A slice genus lower bound from $sl(n)$ Khovanov-Rozansky homology

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Abstract

We show that a generic perturbation of the doubly-graded Khovanov-Rozansky knot homology gives rise to a lower-bound on the slice genus of a knot. We prove a theorem about obtainable presentations of surfaces embedded in 4-space, which we use to simplify significantly our algebraic computations.

1 Introduction

1.1 $sl(2)$ Khovanov homology

Given the data of an oriented link diagram D, one can compute the HOMFLY polynomial $P(D)$ (a polynomial over $\mathbb{Z}$ in the variables $a^{\pm 1}$ and $b^{\pm 1}$) using the local skein relation in Figure 1 (up to an arbitrary choice for the HOMFLY polynomial $P(U)$ of the unknot U).

This polynomial is invariant under the oriented Reidemeister moves and hence defines an invariant of oriented links. If we specialize for $n \geq 2$ by defining

$$P_n(q) = P(a = q^n, b = q + q^{-1}),$$

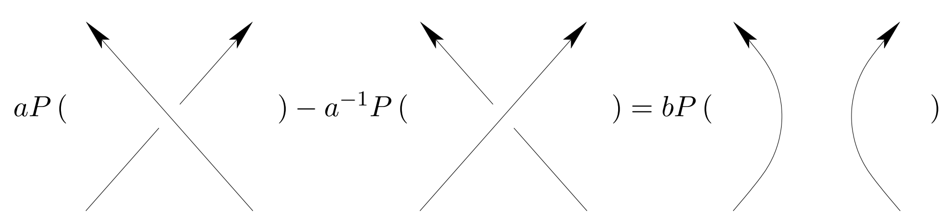

Figure 1: HOMFLY skein relation.

we obtain the $sl(n)$ quantum polynomial of the link given by the diagram D. We will be using the normalization $P_n(U) = (q^n - q^{-n})/(q - q^{-1})$. The polynomial $P_2(q)$ is known as the Jones polynomial.

In [6], Khovanov associated to a link diagram D a bigraded chain complex $CKh_2^{i,j}(D)$ with differential

$$d : CKh_2^{i,j}(D) \to CKh_2^{i+1,j}(D)$$

(i is called the homological grading, j is called the quantum grading). To diagrams D and D' differing by a single Reidemeister move, Khovanov gave a quantum-degree 0 chain homotopy equivalence between $CKh_2(D)$ and $CKh_2(D')$, thus showing that the homology groups $HKh_2^{i,j}(D)$ are knot invariants. Furthermore these homology groups provide a *categorification* of the Jones polynomial $P_2(D)$, by which is meant

$$P_2(q) = \sum_{i,j} (-1)^i q^j dim(HKh_2^{i,j}).$$

A powerful facet of the HKh_2 theory was conjectured by Khovanov (and later proved by Jacobsson [3]), namely that the homology theory should be functorial for link cobordisms up to sign. More explicitly, suppose we start with a smooth embedding of a surface Σ with boundary

$$\Sigma \hookrightarrow [0,1] \times \mathbb{R}^3,$$

$$\partial\Sigma = (L_0 \hookrightarrow \{0\} \times \mathbb{R}^3) \coprod (L_1 \hookrightarrow \{1\} \times \mathbb{R}^3),$$

otherwise known as a *link* or *knot cobordism* $\Sigma : L_0 \to L_1$ between the links L_0 and L_1. Choose link diagrams D_0, D_1 of the links L_0, L_1. Next we take a representation of the surface Σ as a product of *elementary cobordisms.* Elementary cobordisms consist of before-and-after link diagrams where we have made one local change in the before diagram to get to the after diagram. The local changes that we are allowed to make consist of each of the Reidemeister moves and the *Morse moves* which correspond to adding a handle to the surface as shown in Figure 2.

