{i})^n \delta_{\Delta_f} \cdot \delta_{T_{\mathbf{g}}}$. By choosing $\zeta_{\mathbf{f}} = T_{\mathbf{g}} \cap \Delta_f$ we actually cancel this term and get $R'_{\mathbf{f}} = 0$ (no need for $S$).

*Case 3:* Alternation is more involved. We do increasing levels of difficulty.

$\mathbf{n=2}$: $\mathbf{f} = f \otimes g + g \otimes f$. On $X - V$

$$\begin{aligned} R'_{\mathbf{f}} &= \log f \,\mathrm{dlog}\, g - (2\pi\sqrt{-1}) \log g \cdot \delta_{T_f} + \\ &\quad + \log g \,\mathrm{dlog}\, f - (2\pi\sqrt{-1}) \log f \cdot \delta_{T_g} = \mathrm{d}[\log f \, \log g]. \end{aligned}$$

$\mathbf{n=3}$: $\mathbf{f} = f \otimes g \otimes h + g \otimes f \otimes h \Rightarrow R'_{\mathbf{f}} = \mathrm{d}[\log f \, \log g \,\mathrm{dlog}\, h]$

$$\mathbf{f} = f \otimes g \otimes h + f \otimes h \otimes g \Rightarrow R'_{\mathbf{f}} = \mathrm{d}[2\pi\sqrt{-1} \log g \, \log h \cdot \delta_{T_f}]$$

$$\begin{aligned} \mathbf{f} &= f \otimes g \otimes h + h \otimes g \otimes f \Rightarrow R'_{\mathbf{f}} = \log f \,\mathrm{dlog}\, g \wedge \mathrm{dlog}\, h + \\ &\quad + (2\pi\sqrt{-1}) \log g \,\mathrm{dlog}\, h \cdot \delta_{T_f} - 4\pi^2 \log h \cdot \delta_{T_f \cap T_g} + \\ &\quad + \log h \,\mathrm{dlog}\, g \wedge \mathrm{dlog}\, f + (2\pi\sqrt{-1}) \log g \,\mathrm{dlog}\, f \cdot \delta_{T_h} - \\ &\quad - 4\pi^2 \log f \cdot \delta_{T_h \cap T_g} \\ &= \mathrm{d}\big[ -\log f \log h \,\mathrm{dlog}\, g + 2\pi\sqrt{-1} \log f \log g \cdot \delta_{T_h} + \\ &\quad + 2\pi\sqrt{-1} \log h \log g \cdot \delta_{T_f} \big]. \end{aligned}$$

**n > 3**: $\mathbf{f} = f_1 \otimes \cdots \otimes f_i \otimes \cdots \otimes f_j \otimes \cdots \otimes f_n + f_1 \otimes \cdots \otimes f_j \otimes \cdots \otimes f_i \otimes \cdots \otimes f_n$. Then on $X \backslash V_{\mathbf{f}}$, $R'_{\mathbf{f}} = \mathrm{d}[S]$ (choosing $\partial^{-1}_{(X,V)} T_{\mathbf{f}} = \partial^{-1} 0 = 0$) where

$$\begin{aligned} S_{ij} = {} & (2\pi\sqrt{-1})^{i-1} \log f_i \log f_j \,\mathrm{dlog}\, f_{i+1} \\ & \wedge \cdots \wedge \mathrm{dlog}\, f_{j-1} \wedge \mathrm{dlog}\, f_{j+1} \wedge \cdots \wedge \mathrm{dlog}\, f_n \cdot \delta_{T_{f_1} \cap \cdots \cap T_{f_{i-1}}} + \\ & + (2\pi\sqrt{-1})^{i} [\log f_i \log f_{i+1} \,\mathrm{dlog}\, f_{i+2} \wedge \cdots \wedge \mathrm{dlog}\, f_{j-1} \wedge \\ & \wedge \mathrm{dlog}\, f_{j+1} \wedge \cdots \wedge \mathrm{dlog}\, f_n \cdot \delta_{T_{f_1} \cap \cdots \cap T_{f_{i-1}} \cap T_{f_j}} + \\ & + \log f_j \log f_{i+1} \,\mathrm{dlog}\, f_{i+2} \wedge \cdots \wedge \mathrm{dlog}\, f_{j-1} \wedge \mathrm{dlog}\, f_{j+1} \\ & \wedge \cdots \wedge \mathrm{dlog}\, f_n \cdot \delta_{T_{f_1} \cap \cdots \cap T_{f_{i-1}} \cap T_{f_i}}] + \cdots + \\ & + (2\pi\sqrt{-1})^{j-2} [\log f_i \log f_{j-1} \,\mathrm{dlog}\, f_{j+1} \\ & \wedge \cdots \wedge \mathrm{dlog}\, f_n \cdot \delta_{T_{f_1} \cap \cdots \cap T_{f_{i-1}} \cap T_{f_j} \cap T_{f_{i+1}} \cap \cdots \cap T_{f_{j-2}}} \times \\ & \times \log f_j \log f_{j-1} \,\mathrm{dlog}\, f_{j+1} \\ & \wedge \cdots \wedge \mathrm{dlog}\, f_n \cdot \delta_{T_{f_1} \cap \cdots \cap T_{f_{i-1}} \cap T_{f_i} \cap T_{f_{i+1}} \cap \cdots \cap T_{f_{j-2}}}]. \qquad \square \end{aligned}$$

*Remarks.* (i) Let $n$ and $d$ be arbitrary. Clearly the proof also shows that $R'$ induces a map $\{\ker(\mathrm{Tame}) \subseteq K_n^M(\mathbb{C}(X))\} \to H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Q}(n))$. On the other hand, to induce a map on all of $K_n^M$ one needs the *triple*⋆ $((2\pi \mathrm{i})^n T_{\mathbf{f}}, \Omega_{\mathbf{f}}, R_{\mathbf{f}})$. The resulting general version of the Milnor regulator, $K_n^M(\mathbb{C}(X)) \to H^n_{\mathcal{D}}(\eta_X, \mathbb{Z}(n))$, is also the form *in which it is compatible with products*. That is, $K_n^M(\mathbb{C}(X)) \otimes K_m^M(\mathbb{C}(X)) \to K_{n+m}^M(\mathbb{C}(X))$ *never* corresponds to simply wedging together $R'_{\mathbf{f}}$'s — you have to use the multiplication table (in [8] or [19]) for computing $H^n_{\mathcal{D}}(X, \mathbb{Z}(n)) \otimes H^m_{\mathcal{D}}(X, \mathbb{Z}(m)) \to H^{n+m}_{\mathcal{D}}(X, \mathbb{Z}(n+m))$. This boils down to the formula $R_{\mathbf{f} \otimes \mathbf{g}} = R_{\mathbf{f}} \wedge \Omega_{\mathbf{g}} \pm (2\pi \mathrm{i})^n R_{\mathbf{g}} \cdot \delta_{T_{\mathbf{f}}}$ in general.

(ii) Composition $R'$ with the ‘real’ projection $H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Q}(n)) \to H^{n-1}(\eta_X, \mathbb{R}(n-1))$ gives a map $r$: $K_n^M(\mathbb{C}(X)) \to H^{n-1}(\eta_X, \mathbb{R})$, corresponding to the real (imaginary) part of $R'_{\mathbf{f}}$ for $n$ odd (even). [Note that the ambiguous membrane term is killed.] This regulator $r$ agrees (up to factors of $(2\pi \mathrm{i})$) with the induced by the map in [11, Section 2.2], see [18, Section 3.1]. Composing $r$ with further projection to $(\{H^{n-1,0}(X, \mathbb{C}) \oplus H^{0,n-1}(X, \mathbb{C})\} \cap H^{n-1}(X, \mathbb{R}))^\vee$ on the right, and on the left with $H^0(X, K^M_{n,X}) \overset{(\cong_{\otimes \mathbb{Q}})}{\to} \ker(\mathrm{Tame})$ $[\subseteq K_n^M(\mathbb{C}(X))]$ (see [27]), induces the real regulator in [23] (for the case $m$, $k = n$).

⋆See Section 6.1 (and [19, Section 5.9]). Technically for $d \geqslant n$ one needs to replace $X \backslash V_{\mathbf{f}}$ by a normal-crossings pair $\tilde{X} \backslash \tilde{V}_{\mathbf{f}}$ (just for the map on *all* of $K_n^M$). We have carefully pursued lines of reasoning (and cases) in this paper that avoid this complication.

# PART 2. RESIDUES AND POLYLOGARITHMS

## 5. Higher Residues for Currents

### 5.1. DEFINITION OF $\overline{\mathrm{Res}}^i$

Following a suggestion in [9], we now construct a local–global spectral picture which will aid in describing the 'residues' of the Milnor currents. We must avoid the assumption of normal crossings, since even for good $\mathbf{f}$ the situation may not be this nice.

Consider the following inclusions for the '$\mathbf{f}$-substrata' $V_{\mathbf{f}}^k$:

$$\jmath^{(k)}\colon X\backslash V_{\mathbf{f}} \hookrightarrow X, \qquad \jmath^{(k,k+1)}\colon V_{\mathbf{f}}^k\backslash V_{\mathbf{f}}^{k+1} \hookrightarrow V_{\mathbf{f}}^k,$$
$$\iota^{(k)}\colon V_{\mathbf{f}}^k \hookrightarrow X, \qquad \iota^{(k,k+1)}\colon V_{\mathbf{f}}^{k+1} \hookrightarrow V_{\mathbf{f}}^k,$$

where $V_{\mathbf{f}}^0 := X$ and $V_{\mathbf{f}}^1 := V_{\mathbf{f}}$. Pushing forward by $\iota$ we have the exact sequence of sheaves

$$\begin{array}{ccccl} 0 \to \iota_*^{(k+1)}\mathbb{C} \to \iota_*^{(k)}\mathbb{C} \to & \iota_*^{(k)}\mathbb{C}/\iota_*^{(k+1)}\mathbb{C} & \to 0 \\ & (2\pi \mathrm{i})^k\cdot\iota_*^{(k)} \uparrow\cong & \\ & \jmath_*^{(k,k+1)}\mathbb{C} & \end{array}$$

in which we may resolve terms by

$$\iota_*^{(k)}\mathbb{C} \underset{(\simeq)}{\to} {}'\mathcal{D}^\bullet_{(V_{\mathbf{f}}^k)^\infty}, \qquad \begin{array}{cc} \iota_*^{(k)}\mathbb{C}/\iota_*^{(k+1)}\mathbb{C} \underset{(\simeq)}{\to} & {}'\mathcal{D}^\bullet_{(V_{\mathbf{f}}^k)^\infty}/{}'\mathcal{D}^\bullet_{(V_{\mathbf{f}}^{k+1})^\infty} \\ & (2\pi \mathrm{i})^k\cdot\iota_*^{(k)} \uparrow\simeq \\ & {}'\mathcal{D}^\bullet_{(V_{\mathbf{f}}^k\backslash V_{\mathbf{f}}^{k+1})^\infty}[-2k]. \end{array}$$

Here

$${}'\mathcal{D}^\bullet_{(V_{\mathbf{f}}^k\backslash V_{\mathbf{f}}^{k+1})^\infty} :={}' \mathcal{D}^\bullet_{(V_{\mathbf{f}}^k)}/\{{}'\mathcal{D}^\bullet_{(V_{\mathbf{f}}^{k+1})^\infty} \subset {}'\mathcal{D}^\bullet_{(V_{\mathbf{f}}^k)}\}$$

are (quotients of) currents on $V_{\mathbf{f}}^k$ – *not* currents on $X$ with support on $V_{\mathbf{f}}^k$, which is what they are mapping to via $\iota_*^{(k)}$ (essentially multiplication by $\delta_{V_{\mathbf{f}}^k}$). The $(2\pi \mathrm{i})^k$ twist will make sense later. The short-exact sequence reflects a descending filtration by coniveau

$$F_N^{\mathsf{p}}\mathbb{C} := \iota_*^{(\mathsf{p})}\mathbb{C}, \qquad Gr_N^{\mathsf{p}}\mathbb{C} = \frac{F_N^{\mathsf{p}}}{F_N^{\mathsf{p}+1}} \cong \jmath_*^{(\mathsf{p},\mathsf{p}+1)}\mathbb{C},$$

and from the acyclic resolutions we get the initial 'exact triangles'

$$\begin{array}{ccl} \cdots \overset{\mathrm{d}}{\to} H^*_{V_{\mathbf{f}}^{k+1}}(X,\mathbb{C}) \to H^*_{V_{\mathbf{f}}^k}(X,\mathbb{C}) \to & H^*_{V_{\mathbf{f}}^k\backslash V_{\mathbf{f}}^{k+1}}(X,\mathbb{C}) & \overset{\mathrm{d}}{\to} \cdots \\ & (2\pi \mathrm{i})^k\cdot\iota_*^{(k)} \uparrow\cong & \\ & H^{*-2i}(V_{\mathbf{f}}^k\backslash V_{\mathbf{f}}^{k+1},\mathbb{C}). & \end{array} \tag{5.1}$$

for the corresponding spectral sequence.

