CHARACTERISTIC p ANALOGUE OF MODULES WITH FINITE CRYSTALLINE HEIGHT

Victor Abrashkin[1]

Abstract. In the case of local fields of positive characteristic we introduce an analogue of Fontaine's concept of Galois modules with crystalline height $h \in \mathbb{N}$. If $h = 1$ these modules appear as geometric points of Faltings's strict modules. We obtain upper estimates for the largest upper ramification numbers of these modules and prove (under an additional assumption) that these estimates are sharp.

0. Introduction.

Let p be a prime number. Let K be a complete discrete valuation field with perfect residue field k of characteristic p. Choose a separable closure $K_{\rm sep}$ of K and set $\Gamma_K = \mathrm{Gal}(K_{\rm sep}/K)$. Denote by R the valuation ring of K and for any $v > 0$, by $\Gamma_K^{(v)}$ the ramification subgroup of Γ_K with the upper number v.

Suppose, first, that K is of characteristic 0, i.e. K contains $\mathbb{Q}_p$, and consider $e = e(K)$ — the ramification index of K over $\mathbb{Q}_p$. In this situation for $h \in \mathbb{N}$, Fontaine [Fo3] introduced the category of finite $\mathbb{Z}_p[\Gamma_K]$-modules with crystalline height h. Examples of such modules are given by subquotients of crystalline representations of Γ_K with Hodge-Tate filtration of length h or, more specifically, of Galois modules of h-th etale coomology of projective schemes over K with good reduction. If $h = 1$ then the corresponding Galois modules appear as points $G(K_{\rm sep})$ of finite flat p-group schemes (i.e. killed by a power of the endomorphism $p\,\mathrm{id}_G$) G over R . In this case Fontaine [Fo1] proved very important ramification estimate:

if $H \in \mathrm{MG}_K^1$, $p^N H = 0$ *and* $v > e\left(N + \dfrac{1}{p-1}\right) - 1$ *then* $\Gamma_K^{(v)}$ *acts trivially on* H.

This result was generalised in [Ab1] (cf. also [Fo2], [Ab2]):

if H *is a subquotient of crystalline representation of* Γ_K *with the Hodge-Tate filtration of length* $h < p-1$, $p^N H = 0$ *and* $e = 1$ *then for* $v > \left(N + \dfrac{h}{p-1}\right) - 1$, $\Gamma_K^{(v)}$ *acts trivially on* H.

Now suppose that K is of characteristic p and $k \supset \mathbb{F}_q$, where q is a power of p. Introduce an analogue of $\mathbb{Z}_p$. This will be a subring $O = \mathbb{F}_q[[\pi]]$ of R, where $\pi \in R$ is not invertible in R. If E is the fraction field of O in K then denote by $e = e(K/E)$

1991 *Mathematics Subject Classification.* 11S15, 11S20.

Key words and phrases. local fields, ramification filtration, crystalline reprsentations.

[1]Partially supported by EPSRC, GR/S72252/01

the ramification index of K over E. In this situation an analogue of the category of finite flat p-group schemes over R is the category of O-strict modules over R with etale generic fibre. (The concept of O-strict module was introduced in [Fa].) This category was studied in [Ab4], where the following ramification estimate[2] was obtained:

if $H = G(K_{\mathrm{sep}})$, where G is an O-strict module over R and $\pi^N H = 0$ then for $v > e\left(N + \dfrac{1}{q-1}\right) - 1$, $\Gamma_K^{(v)}$ acts trivially on H.

This estimate is a complete analogue of the above Fontaine's estimate for p-group schemes in the mixed characteristic case.

For $h \in \mathbb{N}$, we apply Fontaine's idea from [Fo3] to introduce the category of $O[\Gamma_K]$-modules $\mathrm{MG}^h(O)_K$ with "crystalline height" h. Notice that if $H \in \mathrm{MG}^1(O)_K$ then H appears in the form $G(K_{\mathrm{sep}})$, where G is an O-strict module over R. Then we apply methods from [Ab4] to prove in section 3 the ramification estimate:

if $H \in \mathrm{MG}^h(O)_K$ and $\pi^N H = 0$ then for $v > e\left(N - 1 + \dfrac{hq}{q-1}\right) - 1$, $\Gamma_K^{(v)}$ acts trivially on H.

The proof uses essentially the existence of embedding of any $H \in \mathrm{MG}^h(O)_K$ in a π-divisible group consisting of objects of the category $\mathrm{MG}^h(O)_K$. This statement is parallel to the corresponding statement for $h = 1$ from [Ab4] and is proved in section 2. Finally, we show in section 4 that the above ramification estimates can not be improved if $\binom{-h}{N-1} \not\equiv 0 \bmod p$. Notice that this estimate does not match with the above mentioned estimate for subquotients of crystalline representations in the mixed characteristic case. One can say that in the case of local fields of positive characteristic the Galois modules with finite crystalline height do not give a precise analogue of crystalline representations.

Everywhere in the paper if $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are morphisms then their composition will be denoted as fg, i.e. for any $a \in A$, $(fg)(a) = g(f(a))$.

1. Main notation and results.

1.1. *The categories* $\mathrm{mod}(O)_R$ *and* $\mathrm{MG}(O)_K$.

As in the introduction, let $q \in \mathbb{N}$ be a power of a prime number p. Let $O = \mathbb{F}_q[[\pi]]$ be a ring of formal power series in one fixed indeterminate π and denote by E the fraction field of O. Let R be an O-algebra. Everywhere in the paper R is an integral domain with the fraction field K. Choose a separable closure K_{sep} of K and set $\Gamma_K = \mathrm{Gal}(K_{\mathrm{sep}}/K)$.

Denote by $\sigma = \sigma_q : R \longrightarrow R$ the ring endomorphism of R such that $\sigma(r) = r^q$ for any $r \in R$.

Let $\mathrm{MG}(O)_K$ be the category of O-modules of finite rank with continuous O-linear action of Γ_K. Introduce the category $\mathrm{mod}(O)_R$ of triples $(L, F, [\pi])$, where

- L is a free R-module of finite rank;
- if $L^{(q)} = L \otimes_{(R,\sigma)} R$ then $F : L^{(q)} \longrightarrow L$ is an injective R-linear morphism;

[2]the statement of this result in the Introduction to [Ab4] contains a misprint

• $[\pi] \in \operatorname{End}_R L$ is nilpotent and $F[\pi] = [\pi]^{(q)}F$, where $[\pi]^{(q)} := [\pi] \otimes \mathrm{id} \in \operatorname{End}_R(L^{(q)})$.

If $\mathcal{L} = (L, F, [\pi])$ and $(L_1, F, [\pi])$ are two objects from $\operatorname{mod}(O)_R$ then $\operatorname{Hom}_{\operatorname{mod}(O)_R}(\mathcal{L}, \mathcal{L}_1)$ consists of R-linear morphisms $f : L \longrightarrow L_1$ such that $f^{(q)}F = Ff$ and $f[\pi] = [\pi]f$.

Remark. We have a natural embedding $\operatorname{End}_R L \subset \operatorname{End}_K(L \otimes_R K)$. Therefore, if $\operatorname{rk}_R L = s$ then $[\pi]^s = 0$.

1.2. *Functor* $\mathcal{M}_\Gamma : \operatorname{mod}(O)_R \longrightarrow \operatorname{MG}(O)_K$.

Let $\mathcal{L} = (L, F, [\pi])$ be an object of the category $\operatorname{mod}(O)_R$. Consider the R-algebra $A = A(\mathcal{L}) := \operatorname{Sym}_R L/I$, where the ideal I is generated by the elements $l^q - F(l \otimes 1) \in \operatorname{Sym}_R L$ for all $l \in L$. Because F is injective, for $A_K = A \otimes_R K$, we have $\Omega^1_{A/K} = 0$, A_K is an etale K-algebra and $\operatorname{rk}_R A = \dim_K A_K = q^{\operatorname{rk}_R L}$. In particular, if $G = \operatorname{Spec} A$ then $G(K_{\mathrm{sep}}) = \operatorname{Hom}_{R\text{-alg}}(A, K_{\mathrm{sep}})$ consists of $q^{\operatorname{rk}_R L}$ elements. Notice that G has a natural structure of a group scheme over R given by the comultiplication $\Delta_A : A \longrightarrow A \otimes_K A$ and the counit $e_A : A \longrightarrow K$ such that $\Delta_A(l) = l \otimes 1 + 1 \otimes l$ and $e_A(l) = 0$ for all $l \in L$. Set $[\alpha](l) = \alpha l$ for $\alpha \in \mathbb{F}_q$ and $l \in L$. Introduce $[\pi]_A : A \longrightarrow A$, which is induced by the given above $[\pi] \in \operatorname{End}_R L$. As a result, we obtain a structure of O-comodule on A. Therefore, $G(K_{\mathrm{sep}})$ is an O-module with a natural continuous action of the Galois group Γ_K i.e. $G(K_{\mathrm{sep}}) \in \operatorname{MG}(O)_K$.

