Augmented k-ary n-cubes

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Abstract

We define an interconnection network $AQ_{n,k}$ which we call the augmented k-ary n-cube by extending a k-ary n-cube in a manner analogous to the existing extension of an n-dimensional hypercube to an n-dimensional augmented cube. We prove that the augmented k-ary n-cube $AQ_{n,k}$ has a number of attractive properties (in the context of parallel computing). For example, we show that the augmented k-ary n-cube $AQ_{n,k}$: is a Cayley graph, and so is vertex-symmetric, but not edge-symmetric unless $n = 2$; has connectivity $4n-2$ and wide-diameter at most $max\{(n-1)k-(n-2), k+7\}$; has diameter $\lfloor \frac{k}{3} \rfloor + \lceil \frac{k-1}{3} \rceil$, when $n = 2$; and has diameter at most $\frac{k}{4}(n+1)$, for $n \geq 3$ and k even, and at most $\frac{k}{4}(n+1) + \frac{n}{4}$, for $n \geq 3$ and k odd.

keywords: interconnection networks; parallel computing; k-ary n-cubes; augmented cubes.

1 Introduction

Hypercubes are perhaps the most well known of all interconnection networks for parallel computing, given their basic simplicity, their generally desirable topological and algorithmic properties, and the extensive investigation they have undergone (not just in the context of parallel computing but also in discrete mathematics in general; see, for example, [27] for some essential properties of hypercubes). However, a multitude of different interconnection networks have been devised and developed in a continuing search for improved performance, with many of these networks having hypercubes at their roots. Amongst these generalisations of hypercubes are k-ary n-cubes [14], augmented cubes [12], cube-connected cycles [26], twisted cubes [19], twisted n-cubes [18], crossed cubes [16], folded hypercubes [17], Mcubes [30], Möbius cubes [13], generalised twisted cubes [11], shuffle cubes [24], k-skip enhanced cubes [31], twisted hypercubes [22], supercubes [29], and Fibonacci cubes [20].

Perhaps the most popular of these generalisations are the k-ary n-cubes [14]. A k-ary n-cube Q_n^k is essentially a 'k-bit version' of a '2-bit' hypercube in that vertices are represented by n-tuples of integers from $\{0, 1, \ldots, k-1\}$ so that two vertices are joined by an edge if, and only if, their representations are identical save in one bit position, where in that position the bits differ by 1 modulo k (thus, a k-ary 2-cube, for example, is just a $k \times k$ torus). It turns out that k-ary n-cubes have similar properties to hypercubes yet provide more flexibility with regard to incorporating more processors; for the two parameters available, k and n, allow us to regulate the degree of the nodes yet still incorporate large numbers of processors, although usually at a cost to some other property such as the diameter or the connectivity. Some properties of the k-ary n-cube are that it: has k^n vertices and nk^n edges; has diameter $n\lfloor\frac{k}{2}\rfloor$; has wide-diameter $n\lfloor\frac{k}{2}\rfloor + 1$, when $n \geq 3$ or when $n = 2$ and $k \geq 6$ [21]; has connectivity $2n$ [10]; is a Cayley graph, and so is vertex-symmetric [7], and also edge-symmetric [4]; and has an $O(nk)$ time optimal routing algorithm [3, 15]. A number of distributed memory multiprocessors have been built with a k-ary n-cube forming the underlying topology, such as the Mosaic [28], the iWARP [9], the J-machine [25], the Cray T3D [23], the Cray T3E [2], and the IBM Blue Gene [1].

Another generalisation of hypercubes are augmented cubes, recently proposed by Choudum and Sunitha [12] as improvements over hypercubes. Hypercubes and augmented cubes (of the same dimensions) have the same sets of vertices. However, whereas the recursive construction of an n-dimensional hypercube is to take two copies of an $(n-1)$-dimensional hypercube and join corresponding pairs of vertices, the recursive construction of an n-dimensional augmented cube AQ_n is to take two copies of an $(n-1)$-dimensional augmented cube and as well as joining corresponding pairs of vertices, pairs of vertices of Hamming distance $n-1$ are also joined (that is, vertices that are different in every component). Choudum and Sunitha show that an n-dimensional augmented cube AQ_n: has 2^n vertices and $n2^n$ edges; has diameter $\lceil\frac{n}{2}\rceil$; has connectivity $2n-1$; is a Cayley graph and so is vertex-symmetric; and has an $O(n)$ time optimal routing algorithm.

In this paper, and inspired by [12], we extend a k-ary n-cube in a manner analogous to the extension of an n-dimensional hypercube to an n-dimensional augmented cube. Our definition of an *augmented k-ary n-cube* $AQ_{n,k}$, in comparison with that in [12], is not a straightforward generalisation; however, we believe that it does reflect the essence of the extension in [12], and our structural results bear this out. We give two different definitions of an augmented k-ary n-cube in Section 2 and show that they yield the same interconnection network. In Section 3, we show that an augmented k-ary n-cube $AQ_{n,k}$ is vertex-symmetric and, furthermore, a Cayley graph, though not edge-symmetric unless $n = 2$. In Section 4, we show that an augmented k-ary n-cube $AQ_{n,k}$ has connectivity $4n-2$, and that we can build a set of $4n-2$ mutually disjoint paths joining any two distinct vertices so that the path of maximal length has length at most $max\{(n-1)k-(n-2), k+7\}$; that is, $AQ_{n,k}$ has wide-diameter at most $max\{(n-1)k-(n-2), k+7\}$. In Section 5, we examine the diameter of the augmented k-ary n-cube $AQ_{n,k}$ and show that the diameter of the augmented k-ary 2-cube $AQ_{2,k}$ is $\lfloor\frac{k}{3}\rfloor + \lceil\frac{k-1}{3}\rceil$. We also show that the diameter of the augmented k-ary n-cube $AQ_{n,k}$ is at most $\frac{k}{4}(n+1)$, when $n \geq 3$ and k is even, and at most $\frac{k}{4}(n+1)+\frac{n}{4}$, when $n \geq 3$ and k is odd. Our conclusions are presented in Section 6.

2 Basic definitions

We assume throughout that arithmetic on tuple elements is modulo k, and we denote tuples of elements by bold type. Recall the definition of the *k-ary n-cube* Q_n^k: the vertex set $V(Q_n^k)$ is $\{(a_n, a_{n-1}, \ldots, a_1) : 0 \leq a_i \leq k-1\}$; and the edge set $E(Q_n^k)$ is $\{(\mathbf{u}, \mathbf{v}) : \mathbf{u} = (u_n, u_{n-1}, \ldots, u_1), \mathbf{v} = (v_n, v_{n-1}, \ldots, v_1),$ either $u_i = v_i - 1$ or $u_i =$

$v_i + 1$, for some i, and $u_j = v_j$, for all $i \neq j\}$. Whilst we regard all graphs defined in this paper as undirected, our definitions define all edges from the perspective of a given vertex. Thus, in our definition of Q_n^k we define the (undirected) edge $(\mathbf{u}, \mathbf{v})$ twice: once from the perspective of $\mathbf{u}$, as the edge $(\mathbf{u}, \mathbf{v})$; and once from the perspective of $\mathbf{v}$, as the edge $(\mathbf{v}, \mathbf{u})$. The reason we do this is that later we shall define paths in our graphs and an undirected edge will be regarded differently depending upon the direction it is being traversed in the path. The following definition adheres to this convention.

Definition 1 Let $n \geq 1$ and $k \geq 3$ be integers. The *augmented k-ary n-cube $AQ_{n,k}$* has k^n vertices, each labelled by an n-bit string $(a_n, a_{n-1}, \ldots, a_1)$, with $0 \leq a_i \leq k-1$, for $1 \leq i \leq n$. There is an edge joining vertex $\mathbf{u} = (u_n, u_{n-1}, \ldots, u_1)$ to vertex $\mathbf{v} = (v_n, v_{n-1}, \ldots, v_1)$ if, and only if:

- $v_i = u_i - 1$ (resp. $v_i = u_i + 1$), for some $1 \leq i \leq n$, and $v_j = u_j$, for all $1 \leq j \leq n, j \neq i$; call the edge $(\mathbf{u}, \mathbf{v})$ an $(i, -1)$-*edge* (resp. an $(i, +1)$-*edge*); or
- for some $2 \leq i \leq n$, $v_i = u_i - 1, v_{i-1} = u_{i-1} - 1, \ldots, v_1 = u_1 - 1$ (resp. $v_i = u_i + 1, v_{i-1} = u_{i-1} + 1, \ldots, v_1 = u_1 + 1$), $v_j = u_j$, for all $j > i$; call the edge $(\mathbf{u}, \mathbf{v})$ a $(\leq i, -1)$-*edge* (resp. a $(\leq i, +1)$-*edge*).

We emphasise that the graph $AQ_{n,k}$ is undirected but that edges are *labelled* differently, as an $(i, +1)$-edge or as an $(i, -1)$-edge, for example, according to the perceived orientation.

The augmented k-ary n-cube $AQ_{n,k}$ can also be recursively defined as follows (the proof of this fact is a simple induction).

Definition 2 Fix $k \geq 3$. The augmented k-ary 1-cube $AQ_{1,k}$ has vertex set $\{0, 1, \ldots, k-1\}$ and there is an edge joining vertex u to vertex v if, and only if, $v = u + 1$ or $v = u - 1$. Fix $n \geq 2$. Take k copies of an augmented k-ary $(n-1)$-cube $AQ_{n-1,k}$ and for the ith copy, add an extra number i as the nth bit of each vertex (all vertices have the same nth bit if they are in the same augmented k-ary $(n-1)$-cube). Four more edges are added for each vertex, namely the $(n, -1)$-edge, the $(n, +1)$-edge, the $(\leq n, -1)$-edge and the $(\leq n, +1)$-edge (as defined in Definition 1).

With respect to the above definition, we refer to the subgraph of $AQ_{n,k}$ induced by the vertices whose first component is i, for some fixed $i \in \{0, 1, \ldots, k-1\}$, as $AQ_{n-1,k}^i$ (this subgraph is clearly a copy of $AQ_{n-1,k}$).

Clearly, when $n \geq 2$, $AQ_{n,k}$ has k^n vertices, $(2n-1)k^n$ edges, and every vertex has degree $4n - 2$.

We adopt the following notation with regard to identifying specific vertices relevant to a given vertex in $AQ_{n,k}$. Let $\mathbf{v} = (v_n, v_{n-1}, \ldots, v_1)$ be some vertex of $AQ_{n,k}$. For each $i \in \{0, 1, \ldots, k-1\}$ and each $j \in \{1, 2 \ldots, n\}$, we denote the vertex $(v_n, v_{n-1}, \ldots, v_{j+1}, i, v_{j-1}, \ldots, v_1)$ by $\mathbf{v}|_j^i$. For $j \in \{1, 2, \ldots, n\}$, we refer to the neighbour $(v_n, \ldots, v_{j+1}, v_j + 1, v_{j-1}, \ldots, v_1)$ (resp. $(v_n, \ldots, v_{j+1}, v_j - 1, v_{j-1}, \ldots, v_1)$, $(v_n, \ldots, v_{j+1}, v_j + 1, v_{j-1} + 1, \ldots, v_1 + 1)$, $(v_n, \ldots, v_{j+1}, v_j - 1, v_{j-1} - 1, \ldots, v_1 - 1)$) as $\mathbf{v}_{(j,+1)}$ (resp. $\mathbf{v}_{(j,-1)}$, $\mathbf{v}_{(\leq j,+1)}$, $\mathbf{v}_{(\leq j,-1)}$). We can combine our notation as the following example shows: $\mathbf{v}_{(j,+1)}|_n^3$ denotes the vertex obtained by taking the vertex

$\mathbf{v}_{(j,+1)}$ and fixing its nth component at 3 whilst leaving all other components as they were.

Paths in graphs are given as sequences of vertices (on occasion, a path might consist of a solitary vertex). A path in $AQ_{n,k}$ might be specified by the source vertex and a sequence of labels detailing the edges to be traversed, *e.g.*, the path in $AQ_{3,5}$ detailed as having the source vertex $(0,0,0)$ and then following the edges labelled $(\leq 2,+1),(3,-1),(1,+1)$ is actually the path $(0,0,0),(0,1,1),(4,1,1),(4,1,2)$.

The augmented 5-ary 2-cube is depicted in Fig. 1 (in two different ways): in the first drawing, the edges of the underlying 5-ary 2-cube (that is, the $(2,+1)$-edges, the $(2,-1)$-edges, the $(1,+1)$-edges and the $(1,-1)$-edges) are drawn using narrow pen and the "augmented" edges (that is, the $(\leq 2,+1)$-edges and the $(\leq 2,-1)$-edges) are drawn using broad pen; in the second, the $(1,+1)$-edges, the $(1,-1)$-edges, the $(\leq 2,+1)$-edges, and the $(\leq 2,-1)$-edges are drawn using narrow pen and the $(2,+1)$-edges and the $(2,-1)$-edges are drawn using broad pen.