The $E_1$-term is then[⋆]

$$\begin{aligned} E_1^{\mathsf{p},-\mathsf{q}}(n) &:= H^{2n+\mathsf{p}-\mathsf{q}-1}\{\Gamma(X, Gr_N^{\mathsf{p}}\mathbb{C})\} \\ &= H^{2n+\mathsf{p}-\mathsf{q}-1}\{\Gamma(X, {}'\mathcal{D}^\bullet_{(V_{\mathbf{f}}^{\mathsf{p}})\infty}/{}'\mathcal{D}^\bullet_{(V_{\mathbf{f}}^{\mathsf{p}+1})\infty})\} \\ &\xleftarrow[(2\pi \mathrm{i})^{\mathsf{p}}\cdot \iota_*^{(\mathsf{p})}]{\cong} H^{2n-(\mathsf{p}+\mathsf{q})-1}\{\Gamma(V_{\mathbf{f}}^{\mathsf{p}}, {}'\mathcal{D}^\bullet_{(V_{\mathbf{f}}^{\mathsf{p}}\backslash V_{\mathbf{f}}^{\mathsf{p}+1})\infty})\} \\ &\cong H^{2n-(\mathsf{p}+\mathsf{q})-1}(V_{\mathbf{f}}^{\mathsf{p}}\backslash V_{\mathbf{f}}^{\mathsf{p}+1}, \mathbb{C}) \\ &\Longrightarrow Gr_N^{\mathsf{p}} H^{2n+\mathsf{p}-\mathsf{q}-1}(X, \mathbb{C}) =: E_\infty^{\mathsf{p},-\mathsf{q}}. \end{aligned}$$

We use the $\mathrm{d}_i\colon E_i^{0,-n}(n) \to E_i^{i,-n-i+1}(n)$ in this spectral sequence to get 'residue' maps

$$\begin{aligned} &\mathrm{Res}^i\colon \{\ker(\mathrm{Res}^{i-1}) \subseteq H^{n-1}(X\backslash V_{\mathbf{f}}, \mathbb{C})\} \\ &\quad \to \{\text{subquotient of } H^{n-2i}(V_{\mathbf{f}}^i\backslash V_{\mathbf{f}}^{i+1}, \mathbb{C})\}. \end{aligned}$$

Replacing the left-hand column $E_1^{0,\mathsf{q}}(n) \cong H^{2n-\mathsf{q}}(X\backslash V_{\mathbf{f}}, \mathbb{C})$ by zeroes, one has a spectral sequence ${}'E$ converging to ${}'E_\infty^{\mathsf{p},-\mathsf{q}} \cong Gr_N^{\mathsf{p}} H_{V_{\mathbf{f}}}^{2n+\mathsf{p}-\mathsf{q}-1}(X, \mathbb{C})$; a simple algebraic argument then gives the long-exact sequence

$$\to H^*_{V_{\mathbf{f}}}(X) \xrightarrow[\iota_*^{(1)}]{\mathrm{Gy}} H^*(X) \xrightarrow[\jmath^*_{(1)}]{} H^*(X\backslash V_{\mathbf{f}}) \xrightarrow{\mathrm{Res}} H^{*+1}_{V_{\mathbf{f}}}(X) \to$$

in which the $N$-graded pieces of Res are exactly the $\mathrm{Res}^i$.

PROPOSITION 5.1. *If successive* $\mathrm{Res}^i[R'_{\mathbf{f}}]$ *are all trivial*, *then* $[R'_{\mathbf{f}}]$ *comes from* $H^*(X)$.

This means $(a)$ $\int_X R'_{\mathbf{f}}\wedge(\cdot)$ gives a well-defined functional on $H_c^*(X\backslash V_{\mathbf{f}})/\ker(\jmath_*^{(1)})$ and $(b)$ $R'_{\mathbf{f}}$ may be 'completed' to a closed current on all of $X$ by adding currents supported on the $V_{\mathbf{f}}^{i\geqslant 1}$.

In order to work modulo the choice of membrane $\zeta_{\mathbf{f}}$, we need to modify the spectral sequence for $\mathbb{C}/\mathbb{Q}(n)$ coefficients. Just as $\Gamma(X, {}'\mathcal{D}^\bullet_{(V_{\mathbf{f}}^p)\infty})$ computed $H^*(X, F_N^{\mathsf{p}}\mathbb{C})$ above, cohomologies of $F_N^{\mathsf{p}}(\mathbb{C}/\mathbb{Q}(n))$ are given by taking $H^*$ of

$$\begin{aligned} &\mathrm{Cone}\big\{C_{2d-\bullet}^{V_{\mathbf{f}}^{\mathsf{p}}}(X, \mathbb{Q}(n)) \xrightarrow{\delta_{(\cdot)}} \Gamma(X, {}'\mathcal{D}^\bullet_{(V_{\mathbf{f}}^{\mathsf{p}})\infty})\big\} \\ &\quad \xleftarrow[(2\pi \mathrm{i})^{\mathsf{p}}\cdot\iota_*^{(\mathsf{p})}]{\simeq} \mathrm{Cone}\big\{C_{2d-\bullet}(V_{\mathbf{f}}^{\mathsf{p}}, \mathbb{Q}(n-\mathsf{p})) \xrightarrow{\delta_{(\cdot)}} \Gamma(V_{\mathbf{f}}^{\mathsf{p}}, {}'\mathcal{D}^\bullet_{V_{\mathbf{f}}^{\mathsf{p}}})\big\}. \end{aligned} \tag{5.2}$$

[⋆]Similarly, the choice of indices will also make sense later.

The corresponding $\overline{\mathrm{Res}}^i$ map from $\ker(\overline{\mathrm{Res}}^{i-1}) \subseteq H^{n-1}(X\backslash V_{\mathbf{f}}, \mathbb{C}/\mathbb{Q}(n))$ to a subquotient of $H^{n-2i}(V_{\mathbf{f}}^i\backslash V_{\mathbf{f}}^{i+1}, \mathbb{C}/\mathbb{Q}(n-i))$.

COROLLARY 5.2. *If the* $\overline{\mathrm{Res}}^i[R'_{\mathbf{f}}]$ *are trivial, then a suitable modification of the choice of* $\zeta_{\mathbf{f}}$ *gives an* $R''_{\mathbf{f}}$ *completable to a closed section* $\in \Gamma('\mathcal{D}_X^{n-1})$. *All* $\mathrm{Res}^i[R''_{\mathbf{f}}]$ *are zero.*

*Remark.*

(i) We frequently work with $d(= \dim X) = n-1$. To get nontrivial $\overline{\mathrm{Res}}^i$ in this case, we must take $d \geqslant 2i$.
(ii) By similar reasoning, if $\overline{\mathrm{Res}}^i[R'_{\mathbf{f}}] \equiv 0$ for all $i \leqslant r$, $\exists R''_{\mathbf{f}}$ with $\mathrm{Res}^i[R''_{\mathbf{f}}] = 0$ for all $i \leqslant r$.

### 5.2. INTERPRETATION OF $\mathrm{Res}^1$

This is essentially just d[·]. Tracing through $d_1$ in the spectral sequence, $[R'_{\mathbf{f}}] \in H^{n-1}(X\backslash V_{\mathbf{f}}, \mathbb{C})$ lifts to a closed section $R'_{\mathbf{f}} \in \Gamma(X, '\mathcal{D}_X^{n-1}/'\mathcal{D}^{n-1}_{(V_{\mathbf{f}})^\infty})$, then to a *non*-closed section $\in \Gamma(X, '\mathcal{D}_X^{n-1})$, so that $\mathrm{d}[R'_{\mathbf{f}}] \in \Gamma(X, '\mathcal{D}^n_{(V_{\mathbf{f}})^\infty}) \to \Gamma(X, '\mathcal{D}^n_{(V_{\mathbf{f}})^\infty}/ '\mathcal{D}^n_{(V_{\mathbf{f}}^2)^\infty})$. Modulo currents exact in the last complex, the image (of $\mathrm{d}[R'_{\mathbf{f}}]$) lifts to $(1/2\pi\mathrm{i})\cdot \mathrm{d}[R'_{\mathbf{f}}] \in \Gamma(V, '\mathcal{D}^{n-2}_{V_{\mathbf{f}}}/'\mathcal{D}^{n-2}_{(V_{\mathbf{f}}^2)^\infty})$ under $(2\pi\mathrm{i})\cdot\iota_*^{(1)}$; then $\mathrm{Res}^1[R'_{\mathbf{f}}]$ is the class of the latter, in $H^{n-2}(V_{\mathbf{f}}\backslash V_{\mathbf{f}}^2, \mathbb{C})$.

If $\mathbf{f} = f_1 \otimes \cdots \otimes f_n$ is good, then in $\Gamma('\mathcal{D}_X^\bullet)$

$$\begin{aligned}
\mathrm{d}[R_{\mathbf{f}}] &= \mathrm{d}\left[\sum_{k=1}^{n} (\pm 2\pi\mathrm{i})^{k-1} \log f_k \,\mathrm{dlog}\, f_{k+1} \wedge \cdots \wedge \mathrm{dlog}\, f_n \cdot \delta_{T_{f_1}\cap\cdots\cap T_{f_{k-1}}}\right] \\
&= \sum_{\ell=1}^{n}\left[\sum_{k=1}^{\ell-1} (\pm 2\pi\mathrm{i})^{k-1} 2\pi\mathrm{i}(-1)^{\ell-k-1} \log f_k \,\mathrm{dlog}\, f_{k+1} \wedge\cdots\wedge \times\right. \\
&\quad \times \widehat{\mathrm{dlog}\, f_\ell} \wedge \cdots \wedge \mathrm{dlog}\, f_n \cdot \delta_{T_{f_1}\cap\cdots\cap T_{f_{k-1}}} + \sum_{k=\ell+1}^{n} (\pm 2\pi\mathrm{i})^{k-1} \times \\
&\quad \times (-1)^{n+\ell-k-1} \log f_{k+1} \,\mathrm{dlog}\, f_{k+2} \wedge \cdots \wedge \mathrm{dlog}\, f_n \cdot \\
&\quad \left.\cdot\, \delta_{T_{f_1}\cap\cdots\cap \widehat{T_{f_\ell}} \cap \cdots \cap T_{f_{k-1}}}\right] \cdot \delta_{(f_\ell)} \pm (2\pi\mathrm{i})^n \delta_{T_f} \\
&= 2\pi\mathrm{i} \sum_{\ell=1}^{n} (-1)^\ell R_{f_1\otimes\cdots\otimes \hat{f}_\ell \otimes\cdots\otimes f_n} \cdot \delta_{(f_\ell)} \pm (2\pi\mathrm{i})^n \delta_{T_f}.
\end{aligned}$$

Now writing $V_{\mathbf{f}} = \cup\{|(f_\ell)| =: V_\ell\}$, $W_\ell := V_\ell \cap V_{\mathbf{f}}^2$, the boundary of the membrane is

$$\partial_X(\partial^{-1}_{(X,V_{\mathbf{f}})} T_{\mathbf{f}}) = T_{\mathbf{f}} - \sum_\ell (-1)^\ell \partial^{-1}_{(V_\ell, W_\ell)} T_{\partial_\ell \mathbf{f}},$$

and so

$$\mathrm{d}[R'_{\mathbf{f}}] = 2\pi\mathrm{i}\sum_{\ell=1}^{n}(-1)^{\ell} R'_{\partial_\ell \mathbf{f}} \cdot \delta_{(f_\ell)};$$

dividing by $2\pi\mathrm{i}$ gives $\mathrm{Res}^1$.

Noticing that

$$\{\mathbf{g}\} = 0 \Longrightarrow R'_{\mathbf{g}} = \mathrm{d}[S] \text{ on } X\backslash V_{\mathbf{f}} \Longrightarrow \text{all } \mathrm{Res}^i[R'_{\mathbf{g}}] = 0$$

then gives the

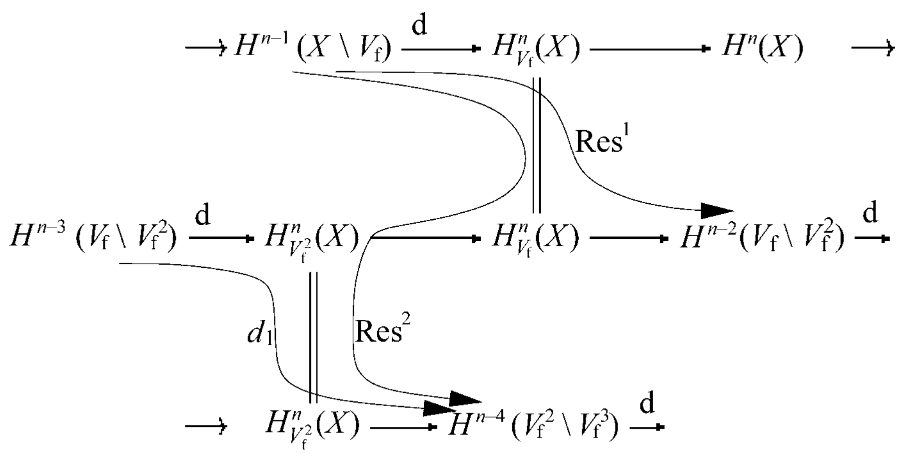
PROPOSITION 5.3. *For $n > d$ we have a commutative diagram*

$$\begin{array}{ccc}
K_n^M(\mathbb{C}(X)) & \xrightarrow{R'} & H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Q}(n)) \\
\Big\downarrow \text{Tame} & & \Big\downarrow \overline{\mathrm{Res}}^1 \\
\coprod_{x \in X^1} K_{n-1}^M(\mathbb{C}(x)) & \xrightarrow{\coprod R'} & \coprod_{x\in X^1} H^{n-2}(\eta_x, \mathbb{C}/\mathbb{Q}(n-1)),
\end{array}$$