Clearly, the correspondence $\mathcal{L} \mapsto G(K_{\mathrm{sep}})$ determines a functor $\mathcal{M}_\Gamma$ from $\operatorname{mod}(O)_R$ to $\operatorname{MG}(O)_K$. As a matter of fact, with the above notation the correspondence $\mathcal{L} \mapsto G$ induces an antiequivalence of the category $\operatorname{mod}(O)_R$ and the category of finite flat p-group schemes G over R with etale generic fibre, zero Verschiebung V_G and a structure of O-module scheme.

1.3. *The categories* $\operatorname{mod}^h(O)_R$ *and* $\operatorname{MG}^h(O)_K$.

Let $h \in \mathbb{N}$. Introduce the category $\operatorname{mod}^h(O)_R$ as a full subcategory in $\operatorname{mod}(O)_R$ consisted of $\mathcal{L} = (L, F, [\pi])$ such that $(\pi \,\mathrm{id}_L - [\pi])^h(L) \subset \operatorname{Im} F$.

Denote by $\operatorname{MG}^h(O)_K$ the full subcategory in $\operatorname{MG}(O)_K$ consisting of $O[\Gamma_K]$-modules $\mathcal{M}_\Gamma(\mathcal{L})$, where $\mathcal{L}$ is an object of the category $\operatorname{mod}^h(O)_R$.

We are going to prove the following three results:

• if $H \in \operatorname{MG}^h(O)_K$ then H can be embedded into a π-divisible group of finite height, consisting of objects of the category $\operatorname{MG}^h(O)_K$;

• if $H \in \operatorname{MG}^h(O)_K$ and $\pi^N H = 0$ then the ramification subgroups $\Gamma_K^{(v)}$ act trivially on H for $v > e(N - 1 + \dfrac{qh}{q-1}) - 1$;

• with the above notation if $\dbinom{-h}{N-1} \not\equiv 0 \bmod p$ then the above ramification estimate is sharp.

Remark. In the context of classical p-group schemes an analogue of the above first result is Raynaud's theorem stating that any finite flat group schems admits embedding into a p-divisible group (even into an abelian scheme). In the context of O-strict modules (the case $h = 1$) this result was proved in [Ab4]. The case of arbitrary h will be proved in the next section by esssentially the same method. It seems our method can be also applied to prove an analogue of this statement for Fontaine's modules of finite cryatalline height in the mixed characteristic case.

2. Embedding into a π-divisible group.

2.1. *The concept of π-divisible group.*

Tate's definition of p-divisible groups in the category of finite flat p-group schemes admits the following interpretation in the categories $\mathrm{MG}^h(O)_K$ and $\mathrm{mod}^h(O)_R$.

A π-divisible group in the category $\mathrm{MG}^h(O)_K$ is an inductive system $\{H_n, i_n\}_{n\geqslant 1}$, where for any $n\in\mathbb{N}$, $H_n\in\mathrm{MG}^h(O)_R$ and $i_n: H_n\longrightarrow H_{n+1}$ are embeddings of $O[\Gamma_K]$-modules such that if $n>m$ and $i_{mn}: H_m\longrightarrow H_n$ is the composition of $i_m,\dots,i_{n-1}$, then we have the short exact sequence

$$0\longrightarrow H_m\xrightarrow{i_{mn}} H_n\xrightarrow{j_{nm}} H_{n-m}\longrightarrow 0$$

and $j_{nm}i_{n-m,n}=\pi^n\,\mathrm{id}_{H_n}$.

The above definition can be also adjusted to the category $\mathrm{mod}^h(O)_R$ by introducing the concept of strict embedding. If $\mathcal{L}=(L,F,[\pi])$ and $\mathcal{L}_1=(L_1,F,[\pi])$ then $i\in\mathrm{Hom}_{\mathrm{mod}(O)_R}(\mathcal{L}_1,\mathcal{L})$ is a strict embedding if it is induced by $i: L_1\longrightarrow L$ such that $L/i(L_1)$ has no R-torsion. Such i gives rise to a natural short exact sequence $0\longrightarrow\mathcal{L}_1\longrightarrow\mathcal{L}\longrightarrow\mathcal{L}_2\longrightarrow 0$ in the category $\mathrm{mod}(O)_R$. Then we can proceed similarly to introduce a $[\pi]$-divisible group as an inductive system $\{\mathcal{L}_n, i_n\}_{n\geqslant 1}$ of objects of the category $\mathrm{mod}^h(O)_R$, where all i_n are strict embeddings.

2.2. *The statement of the first main theorem.*

Theorem A. *If $H\in\mathrm{MG}^h(O)_K$ then there is a π-divisible group $\{H_n,i_n\}_{n\geqslant 1}$ in the category $\mathrm{MG}^h(O)_K$ such that if $N\in\mathbb{N}$ is such that $\pi^N\,\mathrm{id}_H=0$ then there is an embedding of H into H_N in the category $\mathrm{MG}(O)_K$.*

The above theorem is implied by the following theorem.

Theorem A'. *If $\mathcal{L}\in\mathrm{mod}^h(O)_R$ then there is a $[\pi]$-divisible group $\{\mathcal{L}_n,i_n\}_{n\geqslant 1}$ in $\mathrm{mod}^h(O)_R$ such that if $N\in\mathbb{N}$ is such that $[\pi^N]\mathcal{L}=0$ then there is an epimorphic map from $\mathcal{L}_N$ to $\mathcal{L}$ in the category $\mathrm{mod}(O)_R$.*

The proof of theorem A' will be given in nn.2.3-2.6 below.

2.3. Suppose $\mathcal{L}=(L,F,[\pi])\in\mathrm{mod}^h(O)_R$.

Lemma. *There is a unique R-linear $V=V_{\mathcal{L}}: L\longrightarrow L^{(q)}$ such that*
a) $V[\pi]^{(q)}=[\pi]V$;
b) $VF=(\pi\,\mathrm{id}_L-[\pi])^h$;
c) $FV=(\pi\,\mathrm{id}_{L^{(q)}}-[\pi]^{(q)})^h$.

Proof. Because F is injective and $\mathrm{Im}\,F\supset\mathrm{Im}(\pi\,\mathrm{id}_L-[\pi])^h$, there is a unique R-linear V such that $VF=(\pi\,\mathrm{id}_L-[\pi])^h$. Then $V[\pi]^{(q)}F=VF[\pi]=[\pi]VF$ implies that $V[\pi]^{(q)}=[\pi]V$, because F is injective. Similarly,

$$FVF=F(\pi\,\mathrm{id}_L-[\pi])^h=(\pi\,\mathrm{id}_{L^{(q)}}-[\pi]^{(q)})^hF$$

implies the part c) of our lemma.

2.4. *Matrix identities.*

Suppose $\mathcal{L} = (L, F, [\pi])$ is an object of the category $\mathrm{mod}^h(O)_R$ and $V = V_{\mathcal{L}}$ is the morphism from n.2.3. Choose an R-basis $\bar{e} = \{e_b\}_{1\leqslant b\leqslant s}$ of L and consider square matrices $C = (c_{ab}), D = (d_{ab}), \Pi = (\gamma_{ab}) \in M_s(R)$ such that for $1 \leqslant b \leqslant s$,

$$V(e_b) = \sum_a e_a \otimes c_{ab},\ F(e_b \otimes 1) = \sum_a e_a d_{ab},\ \Pi(e_b) = \sum_b e_a \gamma_{ab}.$$

Then in obvious notation

$$V(\bar{e}) = \bar{e} \otimes C,\ F(\bar{e} \otimes 1) = \bar{e}D,\ [\pi](\bar{e}) = \bar{e}\Pi$$

and we have the following rules of composition

$$VF : \bar{e} \xrightarrow{V} \bar{e} \otimes C \xrightarrow{F} \bar{e}DC$$

$$FV : \bar{e} \otimes 1 \xrightarrow{F} \bar{e}D \xrightarrow{V} \bar{e} \otimes CD.$$

The proof of the following proposition is quite straightforward.