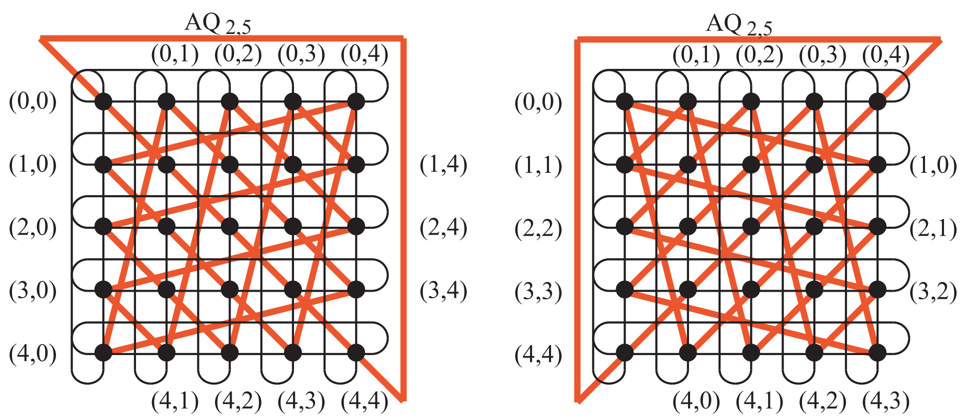

Figure 1. Two views of an augmented 5-ary 2-cube.

3 Symmetry

In this section, we examine $AQ_{n,k}$ as to any symmetric properties it might have. We begin with a useful lemma which will be used to reduce case analyses in subsequent proofs, and the proof of which is trivial (especially given Fig. 1).

Lemma 3 (a) The following are automorphisms of $AQ_{n,k}$:

- (i) the mapping taking the vertex $(v_n, v_{n-1}, \ldots, v_1)$ to $(v_n - a_n, v_{n-1} - a_{n-1}, \ldots, v_1 - a_1)$, where $(a_n, a_{n-1}, \ldots, a_1) \in \{0, 1, \ldots, k-1\}^n$ is fixed;
- (ii) the mapping taking the vertex $(v_n, v_{n-1}, \ldots, v_1)$ to $(\epsilon v_n, \epsilon v_{n-1}, \ldots, \epsilon v_1)$, where $\epsilon \in \{+1, -1\}$ is fixed.

(b) For $i, j \in \{0, 1, \ldots, k-1\}$, the mapping taking the vertex $(i, v_{n-1}, v_{n-2}, \ldots, v_1)$ to $(j, v_{n-1}, v_{n-2}, \ldots, v_1)$ is an isomorphism of $AQ^i_{n-1,k}$ to $AQ^j_{n-1,k}$.

(c) The mapping taking the vertex (i, j) to the vertex (j, i) is an automorphism of $AQ_{2,k}$.

(d) The mapping taking the vertex (i,j) to the vertex $(j-i,j)$, if $i \leq j$, and the vertex (i,j) to the vertex $(k-(i-j),j)$, if $i > j$, is an automorphism.

A graph is *vertex-symmetric* (also known as *vertex-transitive*) if it has an automorphism mapping any given vertex to any other given vertex. The property of a graph being vertex-symmetric is important when that graph is used as an interconnection network for parallel computing, for having a vertex-symmetric interconnection network makes parallel algorithm design and topological analysis easier, as well as allowing flexibility in, for example, linear array simulations.

An immediate corollary of Lemma 3 is the following.

Corollary 4 The augmented k-ary n-cube $AQ_{n,k}$ is vertex-symmetric.

Proof Given vertices $\mathbf{u} = (u_n, u_{n-1}, \ldots, u_1)$ and $\mathbf{v} = (v_n, v_{n-1}, \ldots, v_1)$ of $AQ_{n,k}$, by Lemma 3, the mapping taking an arbitrary vertex $(w_n, w_{n-1}, \ldots, w_1)$ to $(w_n - (u_n - v_n), w_{n-1} - (u_{n-1} - v_{n-1}), \ldots, w_1 - (u_1 - v_1))$ is an automorphism mapping $\mathbf{u}$ to $\mathbf{v}$. $\square$

However, we can do better. Let Γ be a finite group and let $S \subseteq \Gamma$ be a set of generators of Γ not containing the identity and closed under inversion; that is, $s^{-1} \in S$ whenever $s \in S$. The simple undirected graph $G(\Gamma, S)$ with vertex set Γ and where two vertices g and h are adjacent if, and only if, $gh^{-1} \in S$, is called the *Cayley graph of* Γ (with generating set S). Knowledge that an interconnection network is a Cayley graph not only immediately yields that the graph is vertex-symmetric but also provides an algebraic description of the graph that will be useful in, for example, developing routing algorithms.

Let $(Z_k)^n$ denote the n-fold Cartesian product of the group $(Z_k, \oplus_k)$, where $Z_k = \{0, 1, \ldots, k-1\}$ and where $\oplus_k$ denotes addition modulo k. Let $\mathbf{x} = (x_n, x_{n-1}, \ldots, x_1) \in (Z_k)^n$; so $\mathbf{x}^{-1} = (k - x_n, k - x_{n-1}, \ldots, k - x_1)$.

Proposition 5 For every $n \geq 1$, $AQ_{n,k} \cong G((Z_k)^n, S)$, where S is the set

$$\begin{aligned}&\{(0,\ldots,0,0,k-1,k-1),(0,\ldots,0,k-1,k-1,k-1),\ldots,(k-1,\ldots,k-1,k-1),\\ &\quad(0,\ldots,0,0,1,1),(0,\ldots,0,1,1,1),\ldots,(1,\ldots,1,1),\\ &\quad(k-1,0,0,\ldots,0),(0,k-1,0,\ldots,0),\ldots,(0,\ldots,0,k-1),\\ &\quad(1,0,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\}.\end{aligned}$$

Proof By definition, $V(AQ_{n,k}) = Z_k \times Z_k \times \ldots \times Z_k$ (repeated n times). Let $\mathbf{u} = (u_n, u_{n-1}, \ldots, u_1)$ and $\mathbf{v} = (v_n, v_{n-1}, \ldots, v_1)$ be vertices of $AQ_{n,k}$.

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are adjacent in $AQ_{n,k}$. So, for some i, one of the following holds:

1. $\mathbf{v} = (u_n, u_{n-1}, \ldots, u_{i+1}, u_i \oplus_k 1, u_{i-1}, \ldots, u_1)$
2. $\mathbf{v} = (u_n, u_{n-1}, \ldots, u_{i+1}, u_i \oplus_k 1, u_{i-1} \oplus_k 1, \ldots, u_1 \oplus_k 1)$
3. $\mathbf{v} = (u_n, u_{n-1}, \ldots, u_{i+1}, u_i \oplus_k (k-1), u_{i-1}, \ldots, u_1)$
4. $\mathbf{v} = (u_n, u_{n-1}, \ldots, u_{i+1}, u_i \oplus_k (k-1), u_{i-1} \oplus_k (k-1), \ldots, u_1 \oplus_k (k-1))$

Thus, we have (respectively):

1. $\mathbf{u} \oplus_k \mathbf{v}^{-1} = (u_n \oplus_k (k - u_{n-1}), \ldots, u_{i+1} \oplus_k (k - u_{i+1}), u_i \oplus_k (k - (u_i + 1)),$
   $u_{i+1} \oplus_k (k - u_{i+1}), \ldots, u_0 \oplus_k (k - u_0))$
   $= (0, \ldots, 0, k-1, 0, \ldots, 0) \in S$
2. $\mathbf{u} \oplus_k \mathbf{v}^{-1} = (0, \ldots, 0, k-1, \ldots, k-1) \in S$
3. $\mathbf{u} \oplus_k \mathbf{v}^{-1} = (0, \ldots, 0, 1, 0, \ldots, 0) \in S$
4. $\mathbf{u} \oplus_k \mathbf{v}^{-1} = (0, \ldots, 0, 1, \ldots, 1) \in S$.

Hence, $\mathbf{u} \oplus_k \mathbf{v}^{-1} \in S$.

Conversely, suppose that $\mathbf{u} \oplus_k \mathbf{v}^{-1} \in S$. So, $\mathbf{u} \oplus_k \mathbf{v}^{-1}$ is of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$ or $(0, \ldots, 0, 1, \ldots, 1)$ or $(0, \ldots, 0, k-1, 0, \ldots, 0)$ or $(0, \ldots, 0, k-1, \ldots, k-1)$. Hence, for some i, one of the following holds:

1. $\mathbf{u} = (u_n, \ldots, u_{i+1}, u_i \oplus_k (k-1), u_{i-1}, \ldots, u_1)$
2. $\mathbf{v} = (u_n, \ldots, u_{i+1}, u_i \oplus_k (k-1), u_{i-1} \oplus_k (k-1), \ldots, u_1 \oplus_k (k-1))$
3. $\mathbf{v} = (u_n, \ldots, u_{i+1}, u_i \oplus_k 1, u_{i-1}, \ldots, u_1)$
4. $\mathbf{v} = (u_n, \ldots, u_{i+1}, u_i \oplus_k 1, u_{i-1} \oplus_k 1, \ldots, u_1 \oplus_k 1)$.

So $\mathbf{u}$ and $\mathbf{v}$ are adjacent in $AQ_{n,k}$. □

As remarked earlier, (by definition) all Cayley graphs are vertex-symmetric and so we obtain an alternative proof of Corollary 4.

We end this section on symmetry by noting that although the augmented k-ary n-cube is vertex-symmetric (and, indeed, a Cayley graph), it is edge-symmetric only when $n = 2$ (recall that a graph is *edge-symmetric* if given any two edges (a, b) and (a', b'), either there is an automorphism mapping a to a' and b to b' or there is an automorphism mapping a to b' and b to a'). Our key observation is that if a graph is edge-symmetric then the number of common neighbours of end-vertices of any edge must be constant.

Suppose that $n \geq 3$ and $k \geq 3$. Consider the edges $((0, 0, \ldots, 0), (1, 0, \ldots, 0))$ and $((0, 0, \ldots, 0), (0, 1, \ldots, 1))$ of $AQ_{n,k}$. It is easy to see that $(0, 0, \ldots, 0)$ and $(1, 0, \ldots, 0)$ have 2 common neighbours (namely $(1, 1, \ldots, 1)$ and $(0, k-1, \ldots, k-1)$), if $k \geq 4$, and 3 common neighbours (namely $(1, 1, \ldots, 1)$, $(0, 2, \ldots, 2)$, and $(2, 0, \ldots, 0)$), if $k = 3$, while $(0, 0, \ldots, 0)$ and $(0, 1, \ldots, 1)$ have 4 common neighbours (namely $(1, 1, \ldots, 1)$, $(0, 0, 1, \ldots, 1)$, $(0, 1, 0, \ldots, 0)$, and $(k-1, 0, \ldots, 0)$, if $k \geq 4$, and 5 common neighbours (namely $(1, 1, \ldots, 1)$, $(0, 0, 1, \ldots, 1)$, $(0, 1, 0, \ldots, 0)$, $(2, 0, \ldots, 0)$, and $(0, 2, \ldots, 2)$), if $k = 3$.

Suppose that $n = 2$ and $k \geq 3$. By Lemma 3, there are automorphisms fixing the vertex $(0, 0)$ of $AQ_{2,k}$ and mapping the vertex $(0, 1)$ to $(0, k-1)$, $(1, 0)$, $(k-1, 0)$, $(1, 1)$ and $(k-1, k-1)$. Thus, there is an automorphism of $AQ_{2,k}$ mapping any edge incident with $(0, 0)$ to any other edge incident with $(0, 0)$. As $AQ_{2,k}$ is vertex-symmetric, this yields that $AQ_{2,k}$ is edge-symmetric. Thus, we obtain the following result.

Proposition 6 Fix $k \geq 3$. The augmented k-ary 2-cube $AQ_{2,k}$ is edge-symmetric, but the k-ary n-cube $AQ_{n,k}$ is not edge-symmetric when $n \geq 3$. □

4 Connectivity

In this section, we examine the connectivity of $AQ_{n,k}$. The *connectivity* of a graph $G = (V, E)$ is the minimum number of vertices (and their incident edges) needing to be removed so that what remains is a disconnected graph. By Menger's Theorem (see, for example, [8]), a graph $G = (V, E)$ has connectivity at least c if, and only if, given any two distinct vertices of V, there are c vertex-disjoint paths joining them. Having a high connectivity is a desirable property of any interconnection network as it provides fault-tolerance with regard to message routing, allows for hot-spots to be avoided, and allows large messages to be split up into smaller ones and routed in parallel along vertex-disjoint paths.

The connectivity of a graph G is denoted $\kappa(G)$. Henceforth, we write that two paths (which may have common start vertices or common end vertices) are *disjoint* to mean that they are vertex-disjoint. We show that $\kappa(AQ_{n,k}) = 4n - 2$, whenever $n \geq 2$ and $k \geq 3$. We begin by proving this result for $AQ_{2,k}$ and then for the general case using a proof by induction (on n).

4.1 The base case of our induction

The base case of our forthcoming induction is provided by the following result.

Lemma 7 The connectivity of $AQ_{2,k}$ is 6; that is, $\kappa(AQ_{2,k}) = 6$.

Proof We prove our result by constructing 6 disjoint paths joining any two distinct vertices of $AQ_{2,k}$. By Lemma 3, w.l.o.g. we may suppose that our two given vertices of $AQ_{2,k}$ are $u = (0, 0)$ and $v = (i, j)$, where $0 \leq i < j$. For the case when $k = 3$, Lemma 3 tells us that we need only consider the cases when v is $(1, 2)$ and $(2, 2)$. The 6 disjoint paths between $(0, 0)$ and $(1, 2)$ are as follows:

1. $(0, 0), (2, 2), (1, 2)$;
2. $(0, 0), (2, 0), (1, 2)$;
3. $(0, 0), (0, 2), (1, 2)$;
4. $(0, 0), (0, 1), (1, 2)$;
5. $(0, 0), (1, 0), (1, 2)$;
6. $(0, 0), (1, 1), (1, 2)$.