*and for all $n$, $d\{\mathbf{f}\} \in \ker(\mathrm{Tame}) \Longrightarrow \overline{\mathrm{Res}}^1[R'_{\mathbf{f}}] \equiv 0$.*

## 5.3. INTERPRETATION OF $\mathrm{Res}^2$

This is defined when codimension-1 residues are trivial. From the remark to Corollary 5.2, $\overline{\mathrm{Res}}^1[R'_{\mathbf{f}}] \equiv 0 \Longrightarrow \mathrm{Res}^1[R''_{\mathbf{f}}] = 0 \Longrightarrow \mathrm{d}[R''_{\mathbf{f}}]$ exact in $\Gamma(X, {}'\mathcal{D}^{\bullet}_{(V_{\mathbf{f}})^\infty} / {}'\mathcal{D}^{\bullet}_{(V_{\mathbf{f}}^2)^\infty})$. That is, $\exists S \in \Gamma(X, {}'\mathcal{D}^{n-1}_{(V_{\mathbf{f}})^\infty})$ such that $\mathrm{d}[R''_{\mathbf{f}}] = \mathrm{d}[S]$ modulo ${}'\mathcal{D}^{n}_{(V_{\mathbf{f}}^2)^\infty}$. Now to see how to compute $\mathrm{Res}^2[R''_{\mathbf{f}}]$, we paste together exact triangles (5.1):



First use $\mathrm{d}[S]$ to 'move' support of $\mathrm{d}[R''_{\mathbf{f}}]$ to $V^2_{\mathbf{f}}$, obtaining $\mathrm{d}[R''_{\mathbf{f}} - S] \in \Gamma(X, '\mathcal{D}^n_{(V^2_{\mathbf{f}})^\infty}) \to \Gamma\big(X, '\mathcal{D}^n_{(V^2_{\mathbf{f}})^\infty}/'\mathcal{D}^n_{(V^3_{\mathbf{f}})^\infty}\big)$; then move the image by a coboundary and lift via $(2\pi\mathrm{i})^2 \cdot \iota^{(2)}_*$ to

$$\frac{-1}{4\pi^2}\mathrm{d}[\widetilde{R''_{\mathbf{f}} - S}] \in \Gamma(V^2_{\mathbf{f}}, '\mathcal{D}^{n-4}_{V^2_{\mathbf{f}}}/'\mathcal{D}^{n-4}_{(V^3_{\mathbf{f}})^\infty}),$$

finally going modulo $\mathrm{im}(d_1)$ (and exact currents).

EXAMPLE 5.4. Take the case $n = 4$, $X =$ a 3-fold, $\mathbf{f}$ good $\in \ker(\mathrm{Tame})$. For simplicity we restrict to a Zariski-open neighborhood $U \subseteq X$ of a normal crossing in $V_{\mathbf{f}}$, so that $V_{\mathbf{f}} = V_1 \cup V_2$, $V^2_{\mathbf{f}} = V_1 \cap V_2 =: W$. Now $R''_{\mathbf{f}}$ is closed in codimension 0, and $\mathrm{Tame}\{\mathbf{f}\} = 0 \Longrightarrow \mathrm{d}[R''_{\mathbf{f}}]$ is exact ($= \mathrm{d}[S_j]$ on $V_j$) in codimension 1. So $R''_{\mathbf{f}} - S_j$ (on $V_j$) is closed in codimension 1, and $(1/4\pi^2)\,\mathrm{d}[S_j - R''_{\mathbf{f}}] = (1/4\pi^2)\,\mathrm{d}[S_j]$ is our answer in codimension 2. Intuitively

$$\mathrm{Res}^2[R''_{\mathbf{f}}] = \mathrm{Res}_W\{\mathrm{d}^{-1}(\mathrm{Res}_{V_1}[R''_{\mathbf{f}}]) - \mathrm{d}^{-1}(\mathrm{Res}_{V_2}[R''_{\mathbf{f}}])\}$$

modulo $\mathrm{Res}_W$ of currents on $V_j$ closed in codimension 1; this formula is the key to understanding what these higher residues are.

Notice that if $S_j$ has terms of the form $2\pi\mathrm{i}\,\mathrm{Li}_2(f_j)\,\mathrm{dlog}\, g_j \cdot \delta_{V_j}$ [which one expects from *Case 1*, proof of Proposition 4.4] and $g_j = 0$ or $\infty$ at $W$, then the 0-current $\mathrm{Res}^2[R''_{\mathbf{f}}]$ has terms $\mathrm{Li}_2(f_j)$ on $W$. One expects a certain rigidity here: namely, $\mathrm{Res}^2[R''_{\mathbf{f}}]$ must actually (up to $4\pi^2\mathbb{Q}$ 'jumps') be constant on $W$. That is because we have just locally traced through the composition

$$\begin{array}{ccccccc} \ker(\mathrm{Tame}) & \xrightarrow{R'} & \ker(\overline{\mathrm{Res}}^1) & \xrightarrow{\overline{\mathrm{Res}}^2} & \dfrac{\coprod_{y\in X^2} H^0(\eta_y, \mathbb{C}/\mathbb{Q}(2))}{\mathrm{im}(d_1)} & = & \dfrac{\coprod_{y\in X^2} \mathbb{C}/\mathbb{Q}(2)}{\mathrm{im}(d_1)}. \\ \cap & & \cap & & & & \\ K^M_4(\mathbb{C}(X)) & & H^3(\eta_X, \mathbb{C}/\mathbb{Q}(4)) & & & & \end{array}$$

Logically, the next step should be to define 'Tame$^2$' on $\ker(\mathrm{Tame}) \subseteq K^M_4(\mathbb{C}(X))$, so as to get a diagram for $\mathrm{Res}^2$ analogous to Proposition 5.3. First recall the Bloch group $\mathcal{B}_2(\mathbb{C}(X)) := \mathbb{Z}\{\mathbb{C}(X)\backslash\{0, 1, \infty\}\}$ modulo the 'dilogarithm' relations*

$$[f] - [g] + [g/f] - [(1-g)/(1-f)] + [(1-g^{-1})/(1-f^{-1})],$$

and st: $\mathcal{B}_2(\mathbb{C}(X)) \to \bigwedge^2 \mathbb{C}(X)^*$ maps $\{f\}_2 \mapsto (1-f) \wedge f$. Clearly

$$K^M_2(\mathbb{C}(X)) = \mathrm{coker}(\mathrm{st}),$$

$$K^M_3(\mathbb{C}(X)) = \mathrm{coker}\Big\{\mathrm{st} \otimes \mathrm{id}\colon \mathcal{B}_2(\mathbb{C}(X)) \otimes \mathbb{C}(X)^* \to \bigwedge^3 \mathbb{C}(X)^*\Big\};$$

let us also suppose the existence of $T\colon \bigwedge^4 \mathbb{C}(X)^* \to \coprod_{x\in X^1} \bigwedge^3 \mathbb{C}(X)^*$ inducing Tame.

*Note: these are relations on $\mathcal{L}_2(x) = \Im\{\mathrm{Li}_2(x)\} + \log|x|\arg(1-x)$.

Now start by lifting $\{\mathbf{f}\} \in \ker(\mathrm{Tame}) \subseteq K_4^M(\mathbb{C}(X))$ to $\tilde{\mathbf{f}} \in \bigwedge^4 \mathbb{C}(X)^*$, and assume one can trace through a diagram like the following (with $T \circ T = 0$)

$$\begin{array}{ccccc}
\bigwedge^4 \mathbb{C}(X)^* & \xrightarrow{T} & \coprod_{x\in X^1} \bigwedge^3 \mathbb{C}(x)^* & \xrightarrow{T} & \coprod_{y\in X^2} \bigwedge^2 \mathbb{C}(y)^* \\
 & & \uparrow \mathbf{st}\otimes \mathbf{id} & & \uparrow \mathbf{st} \\
 & & \coprod_{x\in X^1} \mathcal{B}_2(\mathbb{C}(x)) \otimes \mathbb{C}(x)^* & \xrightarrow{T} & \coprod_{y\in X^2} \mathcal{B}_2(\mathbb{C}(y))
\end{array}$$

to get an element $\sum a_i\{g_i\}_2 =: \mathrm{Tame}^2\{\mathbf{f}\} \in \ker(\mathrm{st})$. It certainly seems plausible that (for some variant $\widetilde{\mathrm{Li}}_2$ of the dilogarithm)

$$\mathrm{Res}_y^2[R'_{\mathbf{f}}] \stackrel{?}{=} \widetilde{\mathrm{Li}}_2(\mathrm{Tame}_y^2\{\mathbf{f}\}), \tag{5.3}$$

especially in light of the above 'rigidity' (as ker(st) is generated by algebraic numbers, see [29]). Similar arguments suggest an $i$-logarithmic behavior for $\mathrm{Res}^i[R'_{\mathbf{f}}]$.

These suggestions are not entirely wrong, but (except perhaps for individual computations) the approach outlined here would not work. To prove something like (5.3), one would need maps directly from the groups in the diagram, to corresponding currents. This is fine for Goncharov's real currents but ours are only well defined on the level $\otimes^4\mathbb{Z}\{\mathbb{C}(X)^*\}$ and so on. Moreover, while there is no problem with defining the first $T$, issues involving norm maps (encountered in the passage from codimension 1 to 2, see [12] or [18]) have so far proved fatal for defining the other two.

## 6. Higher Residues for $K_n^M$

### 6.1. AJ FOR HIGHER CHOW GROUPS

Since one therefore cannot define the *higher* residue maps for Milnor $K$-theory on the level of functions, we are forced to use the higher Chow complex.

Recall that on $\square^n$ we have $T_n = T_{z_1} \cap \cdots \cap T_{z_n} \in C_n(\square^n)$, $\Omega_n = \mathrm{dlog}\, z_n \wedge \cdots \wedge \mathrm{dlog}\, z_n \in \Gamma(F^n{}'\mathcal{D}^n_{\square n})$, and set $R_n = R_{z_1\otimes\cdots\otimes z_n} \in \Gamma('\mathcal{D}^{n-1}_{\square n})$. Given $\mathcal{Z} \in Z^p(X, n)$ [$X$ smooth projective / $\mathbb{C}$] with irreducible components $\mathcal{Z}_j$, let $\mathcal{Z}_* := \sum \pi_{X\,*}^{\mathcal{Z}_j} \circ \pi_{\square}^{\mathcal{Z}_j *}$; and define

$$R_{\mathcal{Z}} := \mathcal{Z}_* R_n \in \Gamma('\mathcal{D}_X^{2p-n-1}), \qquad \Omega_{\mathcal{Z}} := \mathcal{Z}_*\Omega_n \in \Gamma(F^{p\prime}\mathcal{D}_X^{2p-n}),$$

and

$$T_{\mathcal{Z}} := \pi_X((X \times T_n) \cap \mathcal{Z}) \in C_{2d-2p+n}(X, \mathbb{Z}).$$

Note that $\pi^{\mathcal{Z}_j}_{X\,*}$ involves 'fiberwise integration' when fiber dimension of $\mathcal{Z}_i \to X$ is $\geqslant 1$.

Cohomology of the complex*

$$C^{\bullet}_{\mathcal{D}}(X,\mathbb{Z}(p)) := \mathrm{Cone}\{C_{2d-\bullet}(X,\mathbb{Z}(p)) \oplus \Gamma(X, F^{p\prime}\mathcal{D}^{\bullet}_X) \to \Gamma(X, {}'\mathcal{D}^{\bullet}_X)\}[-1]$$

computes Deligne cohomology: $H^*(C^{\bullet}_{\mathcal{D}}(X,\mathbb{Z}(p))) \cong H^*_{\mathcal{D}}(X,\mathbb{Z}(p))$. Since

$$\begin{aligned}&\partial T_{\mathcal{Z}} = T_{\partial_{\mathcal{B}}\mathcal{Z}}, \qquad \mathrm{d}[\Omega_{\mathcal{Z}}] = \Omega_{\partial_{\mathcal{B}}\mathcal{Z}}, \quad \text{and}\\ &\mathrm{d}[R_{\mathcal{Z}}] = -2\pi\mathrm{i}R_{\partial_{\mathcal{B}}\mathcal{Z}} + \Omega_{\mathcal{Z}} - (2\pi\mathrm{i})^n\delta_{T_{\mathcal{Z}}},\end{aligned}$$

sending (for $-\bullet = n$)

$$\mathcal{Z} \longmapsto (-2\pi\mathrm{i})^{p-n}((2\pi\mathrm{i})^n T_{\mathcal{Z}}, \Omega_{\mathcal{Z}}, R_{\mathcal{Z}}) =: \mathcal{R}(\mathbb{Z})$$

induces a map of complexes

$$\mathcal{R}\colon Z^p(X,-\bullet) \longrightarrow C^{2p+\bullet}_{\mathcal{D}}(X,\mathbb{Z}(p)),$$

and thus

$$\mathrm{AJ}^{p,n}_X\colon \mathrm{CH}^p(X,n) \longrightarrow H^{2p-n}_{\mathcal{D}}(X,\mathbb{Z}(p)).$$

*Remark.*

(i) Whenever $\Omega_{\mathcal{Z}} = 0$ (e.g. $p > d$, $p < n$) and $\partial_{\mathcal{B}}\mathcal{Z} = 0$, $\mathrm{d}[R_{\mathcal{Z}}] = (2\pi\mathrm{i})^n\delta_{T_{\mathcal{Z}}} \Longrightarrow$ $\exists$ topological $\mathbb{Q}$-chain $\zeta$ with $\partial\zeta = T_{\mathcal{Z}}$. So $R_{\mathcal{Z}} - (2\pi\mathrm{i})^n\delta_{\zeta} =: R'_{\mathcal{Z}}$ is a closed $(2p-n-1)$-current and $[(1/(-2\pi\mathrm{i})^{n-p})R'_{\mathcal{Z}}] = \mathrm{AJ}^{p,n}_X(\mathcal{Z})$ (modulo torsion).