Proposition. *Suppose $\bar{e} = (e_b)_{1\leqslant b\leqslant s}$ is a basis of a free R-module L and $D = (d_{ab}), \Pi = (\gamma_{ab}) \in M_s(R)$. Suppose $F : L^{(q)} \longrightarrow L$ is given by the correspondence $\bar{e} \otimes 1 \mapsto \bar{e}D$ and $[\pi] : L \longrightarrow L$ is given via $\bar{e} \mapsto \bar{e}\Pi$. Then $\mathcal{L} = (L, F, [\pi]) \in \mathrm{mod}^h(O)_R$ if and only if*
(1) $D\Pi = \Pi^{(q)}D$, *where* $\Pi^{(q)} = (\gamma_{ab}^q)$;
(2) $\det D \neq 0$;
(3) $C := D^{-1}(\pi E - \Pi)^h \in M_s(R)$;
(4) Π *is nilpotent.*

Remark. a) If above conditions (1)-(3) hold then $V : \bar{e} \mapsto \bar{e}\otimes C$, $CD = (\pi E - \Pi^{(q)})^h$ and $C\Pi^{(q)} = \Pi C$.
b) Because Π is nilpotent, $\det(\pi E - \Pi) \neq 0$ and, therefore, $\det C \neq 0$.

2.5. *Construction of a π-divisible group in* $\mathrm{mod}^h(O)_R$.

For $m \geqslant 1$, let $\bar{e}_m$ be a copy of $\bar{e} = (e_b)_{1\leqslant b\leqslant s}$. Set by definition $\bar{e}_m = \bar{0}$ if $m \leqslant 0$.

For $n \geqslant 1$, construct objects $\mathcal{L}_n = (L_n, F_n, [\pi]_n)$ of the category $\mathrm{mod}^h(O)_R$ as follows.

L_n will be the free R-module of rank 2ns with the basis consisting of all coordinates of the vectors $\bar{e}_1, \ldots, \bar{e}_{2n}$. Define the linear maps $F_n : L_n^{(q)} \longrightarrow L_n$ and $V_n : L_n \longrightarrow L_n^{(q)}$ by the following relations, where $1 \leqslant m \leqslant n$:

$$F_n(\bar{e}_{2m} \otimes 1) = \bar{e}_{2m}D + \bar{e}_{2m-1}$$

$$V_n(\bar{e}_{2m}) = \bar{e}_{2m} \otimes C + \bar{e}_{2m-1} \otimes 1;$$

$$F_n(\bar{e}_{2m-1} \otimes 1) = -\bar{e}_{2m-1}C + \sum_{i\geqslant 0} \bar{e}_{2m-2i}Y_i;$$

$$V_n(\bar{e}_{2m-1} \otimes 1) = -\bar{e}_{2m-1} \otimes D + \sum_{i\geqslant 0} \bar{e}_{2m-2i} \otimes X_i$$

where for $i \geqslant 0$, the matrices $X_i, Y_i \in M_s(R)$ are such that

- $CD + X_0 = \pi^h E$ and $DC + Y_0 = \pi^h E$;
- for $1 \leqslant i \leqslant h$, $X_i = Y_i = (-1)^i \binom{h}{i} \pi^{h-i} E$;
- for $i > h$, $X_i = Y_i = 0$.

Lemma 1. *For $i \geqslant 0$, $DX_i = Y_i D$.*

Proof. It is obviously true if $i \geqslant 1$, because in this case $X_i = Y_i$ are just scalar matrices. If $i = 0$ then

$$Y_0 D = (\pi^h E - DC)D = D(\pi^h E - CD) = DX_0.$$

The lemma is proved.

Lemma 2. $\sum_{i\geqslant 0}[\pi]^i(\bar{e})Y_i = 0.$

Proof. We must prove that $\sum_{i\geqslant 0}\Pi^i Y_i = 0$. But

$$Y_0 = -DC + \pi^h E = -(\pi E - \Pi)^h + \pi^h E = -\sum_{i\geqslant 1}(-1)^i \pi^{h-i}\binom{h}{i}\Pi^i = -\sum_{i\geqslant 1}\Pi^i Y_i.$$

The lemma is proved.

For $1 \leqslant i \leqslant 2n$, set $[\pi]_n(\bar{e}_i) = \bar{e}_{i-2}$.

Proposition. *For any $n \geqslant 1$, $\mathcal{L}_n = (L_n, F_n, [\pi]_n)$ is an object of the category* $\mathrm{mod}^h(O)_R$.

Proof. Clearly, F_n is injective (use that $\det D \neq 0$ and $\det C \neq 0$). It will be sufficient to verify the following two properties:

a) $F_n[\pi]_n = [\pi]_n^{(q)} F_n$;

b) $V_n F_n = (\pi \,\mathrm{id}_{L_{2n}} - [\pi]_n)^h$.

Let $1 \leqslant m \leqslant n$.
Verify a):

$$(F_n[\pi]_n)(\bar{e}_{2m}\otimes 1) = [\pi]_n(F_n(\bar{e}_{2m}\otimes 1) = [\pi]_n(\bar{e}_{2m}D + \bar{e}_{2m-1}) = \bar{e}_{2m-2}D + \bar{e}_{2m-3}$$
$$([\pi]_n^{(q)}F_n)(\bar{e}_{2m}\otimes 1) = F_n(\bar{e}_{2m-2}\otimes 1) = \bar{e}_{2m-2}D + \bar{e}_{2m-3}$$
$$(F_n[\pi]_n)(\bar{e}_{2m-1}\otimes 1) = [\pi]_n(-\bar{e}_{2m-1}C + \sum_{i\geqslant 0}\bar{e}_{2m-2i}Y_i) = -\bar{e}_{2m-3}C + \sum_{i\geqslant 0}\bar{e}_{2(m-1)-2i}Y_i$$
$$([\pi]_n^{(q)}F_n)(\bar{e}_{2m-1}\otimes 1) = F_n(\bar{e}_{2m-3}\otimes 1) = -\bar{e}_{2m-3} + \sum_{i\geqslant 0}\bar{e}_{2(m-1)-2i}Y_i$$

Now verify b):

$$\begin{aligned}(V_nF_n)(\bar{e}_{2m}) &= F_n(V_n(\bar{e}_{2m})) = F(\bar{e}_{2m}\otimes C) + F(\bar{e}_{2m-1}\otimes 1)\\ &= \bar{e}_{2m}DC + \bar{e}_{2m-1}C - \bar{e}_{2m-1}C + \sum_{i\geqslant 0}\bar{e}_{2m-2i}Y_i\\ &= \bar{e}_{2m}(DC + Y_0) + \sum_{i\geqslant 1}\bar{e}_{2m-2i}(-1)^i\pi^{h-i}\binom{h}{i}\\ &= \sum_{h\geqslant i\geqslant 0}\bar{e}_{2m-2i}(-1)^i\pi^{h-i}\binom{h}{i} = (\pi\,\mathrm{id}_{L_{2n}} - [\pi]_n)^h(\bar{e}_{2m});\end{aligned}$$

$$(V_nF_n)(\bar e_{2m-1}) = F_n(V_n\bar e_{2m-1}) = -F(\bar e_{2m-1}\otimes D) + \sum_{i\geqslant 0} F(\bar e_{2m-2i}\otimes X_i)$$

$$= \bar e_{2m-1}CD - \sum_{i\geqslant 0}\bar e_{2m-2i}Y_iD + \sum_{i\geqslant 0}(\bar e_{2m-2i}DX_i + \bar e_{2m-2i-1}X_i)$$

$$= \bar e_{2m-1}(CD+X_0) + \sum_{i\geqslant 1}\bar e_{2m-2i-1}X_i + \sum_{i\geqslant 0}\bar e_{2m-2i}(-Y_iD + DX_i)$$

$$= \sum_{i\geqslant 0}\bar e_{2m-1-2i}(-1)^i\pi^{h-i}\binom{h}{i} = (\pi\,\mathrm{id}_{L_{2n}} - [\pi]_n)^h(\bar e_{2m-1}).$$

The proposition is proved.

Notice that for any $n\geqslant 1$ we have natural strict embeddings $i_n:\mathcal{L}_n\longrightarrow\mathcal{L}_{n+1}$ in the category $\mathrm{mod}^h(O)_R$. Then the above proposition implies the following corollary.

Corollary. *The inductive system $\{\mathcal{L}_n, i_n\}_{n\geqslant 1}$ is a $[\pi]$-divisible group in the category* $\mathrm{mod}^h(O)_R$.

2.6. *Epimorphic map $f:\mathcal{L}_N\longrightarrow\mathcal{L}$.*

For $1\leqslant m\leqslant N$, set $f(\bar e_{2m}) = [\pi]^{N-m}(\bar e) = \bar e\Pi^{N-m}$ and $f(\bar e_{2m-1}) = \bar 0$. This gives an R-linear map $f: L_N\longrightarrow L$. This map is epimorphic because $f(\bar e_{2N}) = \bar e$. It remains to verify that f is a morphism in the category $\mathrm{mod}^h(O)_R$.