The 6 disjoint paths between $(0, 0)$ and $(2, 2)$ are as follows:

1. $(0, 0), (2, 2)$;
2. $(0, 0), (1, 1), (2, 2)$;
3. $(0, 0), (0, 2), (2, 2)$;
4. $(0, 0), (2, 0), (2, 2)$;
5. $(0, 0), (1, 0), (2, 1), (2, 2)$;
6. $(0, 0), (0, 1), (1, 2), (2, 2)$.

For $k > 3$, we have 2 different cases to consider. Recall, $0 \leq i < j$.

Case *(i)* $0 < i < j$. Consider the following 6 paths:

α_1: $\mathbf{u}, (k-1, 0), (k-2, 0), \ldots, (k-j+i, 0), (k-j+i-1, k-1), (k-j+i-2, k-2), \ldots, (i+1, j+1), \mathbf{v}$;

α_2: $\mathbf{u}, (k-1, k-1), (k-2, k-2), \ldots, (j, j), (j-1, j), (j-2, j), \ldots, (i+1, j), \mathbf{v}$;

α_3: $\mathbf{u}, (0, 1), (0, 2), \ldots, (0, j-i), (1, j-i+1), (2, j-i+2), \ldots, (i-1, j-1), \mathbf{v}$;

α_4: $\mathbf{u}, (0, k-1), (0, k-2), \ldots, (0, j+1), (0, j), (1, j), (2, j), \ldots, (i-1, j), \mathbf{v}$;

α_5: $\mathbf{u}, (1,1), (2,2), \ldots, (i,i), (i,i+1), (i,i+2), \ldots, (i,j-1), \mathbf{v}$;

α_6: $\mathbf{u}, (1,0), (2,0), \ldots, (i,0), (i,k-1), (i,k-2), \ldots, (i,j+1), \mathbf{v}$.

These paths are clearly mutually disjoint.

Case (ii) $i = 0$ and $1 \leq j$. Consider the following 6 paths:

α_1: $\mathbf{u}, (k-1,0), (k-1,1), \ldots, (k-1,j-1), \mathbf{v}$;

α_2: $\mathbf{u}, (k-1,k-1), (k-1,k-2), \ldots, (k-1,j), \mathbf{v}$;

α_3: $\mathbf{u}, (0,1), (0,2), \ldots, (0,j-1), \mathbf{v}$;

α_4: $\mathbf{u}, (0,k-1), (0,k-2), \ldots, (0,j+1), \mathbf{v}$;

α_5: $\mathbf{u}, (1,1), (1,2), \ldots, (1,j), \mathbf{v}$;

α_6: $\mathbf{u}, (1,0), (1,k-1), (1,k-2), \ldots, (1,j+1), \mathbf{v}$.

These paths are clearly mutually disjoint. The result follows. □

For any graph G and any two distinct vertices u and v of G, a *c-container* $C_c(u,v)$, for some $c \geq 1$, is a collection of c vertex-disjoint paths joining u and v in G. The *width* of $C_c(u,v)$ is c, the number of paths, and the *length* of $C_c(u,v)$ is the length of the longest path. Suppose further that G has connectivity c. We say that the *wide-diameter* of G is at most d' if for every pair of distinct vertices u and v of G, there is a container $C_c(u,v)$ of width c and of length at most d'. By examining each of the different constructions in the proof of Lemma 7, we see that the maximal length path joining $\mathbf{u} = (0,0)$ and $\mathbf{v} = (i,j)$ is k. Thus, we obtain the following result.

Corollary 8 $AQ_{2,k}$ has wide-diameter at most k.

4.2 The induction step

We now prove our general connectivity result.

Theorem 9 $\kappa(AQ_{n,k}) = 4n-2$, whenever $k \geq 3$ and $n \geq 2$, and given any two distinct vertices of $AQ_{n,k}$, there are $4n-2$ mutually disjoint paths joining these two vertices so that the length of the longest of these paths is at most $max\{(n-1)k-(n-2), k+7\}$; that is, $AQ_{n,k}$ has wide-diameter at most $max\{(n-1)k-(n-2), k+7\}$.

Proof When $n = 2$ and $k \geq 3$, the result holds by Lemma 7. We proceed by induction on n. Our induction hypothesis is that any two distinct vertices of $AQ_{n-1,k}$ are joined by a set of $4n-6$ mutually disjoint paths (the base case of the induction is covered by Lemma 7).

We shall also calculate the length of a longest path as constructed according to this proof. Let $d_n(\mathbf{w}, \mathbf{w}')$ be the maximal length of any path as constructed according to this proof joining any two vertices $\mathbf{w}$ and $\mathbf{w}'$ of $AQ_{n,k}$, and let $\delta_n = max\{d_n(\mathbf{w}, \mathbf{w}') :$ $\mathbf{w}$ and $\mathbf{w}'$ are distinct vertices of $AQ_{n,k}\}$. We shall obtain a recursive estimate of δ_n.

Fix $k, n \geq 3$. Given any two distinct vertices $\mathbf{u}$ and $\mathbf{v}$ of $AQ_{n,k}$, we shall construct $4n-2$ disjoint paths joining them. By Lemma 3, w.l.o.g. we may assume that $\mathbf{u} = (0,0,\ldots,0)$ and $\mathbf{v} = (v_n, v_{n-1}, \ldots, v_1)$, with $0 \leq v_n \leq \lfloor \frac{k}{2} \rfloor$.

Case 1: $\mathbf{v} = (v_n, 0, 0, \ldots, 0)$, where $1 \leq v_n \leq \lfloor \frac{k}{2} \rfloor$; so, $\mathbf{v}|_n^0 = \mathbf{u}$.

The vertex $\mathbf{u}$ has $4n-6$ neighbours in $AQ^0_{n-1,k}$. For each of these neighbours $\mathbf{w}$, apart from $(0, 1, 1, \ldots, 1)$ and $(0, k-1, k-1, \ldots, k-1)$, build the path from $\mathbf{w}$ by traversing $(n, +1)$-edges until $AQ^{v_n}_{n-1,k}$ is reached, before moving to $\mathbf{v}$. This accounts for $4n-8$ mutually disjoint paths from $\mathbf{u}$ to $\mathbf{v}$. From the neighbour $(0, k-1, k-1, \ldots, k-1)$, build the path by traversing $(n, +1)$-edges until $AQ^{v_n-1}_{n-1,k}$ is reached, before moving to $\mathbf{v}$. From the neighbour $(0, 1, 1, \ldots, 1)$, traverse $(n, -1)$-edges until $AQ^{v_n+1}_{n-1,k}$ is reached, before moving to $\mathbf{v}$. This accounts for another 2 paths from $\mathbf{u}$ to $\mathbf{v}$ that are mutually disjoint and disjoint from all the other paths constructed above.

From the neighbour $(k-1, k-1, \ldots, k-1)$ of $\mathbf{u}$, traverse $(n, -1)$-edges until $AQ^{v_n}_{n-1,k}$ is reached, before moving to $\mathbf{v}$. From the neighbour $(1, 1, \ldots, 1)$ of $\mathbf{u}$, traverse $(n, +1)$-edges until $AQ^{v_n}_{n-1,k}$ is reached, before moving to $\mathbf{v}$. Finally, two additional paths are obtained by traversing $(n, +1)$-edges from $\mathbf{u}$ until $\mathbf{v}$ is reached, and by traversing $(n, -1)$-edges from $\mathbf{u}$ until $\mathbf{v}$ is reached. All paths constructed are mutually disjoint and can be visualized as in Fig. 2. Note that the length of the longest constructed path is $max\{v_n + 2, k - v_n + 1\}$; so, $d_n(\mathbf{u}, \mathbf{v}) \leq k$.

Having dealt with Case 1, let us henceforth assume that $\mathbf{v}|_n^0 \neq \mathbf{u}$. We now define some paths which we shall use throughout the subsequent cases.

Our induction hypothesis is that there are $4n-6$ disjoint paths joining any two distinct vertices of $AQ_{n-1,k}$. So, by our induction hypothesis, there is a set Π of $4n-6$ disjoint paths joining $\mathbf{u}$ and $\mathbf{v}|_n^0$ in $AQ^0_{n-1,k}$ (by assumption $\mathbf{u}$ and $\mathbf{v}|_n^0$ are distinct). Let us denote 4 of these paths as follows:

- π_1 is the path passing through the neighbour $\mathbf{u}_{(\leq n-1,-1)}$ of $\mathbf{u}$;
- π_2 is the path passing through the neighbour $\mathbf{u}_{(\leq n-1,+1)}$ of $\mathbf{u}$;
- π_3 is the path passing through the neighbour $\mathbf{v}_{(\leq n-1,-1)}|_n^0$ of $\mathbf{v}|_n^0$;
- π_4 is the path passing through the neighbour $\mathbf{v}_{(\leq n-1,+1)}|_n^0$ of $\mathbf{v}|_n^0$.

Note that although π_1 and π_2 are always distinct, as are π_3 and π_4, it may be the case that either π_1 or π_2 is identical to either π_3 or π_4 (note also that any one of the above paths may consist of a solitary edge). We examine each of these circumstances separately. Moreover, there are two distinct situations: when $v_n = 0$; and when $v_n \neq 0$.

Note that every path π in Π, from $\mathbf{u}$ to $\mathbf{v}|_n^0$, is such that there is a path π^i in $AQ^i_{n-1,k}$, where $i \in \{1, 2, \ldots, k-1\}$, from $\mathbf{u}|_n^i$ to $\mathbf{v}|_n^i$ obtained by taking the isomorphic image of π under the natural isomorphism (which takes $(0, a_{n-1}, a_{n-2}, \ldots, a_1)$ to $(i, a_{n-1}, a_{n-2}, \ldots, a_1)$; see Lemma 3). Throughout this proof, we extend this notation to arbitrary paths in $AQ^0_{n-1,k}$.

Consider the situation when $v_n = 0$ (and so $\mathbf{v}|_n^0 = \mathbf{v}$). For each path π_j, where $j \in \{1, 2, 3, 4\}$, that is not the path $\mathbf{u}, \mathbf{v}|_n^0$, truncate π_j at the penultimate vertex (that is, the vertex of the path that is a neighbour of $\mathbf{v}|_n^0$) and also remove the first edge: denote this truncated path by ρ_j (note that a path might be truncated so that it consists of a solitary vertex). Do likewise with all isomorphic images of π_1, π_2, π_3 and π_4 (in $AQ^1_{n-1,k}$, $AQ^2_{n-1,k}$, and so on).

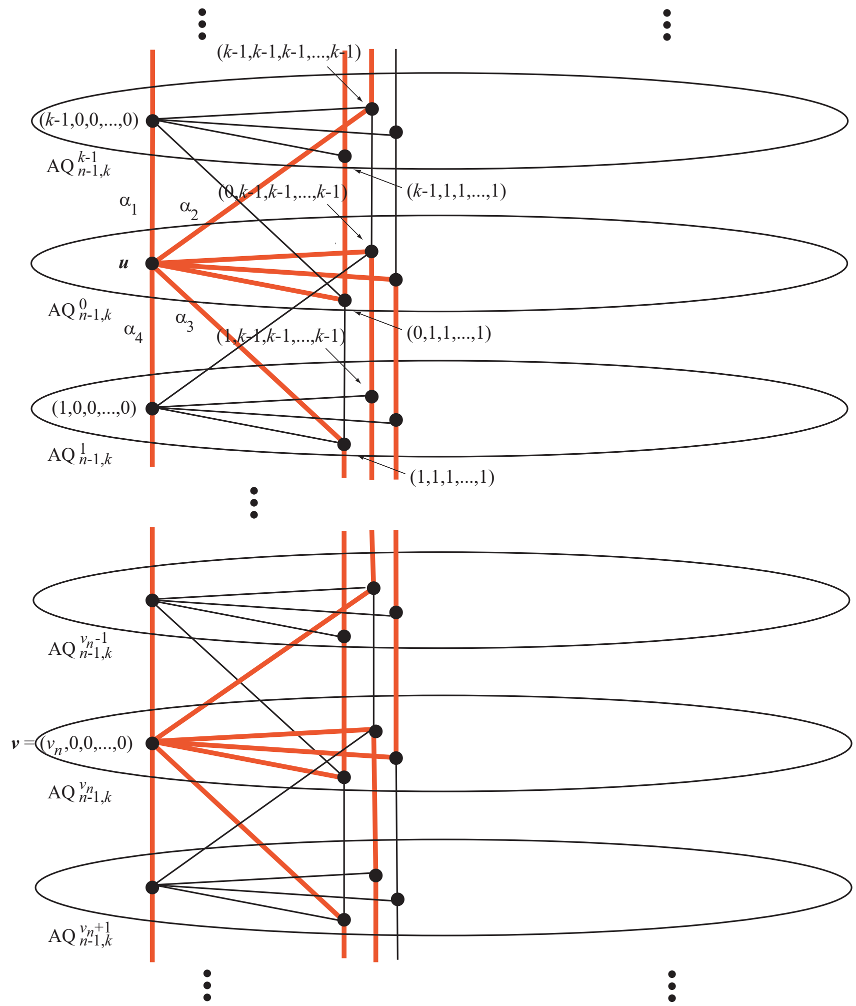

Figure 2. The $4n-2$ disjoint paths in Case 1.