(ii) The composition of AJ with the real projection $\pi^p_{\mathbb{R}}\colon H^{2p-n}_{\mathcal{D}}(X,\mathbb{Z}(n)) \to H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(n))$, coincides with Goncharov's regulator map (in [11]); see [18, Section 3.1.1] for the proof.

### 6.2. DEFINITION OF TAME$^i$

Now $\mathcal{R}$ essentially respects coniveau, and we can use this to induce a map of local–global spectral sequences for CH and $H_{\mathcal{D}}$. The general case requires reduction to a normal-crossings/log-poles setting (as does AJ for quasi-projectives, e.g. $X\backslash V$); see [18, Section 2.4.1] and [19, Section 5.9]. An exception is the case $p > d$, where $H^{2p-n}_{\mathcal{D}}(X,\mathbb{Q}(p)) \cong H^{2p-n-1}(X,\mathbb{C}/\mathbb{Q}(p))$, the middle term of the cone complex drops out, and

$$C^{\bullet}_{\mathcal{D}}(X,\mathbb{Q}(p)) = \mathrm{Cone}\{C_{2d-\bullet}(X,\mathbb{Q}(p)) \to \Gamma(X, {}'\mathcal{D}^{\bullet}_X)\}[-1]$$

*Equivalently $\{C_{2d-\bullet}(X,\mathbb{Z}(p)) \oplus \Gamma(X, F^{p\prime}\mathcal{D}^{\bullet}_X) \oplus \Gamma(X, {}'\mathcal{D}^{\bullet-1}_X)\}$ with differential $\mathrm{D}(a,b,c) := (-\partial a, -\mathrm{d}[b], \mathrm{d}[c] - b + a)$. (See [17] for a discussion of Deligne homology.)

is easily filtered by $F^i_N C^\bullet_{\mathcal{D}}(X, \mathbb{Q}(p)) :=$ limit of

$$\mathrm{Cone}\{C^{V^i}_{2d-\bullet}(X, \mathbb{Q}(p)) \to \Gamma(X, {}'\mathcal{D}^\bullet_{(V^i)^\infty})\}[-1] \xleftarrow[(2\pi\mathrm{i})^i \cdot \iota^{(i)}_*]{\simeq}$$
$$\mathrm{Cone}\{C_{2d-\bullet}(V^i, \mathbb{Q}(p-i)) \to \Gamma(V^i, {}'\mathcal{D}^\bullet_{V^i})\}[-1].$$

The resulting

$$\begin{aligned} E^{i,-j}_1(p) &= H^{2n-(i+j)}\{\mathrm{Gr}^i_N C^\bullet_{\mathcal{D}}(X, \mathbb{Q}(p))\} \\ &\cong \coprod_{x \in X^i} H^{2p-(i+j)-1}(\eta_X, \mathbb{C}/\mathbb{Q}(p-i)) \end{aligned}$$

is the (modified) spectral sequence from Section 5.1; recall the $d_i$: $E^{0,-n}_i(p) \to E^{i,-n-i+1}_i(p)$ are called $\overline{\mathrm{Res}}^i$. Let [4]

$${}'E^{i,-j}_1(p) := H^{i-j}\{\mathrm{Gr}^i_N Z^p(X, -\bullet)\} \cong \coprod_{x \in X^i} \mathrm{CH}^{p-i}(\eta_X, j-i);$$

then commutativity of the square

$$\begin{array}{ccc} H^*\left(\mathrm{Gr}^i_N Z^p(X,-\bullet)\right) & \xrightarrow{\Delta} & H^{*+1}\left(F^{i+1}_N Z^p(X,-\bullet)\right) \\ \downarrow \mathcal{R} & & \downarrow \mathcal{R} \\ H^*\left(\mathrm{Gr}^i_N C^{2p+\bullet}_{\mathcal{D}}(X,\mathbb{Q}(p))\right) & \xrightarrow{\Delta} & H^{*+1}\left(F^{i+1}_N C^{2p+\bullet}_{\mathcal{D}}(X,\mathbb{Q}(p))\right) \end{array}$$

establishes the map ${}'E \to E$ induced by $\mathcal{R}$; in particular this enjoys the property that

$$\overline{\mathrm{Res}}^i \circ \mathrm{AJ}^{p,n}_{\eta X} = \left( \coprod_{x \in X^i} \mathrm{AJ}^{p-i,n-1}_{\eta X} \right) \circ {}'d_i \tag{6.1}$$

as maps from ${}'E^{0,n}_i(p) \to E^{i,-n-i+1}_i(p)$.

If furthermore $n = p (> d)$ then under $\gamma$: $K^M_n(\mathbb{C}(X)) \xrightarrow{\cong} \mathrm{CH}^n(\eta_X, n)$, $\mathrm{AJ}^{n,n}_{\eta X}$ is identified with $R'$ and ${}'d_1$ with Tame. So we generalize Tame(=: Tame$^1$) by setting $\mathrm{Tame}^i := {}'d_i \circ \gamma$. The ker(Tame$^i$) then give a filtration on $K^M_n(\mathbb{C}(X))$ and we define

$$K^M_n(X) := \bigcap \ker(\mathrm{Tame}^i).$$

This consists of those $\{\mathbf{f}\} \in K^M_n(\mathbb{C}(X))$ whose [good representative's] graph $\overline{\gamma}_{\mathbf{f}} \in Z^n(X, n)$ may be completed to a higher Chow ($\partial_{\mathcal{B}}$-) cycle by the addition of components with support (over $X$) of codimension $\geqslant 1$.

Now (6.1) becomes the ultimate generalization of Proposition 5.3:

PROPOSITION 6.1.

(i) *The following diagram commutes:*

$$\begin{array}{ccc} K_n^M(\mathbb{C}(X)) \supseteq \ker(\mathrm{Tame}^{i-1}) & \xrightarrow{\mathrm{Tame}^i} & \coprod_{x\in X^i} \mathrm{CH}^{n-i}(\mathbb{C}(x), n-1) \Big/ \bigcup_{j<i} \mathrm{im}(d_j) \\ \Big\downarrow R' & & \Big\downarrow \coprod \mathrm{AJ}^{n-i,n-1}_{\eta_x} \\ H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Q}(n)) \supseteq \ker(\overline{\mathrm{Res}}^{i-1}) & \xrightarrow{\overline{\mathrm{Res}}^i} & \coprod_{x\in X^i} H^{n-2i}(\eta_X, \mathbb{C}/\mathbb{Q}(n-i)) \Big/ \bigcup_{j<i} \mathrm{im}(\overline{\mathrm{Res}}^j). \end{array}$$

*In terms of currents*, $\overline{\mathrm{Res}}_x^i[R'_{\mathbf{f}}] \equiv (1/(-2\pi \mathrm{i})^{i-1}) R'_{d_i(\gamma_{\mathbf{f}})}$ *up to coboundaries on* $\eta_x$.

(ii) *Since* $\bigcap \ker(\overline{\mathrm{Res}}^i) = \mathrm{im}\{H^{n-1}(X, \mathbb{C}/\mathbb{Q}(n))\}$, *the Milnor regulator has a 'holomorphic' restriction*

$$\begin{aligned} R\colon K_n^M(X) &\longrightarrow \mathrm{im}\{H^{n-1}(X, \mathbb{C}/\mathbb{Q}(n)) \to H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Q}(n))\} \\ &\cong H^{n-1}(X, \mathbb{C}/\mathbb{Q}(n))/\mathrm{im}(\mathrm{Gy}). \end{aligned}$$

*Remark.* Here Gy: $\varinjlim_{V\subset X} H_V^{n-1}(X, \mathbb{C}/\mathbb{Q}(n)) \to H^{n-1}(X, \mathbb{C}/\mathbb{Q}(n))$ is the Gysin homomorphism ($V \subset X$ divisors); we are obviously concerned with situations where it is not surjective, e.g. $d = n-1$ and $K_X \geqslant 0$.

## 6.3. SOME QUESTIONS

(1) *How deep is the filtration on* $K_n^M(\mathbb{C}(X))$?

CONJECTURE 6.2 [Beilinson–Soulé]. $\mathrm{CH}^q(\mathbb{C}(x), m) = 0$ *for* $m \geqslant 2q$. (*Known for* $q = 0, 1$.)

(See [24].) Assuming $n \geqslant 2$ one has this easy.

COROLLARY 6.3. $K_n^M(X) = \ker(\mathrm{Tame}^i)$ *if* $2i + 1 \geqslant n$ (*or* $i \geqslant d$). *Known for* $n = 2, 3, 4$; *in particular* $\ker(\mathrm{Tame}) = K_n^M(X)$ *for* $n = 2, 3$.

(2) *Does* $\mathrm{AJ}^{n-i,n-1}_{\eta X}$ *exhibit '$i$-logarithmic' behavior?*
Let us specialize to the case $n = 2i$, where this is obviously tied to Goncharov's conjecture that Chow polylogarithms are expressible in terms of classical polylogarithms. One easily writes an element $\mathcal{W} \in Z^i(\mathrm{pt.}, 2i-1)$ for which $R_{\mathcal{W}}$ is an $i$-logarithm value: namely,

$$\begin{aligned} \mathcal{W}_i(a) = \Bigg(&1 - \frac{a}{w_{i-1}}, 1 - \frac{w_{i-1}}{w_{i-2}}, \ldots, 1 - \frac{w_2}{w_1}, \\ &1 - w_1, w_1, w_2, \ldots, w_{i-1}\Bigg) \subseteq \square^{2i-1} \end{aligned}$$

has $R_{\mathcal{W}_i(a)} = \int_{\mathcal{W}_i(a)} R_{2i-1} = \mathrm{Li}_i(a)$. But you cannot build *higher Chow cycles* out of the $\mathcal{W}_i(a)$ without alternating them, and alternating destroys the computation.

Our feeling is that the '$i$-logarithms' involved should be not the $\mathrm{Li}_i$, but $\mathbb{C}/\mathbb{Q}(i)$-valued 'lifts' of the real-valued $\mathcal{L}_i$, defined only on small subsets of $\mathbb{Z}\{\mathbb{C}\backslash\{0,1,\infty\}\}$. We work this out for $i=2$ in the following example.

EXAMPLE 6.4. $n=4$, $X=$ 3-fold, $\{\mathbf{f}\}\in\ker(\mathrm{Tame})$. Recall that in Example 5.4 we attempted to relate $\overline{\mathrm{Res}}^2[R'_{\mathbf{f}}]\in\{\coprod_{y\in X^2}\mathbb{C}/\mathbb{Q}(2)\}/\mathrm{im}(d_1)$ to $\mathcal{B}_2(\mathbb{C}(y))$ and the dilogarithm. We accomplish this now (in a weaker sense) by relating $\mathcal{B}_2$ and $\mathcal{L}_2$ to $\mathrm{AJ}^{2,3}_{\eta_y}$ (in the next proposition) on at least certain types of higher Chow cycles.

For $g\in\mathbb{C}(y)\backslash\{0,1\}$ let $\rho(g)=\mathrm{Alt}_3(1-g/z,1-z,z)\in Z^2(\mathbb{C}(y),3)$.

PROPOSITION 6.5 ([18, Section 3.1.2]).

(i) *Given any element* $\xi=\sum m_j\{g_j\}_2\in\ker(\mathrm{st})\subset\mathcal{B}_2(\mathbb{C}(y))$, $\sum m_j\rho(g_j)\in Z^2(\mathbb{C}(y),3)$ *can be completed to a higher Chow cycle* $\mathcal{Z}_\xi$ *by adding decomposable elements* $\in Z^1(\mathbb{C}(y),2)\wedge Z^1(\mathbb{C}(y),1)$.

(ii) *Moreover, the 'real' projection of* $\mathrm{AJ}^{2,3}_{\eta_y}(\mathcal{Z}_\xi)$ *is the constant* $\Im(R_{\mathcal{Z}_\xi})=\sum m_j\mathcal{L}_2(g_j(p))$, $p\in y$ *a general point.*

## 6.4. BLOCH'S CONSTRUCTION

There is a nice geometric perspective on the holomorphic Milnor regulator, which exhibits it as the correct generalization to higher $n(>d)$ of Bloch's construction (in Chapter 8 of [2], for $n=2$, $d=1$). Given a compact Riemann surface $\mathcal{E}$, and $\mathbf{f}\in\otimes^2\mathbb{Z}\{\mathbb{C}(\mathcal{E})^*\}$ (not necessarily good) such that $\{\mathbf{f}\}\in\ker(\mathrm{Tame})\subseteq K_2^M(\mathbb{C}(\mathcal{E}))$, he shows how to complete $\overline{\gamma_{\mathbf{f}}}$ to a certain kind of relative cycle $\Gamma_{\mathbf{f}}$ in $(\mathcal{E}\times\square^n,\mathcal{E}\times\partial\square^2)$ by adding curves (in $\square^2$) over points of $\mathcal{E}$. Then he constructs a bounding chain, and integrates $\pi^*\omega_X\wedge\Omega_2$ over this bounding chain to get a map $\ker(\mathrm{Tame})\to H^1(\mathcal{E},\mathbb{C}/\mathbb{Z}(2))$ by duality.