Proposition. a) $f[\pi] = [\pi]_N f$;
b) $f^{(q)}F = F_N f$.

Proof. Let $1\leqslant m\leqslant N$. Verify a):

$$(f[\pi])(\bar e_{2m}) = [\pi]^{N+1-m}(\bar e) = f(\bar e_{2m-2}) = ([\pi]_Nf)(\bar e_{2m})$$
$$(f[\pi])(\bar e_{2m-1}) = \bar 0 = f(\bar e_{2m-3}) = ([\pi]_Nf)(\bar e_{2m-1}).$$

Now verify b):

$$(F_Nf)(\bar e_{2m}\otimes 1) = f(\bar e_{2m}D) + f(\bar e_{2m-1}\otimes 1) = [\pi]^{N-m}(\bar e)D = \bar e\Pi^{N-m}D$$
$$(f^{(q)}F)(\bar e_{2m}\otimes 1) = F([\pi]^{N-m}\bar e\otimes 1) = F(\bar e\otimes {\Pi^{(q)}}^{N-m}) = \bar eD{\Pi^{(q)}}^{N-m}$$

and use that $\Pi D = D\Pi^{(q)}$.

Finally, $(f^{(q)}F)(\bar e_{2m-1}\otimes 1) = 0$ and

$$(F_Nf)(\bar e_{2m-1}\otimes 1) = f(-\bar e_{2m-1}C + \sum_{i\geqslant 0}\bar e_{2m-2i}Y_i) = [\pi]^{N-m}\left(\sum_{i\geqslant 0}[\pi]^i(\bar e)Y_i\right) = 0$$

by Lemma 2 from n.2.5. The proposition is proved.

3. Ramification estimates.

Suppose $h\in\mathbb{N}$ and $H\in\mathrm{MG}^h(O)_K$, where $O=O_E$ is the valuation ring of the field of formal Laurent series $E=\mathbb{F}_q((\pi))$, $K=k((u))$ is an extension of E with perfect residue field k, $R=O_K$ is the valuation ring of K and e is the ramification index of the extension K/E.

Theorem B. *If $N\in\mathbb{N}$ is such that $\pi^N H=0$ then for*

$$v>e\left(N-1+\frac{hq}{q-1}\right)-1$$

the ramification subgroup $\Gamma_K^{(v)}$ acts trivially on H.

The proof of Theorem B follows the strategy from [Ab4] (where the case $h=1$ was considered) and will be given in nn.3.1-3.4 below.

3.1. We can assume that H is a π^N-torsion part of a π-divisible group in the category $\mathrm{MG}^h(O)_K$. So, if H comes from $\mathcal{L}=(L,F,[\pi])\in\mathrm{mod}^h(O)_R$ then we can choose an R-basis of L consisting of elements of vectors $\bar e_1,\dots,\bar e_N$, where for $i=1,\dots,N$, each $\bar e_i$ is a copy of $\bar e=(e_1,\dots,e_s)$ and it holds $[\pi](\bar e_1)=\bar 0$, $[\pi](\bar e_2)=\bar e_1$, … ,$[\pi](\bar e_N)=\bar e_{N-1}$. Then the structure of an object of the category $\mathrm{mod}^h(O)_R$ on $\mathcal{L}$ is given via matrices $C_1,\dots,C_N\in M_s(R)$, where $\det(C_1)\neq 0$ and

$$\begin{aligned}
&F(\bar e_1\otimes 1)C_1=\pi^h\bar e_1\\
&F(\bar e_1\otimes 1)C_1+F(\bar e_1\otimes 1)C_2=\pi^h\bar e_2-\binom{h}{1}\pi^{h-1}\bar e_1\\
&\dots\dots\dots\dots\dots\dots\\
&F(\bar e_N\otimes 1)C_1+\dots+F(\bar e_1\otimes 1)C_N=\pi^h\bar e_N\\
&\qquad\qquad+\dots+(-1)^i\binom{h}{i}\pi^{h-i}\bar e_{N-i}+\dots+(-1)^h\bar e_{N-h}
\end{aligned}$$

with the agreement $\bar e_i=\bar 0$ if $i\leqslant 0$.

Then H is a set of K_{sep}-points of the K-scheme $\mathcal{B}$ given by the equations

$$\begin{aligned}
&\bar X_1^qC_1=\pi^h\bar X_1\\
&\bar X_2^qC_1+\bar X_1^qC_2=\pi^h\bar X_2-\binom{h}{1}\pi^{h-1}\bar X_1\\
&\dots\dots\dots\dots\dots\dots\\
&\bar X_N^qC_1+\dots+\bar X_1^qC_N=\pi^h\bar X_N\\
&\qquad\qquad+\dots+(-1)^i\binom{h}{i}\pi^{h-i}\bar X_{N-i}+\dots+(-1)^h\bar X_{N-h}
\end{aligned}\tag{1}$$

Here $\bar X_1,\dots,\bar X_N$ are copies of the vector $\bar X$, which contains as its coordinates s independent variables and by definition $\bar X_i=\bar 0$ if $i\leqslant 0$.

3.2 *Auxilliary field K_α*, [Ab3].

Let α be a rational positive number with zero p-adic valuation. Then there are $m\in\mathbb{N}$, $\gcd(m,p)=1$, and $M\in\mathbb{N}$ such that $\alpha=m/(q^M-1)$. Notice that for a given α, the corresponding numbers m and M are not unique and can be chosen to be arbitrarily large if neccessary.

Then there is a field extension K_α of K with the same residue field k such that $[K_\alpha : K] = q^M$ and its Herbrand function equals

$$\varphi_{K_\alpha/K}(x) = \begin{cases} x, & \text{if } 0 \leqslant x \leqslant \alpha; \\ \alpha + \dfrac{x-\alpha}{q^M}, & \text{if } x \geqslant \alpha. \end{cases}$$

From the construction of this field extension K_α/K, cf.[Ab3], it follows the existence of a uniformising element u_α of K_α such that $v_K(u - u_\alpha^{q^M}) = \alpha(1-q^{-M})$, where v_K is the valuation on K such that $v_K(u) = 1$.

Introduce the field isomorphism $h_\alpha : K \longrightarrow K_\alpha = k((u_\alpha))$ such that $h_\alpha(u) = u_\alpha$ and $h|_k = \text{id}$. We have the following property:

$$\forall a \in R,\ v_K(a - h_\alpha(a)^{q^M}) \geqslant 1 + \alpha(1 - q^{-M}).$$

Agree to use the following notation:
- if $a \in K$ then $\tilde{a} := a - h_\alpha(a)^{q^M}$;
- $\pi_\alpha := h_\alpha(\pi) = \pi_\alpha$;
- $\alpha^* = \alpha(1 - q^{-M})$.

Notice that for any $a \in K$, it holds $v_K(\tilde{a}) \geqslant v_K(a) + \alpha^*$.

3.3. Consider the K_α-scheme $\mathcal{B}_\alpha$ given by the equations

$$\begin{aligned} &\bar{Y}_1^q h_\alpha(C_1) = \pi_\alpha^h \bar{Y}_1 \\ &\bar{Y}_2^q h_\alpha(C_1) + \bar{Y}_1^q h_\alpha(C_2) = \pi_\alpha^h \bar{Y}_2 - \binom{h}{1} \pi_\alpha^{h-1} \bar{Y}_1 \\ (2) \qquad &\dots\dots\dots\dots\dots\dots \\ &\bar{Y}_N^q h_\alpha(C_1) + \dots + \bar{Y}_1^q h_\alpha(C_N) = \pi_\alpha^h \bar{Y}_N + \dots + (-1)^i \binom{h}{i} \pi_\alpha^{h-i} \bar{Y}_{N-i} \\ &\qquad\qquad + \dots + (-1)^h \bar{Y}_{N-h} \end{aligned}$$

Here $\bar{Y}_1, \dots, \bar{Y}_N$ are copies of the vector $\bar{Y}$, which contains as its coordinates s independent variables, and by definition $\bar{Y}_i = \bar{0}$ if $i \leqslant 0$.