Suppose that $\rho_1 \neq \rho_3$. If neither ρ_1 nor ρ_3 is the path $\mathbf{u}, \mathbf{v}$ then we construct additional paths $\mathbf{u}, \rho_1^{k-1}, \mathbf{v}|_n^{k-1}, \mathbf{v}$ and $\mathbf{u}, \mathbf{u}|_n^{k-1}, \rho_3^{k-1}, \mathbf{v}$ through $AQ_{n-1,k}^{k-1}$. If $\rho_1 = \mathbf{u}, \mathbf{v}$ then we have that $\mathbf{v} = (0, k-1, k-1, \ldots, k-1)$. In this case, we construct additional paths $\mathbf{u}, \mathbf{u}|_n^{k-1}, \rho_3^{k-1}, \mathbf{v}$ and $\mathbf{u}, \mathbf{v}|_n^{k-1}, \mathbf{v}$ through $AQ_{n-1,k}^{k-1}$. If $\rho_3 = \mathbf{u}, \mathbf{v}$ then we have that $\mathbf{u} = (0, v_{n-1}-1, v_{n-2}-1, \ldots, v_1-1)$, with $v_{n-1} = v_{n-2} = \ldots = v_1 = 1$. In this case, we construct additional paths $\mathbf{u}, \mathbf{u}|_n^{k-1}, \mathbf{v}$ and $\mathbf{u}, \rho_1^{k-1}, \mathbf{v}|_n^{k-1}, \mathbf{v}$ through $AQ_{n-1,k}^{k-1}$.

Suppose that $\rho_1 = \rho_3$. We have that $\rho_1 \neq \rho_2$. In this case, we construct additional paths $\mathbf{u}, \rho_1^{k-1}, \mathbf{v}$ and $\mathbf{u}, \mathbf{u}|_n^{k-1}, \rho_2^{k-1}, \mathbf{v}|_n^{k-1}, \mathbf{v}$ through $AQ_{n-1,k}^{k-1}$.

We proceed in an analogous fashion by considering ρ_2 and ρ_4 in the same way, and constructing disjoint paths from $\mathbf{u}$ to $\mathbf{v}$ through $AQ_{n-1,k}^{1}$. Consequently, we obtain $4n-2$ disjoint paths from $\mathbf{u}$ to $\mathbf{v}$ in $AQ_{n,k}$. We clearly have that $d_n(\mathbf{u}, \mathbf{v}) = d_{n-1}((0, 0, \ldots, 0), (v_{n-1}, v_{n-2}, \ldots, v_1)) + 2 \leq \delta_{n-1} + 2$.

Henceforth, we shall assume that $v_n \neq 0$.

Case 2: $\mathbf{u} \neq \mathbf{v}|_n^0$, $\mathbf{u}$ is not adjacent to $\mathbf{v}|_n^0$, and $\mathbf{u}$ and $\mathbf{v}|_n^0$ do not have a neighbour of $AQ_{n-1,k}^0$ in common.

In particular, $\mathbf{u}, \mathbf{v}$ is not a path in Π.

Sub-case 2.1: $\rho_1 \neq \rho_4$ and $\rho_2 \neq \rho_3$.

We begin by building 6 specific paths:

α_1: $\mathbf{u}, \rho_1^{k-1}, \mathbf{v}|_n^{k-1}, \mathbf{v}|_n^{k-2}, \ldots, \mathbf{v}|_n^{v_n+1}, \mathbf{v}$;

α_2: $\mathbf{u}, \mathbf{u}|_n^{k-1}, \rho_4^{k-1}, \mathbf{v}_{(\leq n,+1)}|_n^{k-2}, \mathbf{v}_{(\leq n,+1)}|_n^{k-3}, \ldots, \mathbf{v}_{(\leq n,+1)}, \mathbf{v}$;

α_3: $\mathbf{u}, \mathbf{u}|_n^1, \mathbf{u}|_n^2, \ldots, \mathbf{u}|_n^{v_n}, \rho_3^{v_n}, \mathbf{v}$;

α_4: $\mathbf{u}, \mathbf{u}_{(\leq n,+1)}, \mathbf{u}_{(\leq n,+1)}|_n^2, \mathbf{u}_{(\leq n,+1)}|_n^3, \ldots, \mathbf{u}_{(\leq n,+1)}|_n^{v_n-1}, \rho_2^{v_n}, \mathbf{v}$;

α_5: $\mathbf{u}, \rho_2, \mathbf{v}|_n^0, \mathbf{v}|_n^1, \ldots, \mathbf{v}|_n^{v_n-1}, \mathbf{v}$;

α_6: $\mathbf{u}, \rho_3, \mathbf{v}_{(\leq n,-1)}|_n^1, \mathbf{v}_{(\leq n,-1)}|_n^2, \ldots, \mathbf{v}_{(\leq n,-1)}, \mathbf{v}$.

These paths can be visualized as in Fig. 3, and can easily be seen to be mutually disjoint.

There are $4n-8$ paths in Π apart from π_2 and π_3; let π be any one of them. We truncate π at the penultimate vertex, and then extend this path along $(n,+1)$-edges until we reach $AQ_{n-1,k}^{v_n}$. Finally, we extend the path by an edge to $\mathbf{v}$. Again, it is easy to see that the resulting set of $4n-2$ paths are mutually disjoint. Furthermore, we have that $d_n(\mathbf{u}, \mathbf{v}) = d_{n-1}((0,0,\ldots,0),(v_{n-1}, v_{n-2}, \ldots, v_1)) + max\{k-v_n-1, v_n\} \leq \delta_{n-1} + k - 2$.

Sub-case 2.2: $\rho_1 = \rho_4$ and $\rho_2 \neq \rho_3$.

Note that, by definition, ρ_1, ρ_2 and ρ_3 are distinct. Referring to Sub-case 2.1 (and Fig. 3), if we can amend paths α_1 and α_2 so that they remain disjoint and also disjoint from all of the other $4n-4$ paths then we are done. Replace α_1 and α_2 with the paths α_1' and α_2' defined as:

α_1': $\mathbf{u}, \rho_1^{k-1}, \mathbf{v}_{(\leq n,+1)}|_n^{k-2}, \mathbf{v}_{(\leq n,+1)}|_n^{k-3}, \ldots, \mathbf{v}_{(\leq n,+1)}, \mathbf{v}$;

α_2': $\mathbf{u}, \mathbf{u}|_n^{k-1}, \rho_2^{k-1}, \mathbf{v}|_n^{k-1}, \mathbf{v}|_n^{k-2}, \ldots, \mathbf{v}|_n^{v_n+1}, \mathbf{v}$.

Again, it is easy to see that the resulting set of $4n-2$ paths are mutually disjoint. The amendments made can be visualized as in Fig. 4. Furthermore, we have that $d_n(\mathbf{u}, \mathbf{v}) = d_{n-1}((0,0,\ldots,0),(v_{n-1}, v_{n-2}, \ldots, v_1)) + max\{k-v_n, v_n\} \leq \delta_{n-1} + k - 1$.

Sub-case 2.3: $\rho_1 \neq \rho_4$ and $\rho_2 = \rho_3$.

Note that, by definition, ρ_1, ρ_2 and ρ_4 are distinct. Referring to Sub-case 2.1 (and Fig. 3), if we can amend paths α_3, α_4, α_5 and α_6 so that they remain disjoint and also disjoint from all of the other $4n-6$ paths then we are done. Replace α_3, α_4, α_5 and α_6 with the paths α_3', α_4', α_5' and α_6' defined as:

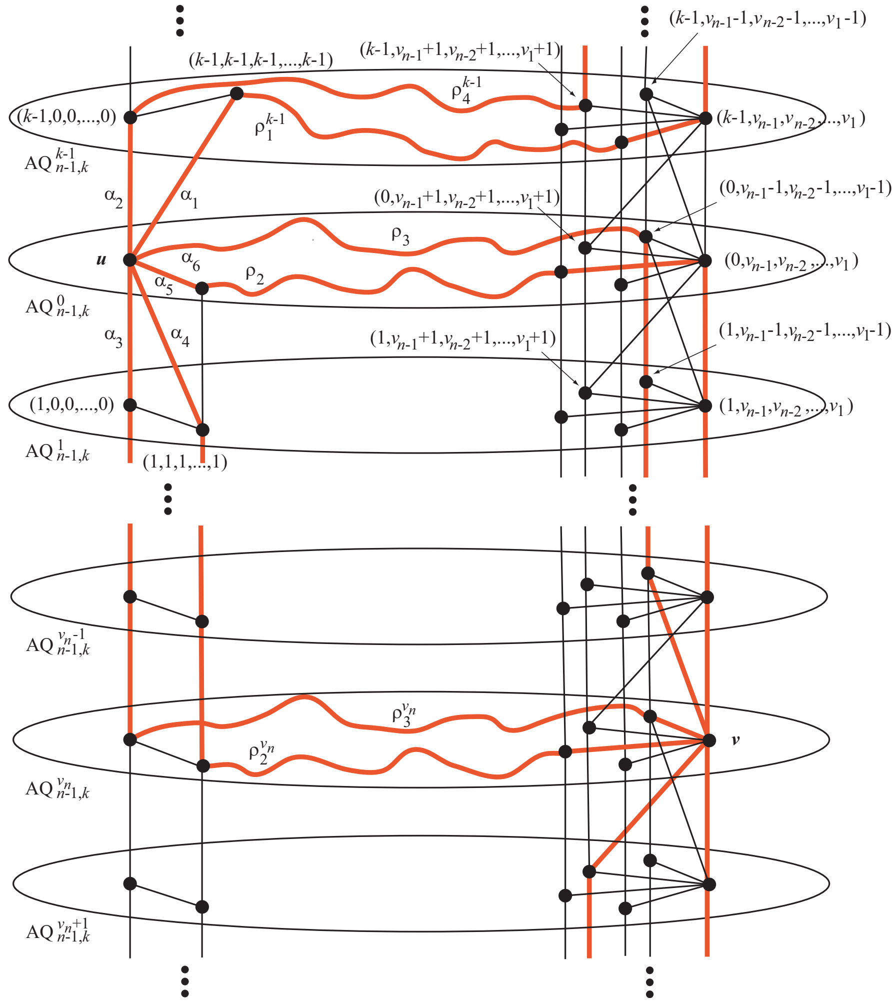

Figure 3. The 6 disjoint paths in Sub-case 2.1.

α_3': $\mathbf{u}, \mathbf{u}|_n^1, \mathbf{u}|_n^2, \ldots, \mathbf{u}|_n^{v_n}, \rho_1^{v_n}, \mathbf{v}$;

α_4': $\mathbf{u}, \mathbf{u}_{(\leq n,+1)}, \mathbf{u}_{(\leq n,+1)}|_n^2, \mathbf{u}_{(\leq n,+1)}|_n^3, \ldots, \mathbf{u}_{(\leq n,+1)}|_n^{v_n-1}, \rho_2^{v_n}, \mathbf{v}$;

α_5': $\mathbf{u}, \rho_2, \mathbf{v}_{(\leq n,-1)}|_n^1, \mathbf{v}_{(\leq n,-1)}|_n^2, \ldots, \mathbf{v}_{(\leq n,-1)}, \mathbf{v}$;

α_6': $\mathbf{u}, \rho_1, \mathbf{v}|_n^0, \mathbf{v}|_n^1, \ldots, \mathbf{v}|_n^{v_n-1}, \mathbf{v}$.

Again, it is easy to see that the resulting set of $4n-2$ paths are mutually disjoint. The amendments made can be visualized as in Fig. 5. Furthermore, we have that $d_n(\mathbf{u},\mathbf{v}) = d_{n-1}((0,0,\ldots,0),(v_{n-1},v_{n-2},\ldots,v_1)) + max\{k-v_n-1, v_n\} \leq \delta_{n-1}+k-2$.

<u>Sub-case 2.4</u>: $\rho_1 = \rho_4$ and $\rho_2 = \rho_3$.

By making the amendments in Sub-cases 2.2 and 2.3, we obtain a set of $4n-2$ mutually disjoint paths. Furthermore, we have that $d_n(\mathbf{u},\mathbf{v}) = d_{n-1}((0,0,\ldots,0),(v_{n-1},v_{n-2}, \ldots, v_1)) + max\{k-v_n, v_n\} \leq \delta_{n-1}+k-1$.

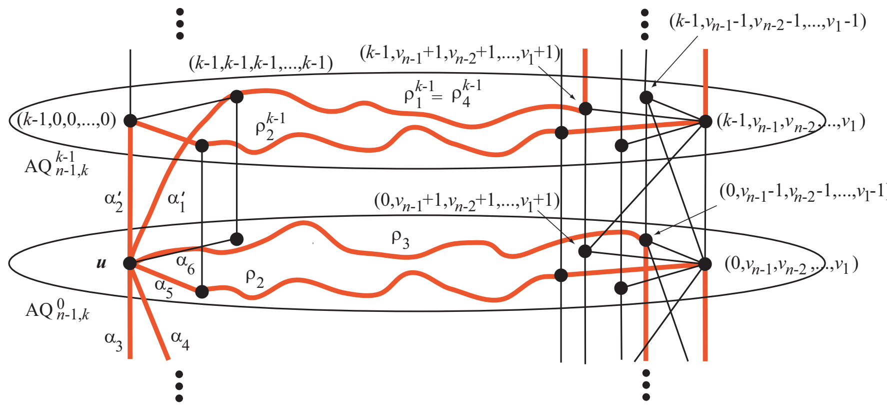

Figure 4. The amendments in Sub-case 2.2.