He allows certain corner intersections (satisfying some integrability conditions) for his relative cycles; to get rid of the $\int$ conditions we disallow corners by taking $Z^n(X\times\square^n,X\times\partial\square^n)$ to be the *subset* of $c^n(X,n)$ consisting of $\{\mathcal{Z}|\mathcal{Z}\cdot(X\times\partial\square^n)=0\}$. We then choose a *good* representative $\mathbf{f}$ of $\{\mathbf{f}\}\in K_n^M(X)$ (not ker(Tame), of course), complete $\overline{\gamma_{\mathbf{f}}}$ to a higher Chow cycle $\Gamma_{\mathbf{f}}\in Z^n(X,n)$ ($\partial_{\mathcal{B}}\Gamma_{\mathbf{f}}=0$), and alternate to a relative cycle* $\mathrm{Alt}_n\Gamma_{\mathbf{f}}\in Z^n(X\times\square^n,X\times\partial\square^n)_{\mathbb{Q}}$. Finally we apply the homotopy $\theta$ from Section 3.3 [formula (3.1)] over all of $X$, to get a [limit of] bounding chain[s] and integrate $\pi^*\omega_X\wedge\Omega_n$ over the result. We lose torsion information from the alternation, though; it is better** (and equivalent) to take $\mathrm{AJ}(\Gamma_{\mathbf{f}})$. In either case we must go modulo AJ of ambiguities in codim. $\geqslant 1$, arising from the completion $\gamma_{\mathbf{f}}\mapsto\Gamma_{\mathbf{f}}$; so the target is $H^{n-1}(X,\mathbb{C}/\mathbb{Q}(n))/\mathrm{im}(\mathrm{Gy})$.

*That $\mathrm{Alt}_n\Gamma_{\mathbf{f}}$ has the same higher Chow class as $\Gamma_{\mathbf{f}}$ (mod torsion) follows from [21].

**One could also use normalization of chain complexes ([10, Section III.2]) to produce (from $\Gamma_{\mathbf{f}}$) a relative cycle *without* losing torsion.

## PART 3. BEHAVIOR OF $[R_{\mathbf{f}}]$ IN FAMILIES

### 7. Rigidity

#### 7.1. THE STATEMENT

We now investigate the behavior of the Milnor regulator on smooth $(n-1)$-dimensional complete intersections in $\mathbb{P}^{n+r}$ of multidegree $(D_0, \ldots, D_r)[r \geqslant 0]$; to avoid redundancy take all $D_j \geqslant 2$. The family of all such is denoted

$$\mathcal{X} \xrightarrow{\pi} \mathcal{S} := \mathbb{P}H^0\,(\mathbb{P}^{n+r}, \mathcal{O}(D_0) \oplus \cdots \oplus \mathcal{O}(D_r)) \backslash \Delta,$$

where $\Delta$ is the discriminant locus and $\pi^{-1}(s) =: X_s$. We are interested in the situation where

$$\{\mathbf{f}_s\} \in \ker(\text{Tame}) \subseteq K_n^M(\mathbb{C}(X_s))$$

is an analytic family over an open ball $\mathcal{U} \subset \mathcal{S}$; in particular, we assume $\{\mathbf{f}_s\}$ comes from the fiberwise restriction of $\{\mathbf{F}\} \in \ker(\text{Tame}) \subseteq K_n^M(\mathbb{C}(\tilde{\mathcal{X}}))$, where $\tilde{\mathcal{S}} \xrightarrow{\rho} {}'\mathcal{S} \overset{\text{Zar. op.}}{\subset} \mathcal{S}$ is a finite cover and $\tilde{\mathcal{X}} = \mathcal{X} \times_{\mathcal{S}} \tilde{\mathcal{S}}$. (We are implicitly identifying $\mathcal{U}$ with a component of $\rho^{-1}(\mathcal{U})$.)

For convenience write $\tilde{V} = V_{\mathbf{F}} \subset \tilde{\mathcal{X}}$, $V_s = V_{\mathbf{f}_s} = \tilde{V} \cap X_s$, $\mathcal{X}_{\mathcal{U}} = \pi^{-1}\mathcal{U}$ and $V_{\mathcal{U}} = \tilde{V} \cap \mathcal{X}_{\mathcal{U}}$, shrinking $\mathcal{U}$ if necessary so that $\mathcal{X}_{\mathcal{U}} \backslash V_{\mathcal{U}}$ and $(X_s \backslash V_s) \times \mathcal{U}$ are diffeomorphic for any $s \in \mathcal{U}$. This ensures $\text{rank}\{H^{n-1}(X_s \backslash V_s, \mathbb{C})\}$ is constant so that it makes sense to apply the Gauss–Manin connection to a section$^\star$ of $H^{n-1}_{X_s \backslash V_S}$ on $\mathcal{U}$. Let $0 \in \mathcal{U}$ be a point.

Corresponding to $\{\mathbf{F}\}$ and $\{\mathbf{f}_s\}$ there are regulator currents $R_{\mathbf{F}} \in \Gamma({}'\mathcal{D}^{n-1}_{\tilde{\mathcal{X}}})$ generating by fiberwise pullback $\iota^*_{X_s} R_{\mathbf{F}} = R_{\mathbf{f}_s} \in \Gamma({}'\mathcal{D}^{n-1}_{X_s})$, so that $[R_{\mathbf{f}_s}] \in \Gamma(\mathcal{U}, \mathcal{H}^{n-1}_{X_s \backslash V_s} \otimes \mathbb{C}/\mathbb{Q}(n))$. By differentiating the periods of $R_{\mathbf{f}_s}$ (modulo $\mathbb{Q}(n)$) on $(n-1)$-cycles $\mathcal{C}_s \subset X_s \backslash V_s$ to obtain $\nabla[R_{\mathbf{f}_s}] \in \Gamma(\mathcal{U}, \Omega^1_{\mathcal{S}} \otimes \mathcal{H}^{n-1}_{X_S \backslash V_s})$, we will prove the following theorem.

THEOREM 7.1 [Rigidity]. *In the situation just described with* $\{\mathbf{f}_s\} \in \ker(\text{Tame}) \subseteq K_n^M(\mathbb{C}(X_s))$, $\nabla[R_{\mathbf{f}_s}]$ *is zero unless* $X_s$ *are elliptic curves*; *that is*, *unless* $n = 2$ $(\dim X_s = 1)$ *and* $\deg(K_{X_s})[= \sum D_j - (n + r + 1)] = 0$, $[R_{\mathbf{f}_s}]$ *is flat*.

*Remark.* In the case of elliptic curves Collino [6] has constructed a family $\{\mathbf{f}_s\} \in \ker(\text{Tame})$, whose infinitesimal invariant $\nabla[R_{\mathbf{f}_s}]$ he shows to be nonzero by means of theta nulls.

#### 7.2. THE PROOF

The first step is to lift $[R_{\mathbf{f}_s}]$ to $[R'_{\mathbf{f}_s}] \in \Gamma(\mathcal{U}, \mathcal{H}^{n-1}_{X_s \backslash V_s})$, in such a way that they come from fiberwise pullback of some $R'_{\mathbf{F}} \in \Gamma(\mathcal{X}_{\mathcal{U}}, {}'\mathcal{D}^{n-1}_{\tilde{\mathcal{X}}})$. Now while $\Omega_{\mathbf{f}_s} = 0$ $(n >$

$^\star$We avoid the $R^{n-1}_{\pi *}$ notation to prevent confusion between $\mathcal{H}^{n-1}_{X_s}$ and $\mathcal{H}^{n-1}_{X_s \backslash V_s}[= R^{n-1}_{\pi *}\mathbb{C}$ for $\tilde{\mathcal{X}} \to \tilde{\mathcal{S}}$ and $\tilde{\mathcal{X}} \backslash \tilde{V} \to \tilde{\mathcal{S}}$, resp.].

$d = n-1$), $\{\mathbf{F}\} \in \ker(\mathrm{Tame})$ does *not* imply $\Omega_{\mathbf{F}} = 0$ [compare Lemma 2.2]. This is because $\tilde{\mathcal{X}}$ is quasi-projective; in fact we have (in taking $'\mathcal{S} \subset \mathcal{S}$) purposely omitted from $\tilde{\mathcal{X}}$ fibers (which would have been 'vertical' components of $\tilde{V}$) along which Tame$\{\mathbf{F}\}$ may not be 0. On the other hand, $\{\mathbf{F}\} \in \ker(\mathrm{Tame})$ does imply $\mathrm{d}[\Omega_{\mathbf{F}}] = 0$ [as in proof of 2.2], and so $\Omega_{\mathbf{F}}$ gives a class $[\Omega_{\mathbf{F}}]_{\mathcal{U}} \in H^n(\mathcal{X}_{\mathcal{U}} \backslash V_{\mathcal{U}}, \mathbb{C})$ with $\iota^*_{X_0}[\Omega_{\mathbf{F}}]_{\mathcal{U}} = [\Omega_{\mathbf{f}_0}] = 0$; by acyclicity of $\mathcal{U}$ and rigidity of $H^*_{\mathrm{DR}}$, $[\Omega_{\mathbf{F}}]_{\mathcal{U}} = 0$. Since (on $\mathcal{X}_{\mathcal{U}} \backslash V_{\mathcal{U}}$) $\mathrm{d}[R_{\mathbf{F}}] = \Omega_{\mathbf{F}} - (2\pi \mathrm{i})^n \delta_{T_{\mathbf{F}}}$, $[T_{\mathbf{F}}] = 0$ in $H_*(\mathcal{X}_{\mathcal{U}}, V_{\mathcal{U}} \cup \mathcal{X}_{\partial \mathcal{U}})$. Set $R'_{\mathbf{F}} := R_{\mathbf{F}} + (2\pi \mathrm{i})^n \delta_{\partial^{-1} T_{\mathbf{F}}}$, where $\partial^{-1} T_{\mathbf{F}}$ is a relative bounding chain.

Let $\{t_i\}$ be a coordinate system on $\mathcal{U}$ with $t_i(0) = 0$ $(\forall i)$; and let $\epsilon_i \in \mathcal{U}$ denote the point with $t_j(\epsilon_i) = \epsilon \cdot \delta_{ij}$. Paths $[0, \epsilon_i]$ are taken to lie in the disks $\mathcal{U}_i := \mathcal{U} \cap \{t_j = 0 | \forall j \neq i\}$. If $\{\mathcal{C}_s \in C_{n-1}(X_s \backslash V_s)\}_{s \in \mathcal{U}}$ is a continuous family of topological cycles, then

$$\int_{\mathcal{C}_s} R_{\mathbf{f}_s} \equiv \int_{\mathcal{C}_s} R'_{\mathbf{f}_s} = \int_{\mathcal{C}_s} [\iota^*_{X_s}] R'_{\mathbf{F}} \bmod \mathbb{Z}(n) \Longrightarrow$$

$$(\nabla_{\partial/\partial t_i}[R_{\mathbf{f}_s}])_{s=0}(\mathcal{C}_0) := \left( \frac{\partial}{\partial t_i} \int_{\mathcal{C}_s} R_{\mathbf{f}_s} \right)_{s=0}$$

$$= \lim_{\epsilon \to 0} \left( \int_{\mathcal{C}_{\epsilon_i}} R'_{\mathbf{F}} - \int_{\mathcal{C}_0} R'_{\mathbf{F}} \right)$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\mathcal{C}_{[0,\epsilon_i]}} \mathrm{d}[R'_{\mathbf{F}}] = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\mathcal{C}_{[0,\epsilon_i]}} \Omega_{\mathbf{F}} = \int_{\mathcal{C}_0} \iota^*_{X_0} \langle \widetilde{\partial/\partial t_i}, \Omega_{\mathbf{F}} \rangle$$

(since $\mathrm{d}[R'_{\mathbf{F}}] = \Omega_{\mathbf{F}}$ on $\mathcal{X}_{\mathcal{U}}$), and we conclude that

$$\nabla[R_{\mathbf{f}_s}] = \sum_i \mathrm{d}t_i \otimes [\iota^*_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \langle \widetilde{\partial/\partial t_i}, \Omega_{\mathbf{F}} \rangle] \in \Gamma(\mathcal{U}, \Omega^1_{\tilde{\mathcal{S}}} \otimes \mathcal{H}^{n-1}_{X_s \backslash V_s}).$$