Denote by $\bar{h}_\alpha : K_{\text{sep}} \longrightarrow K_{\text{sep}}$ any field isomorphism, which extends the field isomorphism h_α from n.3.2. Clearly, $\bar{h}_\alpha$ induces a one-to-one map between the points $\mathcal{B}(K_{\text{sep}})$ and $\mathcal{B}_\alpha(K_{\text{sep}})$. In particular, if L, resp. L_α, is the field of definition of all points from $\mathcal{B}(K_{\text{sep}})$, resp. $\mathcal{B}_\alpha(K_{\text{sep}})$, then $\bar{h}_\alpha(L) = L_\alpha$.

Lemma. a) *If* $\alpha^* > e\left(N - 1 + \dfrac{hq}{q-1}\right) - 1$ *and* $(\bar{Y}_1^0, \dots, \bar{Y}_N^0) \in \mathcal{B}_\alpha(K_{\text{sep}})$ *then there is a unique* $(\bar{X}_1^0, \dots, \bar{X}_N^0) \in \mathcal{B}(K_{\text{sep}})$ *such that* $v_K(\bar{X}_i^0 - \bar{Y}_i^{0q^M}) > \dfrac{eh}{q-1}$ *for all* $1 \leqslant i \leqslant N$.

b) *With the above notation the correspondence*

$$(\bar{Y}_1^0, \dots, \bar{Y}_N^0) \mapsto (\bar{X}_1^0, \dots, \bar{X}_N^0)$$

gives a one-to-one map between the points of $\mathcal{B}_\alpha(K_{\text{sep}})$ *and* $\mathcal{B}(K_{\text{sep}})$.

This Lemma implies obviously the following corollary.

Corollary. *If* $\alpha^* > e\left(N-1+\dfrac{hq}{q-1}\right)-1$ *then* $LK_\alpha = L_\alpha$.

Proof of Lemma. Prove first the part a).

For $1 \leqslant i \leqslant N$, let $\bar{Z}_i = \bar{X}_i - \bar{Y}_i^{0q^M}$, $\widetilde{C}_i = C_i - h_\alpha(C_i)^{(q^M)}$ and $\tilde{\pi}_{\alpha i} = \pi_\alpha^i - h_\alpha(\pi^i)^{q^M}$. Then $\bar{Z}_1, \dots, \bar{Z}_N$ satisfy the following equations (where by definition $\bar{Z}_i = \bar{0}$ if $i \leqslant 0$)

$$\begin{aligned}
&\bar{Z}_1^q C_1 - \pi^h \bar{Z}_1 = \bar{F}_1\\
&\bar{Z}_2^q C_1 + \bar{Z}_1^q C_2 = \pi^h \bar{Z}_2 - \binom{h}{1}\pi^{h-1}\bar{Z}_1 + \bar{F}_2\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \qquad (3)\\
&\bar{Z}_N^q C_1 + \dots + \bar{Z}_1^q C_N = \pi^h \bar{Z}_N + \dots + (-1)^i\binom{h}{i}\pi^{h-i}\bar{Z}_{N-i}\\
&\qquad\qquad + \dots + (-1)^h \pi^{N-h}\bar{Z}_{N-h} + \bar{F}_N
\end{aligned}$$

where

$$\begin{aligned}
&\bar{F}_1 = \tilde{\pi}_{\alpha h}\bar{Y}_1^{0q^M} - \bar{Y}_1^{0q^{M+1}}\widetilde{C}_1\\
&\bar{F}_2 = \tilde{\pi}_{\alpha h}\bar{Y}_2^{0q^M} - \binom{h}{1}\tilde{\pi}_{\alpha,h-1}\bar{Y}_1^{0q^M} - (\bar{Y}_2^{0q^{M+1}}\widetilde{C}_1 + \bar{Y}_1^{0q^M}\widetilde{C}_2)\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\bar{F}_N = \tilde{\pi}_{\alpha h}\bar{Y}_N^{0q^M} + \dots + \binom{h}{i}\tilde{\pi}_{\alpha,h-i}\bar{Y}_{N-i}^{0q^M} + \dots + (-1)^{h-1}h\tilde{\pi}_{\alpha 1}\bar{Y}_{N-h+1}^{0q^M} -\\
&\qquad - (\bar{Y}_N^{0q^{M+1}}\widetilde{C}_1 + \dots + \bar{Y}_1^{0q^{M+1}}\widetilde{C}_N)
\end{aligned}$$

Notice that by the choice of α,

$$v_K(\bar{F}_i) = \min\{v_K(\text{coordinates of } \bar{F}_i\} > e\left(N-1+\frac{hq}{q-1}\right)$$

Prove that the system (3) has a unique solution $\bar{Z}_1^0, \dots, \bar{Z}_N^0$ such that all $v_K(\bar{Z}_i) > \dfrac{eh}{q-1}$. Let $\pi_1 \in K_{\text{sep}}$ be such that $\pi_1^{q-1} = \pi$ and for $1 \leqslant i \leqslant N$, $\bar{W}_i = \bar{Z}_i/\pi_1^h$ and $\bar{G}_i = \bar{F}_i/\pi_1^{qh}$. Then system (3) can be rewritten in the following form

$$\begin{aligned}
&\bar{W}_1^q C_1 - \bar{W}_1 = \bar{G}_1\\
&\bar{W}_2 C_1 - \bar{W}_2 = -\frac{h}{\pi}\bar{W}_1 - \bar{W}_1^q C_1 + \bar{G}_2\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&\bar{W}_N^q C_1 - \bar{W}_N = -\frac{1}{\pi}\binom{h}{1}\bar{W}_{N-1} + \dots + \frac{(-1)^i}{\pi^i}\binom{h}{i} + \dots + \frac{(-1)^h}{\pi^h}\bar{W}_{N-h}\\
&\qquad - (\bar{W}_{N-1}^q C_2 + \dots + \bar{W}_1^q C_N) + \bar{G}_N
\end{aligned}$$

Then the inequality $v_K(\bar{G}_1) > e(N-1) \geqslant 0$ implies the existence of a unique solution $\bar{W}_1^0$ of the first equation such that $v_K(\bar{W}_1^0) > 0$. If $N = 1$ then the lemma is proved. If $N > 1$ then $v_K(W_1^0) = v_K(\bar{G}_1) > e(N-1) \geqslant e$ and we obtain a unique solution $\bar{W}_2^0$ of the second equation such that $v_K(\bar{W}_2^0) > 0$. Again if $N = 2$ then the lemma is proved. If $N > 2$ use that $v_K(\bar{W}_2^0) > e(N-2) \geqslant e$ and continue similarly. This means that the above system has a unique solution $\bar{W}_1^0, \dots, \bar{W}_N^0$ such that all $v_K(\bar{W}_i) > 0$. But this is equivalent to the statement of our lemma because for $1 \leqslant i \leqslant N$, $\bar{Z}_i = \pi_1^h \bar{W}_i$. The part a) of lemma is proved.

The part b) follows from the following observation. Any two solutions $(\bar{X}_1^0, \dots, \bar{X}_N^0)$ and $(\bar{X}_1^1, \dots, \bar{X}_N^1)$ of (1) such that for $1 \leqslant i \leqslant N$, $v_K(\bar{X}_i^0 - \bar{X}_i^1) > eh/(q-1)$, must coincide.

The lemma is proved.

3.4. For any finite extension $A \subset B$ of complete discrete valuation fields A and B with perfect residue fields denote by $v(B/A)$ the biggest ramification number of this extension. This is the second coordinate of the last corner of the graph of the Herbrand function $\varphi_{B/A}$ and it can be charcaterized by the the following property:

($*$) *the ramification subgroup* $\Gamma_A^{(v)}$ *acts trivially on* B *if and only if* $v > v(B/A)$.

The existence of the field isomorphism $\bar{h}_\alpha$ from n.3.3 implies that we have the equality of the Herbrand functions $\varphi_{L/K} = \varphi_{L_\alpha/K_\alpha}$ and, therefore, $v(L/K) = v(L_\alpha/K_\alpha)$. If $L_\alpha = LK_\alpha$ then clearly the above condition ($*$) implies that (because $\alpha = v(K_\alpha/K)$)

$$v(L_\alpha/K) = \max\{v(L/K), \alpha\}.$$

On the other hand, if we apply the composition property of Herbrand's functions

$$\varphi_{L_\alpha/K}(x) = \varphi_{K_\alpha/K}(\varphi_{L_\alpha/K_\alpha}(x))$$

where $x \geqslant 0$, to their last corner points then we obtain

$$v(L_\alpha/K) = \max\{\alpha, \varphi_{K_\alpha/K}(v(L/K))\}.$$

Now suppose that

$$v = v(L/K) > e\left(N - 1 + \frac{qh}{q-1}\right) - 1.$$

Choose $\alpha \in \mathbb{Q}$ and the corresponding $M \in \mathbb{N}$, cf. n.3.2, such that

$$v > \alpha > \alpha^* > e\left(N - 1 + \frac{qh}{q-1}\right) - 1.$$

Then by Corollary from n.3.3 we have $L_\alpha = LK_\alpha$ and

$$v(L_\alpha/K) = \max\left\{\alpha, \alpha + \frac{v-\alpha}{q^M}\right\} = \alpha + \frac{v-\alpha}{q^M} < v = \max\{v, \alpha\} = v(L_\alpha/K).$$

The contradiction. Therefore, the above assumption about $v = v(L/K)$ is false and our Theorem is completely proved.