<u>Case 3</u>: $\mathbf{u} \neq \mathbf{v}|_n^0$ and $\mathbf{u}$ and $\mathbf{v}|_n^0$ are not adjacent, but $\mathbf{u}$ and $\mathbf{v}|_n^0$ have a neighbour of $AQ_{n-1,k}^0$ in common.

All the constructions in Sub-cases 2.1, 2.2, 2.3 and 2.4 work here unless $(v_{n-1} - 1, v_{n-2} - 1, \ldots, v_1 - 1) = (1, 1, \ldots, 1)$, *i.e.*, unless $\mathbf{v} = (v_n, 2, 2, \ldots, 2)$. Thus, this is the only situation to deal with (note that $k \geq 4$, as otherwise $\mathbf{u}$ and $\mathbf{v}|_n^0$ would be adjacent).

One of the paths in the set Π is the path $\mathbf{u}, (0, 1, 1, \ldots, 1), \mathbf{v}$, and let π be the path passing through $(0, 3, 3, \ldots, 3)$. Truncate π at the penultimate vertex $(0, 3, 3, \ldots, 3)$ and also remove the first edge: denote this truncated path by ρ (note that the path ρ might consist of the solitary vertex $(0, 3, 3, \ldots, 3)$). Define the paths ρ^i, for $i \in \{1, 2, \ldots, k-1\}$, as we did earlier.

<u>Sub-case 3.1</u>: $v_n > 1$.

We begin by building 6 specific paths:

α_1: $\mathbf{u}, \rho^{k-1}, \mathbf{v}_{(\leq n,+1)}|_n^{k-2}, \mathbf{v}_{(\leq n,+1)}|_n^{k-3}, \ldots, \mathbf{v}_{(\leq n,+1)}, \mathbf{v}$;

α_2: $\mathbf{u}, \mathbf{u}|_n^{k-1}, \mathbf{v}_{(\leq n,-1)}|_n^{k-1}, \mathbf{v}|_n^{k-1}, \mathbf{v}|_n^{k-2}, \ldots, \mathbf{v}|_n^{v_n+1}, \mathbf{v}$;

α_3: $\mathbf{u}, \mathbf{u}|_n^1, \mathbf{u}|_n^2, \ldots, \mathbf{u}|_n^{v_n}, \mathbf{v}_{(\leq n,-1)}|_n^{v_n}, \mathbf{v}$;

α_4: $\mathbf{u}, \mathbf{v}_{(\leq n,-1)}|_n^1, \mathbf{v}_{(\leq n,-1)}|_n^2, \mathbf{v}_{(\leq n,-1)}|_n^3, \ldots, \mathbf{v}_{(\leq n,-1)}|_n^{v_n-1}, \mathbf{v}$;

α_5: $\mathbf{u}, \rho, \mathbf{v}_{(\leq n,+1)}|_n^1, \mathbf{v}_{(\leq n,+1)}|_n^2, \ldots, \mathbf{v}_{(\leq n,+1)}|_n^{v_n}, \mathbf{v}$;

α_6: $\mathbf{u}, \mathbf{v}_{(\leq n,-1)}|_n^0, \mathbf{v}|_n^0, \mathbf{v}|_n^1, \ldots, \mathbf{v}|_n^{v_n-1}, \mathbf{v}$.

These paths can be visualized as in Fig. 6, and can easily be seen to be disjoint.

There are $4n-8$ paths in Π apart from π and $\mathbf{u}, (0, 1, 1, \ldots, 1), \mathbf{v}$; let π' be any one of them. We truncate π' at the penultimate vertex, and then extend this path along $(n, +1)$-edges until we reach $AQ_{n-1,k}^{v_n}$. Finally, we extend the path by an edge to $\mathbf{v}$. Again, it is easy to see that the resulting set of $4n-2$ paths are mutually disjoint. Furthermore, we have that $d_n(\mathbf{u}, \mathbf{v}) = d_{n-1}((0, 0, \ldots, 0), (2, 2, \ldots, 2)) + max\{k - v_n - 2, v_n\} \leq \delta_{n-1} + max\{k-4, \lfloor \frac{k}{2} \rfloor\}$.

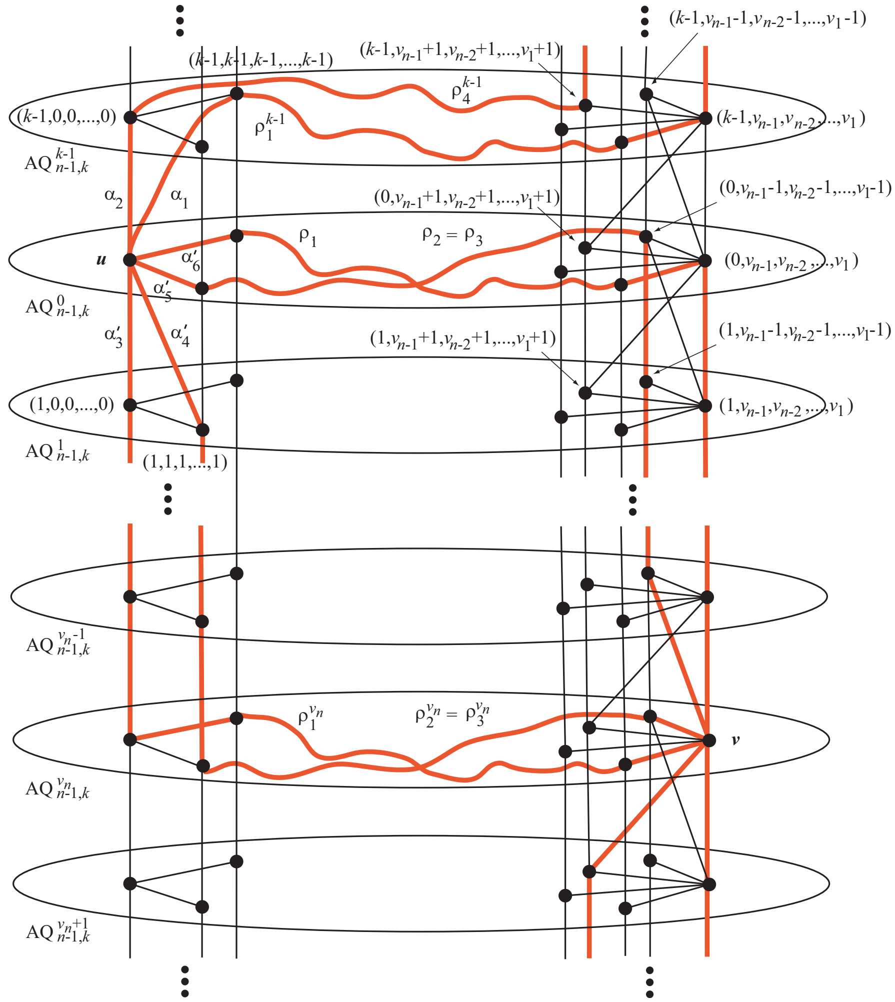

Figure 5. The amendments in Sub-case 2.3.

Sub-case 3.2: $v_n = 1$.

We begin by building 6 specific paths:

α_1: $\mathbf{u}, \rho^{k-1}, \mathbf{v}_{(\leq n,+1)}|_n^{k-2}, \mathbf{v}_{(\leq n,+1)}|_n^{k-3}, \ldots, \mathbf{v}_{(\leq n,+1)}, \mathbf{v}$;

α_2: $\mathbf{u}, \mathbf{u}|_n^{k-1}, \mathbf{v}_{(\leq n,-1)}|_n^{k-1}, \mathbf{v}|_n^{k-1}, \mathbf{v}|_n^{k-2}, \ldots, \mathbf{v}|_n^2, \mathbf{v}$;

α_3: $\mathbf{u}, \mathbf{u}|_n^1, \rho^1, \mathbf{v}$;

α_4: $\mathbf{u}, \mathbf{v}_{(\leq n,-1)}|_n^1, \mathbf{v}$;

α_5: $\mathbf{u}, \rho, \mathbf{v}|_n^0, \mathbf{v}$;

α_6: $\mathbf{u}, \mathbf{v}_{(\leq n,-1)}, \mathbf{v}$.

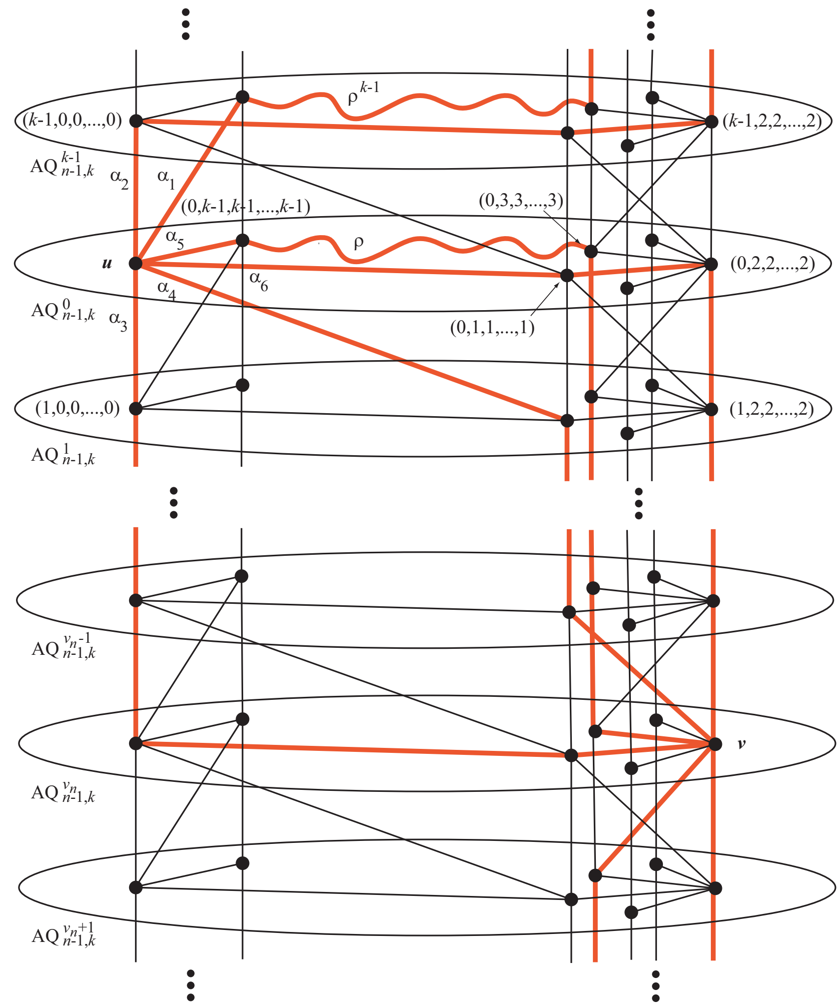

Figure 6. The paths in Sub-case 3.1.

<u>Case 4</u>: $\mathbf{u}$ and $\mathbf{v}|_n^0$ are adjacent.

These paths can be visualized as in Fig. 7, and can easily be seen to be mutually disjoint. There are $4n-8$ paths in Π apart from π and $\mathbf{u}, (0,1,1,\ldots,1), \mathbf{v}$; let π' be any one of them. We truncate π' at the penultimate vertex, and then extend this path along an $(n,+1)$-edge and then an edge to $\mathbf{v}$. Again, it is easy to see that the resulting set of $4n-2$ paths are mutually disjoint. Furthermore, we have that $d_n(\mathbf{u},\mathbf{v}) = max\{d_{n-1}((0,0,\ldots,0),(2,2,\ldots,2)) + k-3, k+1\} \leq \delta_{n-1} + k - 3$.

<u>Sub-case 4.1</u>: $\mathbf{v}|_n^0 \notin \{(0,k-1,k-1,\ldots,k-1),(0,1,1,\ldots,1),(0,2,2,\ldots,2)\}$.

Note that as $(0,k-1,k-1,\ldots,k-1) \neq \mathbf{v}|_n^0 \neq (0,1,1,\ldots,1)$, none of the vertices $(0,k-1,k-1,\ldots,k-1)$, $(0,1,1,\ldots,1)$, $(0,v_{n-1}-1,v_{n-2}-1,\ldots,v_1-1)$ and $(0,v_{n-1}+1,v_{n-2}+1,\ldots,v_1+1)$ is identical to either $\mathbf{u}$ or $\mathbf{v}|_n^0$. Note also that as $\mathbf{u}$ and $\mathbf{v}|_n^0$ are adjacent, so are $(i,1,1,\ldots,1)$ and $(i,v_{n-1}+1,v_{n-2}+1,\ldots,v_1+1)$ and also $(i,k-1,k-1,\ldots,k-1)$ and $(i,v_{n-1}-1,v_{n-2}-1,\ldots,v_1-1)$, for $i \in \{1,2,\ldots,k-1\}$.