We show this lifts to a section $\nu_s \in \Gamma(\mathcal{U}, \Omega^1_{\tilde{\mathcal{S}}} \otimes \mathcal{F}^{n-1} \mathcal{H}^{n-1}_{X_s})$, by checking that $\iota^*_{X_s} \langle \widetilde{\partial/\partial t_i}, \Omega_{\mathbf{F}} \rangle \in Z^0_{\mathrm{d}}(F^{n-1\prime} \mathcal{D}^{n-1}_{X_s})$ at $s = 0$. Take any $\alpha_0 \in \Gamma(\Omega^{n-2}_{(X_0)^\infty})$, $\beta_0 \in Z^0_{\mathrm{d}}(F^1 \Omega^{n-1}_{(X_0)^\infty})$, and let $\tilde{\alpha} \in \Gamma(\Omega^{n-2}_{(\mathcal{X}_{\mathcal{U}})^\infty})$, $\tilde{\beta} \in \Gamma(F^1 \Omega^{n-1}_{(\mathcal{X}_{\mathcal{U}})^\infty})$ be local lifts. Then

$$\pm \int_{X_0} \alpha_0 \wedge \mathrm{d}[\langle \widetilde{\partial/\partial t_i}, \Omega_{\mathbf{F}} \rangle] := \int_{X_0} \mathrm{d}\alpha_0 \wedge \langle \widetilde{\partial/\partial t_i}, \Omega_{\mathbf{F}} \rangle = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\mathcal{X}_{[0,\epsilon_i]}} \mathrm{d}\tilde{\alpha} \wedge \Omega_{\mathbf{F}}$$

$$= \lim_{\epsilon \to 0} \left( \int_{\mathcal{X}_{[0,\epsilon_i]}} \tilde{\alpha} \wedge \mathrm{d}[\Omega_{\mathbf{F}}] + \int_{X_{\epsilon_i}} \alpha_{\epsilon_i} \wedge \Omega_{\mathbf{f}_{\epsilon_i}} - \int_{X_0} \alpha_0 \wedge \Omega_{\mathbf{f}_0} \right) = 0$$

since $\mathrm{d}[\Omega_{\mathbf{F}}] = 0$, while

$$\int_{X_0} \beta_0 \wedge \langle \widetilde{\partial/\partial t_i}, \Omega_{\mathbf{F}} \rangle = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\mathcal{X}_{[0,\epsilon_i]} \subset \mathcal{X}_{\mathcal{U}_i}} \tilde{\beta} \wedge \Omega_{\mathbf{F}} = 0$$

since $\mathcal{X}_{\mathcal{U}_i}$ can only support $n$ $\mathrm{d}z$'s

So if $K_{X_s} < 0$, then $\mathcal{F}^{n-1} \mathcal{H}^{n-1}_{X_s} = 0 \Longrightarrow \nu_s$ trivial $\Longrightarrow$ its image $\nabla[R_{\mathbf{f}_s}]$ is also trivial (the desired result). Henceforth assume $\sum D_j \geqslant n + r + 1 (K_X \geqslant 0)$.

Now since $\mathcal{F}^{n-1}\mathcal{H}^{n-1}_{X_s} \subset H^{n-1}_{X_s,\mathrm{pr}}$, $\nabla\nu_s \in \Gamma(\mathcal{U}, \Omega^2_{\tilde{\mathcal{S}}} \otimes \mathcal{H}^{n-1}_{X_s,\mathrm{pr}})$; moreover, $\nabla \circ \nabla[R_{\mathbf{f}_s}] = 0 \in \Gamma(\mathcal{U}, \Omega^2_{\tilde{\mathcal{S}}} \otimes \mathcal{H}^{n-1}_{X_s\setminus V_s})$, so $\nu_s$ is actually a section of

$$\Omega^2_{\tilde{\mathcal{S}}} \otimes \ker\{\mathcal{H}^{n-1}_{X_s,\mathrm{pr}} \to \mathcal{H}^{n-1}_{X_s\setminus V_s}\} = \Omega^2_{\tilde{\mathcal{S}}} \otimes \ker\{\mathcal{H}^{n-1}_{X_s,\mathrm{pr}} \xrightarrow{\theta} \mathcal{H}^{n-1}_{X_s}/\mathcal{K}_s\},$$

where

$$\mathcal{K}_s := \mathrm{im}\{H^{n-1}_{V_s}(X_s, \mathbb{C}) \to H^{n-1}(X_s, \mathbb{C})\}.$$

If in fact this $\mathcal{K}_s = \mathrm{im}\{H^{n-1}(\mathbb{P}^{n+r}) \to H^{n-1}(X_s)\}$,* then $\theta : \mathcal{H}^{n-1}_{X_s,\mathrm{pr}} \to \mathcal{H}^{n-1}_{X_s,\mathrm{var}}$ is an isomorphism and $\nabla\nu_s = 0$.

Since $\mathcal{K}_s$ is generated over $\mathbb{Q}$, one can choose a local basis of (flat!) rational sections $\{\sigma_\ell\} \in \Gamma(\mathcal{U}, R^{n-1}_{\pi_*}\mathbb{Q})$ for $\Gamma(\mathcal{U}, \mathcal{K}_s)$. Since $\mathcal{K}_s \subseteq \mathcal{F}^1\mathcal{H}^{n-1}_{X_s}$, the $\{\sigma_\ell\}$ are in fact sections of $\ker\{\mathcal{F}^1\mathcal{H}^{n-1}_{X_s} \xrightarrow{\nabla} \Omega^1_{\tilde{\mathcal{S}}} \otimes (\mathcal{F}^0)\mathcal{H}^{n-1}_{X_s}\}$. Taking the quotient by $\mathrm{im}\{\mathcal{H}^{n-1}_{\mathbb{P}^{n+r}} \to \mathcal{H}^{n-1}_{X_s}\}$ gives sections $\{\overline{\sigma_\ell}\}$ of the middle term in the short exact sequence

$$\begin{aligned} 0 \longrightarrow\ & \ker\{\mathcal{F}^2\mathcal{H}^{n-1}_{X_s,\mathrm{var}} \xrightarrow{\nabla} \Omega^1_{\tilde{\mathcal{S}}} \otimes \mathcal{F}^1\mathcal{H}^{n-1}_{X_s,\mathrm{var}}\} \\ \longrightarrow\ & \ker\{\mathcal{F}^1\mathcal{H}^{n-1}_{X_s,\mathrm{var}} \xrightarrow{\nabla} \Omega^1_{\tilde{\mathcal{S}}} \otimes (\mathcal{F}^0)\mathcal{H}^{n-1}_{X_s,\mathrm{var}}\} \\ \longrightarrow\ & \ker\{\mathrm{Gr}^1_{\mathcal{F}}\mathcal{H}^{n-1}_{X_s,\mathrm{var}} \xrightarrow{\bar{\nabla}^1_{(1)}} \Omega^1_{\tilde{\mathcal{S}}} \otimes \mathrm{Gr}^0_{\mathcal{F}}\mathcal{H}^{n-1}_{X_s,\mathrm{var}}\} \longrightarrow 0. \end{aligned}$$

LEMMA 7.2. $\bar{\nabla}^1_{(k)} : \mathrm{Gr}^k_{\mathcal{F}}\mathcal{H}^{n-1}_{X_s,\mathrm{var}} \to \Omega^1_{\tilde{\mathcal{S}}} \otimes \mathrm{Gr}^{k-1}_{\mathcal{F}}\mathcal{H}^{n-1}_{X_s,\mathrm{var}}$ *is injective for* $1 \leqslant k \leqslant n-1$, *provided* $\sum D_j \geqslant n+r+1 [\Longrightarrow \mathrm{Gr}^0_{\mathcal{F}}\mathcal{H}^{n-1}_{X_s} \neq 0]$.

So the last term is zero and the $\{\overline{\sigma_\ell}\}$ pull up to the first term. Now one simply increases all the $\mathcal{F}$ and $\mathrm{Gr}_{\mathcal{F}}$ superscripts by 1, writes this out again with $\{\overline{\sigma_\ell}\}$ in the middle and repeats the bootstrapping procedure. This continues, using injectivity of $\bar{\nabla}^1_{(2)}$, $\bar{\nabla}^1_{(3)}$, etc. until the $\{\overline{\sigma_\ell}\}$ wind up in $\mathcal{F}^{\frac{n}{2}}\mathcal{H}^{n-1}_{X_s,\mathrm{pr}} \cap R^{n-1}_{\pi_*}\mathbb{Q}$ (for $n$ even) or $\mathcal{F}^{(n+1)/2}\mathcal{H}^{n-1}_{X_s,\mathrm{var}} \cap R^{n-1}_{\pi_*}\mathbb{Q}$ (for $n$ odd), which are both zero. So there was only one $\sigma_\ell$ (as they were a basis) and it was in $\Gamma(\mathcal{U}, \mathrm{im}\{\mathcal{H}^{n-1}_{\mathbb{P}^{n+r}} \to \mathcal{H}^{n-1}_{X_s}\})$. We have established $\nabla\nu_s = 0$.

Therefore $\nu_s$ is a section of

$$\ker\left\{\Omega^1_{\tilde{\mathcal{S}}} \otimes \mathcal{F}^{n-1}\mathcal{H}^{n-1}_{X_s(,\mathrm{var})} \xrightarrow{\bar{\nabla}^2_{(n-1)}} \Omega^2_{\tilde{\mathcal{S}}} \otimes \mathrm{Gr}^{n-2}_{\mathcal{F}}\mathcal{H}^{n-1}_{X_s,\mathrm{var}}\right\}.$$

---

*In fact, $\bar{\mathcal{K}}_s := \mathcal{K}_s/\mathrm{im}\{H^{n-1}(\mathbb{P}^{n+r})\}$ can also be shown zero by means of a monodromy argument over $\tilde{\mathcal{S}}$ (since we are working over $\mathcal{U}$ we have opted for the local argument given above). Here is the idea: one can show that $\bar{\mathcal{K}}_0 \subseteq \mathcal{H}^{n-1}_{X_s,\mathrm{var}}$ is an invariant $\pi_1(\tilde{\mathcal{S}})$-module, and clearly $\bar{\mathcal{K}}_0 \neq \mathcal{H}^{n-1}_{X_0,\mathrm{var}}$ if $\deg(K_{X_0}) \geqslant 0$. Since the action of $\pi_1(\tilde{\mathcal{S}})$ on $\mathcal{H}^{n-1}_{X_0,\mathrm{var}}$ is irreducible, $\bar{\mathcal{K}}_s = 0$.

LEMMA 7.3. $\bar{\nabla}^2_{(n-1)}$ *is injective for*

$$\sum D_j > n+r+1, \quad n=2, \qquad \sum D_j \geqslant n+r+1, \quad n \geqslant 3.$$

So unless $n=2$ and $\deg(K_{X_s})=0$, $\nu_s=0$ [$\Longrightarrow \nabla[R_{\mathbf{f}_s}] = 0$]. This completes the proof in the case $K_{X_s} \geqslant 0$.

### 7.3. THE ALGEBRAIC LEMMAS

We give a brief indication of how the two algebraic lemmas (due to Nagel) are proved. See [18] for more details. Notation is as follows: we write $S^{a,b}$ for the elements in $\mathbb{C}[z_0, \dots, z_{n+r}; x_0, \dots, x_r]$ of bihomogeneous bidegree $(a, b)$, where $z_i$ and $x_j$ are taken to have, respectively, bidegrees (0, 1) and $(1, -D_j)$. If $\mathsf{F}_j(\mathbf{z}) \in S^{D_j}$ are the homogeneous polynomials cutting out $X_0 \subseteq \mathbb{P}^{n+r}$, then set $\mathsf{F} = \mathsf{F}(\mathbf{x}, \mathbf{z}) := x_0\mathsf{F}_0(\mathbf{z}) + \cdots + x_r\mathsf{F}_r(\mathbf{z}) \in S^{1,0}$; the corresponding bigraded Jacobi rings are defined by

$$R_{\mathsf{F}}^{a,b} := S^{a,b} \Big/ \left( \frac{\partial \mathsf{F}}{\partial z_0}, \dots, \frac{\partial \mathsf{F}}{\partial z_{n+r}}; \frac{\partial \mathsf{F}}{\partial x_0}, \dots, \frac{\partial \mathsf{F}}{\partial x_r} \right).$$

Finally put $D := \sum D_j$.

*Proof of Lemma* 7.2. The dual $\theta^1_{\tilde{\mathcal{S}}} \otimes \mathrm{Gr}^{n-k}_{\mathcal{F}} \mathcal{H}^{n-1}_{X_s,\mathrm{pr}} \to \mathrm{Gr}^{n-k-1}_{\mathcal{F}} \mathcal{H}^{n-1}_{X_s,\mathrm{pr}}$ of $\bar{\nabla}^1_{(k)}$ is isomorphic (at $s = 0$) to $S^{1,0}/(\mathsf{F}) \otimes R_{\mathsf{F}}^{k-1,D} \stackrel{\mu^1_{(k)}}{\to} R_{\mathsf{F}}^{k,D}$, surjectivity of which follows from [25] Lemma 3.4. □

*Proof of Lemma* 7.3. This lemma generalizes [7]. The dual $\theta^2_{\tilde{\mathcal{S}}} \otimes \mathrm{Gr}^1_{\mathcal{F}} \mathcal{H}^{n-1}_{X_s,\mathrm{pr}} \to \theta^1_{\tilde{\mathcal{S}}} \otimes \mathrm{Gr}^0_{\mathcal{F}} \mathcal{H}^{n-1}_{X_s,\mathrm{pr}}$ of $\bar{\nabla}^2_{(n-1)}$ is isomorphic (at $s = 0$) to $\wedge^2 S^{1,0}/(\mathsf{F}) \otimes R_{\mathsf{F}}^{n-2,D} \stackrel{\mu^2_{(n-1)}}{\to} S^{1,0}/(\mathsf{F}) \otimes R_{\mathsf{F}}^{n-1,D}$, surjectivity of which follows* from [25] Lemma 3.5 and vanishing of $R_{\mathsf{F}}^{n,D}$. □

## 8. Vanishing

### 8.1. TARGET OF THE HOLOMORPHIC REGULATOR

We now return to the family $\mathcal{X} \stackrel{\pi}{\to} \mathcal{S}$ of smooth complete intersections, and take $X_0$ to be a very general member of this family. Given $\{\mathbf{f}_0\} \in \ker(\mathrm{Tame}) \subseteq K^m_n(\mathbb{C}(X_0))$,

*With his $p$, $n =$ the present $n$, $n + r$ (resp.). For both lemmas one needs to check that Nagel's restriction (his $r \leqslant n-3$) to $\dim(X) \geqslant 2$ is not essential.

there exists an extension to $\{\mathbf{F}\} \in \ker(\mathrm{Tame}) \subseteq K_n^M(\mathbb{C}(\tilde{\mathcal{X}}))$ as in Section 7, and $[\tilde{\mathcal{S}} \supset]\mathcal{U} \ni 0$ sufficiently small that $\mathcal{X}_{\mathcal{U}} \backslash V_{\mathcal{U}} \simeq (X_0 \backslash V_0) \times \mathcal{U}$. (This is proved by a standard 'spread' argument, see [18].) Section 7.2 then shows $\nabla[R_{\mathbf{f}_s}] = 0$, where $\mathbf{f}_s$ are the restrictions of $\mathbf{F}$ to $X_{s\in\mathcal{U}}$; but this is not enough to accomplish the monodromy argument that will give the vanishing theorem below.