4. Computation of an upper ramification number.

As earlier, $h, N \in \mathbb{N}$, $K = k((u))$ with perfect k of characteristic p, q is a power of p, $E = \mathbb{F}_q((\pi))$ is a subfield in K, R and O are valuation rings in K and, resp. E, and $e = e(K/E)$ is the ramification index of the field extension K/E.

4.1 *The statement of the main result.*

Introduce $\mathcal{L} = (L, F, [\pi]) \in \mathrm{mod}^h(O)_R$ as follows.

Let L be a free R-module with the basis $e_1, \dots, e_N, e_1^0, \dots, e_N^0$. Define $[\pi] \in \mathrm{End}_R L$ by the relations $[\pi](e_n^0) = e_{n-1}^0$ and $[\pi](e_n) = e_{n-1}$, where $1 < n \leqslant N$. Define an R-linear morphism $F : L^{(q)} \longrightarrow L$ by the following relations:

$$F(e_n^0 \otimes 1) = e_n^0, \quad F(e_n \otimes 1) = \sum_{0 \leqslant j \leqslant h} (-1)^j \binom{h}{j} \pi^{h-j} e_{n-j} - u e_n^0,$$

where $1 \leqslant n \leqslant N$, and by definition $e_n = e_n^0 = 0$ if $n \leqslant 0$. Clearly, for any $1 \leqslant n \leqslant N$, $(\pi \,\mathrm{id}_L - [\pi])^h (e_n^0)$ is an R- linear combination of $F(e_i^0 \otimes 1) = e_i^0$, $1 \leqslant i \leqslant n$, and also $(\pi \,\mathrm{id}_L - [\pi])^h (e_n) = F(e_n \otimes 1) - u e_n^0 \in \mathrm{Im}\, F$. Therefore, $\mathcal{L} = (L, F, [\pi]) \in \mathrm{mod}^h(O)_R$.

Let $\mathcal{L}^{\mathrm{et}} := (L^{\mathrm{et}}, F^{\mathrm{et}}, [\pi]^{\mathrm{et}})$, where L^{et} is the submodule of L generated by $e_1^0, \dots, e_N^0$ and F^{et} and $[\pi]^{\mathrm{et}}$ are induced by F and, resp., $[\pi]$. Then $\mathcal{L}^{\mathrm{et}} \in \mathrm{mod}^0(O)_R \subset \mathrm{mod}^h(O)_R$ and we have a natural embedding of $\mathcal{L}^{\mathrm{et}}$ into $\mathcal{L}$ in $\mathrm{mod}^h(O)_R$. This embedding is strict and gives rise to the following short exact sequence in the category $\mathrm{mod}^h(O)_R$

$$0 \longrightarrow \mathcal{L}^{\mathrm{et}} \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}^{(h)} \longrightarrow 0,$$

where $\mathcal{L}^{(h)} = (L^{(h)}, F^{(h)}, [\pi]^{(h)}) \in \mathrm{mod}^h(O)_R$, $L^{(h)}$ is the free R-module with the basis $e_1^1, \dots, e_N^1$, $[\pi]^{(h)}(e_n^1) = e_{n-1}^1$ and $F^{(h)}(e_n^1) = (\pi \,\mathrm{id}_{L^{(h)}} - [\pi]^{(h)})^h e_n^1$ with $1 \leqslant n \leqslant N$ and $e_0^1 = 0$.

Theorem C. *Suppose* $H = \mathcal{M}_\Gamma(\mathcal{L})$ *and* $\binom{-h}{N-1} \not\equiv 0 \bmod p$. *Then the ramification subgroups* $\Gamma_K^{(v)}$ *act trivially on* H *if and only if* $v > e(N - 1 + hq/(q-1)) - 1$.

Remark. The above result has an analogue in the mixed characteristic case. Namely, if $K \supset \mathbb{Q}_p$ then there is a finite flat group scheme G over $R = O_K$ such that $G = \mathrm{Spec}\, A$, $A = \oplus_{0 \leqslant i < p^N} R[\sqrt[p^N]{v^i}]$ and $v = 1 + u$, where u is a uniformising element of K. This group scheme appears as an extension of the constant etale group scheme $(\mathbb{Z}/p^N\mathbb{Z})_R$ over R via the constant multiplicative group scheme $\mu_{p^N} = \mathrm{Spec}\, R[X]/(X^{p^N} - 1)$ over R. One can verify that $\Gamma_K^{(v)}$ acts trivially on $G(K_{\mathrm{sep}})$ if and only if $v > e(N + 1/(p-1)) - 1$, i.e. Fontaine's estimate from [Fo1] is sharp. The above Theorem C shows that under the additional condition $\binom{-h}{N-1} \not\equiv 0 \bmod p$ the estimate from Theorem B is sharp. If this additional condition does not hold one should expect the existence of better estimates.

The proof of theorem C will be given in the remaining part of the paper.

4.2. Let $X_1, \dots, X_N, Y_1, \dots, Y_N$ be independent variables. Then the $O[\Gamma_K]$-module H appears as the set of all solutions of the following system of equations:

$$X_n^q = X_n, \quad Y_n^q = \sum_{0 \leqslant j \leqslant h} (-1)^j \binom{h}{j} \pi^{h-j} Y_{n-j} - u X_n$$

where $1 \leqslant n \leqslant N$ and by definition $X_n = Y_n = 0$ if $n \leqslant 0$. Notice that the structure of $O[\Gamma_K]$-module on this set of solutions is induced by the usual addition and the action of O is given by the relations $[\pi](X_n) = X_{n-1}$, $[\pi](Y_n) = Y_{n-1}$ and for any $\alpha \in \mathbb{F}_q$, $[\alpha](X_n) = \alpha X_n$, $[\alpha](Y_n) = \alpha Y_n$.

Let $\pi_1 \in K_{\text{sep}}$ be such that $\pi_1^{q-1} = \pi$. For $1 \leqslant n \leqslant N$, set $T_n = Y_n \pi_1^{-h}$. Then the field L_1 of definition of all points of H over $K_1 = K(\pi_1)$ is generated by the coordinates of any solution $(T_1^0, \dots, T_N^0)$ of the following system of equations

$$T_n - T_n^q = \sum_{1 \leqslant j \leqslant h} (-1)^{j-1} \binom{h}{j} \frac{1}{\pi^j} T_{n-j} - \frac{u}{\pi_1^{qh}}, \quad 1 \leqslant n \leqslant N, \tag{4}$$

where by definition $T_n = 0$ if $n \leqslant 0$. Notice that L_1 is the composite of K_1 and the field L of definition of all points of H over K. Because K_1 is tamely ramified over K, our theorem will be proved if we show that

$$v(L_1/K) = e(N - 1 + hq/(q-1)) - 1.$$

4.3. Let v_K be the valuation of K_{sep} normalised by the condition $v_K(u) = 1$.

Proposition. *Suppose* $1 \leqslant n \leqslant N$ *and* $r \in \mathbb{Z}_{\geqslant 0}$ *is such that* $rh < n \leqslant (r+1)h$. *Then*

$$v_K(T_n^0) = -\frac{eh}{q-1} + \frac{1}{q^{r+1}}.$$

Proof. Use induction on $1 \leqslant n \leqslant N$.

If $n = 1$ the statement is obviously true, because $T_1^0 - T_1^{0q} = -u\pi_1^{-qh}$, $v_K(T_1^0) = \dfrac{1}{q} v_K(u\pi_1^{-qh})$ and $r = 0$.

Suppose the proposition is proved for all $n' < n$.

If $n \leqslant h$ then the proposition follows from the equation for T_h^0 and the inequalities $v_K\left(\dfrac{1}{\pi} T_{n-1}^0\right), \dots, v_K\left(\dfrac{1}{\pi^{n-1}} T_1\right) \geqslant -(n-1)e - \dfrac{eh}{q-1} + \dfrac{1}{q} > -\dfrac{ehq}{q-1} + 1 = v_K\left(\dfrac{u}{\pi_1^{qh}}\right)$.