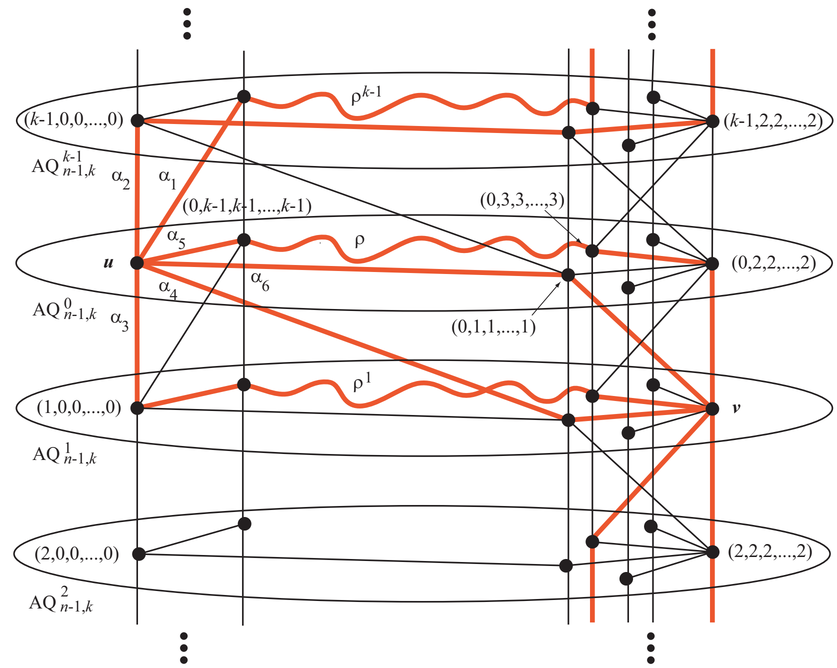

Figure 7. The paths in Sub-case 3.2.

One of the paths in Π is the edge $(\mathbf{u}, \mathbf{v}|_n^0)$. For each path in Π, apart from the edge $(\mathbf{u}, \mathbf{v}|_n^0)$ and the path passing through $(0, v_{n-1}-1, v_{n-2}-1, \ldots, v_1-1)$, truncate this path at the penultimate vertex and extend it using $(n,+1)$-edges until $AQ^{v_n}_{n-1,k}$ is reached before extending it further by an edge to $\mathbf{v}$. As regards the path in Π passing through $(0, v_{n-1}-1, v_{n-2}-1, \ldots, v_1-1)$, truncate this path at $(0, v_{n-1}-1, v_{n-2}-1, \ldots, v_1-1)$ and extend it using $(n,+1)$-edges until $AQ^{v_n-1}_{n-1,k}$ is reached before extending it further by an edge to $\mathbf{v}$. Also, extend the edge $(\mathbf{u}, \mathbf{v}|_n^0)$ using $(n,+1)$-edges to $\mathbf{v}$. These $4n-6$ paths from $\mathbf{u}$ to $\mathbf{v}$ can be visualized as in Fig. 8.

Form the following paths:

α_1: $\mathbf{u}, \mathbf{u}_{(\leq n,+1)}, \mathbf{u}_{(\leq n,+1)}|_n^2, \ldots, \mathbf{u}_{(\leq n,+1)}|_n^{v_n+1}, \mathbf{v}_{(\leq n,+1)}, \mathbf{v}$;

α_2: $\mathbf{u}, \mathbf{u}|_n^1, \mathbf{u}|_n^2, \ldots, \mathbf{u}|_n^{v_n}, \mathbf{v}$;

α_3: $\mathbf{u}, \mathbf{u}|_n^{k-1}, \mathbf{v}|_n^{k-1}, \mathbf{v}|_n^{k-2}, \ldots, \mathbf{v}|_n^{v_n+1}, \mathbf{v}$;

α_4: $\mathbf{u}, \mathbf{u}_{(\leq n,-1)}, \mathbf{v}_{(\leq n,-1)}|_n^{k-1}, \mathbf{v}_{(\leq n,-1)}|_n^{k-2}, \mathbf{v}_{(\leq n,-1)}|_n^{k-3}, \ldots, \mathbf{v}_{(\leq n,-1)}|_n^{v_n}, \mathbf{v}$.

All paths can be visualized in Fig. 8. It is easy to see that as $(0,1,1,\ldots,1) \neq (0, v_{n-1}-1, v_{n-2}-1, \ldots, v_1-1)$, *i.e.*, $\mathbf{v}|_n^0 \neq (0,2,2,\ldots,2)$, the $4n-6$ paths, constructed above, and the paths α_1, α_2, α_3 and α_4 are all mutually disjoint. Furthermore, we have that $d_n(\mathbf{u}, \mathbf{v}) = max\{d_{n-1}((0,0,\ldots,0),(v_{n-1}, v_{n-2}, \ldots, v_1)) + v_n, k-v_n+2, v_n+3\} \leq \delta_{n-1} + \lfloor \frac{k}{2} \rfloor$.

Case 4.2: $\mathbf{v}|_n^0 = (0,1,1,\ldots,1)$.

One of the paths in Π is the edge $(\mathbf{u}, \mathbf{v}|_n^0)$. For each path in Π, apart from the edge $(\mathbf{u}, \mathbf{v}|_n^0)$, truncate this path at the penultimate vertex and extend it using $(n,+1)$-

edges until $AQ^{v_n}_{n-1,k}$ is reached before extending it further by an edge to $\mathbf{v}$. Extend the edge $(\mathbf{u}, \mathbf{v}|^0_n)$ using $(n, -1)$-edges to $\mathbf{v}$.

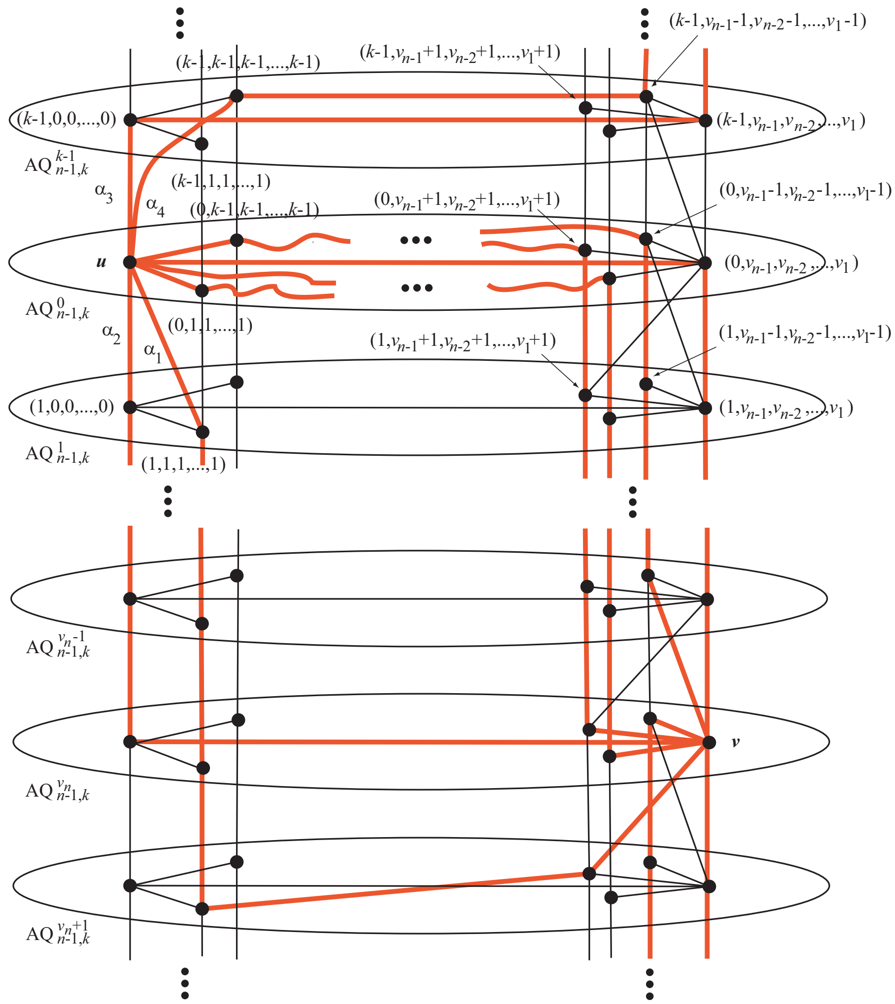

Figure 8. The paths in Sub-case 4.1.

Let the path ρ in $AQ^{k-1}_{n-1,k}$ be defined as $(k-1, k-1, k-1, \ldots, k-1), (k-1, 0, k-1, \ldots, k-1), (k-1, 1, k-1, \ldots, k-1), (k-1, 2, k-1, \ldots, k-1), (k-1, 2, 0, \ldots, 0), (k-1, 2, 1, \ldots, 1), (k-1, 2, 2, \ldots, 2)$ (unless $(k-1, k-1, k-1, \ldots, k-1) = (k-1, 2, 2, \ldots, 2)$ when ρ is just a solitary vertex). Note that ρ avoids $(k-1, 0, 0, \ldots, 0)$ and $(k-1, 1, 1, \ldots, 1)$. Define the paths:

α_1: $\mathbf{u}, \rho, \mathbf{v}_{(\leq n,+1)}|^{k-2}_n, \mathbf{v}_{(\leq n,+1)}|^{k-3}_n, \ldots, \mathbf{v}_{(\leq n,+1)}, \mathbf{v}$;

α_2: $\mathbf{u}, \mathbf{u}|^{k-1}_n, \mathbf{u}|^{k-2}_n, \ldots, \mathbf{u}|^{v_n}_n, \mathbf{v}$;

α_3: $\mathbf{u}, \mathbf{u}|^1_n, \mathbf{u}|^2_n, \ldots, \mathbf{u}|^{v_n-1}_n, \mathbf{v}$;

α_4: $\mathbf{u}, \mathbf{v}|^1_n, \mathbf{v}|^2_n, \ldots, \mathbf{v}|^{v_n-1}_n, \mathbf{v}$.

Our collection of $4n-2$ paths from $\mathbf{u}$ to $\mathbf{v}$ can be visualized as in Fig. 9, and they are clearly mutually disjoint. Furthermore, we have that $d_n(\mathbf{u},\mathbf{v}) = max\{d_{n-1}((0,0,0,\ldots,0),(1,1,\ldots,1)) + v_n, k - v_n + 6\} \leq max\{\delta_{n-1} + \lfloor \frac{k}{2} \rfloor, k+5\}$.

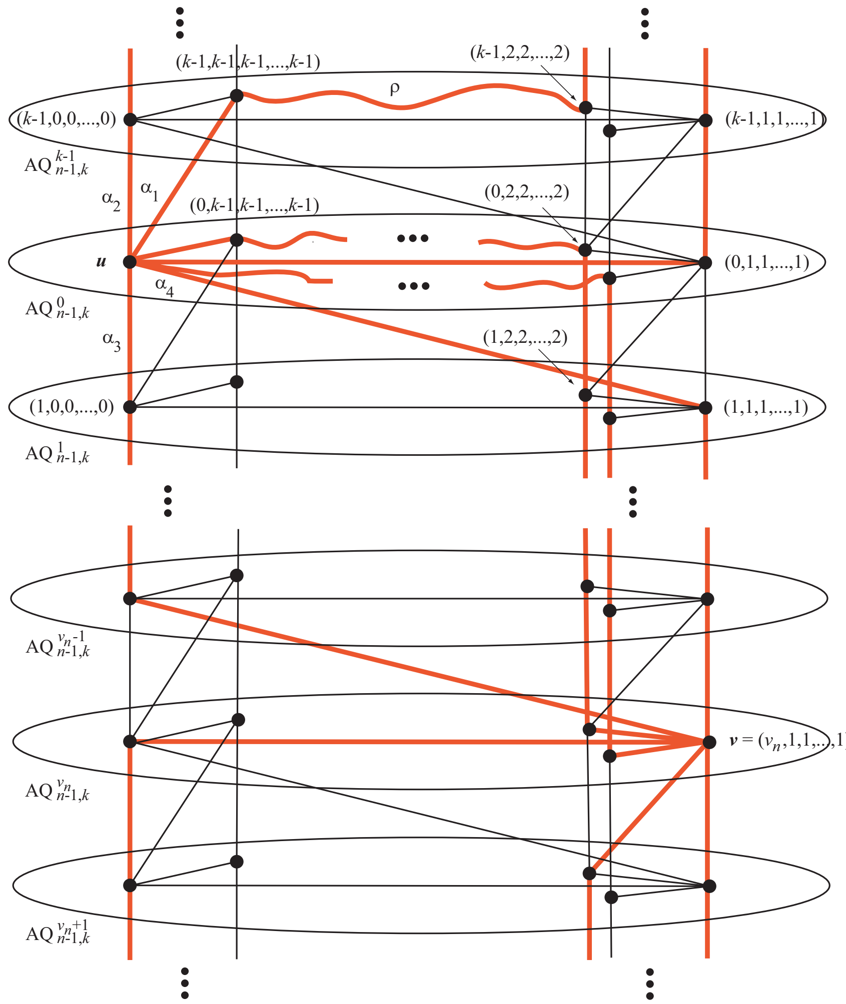

Figure 9. The paths in Sub-case 4.2.

<u>Case 4.3</u>: $\mathbf{v}|_n^0 = (0, k-1, k-1, \ldots, k-1)$.

One of the paths in Π is the edge $(\mathbf{u}, \mathbf{v}|_n^0)$. For each path in Π, apart from the edge $(\mathbf{u}, \mathbf{v}|_n^0)$ and the paths passing through $(0,1,1\ldots,1)$ and $(0,k-2,k-2,\ldots,k-2,$ truncate this path at the penultimate vertex and extend it using $(n,+1)$-edges until $AQ^{v_n}_{n-1,k}$ is reached before extending it further by an edge to $\mathbf{v}$. Extend the edge $(\mathbf{u}, \mathbf{v}|_n^0)$ using $(n,+1)$-edges to $\mathbf{v}$, and extend the truncated path through $(0,k-2,k-2,\ldots,k-2)$ using $(n,+1)$-edges to $(v_n-1,k-2,k-2,\ldots,k-2)$ and then to $\mathbf{v}$. This accounts for $4n-7$ mutually disjoint paths.