Instead we will start from the (for $n \geqslant 4$) stronger assumption that $\{\mathbf{f}_0\} \in K_n^M(X_0)$, and recall from Section 6.2 that

$$[R'_{\mathbf{f}_0}] \in \mathrm{im}\{H^{n-1}(X_0, \mathbb{C}/\mathbb{Q}(n)) \\ \to H^{n-1}(\eta_{X_0}, \mathbb{C}/\mathbb{Q}(n))\} =: \underline{H^{n-1}}(\eta_{X_0}, \mathbb{C}/\mathbb{Q}(n)).$$

An immediate consequence of the argument involving $\mathcal{K}_s$ and Lemma 7.2, is the following statement about the target group of the holomorphic regulator:

PROPOSITION 8.1. *For $X_0$ a very general (smooth) complete intersection $\subseteq \mathbb{P}^{n+r}$ with $K_{X_0} \geqslant 0$,*

$$\underline{H^{n-1}}(\eta_{X_0}, \mathbb{C}/\mathbb{Q}(n)) \cong_{n.c.} \varinjlim_{V\subset X} \frac{H^{n-1}(X_0, \mathbb{C}/\mathbb{Q}(n))}{\mathrm{im}\{H_V^{n-1}(X_0, \mathbb{C}/\mathbb{Q}(n))\}} \\ \cong H_{\mathrm{var}}^{n-1}(X_0, \mathbb{C}/\mathbb{Q}(n))[\neq 0].$$

*Remark.*

(i) The statement here is in the second isomorphism.
(ii) The limit is over all (arbitrary unions of) divisors.
(iii) By definition $H_{\mathrm{var}}^{n-1}(X_0) := \mathrm{coker}\{H^{n-1}(\mathbb{P}^{n+r}) \to H^{n-1}(X_0)\}$.
(iv) Proposition is clearly false for $\sum D_j < n+r+1$; e.g. a general quadric surface $\subset \mathbb{P}^3$ is birational to $\mathbb{P}^1 \times \mathbb{P}^1$, and removing the divisors $\{0\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{0\}$ eliminates $H^2$ completely.

Since $X_0$ is a smooth complete intersection in $\mathbb{P}^{n+r}$,

$$\mathrm{im}\{H_{n-1}(\eta_{X_0}, \mathbb{Q}) \to H_{n-1}(X_0, \mathbb{Q})\} \subset \ker\{H_{n-1}(X_0, \mathbb{Q}) \to H_{n-1}(\mathbb{P}^{n+r}, \mathbb{Q})\}.$$

COROLLARY 8.2. *This inclusion is actually an equality; that is, any topological cycle in the r.h.s. may be 'moved' by a $\mathbb{Q}$-coboundary (on $X_0$) to avoid an arbitrary configuration of divisors.*

It is on a basis (for the r.h.s.) consisting of 'moved' cycles that one computes periods $\in \mathbb{C}/\mathbb{Q}(n)$ of $R_{\mathbf{f}_0}$, in order to determine the class $[R'_{\mathbf{f}_0}]$. Now we prove these periods are essentially always zero.

### 8.2. THE VANISHING ARGUMENT

Let $\mathbf{f}_0 \in \otimes^n \mathbb{Z}\{\mathbb{C}(X)^*\}$ be a good representative of $\{\mathbf{f}_0\} \in K_n^M(X_0) \subseteq K_n^M(\mathbb{C}(X))$, so that $\overline{\gamma_{\mathbf{f}_0}}$ is the codimension-0 component of a ($\partial_{\mathcal{B}}$-closed) higher Chow cycle

$\Gamma_0 \in Z^n(X_0, n)$. Writing $\bar{\mathcal{X}} \to \bar{\mathcal{S}} = \mathbb{P}H^0(\mathbb{P}^{n+r}, \mathcal{O}(D_0) \oplus \cdots \oplus \mathcal{O}(D_r))$ for the full family (including singular fibers), there exists a finite branched cover $\tilde{\bar{\mathcal{X}}} \xrightarrow{\pi} \tilde{\bar{\mathcal{S}}} \ni 0$ and a 'spread' $\tilde{\bar{\Gamma}} \in Z^n(\tilde{\bar{\mathcal{X}}}, n)$ restricting to $\Gamma_0$ at 0. Now $\operatorname{supp}_{\tilde{\bar{\mathcal{X}}}}(\partial_{\mathcal{B}}\tilde{\bar{\Gamma}}) \subseteq \pi^{-1}$ {codim. -1 subset of $\tilde{\bar{\mathcal{S}}}$}, so one may choose a Zariski-open subfamily $\tilde{\mathcal{X}} \to \tilde{\mathcal{S}}$ (with smooth fibers) on which the restriction $\tilde{\Gamma} \in Z^n(\tilde{\mathcal{X}}, n)$ of $\tilde{\bar{\Gamma}}$ is a higher Chow cycle.

Writing $\gamma_s^i$ for components of codimension $i \geqslant 1$ (on $X_s$), $\Gamma$ produces by fiber-wise restriction a family of higher Chow cycles

$$\Gamma_s = \overline{\gamma_{\mathbf{f}_s}} + \sum_{i \geqslant 1} \gamma_s^i \quad \left( \bigcup_i \operatorname{supp}_{X_s}(\gamma_s^i) =: V_s \right)$$

giving rise to currents $R_{\Gamma_s} = R_{\mathbf{f}_s} + \sum_{i \geqslant 1} R_{\gamma_s^i}$ [by the formula in Section 6.1]. Since $\Omega_{\Gamma_s} = 0$ by type considerations, $\mathrm{d}[R_{\Gamma_s}] = -(2\pi \mathrm{i})^n \delta_{T_{\Gamma_s}}$; and so $R'_{\Gamma_s} = R_{\Gamma_s} + (2\pi \mathrm{i})^n \delta_{\partial^{-1} T_{\Gamma_s}}$ is d-closed.

We can put these $R'_{\Gamma_s}$ into families locally, over any sufficiently small ball $\mathcal{U} \subseteq \tilde{\mathcal{S}}$. On $\tilde{\mathcal{X}}$ one has $\mathrm{d}[\Omega_{\tilde{\Gamma}}] = \Omega_{\partial_{\mathcal{B}}\tilde{\Gamma}} = 0$ and $\iota_{X_s}^* \Omega_{\tilde{\Gamma}} = \Omega_{\Gamma_s} = 0$; so $\Omega_{\tilde{\Gamma}}$ has trivial class on $\tilde{\mathcal{X}}_{\mathcal{U}}$, and one can write an $R'_{\tilde{\Gamma}}$ with $\iota_{X_s}^* R'_{\tilde{\Gamma}} = R'_{\Gamma_s}$ ($\forall\, s \in \mathcal{U}$), $\mathrm{d}[R'_{\tilde{\Gamma}}] = \Omega_{\tilde{\Gamma}}$. Moreover $\Omega_{\tilde{\Gamma}}$ has (by further type considerations) no codim. $\geqslant 1$ 'components', and so $\Omega_{\tilde{\Gamma}} = \Omega_{\mathbf{F}}$.

Lifting $\mathbf{f}_s$ to $\Gamma_s$ has saved us from the headache of working away from $V_s$. There are actually two lifts going on here: since $\{\mathbf{f}_s\} \in K_n^M(X_s) \subseteq K_n^M(\mathbb{C}(X_s))$ the class $[R_{\mathbf{f}_s}] \in \operatorname{im}\{H^{n-1}(X_s, \mathbb{C}/\mathbb{Q}(n)) \to H^{n-1}(X_s \backslash V_s, \mathbb{C}/\mathbb{Q}(n))\}$; $[R_{\Gamma_s}] \in H^{n-1}(X_s, \mathbb{C}/\mathbb{Q}(n))$ gives a global lift over $\tilde{S}$, of which $[R'_{\Gamma_s}] \in H^{n-1}(X_s, \mathbb{C})$ is a local lift, e.g. over $\mathcal{U}$. However, it may be analytically continued to a 'multivalued section' over all of $\tilde{\mathcal{S}}$, and one can look at its monodromy in $H^{n-1}(X_s, \mathbb{Q}(n))$.

Before doing this we show locally that $[R'_{\Gamma_s}]$ is flat, i.e. $\nabla R'_{\Gamma_s} = 0$. Since locally $\mathrm{d}[R'_{\tilde{\Gamma}}] = \Omega_{\mathbf{F}}$ we have

$$\begin{aligned} \nabla R'_{\Gamma_s} &= \sum \mathrm{d}t_i \otimes \widetilde{\langle \partial/\partial t_i, \Omega_{\mathbf{F}} \rangle} \\ &\in \ker\{\Omega^1_{\tilde{S}} \otimes \mathcal{F}^{n-1}\mathcal{H}^{n-1}_{X_s} \to \Omega^2_{\tilde{\mathcal{S}}} \otimes \mathrm{Gr}_{\mathcal{F}}^{n-2}\mathcal{H}^{n-2}_{X_s,\mathrm{var}}\}, \end{aligned}$$

which is zero except for $n = 2$, $K_{X_0}$ trivial (by Lemma 7.3).

Describing monodromy by the map

$$\rho : \pi_1(\tilde{\mathcal{S}}, 0) \to \operatorname{Aut}\{H^{n-1}(X_0, \mathbb{C})\},$$

we note the difference of classes

$$\rho(\alpha)[R'_{\Gamma_0}] - [R'_{\Gamma_0}] \in H^{n-1}(X_0, \mathbb{Q}(n)) \tag{8.1}$$

since they both go to the same $[R_{\Gamma_0}] \in H^{n-1}(X_0, \mathbb{C}/\mathbb{Q}(n))$. Now recall that if $\alpha$ goes around a divisor in $\tilde{\bar{\mathcal{S}}} \backslash \tilde{\mathcal{S}}$ over which $X_s$ acquires a node ($[R_{\Gamma_s}]$ need not be defined there), we may speak of the vanishing cycle $\delta \in H_{n-1}(X_0, \mathbb{Q})$ associated to $\alpha$, whose 'flat transport' to the nodal $X_s$ is homologous to zero. It is a fact that such $\delta$ span ker $\{H_{n-1}(X_0, \mathbb{Q}) \to H_{n-1}(\mathbb{P}^{n+r}, \mathbb{Q})\}$; let $\{\delta_i\}$ be a basis (with associated loops $\alpha_i$) and $\{\hat{\delta}_i\}$ a dual basis* for $H^{n-1}_{\mathrm{var}}(X_0, \mathbb{Q})$, which one easily lifts to $H^{n-1}_{\mathrm{pr}}(X_0, \mathbb{Q})$.

Since $[R'_{\Gamma_s}]$ is flat, the Picard–Lefschetz formula (see [15] or [22]) applies to compute

$$\rho(\alpha_i)[R'_{\Gamma_0}] - [R'_{\Gamma_0}] = \pm\left(\int_{\delta_i} R'_{\Gamma_o}\right) \cdot \hat{\delta}_i.$$

Combined with (8.1) this gives *immediately*

$$\int_{\delta} R'_{\Gamma_0} \in \mathbb{Q}(n) \quad \forall \delta \in \ker\{H_{n-1}(X_0, \mathbb{Q}) \to H_{n-1}(\mathbb{P}^{n+r}, \mathbb{Q})\},$$

which is to say

$$[R_{\Gamma_0}] \in \mathrm{im}\{H^{n-1}(\mathbb{P}^{n+r}, \mathbb{C}/\mathbb{Q}(n)) \to H^{n-1}(X_0, \mathbb{C}/\mathbb{Q}(n))\};$$

and so (except for elliptic curves)

$$[R_{\mathbf{f}_0}] = 0 \in H^{n-1}(X_0 \backslash V_0, \mathbb{C}/\mathbb{Q}(n)),$$

since the composition $H^{n-1}(\mathbb{P}^{n+r}) \to H^{n-1}(X_s) \to H^{n-1}(X_s \backslash V_s)$ is zero.

## 8.3. STATEMENT AND INTERPRETATIONS

We formally state what we have proved.