If $n > h$ then $r \geqslant 1$ and we have the following inequalities

$$v_K\left(\frac{1}{\pi} T_{n-1}\right), \dots, v_K\left(\frac{1}{\pi^{h-1}} T_{n-h+1}\right) > v_K\left(\frac{1}{\pi^h} T_{n-h}\right).$$

Indeed, by the induction assumption all terms in the left-hand side are not less than $-(h-1)e + \left(-\dfrac{eh}{q-1} + \dfrac{1}{q^{r+1}}\right)$ and this number is strictly larger than

$$-he + \left(-\frac{eh}{q-1} + \frac{1}{q^r}\right) = v_K\left(\frac{1}{\pi^h} T_{n-h}\right).$$

Therefore, the equation for T_n^0 implies that

$$v_K(T_n^0) = \frac{1}{q} v_K\left(\frac{1}{\pi^h} T_{n-h}\right) = -\frac{eh}{q-1} + \frac{1}{q^{r+1}}.$$

The proposition is proved.

4.4. Let $\alpha=\dfrac{m}{q^M-1}\in\mathbb{Q}$ be a rational number from n.3.2, then we have the field extension K_α of K and a field automorphism $\bar h_\alpha : K_{\rm sep}\longrightarrow K_{\rm sep}$ such that $\bar h_\alpha|_K=h_\alpha$ maps K onto K_α. As earlier, introduce the notation $h_\alpha(\pi)=\pi_\alpha$, $h_\alpha(\pi_1)=\pi_{1\alpha}$, $h_\alpha(u)=u_\alpha$. Obviously, we can assume that $\bar h_\alpha$ is chosen in such a way that $K_{1\alpha}:=K_\alpha(\pi_{1\alpha})=\bar h_\alpha(K_1)$ will coincide with the composite K_1K_α. Set for any $a\in K_1$, $\tilde a=a-h_\alpha(a)^{q^M}$. Then if (as earlier) $\alpha^*=\alpha(1-q^M)$, then $v_K(\tilde u)=1+\alpha^*$. This implies that

$$v_K(\widetilde{u\pi_1^{-qh}})=-\frac{eqh}{q-1}+1+\alpha^*. \tag{5}$$

Also notice that for any $a\in K$, it holds $v_K(\tilde a)\geqslant v_K(a)+\alpha^*$.

If L_1 is the field of definition of all points of $O[\Gamma_K]$-module H over $K_1=K(\pi_1)$ cf. n.4.2, then $\bar h_\alpha(L_1)=L_{1\alpha}$ is the field of definition of all solutions (equivalently, of any solution) of the system of equations

$$T_{n\alpha}-T_{n\alpha}^q=\sum_{1\leqslant j\leqslant h}(-1)^{j-1}\binom{h}{j}\frac{1}{\pi_\alpha^j}T_{j\alpha}-\frac{u_\alpha}{\pi_{1\alpha}^{qh}},\quad 1\leqslant n\leqslant N, \tag{6}$$

over the field $K_{1\alpha}$.

Fix a solution $(T^0_{1\alpha},\dots,T^0_{N\alpha})$ of this system and introduce for $1\leqslant n\leqslant N$, new variables Z_n such that $Z_n=T_n-T_{n\alpha}^{0q^M}$.

Proposition. *For $i\in\mathbb{Z}_{\geqslant0}$, set $\gamma_i=\binom{-h}{i}$. Then $Z_1,\dots,Z_N$ satisfy the following system of equations*

$$Z_n-Z_n^q=\sum_{1\leqslant i<n}\gamma_i\left(\widetilde{\frac{1}{\pi^i}}\right)T^{0q^{M+1}}_{n-i,\alpha}+\sum_{1\leqslant i<n}\frac{\gamma_i}{\pi^i}Z^q_{n-i}+\sum_{0\leqslant i<n}\gamma_i\left(\widetilde{\frac{u}{\pi_1^{qh}\pi^i}}\right), \tag{7}$$

where $1\leqslant n\leqslant N$.

Proof. Notice first that

$$(\pi\,\mathrm{id}_L-[\pi])^{-h}=\pi^h\sum_{i\geqslant0}\gamma_i\pi^{-i}[\pi]^i$$

and, therefore, system (4) from n.4.2 can be rewritten in the following equivalent form

$$T_n-T_n^q=\sum_{1\leqslant i<n}\frac{\gamma_i}{\pi^i}T^q_{n-i}+\frac{u}{\pi_1^{qh}}\sum_{0\leqslant i<n}\frac{\gamma_i}{\pi^i},1\leqslant n\leqslant N$$

Similarly, the system

$$T^0_{n\alpha}-T^{0q}_{n\alpha}=\sum_{1\leqslant i<n}\frac{\gamma_i}{\pi_\alpha^i}T^{0q}_{n-i,\alpha}+\frac{u_\alpha}{\pi_{1\alpha}^{qh}}\sum_{0\leqslant i<n}\frac{\gamma_i}{\pi_\alpha^i},1\leqslant n\leqslant N$$

is equivalent to system (6). It remains only to substitute $T_n=Z_n+T^{0q^M}_{n\alpha}$ and to use that for all $i\geqslant0$, $\pi^{-i}=\pi_\alpha^{-iq^M}+\widetilde{\pi^{-i}}$ and $\dfrac{u}{\pi_1^{qh}\pi^i}=\left(\dfrac{u_\alpha}{\pi_{1\alpha}^{qh}\pi_\alpha^i}\right)^{q^M}+\left(\widetilde{\dfrac{u}{\pi_1^{qh}\pi^i}}\right)$.

The proposition is proved.

Corollary. *If* $(Z_1^0, \dots, Z_N^0) \in K_{\mathrm{sep}}^N$ *is a solution of the above system (7) then* $LL_\alpha = L_\alpha(Z_1^0, \dots, Z_N^0)$.

4.5. Until the end of this paper we assume that $\alpha = \dfrac{m}{q^M - 1} \in \mathbb{Q}$ from the above n.4.4 is chosen such that if $\alpha(1-q^{-M}) = \alpha^* = e(N-1) + \dfrac{eqh}{q-1} - 1 - \varepsilon(\alpha)$, then $\varepsilon(\alpha) < q^{-N}$. (Use that rational numbers with zero p-adic valuation are dense in the set of all rational numbers). Notice that $\alpha^* q^M = m \in \mathbb{N}$ is prime to p and this implies that the p-adic valuation of $\varepsilon(\alpha)q^M$ is zero.

Proposition 1. *System (7) has a solution* $(Z_1^0, \dots, Z_N^0)$ *such that for all* $1 \leqslant n < N$, $Z_n^0 \in L_\alpha$ *and* $v_K(Z_n^0) > e(N-n) - 1$.

Proof. If $N = 1$ there is nothing to prove.

If $N > 1$ use induction on n. If $n = 1$ then $Z_1 - Z_1^q = \left(\widetilde{\dfrac{u}{\pi_1^{qh}}}\right) := A$ and because $v_K(A) = 1 - eqh/(q-1) + \alpha^* = e(N-1) - \varepsilon(\alpha) > e(N-1) - 1 > 0$, we can take $Z_1^0 = \sum_{i \geqslant 0} A^{q^i}$. Clearly, $v_K(Z_1^0) = e(N-1) - \varepsilon(\alpha) > e(N-1) - 1$.

Suppose $1 < n < N$ and we have chosen the corresponding $Z_1^0, \dots, Z_{n-1}^0$. Then by Proposition of n.4.3 for $1 \leqslant i < n$, we have the following estimates:

$$\begin{aligned} v_K\left(\widetilde{\pi^{-i} T^{0q^{M+1}}_{n-i,\alpha}}\right) &\geqslant -ie + \alpha^* + q\left(-\frac{he}{q-1} + \frac{1}{q^{r+1}}\right) \\ &= e(N-1+i) - 1 - \varepsilon(\alpha) + q^{-r} > e(N-n) - 1, \end{aligned}$$

$$v_K(\pi^{-i} Z_{n-i}^q) > -ie + q(e(N-n+i) - 1) > e(N-n) - 1,$$

$$v_K\left(\widetilde{\frac{u}{\pi_1^{qh}\pi_1^i}}\right) = 1 - \frac{eqh}{q-1} + \alpha^* - ie = e(N-i-1) - \varepsilon(\alpha) > e(N-n) - 1.$$

Therefore, $Z_n - Z_n^q \in L_\alpha$ has the v_K-valuation, which is larger than $e(N-n) - 1 > 0$ and we can choose a solution Z_n^0 such that $v_K(Z_n^0) > e(N-n) - 1$.

The proposition is proved.