Let the path ρ in $AQ^{v_n+1}_{n-1,k}$ be defined as $(v_n+1,k-2,k-2,\ldots,k-2),(v_n+1,k-1,k-2,\ldots,k-2),(v_n+1,0,k-2,\ldots,k-2),(v_n+1,1,k-2,\ldots,k-2),(v_n+$

$1, 1, k-1, \ldots, k-1), (v_n+1, 1, 0, \ldots, 0), (v_n+1, 1, 1, \ldots, 1)$ (unless $(v_n+1, k-2, k-2, \ldots, k-2) = (v_n+1, 1, 1, \ldots, 1)$ when ρ is just a solitary vertex). Note that ρ avoids $(v_n+1, 0, 0, \ldots, 0)$ and $(v_n+1, k-1, k-1, \ldots, k-1)$. Define the paths:

α_1: $\mathbf{u}, \mathbf{u}|_n^{k-1}, \mathbf{u}|_n^{k-2}, \ldots, \mathbf{u}|_n^{v_n+1}, \mathbf{v}$;

α_2: $\mathbf{u}, \mathbf{v}|_n^{k-1}, \mathbf{v}|_n^{k-2}, \ldots, \mathbf{v}|_n^{v_n+1}, \mathbf{v}$;

α_3: $\mathbf{u}, \mathbf{u}_{(\leq n,+1)}, \mathbf{u}_{(\leq n,+1)}|_n^2, \mathbf{u}_{(\leq n,+1)}|_n^3, \ldots, \mathbf{u}_{(\leq n,+1)}|_n^{v_n}, \mathbf{v}$;

α_4: $\mathbf{u}, \mathbf{u}|_n^1, \mathbf{u}|_n^2, \ldots, \mathbf{u}|_n^{v_n}, \mathbf{v}$;

α_5: $\mathbf{u}, \mathbf{u}_{(\leq n,+1)}|_n^0, \mathbf{u}_{(\leq n,+1)}|_n^{k-1}, \mathbf{u}_{(\leq n,+1)}|_n^{k-2}, \ldots, \mathbf{u}_{(\leq n,+1)}|_n^{v_n+2}, \rho, \mathbf{v}_{(\leq n,-1)}|_n^{v_n}, \mathbf{v}$.

Our collection of $4n-2$ paths from $\mathbf{u}$ to $\mathbf{v}$ can be visualized as in Fig. 10, and they are clearly mutually disjoint. Furthermore, we have that $d_n(\mathbf{u}, \mathbf{v}) = max\{d_{n-1}((0, 0, \ldots, 0), (k-1, k-1, \ldots, k-1)) + v_n, k - v_n + 8\} \leq max\{\delta_{n-1} + \lfloor \frac{k}{2} \rfloor, k+7\}$.

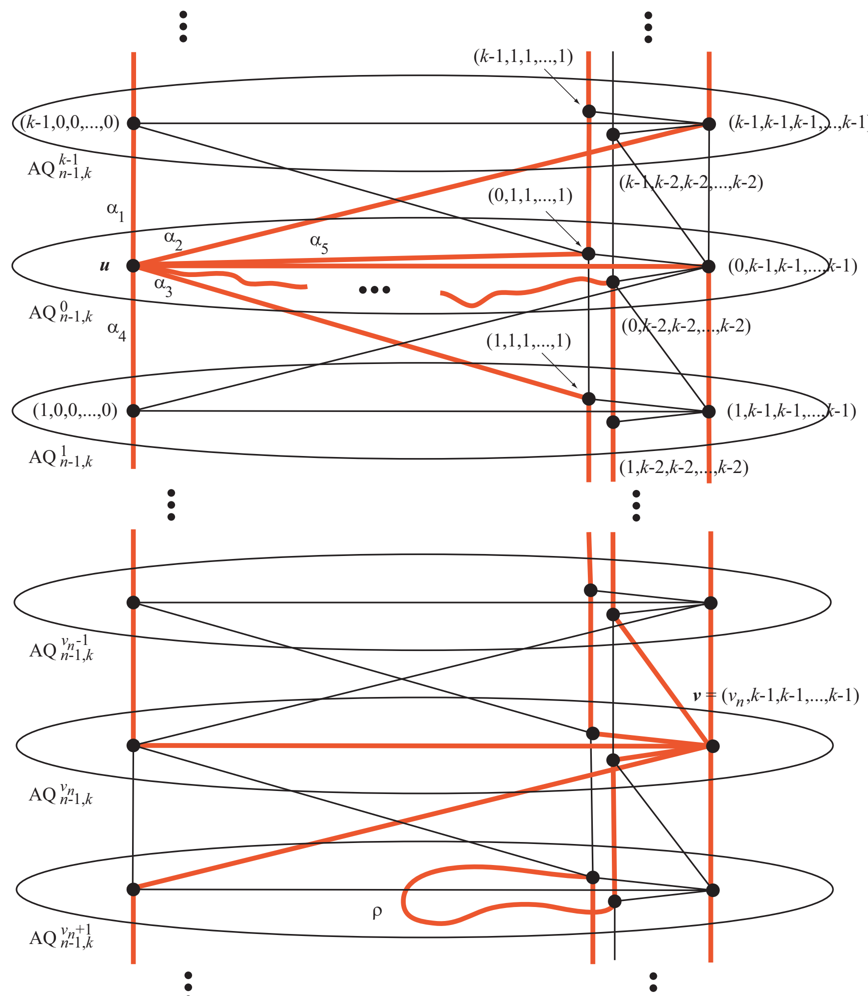

Figure 10. The paths in Sub-case 4.3.

Case 4.4: $\mathbf{v}|_n^0 = (0, 2, 2, \ldots, 2)$.

As $\mathbf{u}$ and $\mathbf{v}|_n^0$ are adjacent, we must have that $k = 3$ and that $\mathbf{v} = (1, 2, 2, \ldots, 2)$. By Lemma 3, there exists an automorphism of $AQ_{n,k}$ mapping $(1, 2, 2, \ldots, 2)$ to $(2, 1, 1, \ldots, 1)$ and fixing $\mathbf{u}$. Thus, this sub-case reduces to Sub-case 4.2.

As regards the length of the longest path constructed, we have that $\delta_n \leq max\{\delta_{n-1} + k - 1, k + 7\}$ and $\delta_2 = k$. Thus, $\delta_n \leq (n-1)k - (n-2)$, unless: $n = 3$ and $k = 3, 4, 5, 6, 7$; $n = 4$ and $k = 3, 4$; or $n = 5$ and $k = 3$, when $\delta_n \leq k + 7$. The result follows by induction. □

5 The diameter

The *diameter* of any graph $G = (V, E)$ is the maximum of the set $\{d_G(x, y) : (x, y) \in V \times V, d_G(x, y)$ is the length of the shortest path joining x and y in $G\}$. Obviously, the smaller the diameter of an interconnection network, the lower the communication latency (be this under store-and-forward or wormhole routing). In this section, we obtain the diameter of $AQ_{2,k}$ and an upper bound on the diameter of $AQ_{n,k}$ when $n \geq 3$.

We begin with some immediate observations as regards the order of edges in paths in $AQ_{n,k}$. Consider some path ρ from some vertex $\mathbf{u}$ of $AQ_{n,k}$ to some vertex $\mathbf{v}$ of $AQ_{n,k}$ within which there is an λ-edge, where $\lambda \in \{(i, +1), (i, -1), (\leq i, +1), (\leq i, -1)\}$, for some i, as the ath edge of the path, and a μ-edge, where $\mu \in \{(j, +1), (j, -1), (\leq j, +1), (\leq j, -1)\}$, for some j, as the bth edge of the path, where $a \neq b$. The path obtained from ρ by traversing a μ-edge as the ath edge of the path and a λ-edge as the bth edge of the path, and leaving the labels of all other edges as they were, is still a path from $\mathbf{u}$ to $\mathbf{v}$. Also, if ρ is a shortest path between $\mathbf{u}$ and $\mathbf{v}$ and there is a $(i, +1)$-edge (resp. $(i, -1)$-edge, $(\leq i, +1)$-edge, $(\leq i, -1)$-edge) in ρ, for some particular i, then there is no $(i, -1)$-edge (resp. $(i, +1)$-edge, $(\leq i, -1)$-edge, $(\leq i, +1)$-edge) in ρ. We use these observations throughout the proof of the following result.

Proposition 10 The diameter of $AQ_{2,k}$ is $\lfloor \frac{k}{3} \rfloor + \lceil \frac{k-1}{3} \rceil$, and for $n \geq 3$ the diameter of $AQ_{n,k}$ is at most $\frac{k}{4}(n+1)$, if k is even, and at most $\frac{k}{4}(n+1) + \frac{n}{4}$, if k is odd.

Proof By Corollary 4, we may restrict our attention to the lengths of paths from an arbitrary vertex of $AQ_{n,k}$ to the vertex $\mathbf{0}$ of $AQ_{n,k}$ when determining the diameter of $AQ_{n,k}$.

Let $\mathbf{v} = (v_2, v_1)$ be a vertex of $AQ_{2,k}$.

Case (i): $k \equiv 0 \pmod 3$.

Sub-case (a): $v_1, v_2 \notin \{\frac{k}{3} + 1, \frac{k}{3} + 2, \ldots, \frac{2k}{3} - 1\}$.

By traversing edges with labels from $\{(i, +1), (i, -1) : i = 1, 2, \ldots, n\}$, we can obtain a path of length at most $\frac{2k}{3}$ from $\mathbf{v}$ to $\mathbf{0}$.

Sub-case (b): exactly one of v_1 and v_2 is in $\{\frac{k}{3} + 1, \frac{k}{3} + 2, \ldots, \frac{2k}{3} - 1\}$.

Suppose that $v_1 \in \{\frac{k}{3} + 1, \frac{k}{3} + 2, \ldots, \frac{2k}{3} - 1\}$. By traversing $(1, +1)$-edges or $(1, -1)$-edges, we can move from $\mathbf{v}$ to (v_2, v_2), and by traversing $(\leq 2, +1)$-edges or $(\leq 2, -1)$-edges we can then move to $\mathbf{0}$. This yields a path of length at most $\frac{2k}{3} - 1$ from $\mathbf{v}$ to $\mathbf{0}$.

If $v_2 \in \{\frac{k}{3}+1, \frac{k}{3}+2, \ldots, \frac{2k}{3}-1\}$ then we proceed similarly except that we first traverse $(2,+1)$-edges or $(2,-1)$-edges to get to (v_1, v_1), before traversing $(\leq 2,+1)$-edges or $(\leq 2,-1)$-edges to get to $\mathbf{0}$.

Sub-case (c): $v_1, v_2 \in \{\frac{k}{3}+1, \frac{k}{3}+2, \ldots, \frac{2k}{3}-1\}$.

Proceeding similarly to as in Sub-case (b) results in a path from $\mathbf{v}$ to $\mathbf{u}$ of length at most $\frac{2k}{3}-1$.

In consequence, when $k \equiv 0$ there is a path from $\mathbf{v}$ to $\mathbf{0}$ of length at most $\frac{2k}{3} = \lfloor\frac{k}{3}\rfloor + \lceil\frac{k-1}{3}\rceil$.

Case (ii): $k \equiv 1 \pmod 3$.

We proceed similarly to as in Case (i) except that we consider the values of v_1 and v_2 as to whether they lie in $\{\lfloor\frac{k}{3}\rfloor+1, \lfloor\frac{k}{3}\rfloor+2, \ldots, \lfloor\frac{k}{3}\rfloor+\lceil\frac{k}{3}\rceil-1\}$. We thus obtain a path from $\mathbf{v}$ to $\mathbf{0}$ of length at most $\lfloor\frac{k}{3}\rfloor+\lceil\frac{k}{3}\rceil-1$. In consequence, when $k \equiv 1 \pmod 3$ there is a path from $\mathbf{v}$ to $\mathbf{0}$ of length at most $\lfloor\frac{k}{3}\rfloor+\lceil\frac{k}{3}\rceil-1 = \lfloor\frac{k}{3}\rfloor+\lceil\frac{k-1}{3}\rceil$.

Case (iii): $k \equiv 2 \pmod 3$.

We proceed similarly to as in Case (i) except that we consider the values of v_1 and v_2 as to whether they lie in $\{\lceil\frac{k}{3}\rceil+1, \lceil\frac{k}{3}\rceil+2, \ldots, 2\lceil\frac{k}{3}\rceil-1\}$. We thus obtain a path from $\mathbf{v}$ to $\mathbf{0}$ of length at most $\lfloor\frac{k}{3}\rfloor+\lceil\frac{k}{3}\rceil$. In consequence, when $k \equiv 2$ there is a path from $\mathbf{v}$ to $\mathbf{0}$ of length at most $\lfloor\frac{k}{3}\rfloor+\lceil\frac{k}{3}\rceil = \lfloor\frac{k}{3}\rfloor+\lceil\frac{k-1}{3}\rceil$.