To diagrams D and D' differing by one elementary cobordism Khovanov associated a chain map $CKh_2(D) \to CKh_2(D')$, inducing a map on homology. For the Reidemeister moves, these maps are just the chain homotopy equivalences referred to earlier. For the 0-handle and 2-handle moves these maps are graded of quantum-degree +1 and for the 1-handle move it is

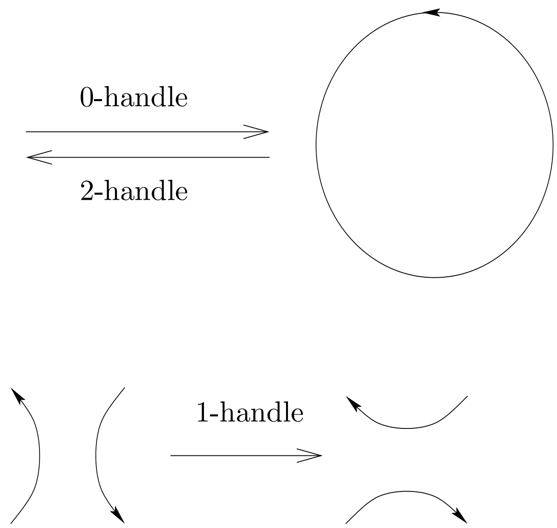

Figure 2: Morse moves for link cobordism. The 0-handle move adds a *cap* to the surface, creating a new unknotted component disjoint from the rest of the link diagram; the 2-handle move adds a *cup* removing such a component; the 1-handle move adds a *saddle* between two arcs of the link diagram.

graded of degree -1. By composing the maps corresponding to these elementary cobordisms we get a map graded of quantum-degree $\chi(\Sigma)$

$$HKh_2(\Sigma) : HKh_2^{i,j}(L_0) \to HKh_2^{i,j+\chi(\Sigma)}(L_1).$$

This map (up to multiplication by -1) is an invariant of Σ, as the notation suggests. In other words, whichever decomposition of Σ into elementary cobordisms is chosen, the induced map $HKh(\Sigma)$ stays the same up to sign.

1.2 Perturbed $sl(2)$ theory

In [9], Lee gives a description of a perturbation HKh' of the Khovanov homology HKh_2. This homology theory is no longer graded in the quantum direction, but filtered: $\ldots \subseteq \mathcal{F}^{j+1}HKh'^i \subseteq \mathcal{F}^j HKh'^i \subseteq \ldots$. The homology $HKh'(L)$ has a particularly simple form: when we take the base ring to be $\mathbb{C}$ it consists of 2^l copies of $\mathbb{C}$ where l is the number of components of the link L. Lee gives explicit chain representatives of these generators defined from any diagram presentation of L. As a formal consequence of the properties of filtered chain complexes, there is a spectral sequence with E_2

page $HKh_2^{i,j}$ converging to E_∞ page $HKh'^{i,j} := \mathcal{F}^j HKh'^i / \mathcal{F}^{j+1} HKh'^i$, the associated graded vector space of HKh' with respect to the filtration $\mathcal{F}$.

Analysis of the behaviour of HKh' under link cobordism was carried out by Rasmussen [12]. In this remarkable paper, Rasmussen showed that the quantum grading of Lee's generators gives rise to a lower bound for the slice genus of a knot. Let us digress to define the slice genus.

Given a knot $K \hookrightarrow \mathbb{R}^3$, a classical knot invariant is the *genus* $g \geq 0$ of the knot. That is, the minimal genus of the surfaces-with-boundary Σ for which there exists an embedding $\Sigma \hookrightarrow \mathbb{R}^3$ with $\partial\Sigma = K$.

If we believe that manifolds with boundary should rightfully live within other manifolds with boundary, we are motivated to make the following definition.

Definition 1.1. *Consider compact, orientable surfaces Σ smoothly embedded in $(-\infty, 0] \times \mathbb{R}^3$ with $K = \partial\Sigma \hookrightarrow \{0\} \times \mathbb{R}^3$. The minimal genus of such surfaces we call the* slice genus $g^*(K)$ *of the knot K.*

By removing a neighbourhood of a point, such a surface provides a knot cobordism between K and the unknot U. Rasmussen showed that the map associated to any presentation (as a composition of elementary cobordisms) of a connected knot cobordism Σ between two (1-component) knots K_0 and K_1 preserves the generators of Lee's homology HKh'. This map is filtered of quantum-degree $\chi(\Sigma)$

$$\mathcal{F}^j HKh'^i(K_0) \to \mathcal{F}^{j+\chi(\Sigma)} HKh'^i(K_1).$$

The Lee homology of the unknot is computable as $HKh'^{0,1} = \mathbb{C}$ and $HKh'^{0,-1} = \mathbb{C}$ with no homology in any other bigrading. Hence as an immediate corollary, $2g^*(K)$ is bounded below by one less than the highest degree in which $HKh'(K)$ is non-zero.