THEOREM 8.3 [Vanishing]. *Let $X_0 \subset \mathbb{P}^{n+r}$ be a very general smooth complete intersection of multidegree $(D_0, \ldots, D_r)$, where if $n = 2$ then $\sum D_j \neq n + r + 1$. Then the image of the holomorphic Milnor regulator*

$$R\colon K_n^M(X) \to \underline{H^{n-1}}(\eta_X, \mathbb{C}/\mathbb{Q}(n)).$$

*is zero.*

*Remark.* This does not rule out interesting images for the holomorphic regulator on very general members of a proper subfamily of $\bar{\mathcal{X}} \to \bar{\mathcal{S}}$. However in the codimension 2 case ($r = 1$) we have the following refinement. If instead of a very general member of the family over $\mathbb{P}H^0(\mathbb{P}^{n+1}, \mathcal{O}(D_1) \oplus \mathcal{O}(D_2))$, one fixes a degree-$D_1$ smooth hypersurface $Y \subseteq \mathbb{P}^{n+1}$ and considers a very general $'X_0$ in the family $\mathbb{P}H^0(Y, \mathcal{O}(D_2))$, the vanishing result holds for $D_2$ sufficiently (possibly very) large. While less spectacular this is in fact harder to prove (see [18] for the argument, which extends techniques from [13]).

*That is, $\int_{\delta_i} \hat{\delta}_j = \delta_{ij}$.

The vanishing theorem has an interesting interpretation in terms of Goncharov's [= Beilinson's] real regulator $\mathsf{r} : \mathrm{CH}^n(X_0, n) \to H^n_{\mathcal{D}}(X_0, \mathbb{R}(n)) \cong H^{n-1}(X_0, \mathbb{R})$. Clearly for $n$ even it is just zero, since $H^{n-1}(\mathbb{P}^{n+r}) = 0$ (and so $H^{n-1}(X_0) = H^{n-1}_{\mathrm{var}}(X_0) \cong \underline{H^{n-1}}(\eta_{X_0})$). For $n$ odd let $\Gamma = \gamma^0 + \sum_{i \geqslant 1} \gamma^i \in Z^n(X, n)$ be a higher Chow cycle ($\mathrm{codim}_X\{\mathrm{supp}_X \gamma^i\} = i$), and write $V_\Gamma := \mathrm{supp}_X(\sum_{i \geqslant 1} \gamma^i)$. To state the result, denote the intersection of $\Gamma \subset X \times \square^n \subset \mathbb{P}^{n+r} \times \square^n$ with $\mathbb{P}^{(n+1)/2+r} \times \square^n$ by $\Gamma \cdot [H]^{(n-1)/2}$; and note that $\pi_\square([H]^{(n-1)/2} \cdot \Gamma) \in Z^{(n+1)/2}(\mathbb{C}, n)$. Also write $[H]$ for the homology class of a hyperplane section of $X$, and $\tilde{\mathsf{r}}^n_\square$ for the real $n$-current on $\square^n$ defined in [11] (as modified slightly in [18]).

Then $\mathsf{r}(\Gamma)$ is computed by a real $(n-1)$-current $\tilde{\mathsf{r}}_\Gamma$ on $X_0$ whose periods are

(i) $\int_{[H]} \tilde{\mathsf{r}}_\Gamma = \int_{\pi_\square([H]^{(n-1)/2} \cdot \Gamma)} \tilde{\mathsf{r}}^n_\square =$ a value of Goncharov's Chow $((n+1)/2)$-logarithm (conjecturally computable in terms of $\mathcal{L}_{(n+1)/2}$),

(ii) $\int_\sigma \tilde{\mathsf{r}}_\Gamma = 0$ for all $\sigma$ avoiding $V_\Gamma \subset X$
(which, together with $[H]$, span $H_{n-1}(X_0, \mathbb{Q})$ by Proposition 8.1).

This result is consistent with the Beilinson conjecture (see for instance [26]), which predicts the nontriviality of [the covolume of] the image of $\mathsf{r}$ for $X$ defined over a number field (and therefore *not* very general). If $X$ is a complete intersection defined over $\mathbb{Q}$, one should except a nontorsion image for the holomorphic Milnor regulator (provided $K_X \geqslant 0$ so that the target group is nonzero).

For $X/\mathbb{Q}$ a version of the Bloch–Beilinson conjecture also predicts injectivity (modulo torsion) of the composition $K^M_n(\mathbb{Q}(X)) \to K^M_n(\mathbb{C}(X)) \to H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Q}(n))$.

## PART 4. RELATIVE MILNOR REGULATORS

For an $(n-1)$-dimensional smooth algebraic variety $X$ with proper subvariety $Y$, one may defined Milnor groups

$$\tilde{K}^M_n(\mathbb{C}(X, Y)) := \frac{\mathbb{Z}\{\mathbb{C}(X, Y)\} \otimes (\otimes^{n-1}\mathbb{Z}\{\mathbb{C}(X)^*\})}{\text{num} \cap \{\text{Steinberg relations } \subset \otimes^n \mathbb{Z}\{\mathbb{C}(X)^*\}\}},$$

where $\mathbb{C}(X, Y) \subseteq \mathbb{C}(X)^*$ is the multiplicative subgroup consisting of functions $\equiv 1$ along $Y$. The association $\mathbf{f} \mapsto R_{\mathbf{f}}$ gives a well-defined map

$$R \colon \tilde{K}^M_n(\mathbb{C}(X, Y)) \to \varinjlim_{V \subset (X,Y)} \mathrm{Hom}\{H_{n-1}(X \backslash V, Y \cap X \backslash V; \mathbb{Z}), \mathbb{C}/\mathbb{Q}(n)\},$$

where $V \subset (X, Y)$ means $V$ intersects $Y$ properly.

### 9. The Simplest Nontrivial Regulator Computation

To give a simple demonstration of a Milnor regulator computation we work on a degenerate elliptic curve – or what is the same, the relative variety $(\mathbb{P}^1, \{0, \infty\})$.

For $\mathbf{f} = \sum \ell_\alpha f_\alpha \otimes g_\alpha \in \mathbb{Z}\{\mathbb{C}(\mathbb{P}^1, \{0, \infty\})\} \otimes \mathbb{Z}\{\mathbb{C}(\mathbb{P}^1)^*\}$, assuming for simplicity $|(f_\alpha)| \cap |(g_\alpha)| = \emptyset \forall \alpha$, we say $\{\mathbf{f}\} \in \ker(\mathrm{Tame})$ if $\prod_\alpha (f_\alpha(p)^{\nu_p(g_\alpha)}/g_\alpha(p)^{\nu_p(f_\alpha)})^{\ell_\alpha} = 1$ for all $p \in V_{\mathbf{f}} = \bigcup_\alpha |(f_\alpha)| \cup |(g_\alpha)| \bigcap \mathbb{C}^*$. The regulator image of such an element is determined entirely by the value* $\int_{\mathcal{C}} R_{\mathbf{f}} \in \mathbb{C}/\mathbb{Q}(2)$, where $\mathcal{C}$ is any generator of $H_1(\mathbb{P}^1, \{0, \infty\}; \mathbb{Z})$ avoiding $V_{\mathbf{f}}$.

We shall compute this value for the element $\{\mathbf{f}\} \in \ker(\mathrm{Tame}) \subseteq \tilde{K}_2^M(\mathbb{C}(\mathbb{P}^1, \{0, \infty\}))$ define by $\mathbf{f} = f^2 \otimes g^2$, where

$$f = \frac{1 - \mathrm{i}/w}{1 + \mathrm{i}/w} \quad \text{and} \quad g = \frac{1 - w}{1 + w}.$$

Let $\gamma_\epsilon$ be the path along $\mathbb{R}^-$ from $\{0\}$ to $\{\infty\}$, perturbed by $\mathrm{e}^{-\mathrm{i}\epsilon}$ to avoid $\{1\} \in |(g)|$. (There will be no need to take $\epsilon \to 0$.)

We want to evaluate

$$\int_{\gamma_\epsilon} R_{\mathbf{f}} = \int_{\gamma_\epsilon} \log f^2 \,\mathrm{dlog}\, g^2 - \sum_{\gamma_\epsilon \cap T_{f^2}} 2\pi\mathrm{i} \log g$$

but do not want to deal with the branch $\log f^2$ or [the point] $\gamma_\epsilon \cap T_{f^2}$. (Here $T_{f^2}$ is the preimage of $\mathbb{R}^-$ under $f^2$, which is the unit circle; $\log g$ blows up very close to the intersection.) Take the $\mathbb{Z}(1)$-valued 0-current $\Delta_f^2 := 2\log f - \log f^2$ with $(1/2\pi\mathrm{i})\,\mathrm{d}[\Delta_f^2] = \delta_{T_{f^2}} - 2\delta_{T_f}$, and observe that on $\mathbb{P}^1$

$$\mathrm{d}[\Delta_f^2 \cdot \log g^2] = 4(\log f \,\mathrm{dlog}\, g - 2\pi\mathrm{i} \log g \cdot \delta_{T_f}) - R_{\mathbf{f}}$$

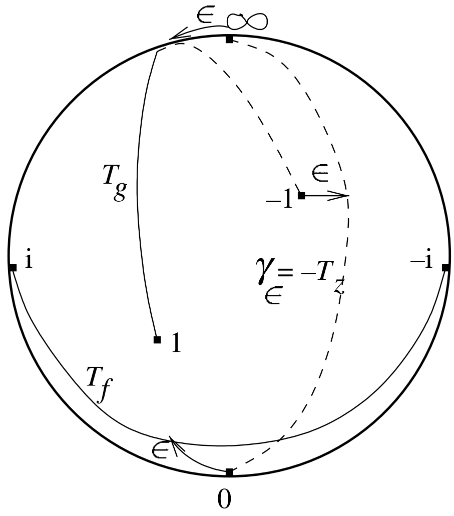
modulo the $\mathbb{Z}(2)$-valued 1-current $2\pi\mathrm{i}\{2\Delta_g^2 \cdot \delta_{T_f} - \Delta_f^2 \cdot \delta_{T_{g^2}}\}$. Since $\gamma_\epsilon \cap T_f = \{0\}$ and $\log g = 0$ there, it follows that modulo $\mathbb{Z}(2)$

$$\int_{\gamma_\epsilon} R_{\mathbf{f}} = 4 \int_{\gamma_\epsilon} \log f \,\mathrm{dlog}\, g.$$

Up to this point our branches of log have strictly had $\arg \in (-\pi, \pi]$. It makes no difference for the computation of the right hand side. if we now perturb $T_f$ and $T_g$ (and the accompanying branches of log) slightly so they avoid** $\{0\}$, $\{\infty\}$, and $\gamma_\epsilon$. We shall also use the slightly altered branch of $\log z$ corresponding to $T_z = -\gamma_\epsilon$. Here is the resulting 'perturbed' picture:

*Note: it is o.k. to replace $\mathbb{Q}(2)$ by $\mathbb{Z}(2)$ here.

**While remaining paths (resp.) from $\{-\mathrm{i}\}$ to $\{\mathrm{i}\}$ and $\{-1\}$ to $\{1\}$.




Now

$$\begin{aligned} 0 &= \int_{\mathbb{P}^1} \mathrm{d}\Big[ -\frac{1}{2\pi \mathrm{i}} \log z \log f \,\mathrm{dlog}\, g + \log z \log g|_{T_f} \Big] \\ &= \int_{\mathbb{P}^1} \{ \log f \,\mathrm{dlog}\, g|_{\gamma_\epsilon} - \log z \,\mathrm{dlog}\, g|_{T_f} + \log z \,\mathrm{dlog}\, g|_{T_f} + \\ &\qquad + \log g \,\mathrm{dlog}\, z|_{T_f} - \log z \log f|_{(g)} + \log z \log g|_{(f)} + \\ &\qquad + 2\pi \mathrm{i} \log z|_{T_f \cap T_g} - 2\pi \mathrm{i} \log\, g|_{T_f \cap \gamma_\epsilon} \} \\ &= \int_{\gamma\epsilon} \log f \,\mathrm{dlog}\, g + \int_{T_f} \log g \,\mathrm{dlog}\, z - \sum_{(g)} \log z \log f + \sum_{(f)} \log z \log g, \end{aligned}$$

since the two intersections are empty. Multiplying by 4, the last two terms are $\mathbb{Z}(2)$. Now $\log g = \log(1-z) - \log(1+z)$ exactly (i.e., *not* mod $\mathbb{Z}(1)$), and so

$$\int_{\gamma_\epsilon} \log f \,\mathrm{dlog}\, g = -\int_{T_f} \log(1-z)\,\mathrm{dlog}\, z + \int_{T_f} \log(1+z)\,\mathrm{dlog}\, z.$$

Since the dilogarithm has no monodromy about 0, this

$$\begin{aligned} &= -2 \int_{T_f} \log(1-z)\,\mathrm{dlog}\, z = 2(\mathrm{Li}(\mathrm{i}) - \mathrm{Li}(-\mathrm{i})) \\ &= 2\Big( \sum_{k=1}^{\infty} \frac{\mathrm{i}^k}{k^2} - \sum_{k=1}^{\infty} \frac{(-\mathrm{i})^k}{k^2} \Big) = 4\mathrm{i} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}, \end{aligned}$$

and four times this is the final result: $16\mathrm{i} \cdot G$, where $G$ is the Catalan constant (a famous transcendental number).

The interested reader may consult [18, Section 3.2] for more general relative regulator computations on $(\mathbb{P}^1, \{0, \infty\})$ and $(\mathbb{P}^2$, triangle).

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