Whilst $\lfloor\frac{k}{3}\rfloor+\lceil\frac{k-1}{3}\rceil$ is an upper bound on the diameter of $AQ_{2,k}$, it is also a lower bound as we now show. Suppose that $k \equiv 0 \pmod 3$ and the length of a shortest path ρ from $(\frac{k}{3}, \frac{2k}{3})$ to $(0,0)$ is less than $\lfloor\frac{k}{3}\rfloor+\lceil\frac{k-1}{3}\rceil = \frac{2k}{3}$. If the edges of ρ are all $(i,+1)$-edges or $(i,-1)$-edges then we immediately obtain a contradiction. Thus, there must be some $(\leq 2,+1)$-edges or $(\leq 2,-1)$-edges in ρ. By symmetry, we may suppose that there are $(\leq 2,-1)$-edges (and so, as ρ is a shortest path, there must be no $(\leq 2,+1)$-edges in ρ). Moreover, we may clearly assume that all these $(\leq 2,-1)$-edges appear as a prefix of ρ.

Suppose that there are at most $\frac{2k}{3}-\lceil\frac{k}{2}\rceil$ $(\leq 2,-1)$-edges in ρ and that traversing these $(\leq 2,-1)$-edges takes us to (v_2', v_1'). For an arbitrary vertex (v_2, v_1) of $AQ_{2,k}$, define $wt(v_2, v_1) = min\{v_2, k-v_2\} + min\{v_1, k-v_1\}$, *i.e.*, the distance of (v_2, v_1) from $(0,0)$ in the k-ary 2-cube Q_2^k. As $wt(\frac{k}{3}, \frac{2k}{3}) = wt(v_2', v_1') = \frac{2k}{3}$, this yields a contradiction (as any path from (v_2', v_1') to $(0,0)$ traversing only edges with labels from $\{(1,+1),(1,-1),(2,+1),(2,-1)\}$ has length at least $wt(v_2', v_1')$). Thus there must be between $\frac{2k}{3}-\lceil\frac{k}{2}\rceil+1$ and $\frac{k}{3}$ $(\leq 2,-1)$-edges in ρ (clearly there cannot exist more than $\frac{k}{3}$ such edges as otherwise we could obtain a shorter path than ρ).

Suppose that there exist $m+\frac{2k}{3}-\lceil\frac{k}{2}\rceil$ $(\leq 2,-1)$-edges in ρ, where $1 \leq m \leq \lceil\frac{k}{2}\rceil-\frac{k}{3}$, and that traversing these edges takes us to the vertex (v_2', v_1'). Then $wt(v_2', v_1') = \frac{2k}{3}-2(m-1)-1$. Any path from (v_2', v_1') to $(0,0)$ not using $(\leq 2,+1)$-edges nor $(\leq 2,-1)$-edges has length at least $\frac{2k}{3}-2(m-1)-1$. Thus, the length of ρ is at least $(\frac{2k}{3}-2(m-1)-1)+(m+\frac{2k}{3}-\lceil\frac{k}{2}\rceil) = \frac{4k}{3}-m+1-\lceil\frac{k}{2}\rceil \geq \frac{4k}{3}-(\lceil\frac{k}{2}\rceil-\frac{k}{3})+1-\lceil\frac{k}{2}\rceil = \frac{5k}{3}-2\lceil\frac{k}{2}\rceil+1 = \frac{2k}{3}$, which yields a contradiction.

Arguing in an analogous fashion with the vertex $(\lfloor\frac{k}{3}\rfloor, \lfloor\frac{k}{3}\rfloor+\lceil\frac{k}{3}\rceil)$ of $AQ_{2,k}$, when $k \equiv 1 \pmod 3$, and with the vertex $(\lceil\frac{k}{3}\rceil, 2\lceil\frac{k}{3}\rceil)$ of $AQ_{2,k}$, when $k \equiv 2 \pmod 3$, yields that the diameter of $AQ_{2,k}$ is $\lfloor\frac{k}{3}\rfloor+\lceil\frac{k-1}{3}\rceil$ irrespective of the value of $k \pmod 3$.

Let $n \geq 3$ and $\mathbf{v} = (v_n, v_{n-1}, \ldots, v_1)$ be a vertex of $AQ_{n,k}$.

Case (i): k is even.

Define $\Sigma_{i=1}^{n}|\frac{k}{2} - v_i| = \alpha$. Traversing $\frac{k}{2}$ $(\leq n, -1)$-edges from $\mathbf{v}$ leads to a vertex $\mathbf{v}' = (v'_n, v'_{n-1}, \ldots, v'_1)$ such that $\Sigma_{i=1}^{n} min\{v'_i, k - v'_i\} = \alpha$, and so by traversing $(i, +1)$-edges and $(i, -1)$-edges, for various i, as appropriate, we obtain a path of length $\frac{k}{2} + \alpha$ from $\mathbf{v}$ to $\mathbf{0}$. Alternatively, we could simply start from $\mathbf{v}$ and traverse $(i, +1)$-edges and $(i, -1)$-edges, as appropriate, to obtain a path of length $\frac{nk}{2} - \alpha$ from $\mathbf{v}$ to $\mathbf{0}$.

Suppose that $\frac{k}{2} + \alpha \leq \frac{nk}{2} - \alpha$, *i.e.*, $2\alpha \leq \frac{k}{2}(n-1)$. So, there is a path of length at most $\frac{k}{2} + \frac{k}{4}(n-1) = \frac{k}{4}(n+1)$ from $\mathbf{v}$ to $\mathbf{0}$. If $2\alpha > \frac{k}{2}(n-1)$ then there is a path of length less than $\frac{nk}{2} - \frac{k}{4}(n-1) = \frac{k}{4}(n+1)$ from $\mathbf{v}$ to $\mathbf{0}$. Thus, when k is even there is a path of length at most $\frac{k}{4}(n+1)$ from $\mathbf{v}$ to $\mathbf{0}$.

Case (ii): k is odd.

We proceed similarly to as in Case (i) but the numerics are slightly messier. Define $\Sigma_{i=1}^{n}|\lceil\frac{k}{2}\rceil - v_i| = \alpha$. Similarly to as in Case (i), we obtain a path from $\mathbf{v}$ to $\mathbf{0}$ of length at most $\lfloor\frac{k}{2}\rfloor + \alpha$ and also one of length at most $n\lceil\frac{k}{2}\rceil - \alpha$.

Suppose that $\lfloor\frac{k}{2}\rfloor + \alpha \leq n\lceil\frac{k}{2}\rceil - \alpha$, *i.e.*, $2\alpha \leq n\lceil\frac{k}{2}\rceil - \lfloor\frac{k}{2}\rfloor$. So, there is a path of length at most $\lfloor\frac{k}{2}\rfloor + \frac{n}{2}\lceil\frac{k}{2}\rceil - \frac{1}{2}\lfloor\frac{k}{2}\rfloor \leq \frac{k}{4}(n+1) + \frac{n}{4}$ from $\mathbf{v}$ to $\mathbf{0}$. If $2\alpha > n\lceil\frac{k}{2}\rceil - \lfloor\frac{k}{2}\rfloor$ then there is a path of length less than $n\lceil\frac{k}{2}\rceil - \frac{n}{2}\lceil\frac{k}{2}\rceil + \frac{1}{2}\lfloor\frac{k}{2}\rfloor \leq \frac{k}{4}(n+1) + \frac{n}{4}$. Thus, when k is odd there is a path of length at most $\frac{k}{4}(n+1) + \frac{n}{4}$ from $\mathbf{v}$ to $\mathbf{0}$. □

Note that we only have an upper bound on the diameter of $AQ_{n,k}$, when $n \geq 3$. Ascertaining the exact value of the diameter appears to be combinatorially quite challenging. However, we conjecture that our upper bound is actually quite close to the true diameter.

6 Conclusions

In this paper, we have defined a new class of graphs, the class of augmented k-ary n-cubes, and we have examined these graphs in relation to some properties pertinent to their use as interconnection networks for parallel computing. We have tabulated our comparison between k-ary n-cubes and augmented k-ary n-cubes in Fig. 11.

	k-ary n-cube Q_n^k	augmented k-ary n-cube $AQ_{n,k}$
number of vertices/edges	k^n/nk^n	$k^n/(2n-1)k^n$
vertex-/edge-symmetric	yes/yes	yes/no unless $n = 2$
connectivity	$2n$	$4n - 2$
wide-diameter ($n \geq 3$)	$n\lfloor\frac{k}{2}\rfloor + 1$	$\leq max\{(n-1)k - (n-2), k+7\}$
wide-diameter ($n = 2$)	$2\lfloor\frac{k}{2}\rfloor + 1$	$\leq k$
diameter ($n \geq 3$)	$n\lceil\frac{k}{2}\rceil$	$\leq \frac{k}{4}(n+1)$ (k even) $\leq \frac{k}{4}(n+1) + \frac{n}{4}$ (k odd)
diameter ($n = 2$)	$2\lceil\frac{k}{2}\rceil$	$\lfloor\frac{k}{3}\rfloor + \lceil\frac{k-1}{3}\rceil$
routing	$O(nk)$ time	$O(nk)$ time

Figure 11. A comparison between Q_n^k and $AQ_{n,k}$.

Both $AQ_{n,k}$ and Q_n^k have k^n vertices, with the former having $(n-1)k^n$ more edges than the latter, and both interconnection networks are Cayley graphs, and so vertex-symmetric; however, $AQ_{n,k}$ is not edge-symmetric, unless $n = 2$, whereas Q_n^k is. $AQ_{n,k}$ has a much improved connectivity of $4n-2$ in comparison with the connectivity of Q_n^k which is $2n$, although this comes at the expense of an increased vertex degree, which is $4n-2$ as opposed to $2n$ for the k-ary n-cube (both $AQ_{n,k}$ and Q_n^k are 'maximally connected', in the sense that if disjoint paths are used to transmit messages from one vertex to another in either network then there are no unused neighbours of the source vertex). We have also shown an upper bound on the diameter of an augmented k-ary n-cube at roughly one half that of a k-ary n-cube.

Recall that both the k-ary n-cube and the augmented k-ary n-cube come with two parameters which are both variable. Suppose that we have a k-ary n-cube, which involves k^n vertices, and we wish to obtain an augmented K-ary N-cube of comparable size, but not necessarily by choosing the parameters $N = n$ and $K = k$, so that the degrees of the two networks are also comparable. Choose

$$K = \frac{k}{2} \text{ and } N = \frac{n}{1 - \frac{1}{\log(k)}}$$

(we assume for simplicity that both N and K are integral). Thus, $k^n = K^N$. Moreover, the degree of the k-ary n-cube Q_n^k is $2n$ and the degree of the augmented K-ary N-cube $AQ_{N,K}$ is

$$4N - 2 = \frac{4n}{1 - \frac{1}{\log(k)}} - 2 \leq \frac{4n}{1 - \frac{1}{\log(3)}} - 2 < 11n - 2,$$

with the diameter of Q_n^k being $\frac{nk}{2}$ in comparison to an upper bound of

$$\frac{K}{4}(N+1) = \frac{k}{8}\left(\frac{n}{1 - \frac{1}{\log(k)}} + 1\right) < \frac{nk}{2}\left(0.68 + \frac{1}{4n}\right)$$

on the diameter of $AQ_{N,K}$ when k is even, and of

$$\begin{aligned}\frac{K}{4}(N+1) + \frac{N}{4} &= N\left(\frac{K}{4} + \frac{1}{4}\right) + \frac{K}{4} = \frac{n}{1 - \frac{1}{\log(k)}}\left(\frac{k}{8} + \frac{1}{4}\right) + \frac{k}{8} \\ &< \frac{nk}{2}\left(0.68 + \frac{1}{k - \frac{k}{\log(k)}} + \frac{1}{4n}\right)\end{aligned}$$

on the diameter of $AQ_{N,K}$ when k is odd (in both the even and odd case, this is asymptotically roughly two-thirds the diameter of Q_n^k). Note that the actual improvement in diameter could well be better than this, given that we have only given an upper bound as to the diameter of a $AQ_{N,K}$. In consequence, we conclude that augmented k-ary n-cubes can be regarded as improvements over k-ary n-cubes.

We have an additional and important comment to make. The augmented k-ary n-cube $AQ_{n,k}$ is 'built on top' of the k-ary n-cube AQ_n^k; that is, Q_n^k is a spanning subgraph of $AQ_{n,k}$. It is not as if we have simply chosen to argue that our network is a viable network without even considering routing and broadcasting; for all routing and broadcasting algorithms which work for Q_n^k also work for $AQ_{n,k}$. Moreover, the constructions used in the proof of Proposition 10 yield a very simple routing algorithm of time complexity $O(nk)$ (albeit possibly non-optimal).

There are numerous directions for further research. One obvious one is an exact characterization of the diameter of an augmented k-ary n-cube. However, even in the absence of this exact characterization, our upper bound results still yield a significant improvement. Whereas the wide-diameter of Q_n^k is $n\lfloor \frac{k}{2} \rfloor + 1$, the wide-diameter of $AQ_{n,k}$ has an upper bound of $\leq max\{(n-1)k-(n-2), k+7\}$, when $n \geq 3$, and k, when $n = 2$. This is possibly to be expected, given that we are constructing $4n-2$ paths in $AQ_{n,k}$ whereas only $2n$ paths need to be constructed in Q_n^k. Nevertheless, it would be interesting to try and improve upon our wide-diameter bound and bring it closer to the diameter of $AQ_{n,k}$.

Finally, there are numerous other aspects relating to augmented k-ary n-cubes which are worthy of study: for example, the embedding of other networks in $AQ_{n,k}$ (*cf.* [4, 5, 12]), the tolerance of faults within $AQ_{n,k}$ (*cf.* [5, 6]), and broadcasting and routing in $AQ_{n,k}$ (*cf.* [3, 12]).

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