Corollary. *With the above notation and assumptions it holds*

$$LL_\alpha = L_\alpha(Z_N^0).$$

Proposition 2. *There is a* $w \in K_{1\alpha}$ *such that* $v_K(w) = -\varepsilon(\alpha)$ *and if* $E_w = K_{1\alpha}(W)$, *where* $W - W^q = w$, *then* $L_\alpha(Z_N^0) = L_\alpha E_w$.

Proof. Z_N^0 satisfies the equation

$$Z_N - Z_N^q = \sum_{1 \leqslant i < N} \gamma_i \widetilde{\pi^{-i} T^{0q^{M+1}}_{N-i,\alpha}} + \sum_{1 \leqslant i < N} \gamma_i \pi^{-i} Z_{N-i}^q + \sum_{0 \leqslant i < N} \gamma_i \left(\widetilde{\frac{u}{\pi_1^{qh}\pi^i}}\right).$$

Use the estimates from above proposition 1 and proposition of n.4.3 to prove that for $i\geqslant 1$, $v_K\left(\widetilde{\pi^{-i}T^{0q^{M+1}}_{N-i,\alpha}}\right)=e(N-i-1)-1-i+q^{-r}>0$ and $v_K\left(\widetilde{\frac{u}{\pi_1^{qh}\pi^i}}\right)>0$ if $i\leqslant N-2$, and $v_K(\pi^{-i}Z^q_{N-i})>e(q-1)i-q>0$ unless $e=i=1$.

Prove that in the remaining case $e=i=1$ we still have that $v_K\left(\frac{1}{\pi}Z^{0q}_{N-1}\right)>0$. Denote by A the right-hand side of the equation for Z_{N-1}. Then $Z^0_{N-1}=\sum_{i\geqslant 0}A^{q^i}$. Using again the above mentioned estimates we obtain that $v_K(A)>1-\varepsilon(\alpha)$. Therefore, $v_K(\pi^{-1}Z^{0q}_{N-1})=-1+qv_K(A)>-1+q(1-\varepsilon(\alpha))>0$.

So, $L_\alpha(Z^0_N)=L_\alpha(W_1)$, where W_1 is a solution of the equation

$$W_1-W_1^q=\gamma_{N-1}\left(\widetilde{\frac{1}{\pi^{N-1}}}\right)T^{0q^{M+1}}_{1\alpha}+\gamma_{N-1}\left(\widetilde{\frac{u}{\pi_1^{qh}\pi^{N-1}}}\right)$$

Denote by m_α the maximal ideal of the valution ring of L_α.

Lemma. *With the above notation*

$$\left(\widetilde{\frac{1}{\pi^{N-1}}}\right)T^{0q^{M+1}}_{1\alpha}+\left(\widetilde{\frac{u}{\pi_1^{qh}\pi^{N-1}}}\right)\equiv\left(\widetilde{\frac{u}{\pi_1^{qh}}}\right)\frac{1}{\pi^{N-1}}\operatorname{mod}\mathrm{m}_\alpha\,.$$

Proof. Notice that $v_K\left(\widetilde{\pi^{1-N}}T^{0q^M}_{1\alpha}\right)=eh-1-\varepsilon(\alpha)+q^{-r-1}>0$. Then the relation $T^0_{1\alpha}-T^{0q}_{1\alpha}=u_\alpha\pi_{1\alpha}^{-qh}$ implies that

$$\left(\widetilde{\frac{1}{\pi^{N-1}}}\right)T^{0q^{M+1}}_{1\alpha}\equiv-\left(\widetilde{\frac{1}{\pi^{N-1}}}\right)\left(\frac{u_\alpha}{\pi_{1\alpha}^{qh}\pi_\alpha^{N-1}}\right)^{q^M}\operatorname{mod}\mathrm{m}_\alpha\,.$$

It remains to notice that with $a=u\pi_1^{-qh}$ and $b=\pi^{1-N}$, we have

$$\widetilde{(ab)}=(a_\alpha^{q^M}+\tilde a)(b_\alpha^{q^M}+\tilde b)=\tilde a b+a_\alpha^{q^M}\tilde b.$$

The lemma is proved.

Finally, notice that if $w=\left(\widetilde{\frac{u}{\pi_1^{qh}}}\right)\frac{1}{\pi_{N-1}}$ then $w\in K_{1\alpha}$ and $v_K(w)=-\varepsilon(\alpha)$.

The proposition is proved.

4.6. Now we can finish the proof of theorem C.

By theorem B, $v:=v(L/K)\leqslant C(h,N):=e(N-1)+\frac{eqh}{q-1}-1$. Suppose this inequality is strict. Choose $\alpha=m/(q^M-1)\in\mathbb{Q}$ from n.4.4, which satisfies the additional conditions from the beginning of n.4.5 and such that
$v<\alpha^*=\alpha(1-q^{-M})<\alpha<\alpha^*+\varepsilon(\alpha)=C(h,N)$ and $\varepsilon(\alpha)q^M>C(h,N)$.

The first above new requirement can be satisfied because $\varepsilon(\alpha)$ can be chosen arbitrarily small and the second one will be satisfied if we choose a sufficiently large M for a given α.

Now proceed as in the proof of theorem B.

Clearly, $v(L/K) = v(L_\alpha/K_\alpha)$. This implies that

$$v(L_\alpha/K) = \max\{\varphi_{K_\alpha/K}(v), \alpha\} = \max\{v, \alpha\} = \alpha$$

and $v(LL_\alpha/K) = \max\{v(L/K), v(L_\alpha/K)\} = \alpha$.

On the other hand, let E_w be the field from proposition 2 of n.4.5. Then $v(L_\alpha E_w/K) = \max(\alpha, v(E/K))$ and $v(E_w/K) = \max(\alpha, \varphi_{K_\alpha/K}(v(E/K_\alpha))$. But $E_w = K_\alpha(W)$ with $W - W^q = w \in K_\alpha$, where $v_{K_\alpha}(w) = q^M v_K(w) = -\varepsilon(\alpha)q^M$ has the zero p-adic valuation. Therefore, $v(E_w/K_\alpha) = \varepsilon(\alpha)q^M > C(h,N) > \alpha$ and

$$\varphi_{K_\alpha/K}(v(E_w/K_\alpha)) = \frac{\varepsilon(\alpha)q^M - \alpha}{q^M} + \alpha = \varepsilon(\alpha) + \alpha^* = C(h,N) > \alpha.$$

Therefore, $\alpha < v(E_w/K) = v(L_\alpha E/K)$. This contradicts to the equality $LL_\alpha = L_\alpha E$.

Therefore, $v = C(h,N) = e(N - 1 + hq/(q-1)) - 1$ and Theorem C is proved.

References

[Ab1] V.Abrashkin, *Ramification in etale cohomology*, Invent. Math. **101** (1990), no. 3, 631-640.

[Ab2] V.Abrashkin, *Modular representations of the Galois group of a local field and a generalization of a conjecture of Shafarevich*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 6, 1135-1182; Engl. transl. in, Math. USSR-Izv. **35** (1990), no. 3, 469-518.

[Ab3] V.Abrashkin, *The ramification filtration of the Galois group of a local field. III*, Izv. Ross. Akad. Nauk Ser. Mat. **62** (1998), no. 5, 3-48; Engl. transl. in, Izv. Math. **62** (1998), no. 5, 857-900.

[Ab4] V.Abrashkin, *Galois modules arising from Faltings's strict modules*, Compositio Math. **142** (2006), 867-888.

[Fa] G.Faltings, *Group schemes with strict $\mathcal{O}$-action*, Moscow Math. J. **2** (2002), no. 2, 249-279.

[Fo1] J.-M.Fontaine, *Il n'y a pas de variete abelienne sur $\mathbb{Z}$*, Inv. Math. **81** (1985), no. 3, 515-538.

[Fo2] J.-M.Fontaine, *Schemas propres et lisses sur $\mathbb{Z}$*, Proc. of Indo-French Conference on Geometry (Bombay, 1989), Hindustan book agency, Delhi, 1993, p. 43-56.

[Fo3] J.-M.Fontaine, *Représentations p-adiques des corps locaux.I.*, The Grothendieck Festschrift, Progr.Math., 87, Birkhauser Boston, Boston, MA, 1990, vol. II, p. 249-309.

[Ga] P. Gabriel, *Etude infinitesimale des schemas en groupes*, Schemas en Groupes I, Lecture notes in Mathematics (Springer-Verlag, eds.), vol. 151, 1970, p. 474-560.

Maths Dept., Durham University, Sci. Laboratories, South Rd., Durham, DH1 3LE, U.K.