In general, to compute HKh' directly is difficult, but it can be carried out for positive knots (those knots that admit a diagram with only positive crossings). In this case we get a tight bound on the slice genus since we can explicitly construct a Seifert surface Σ of the same genus as our lower bound, and then just push this surface into $(-\infty, 0] \times \mathbb{R}^3$ to get a slice surface. In particular we can apply this to torus knots $K_{p,q}$ to get a combinatorial proof of Milnor's conjecture (see Corollary 1.3 for an exact statement) on the value of $g^*(K_{p,q})$.

1.3 $sl(n)$ Khovanov-Rozansky homology

In [7], Khovanov and Rozansky describe bigraded homology theories HKh_n for $n \geq 3$ which categorify the $sl(n)$ polynomials P_n. Again, given a knot diagram D, Khovanov and Rozansky associate to it a chain complex $CKh_n(D)$ (in a much more complicated way than for the case $n = 2$) with differentials

$$d_n : CKh_n^{i,j}(D) \to CKh_n^{i+1,j}(D).$$

And also as before, if D and D' differ by a Reidemeister move, they give a chain homotopy equivalence (graded of quantum-degree 0) between $CKh_n(D)$ and $CKh_n(D')$, thus showing that $HKh_n^{i,j}(D)$ is an invariant of the knot and not just the diagram D. This homology theory is also projectively functorial for knot cobordisms.

Gornik [2] has carried out for HKh_n a direct analogue of Lee's work on HKh_2, describing a perturbation with filtered homology HKh'_n, which can be interpreted as the E_∞ page of a spectral sequence with E_2 page HKh_n. Given an l-component link diagram D, Gornik gives an explicit description of the n^l generators of $HKh'_n(D)$.

1.4 Statement of results

Our intention in this paper is to do for Gornik's work [2] something of what Rasmussen [12] did for that of Lee's [9]. We start by generalizing Gornik's HKh'_n to a theory that we denote HKh_w, where $w \in \mathbb{C}[x]$ is a monic polynomial of degree $n+1$ such that $\partial_x w = dw/dx$ is a product of distinct linear factors. In this set up,

$$HKh'_n = HKh_{x^{n+1}+(n+1)\beta^n x}$$

for $\beta \in \mathbb{C} - \{0\}$. The homology HKh_w is filtered in the quantum direction

$$\ldots \subseteq \mathcal{F}^j HKh_w^i \subseteq \mathcal{F}^{j+1} HKh_w^i \subseteq \ldots .$$

Our procedure essentially starts by following that of [12]. For diagrams D, D' differing by a Reidemeister move we wish to construct a map $\mathcal{F}^j HKh_w^i(D) \to \mathcal{F}^j HKh_w^i(D')$ which preserves our analogues of the generators in [2]. If D, D' differ by a 0- or 2-handle attachment we wish to give maps $\mathcal{F}^j HKh_w^i(D) \to \mathcal{F}^{j-n+1} HKh_w^i(D')$ and if D, D' differ by a 1-handle we wish to give maps $\mathcal{F}^j HKh_w^i(D) \to \mathcal{F}^{j+n-1} HKh_w^i$. We aim to do this so that we can compute that any representation of a connected knot cobordism $\Sigma : D \to D'$ as a product of elementary knot cobordisms preserves the generators of $HKh_w(D)$.

Due to the complexity of the Reidemeister III move, we do not quite fulfill all of our wishes. However a topological argument (Theorem 1.6), by which we only have to consider certain products of elementary cobordisms, is enough for us to get the desired analogue of the slice genus bound in [12].

We write $HKh_w^{i,j}(D) = \mathcal{F}^j HKh_w^i / \mathcal{F}^{j-1} HKh_w^i$ for the associated graded vector space to the quantum filtration.

Our main theorem is

Theorem 1.2. *Let D be a knot diagram of a knot K. If $HKh_w^{0,j}(D) \neq 0$ and $HKh_w^{0,j'}(D) = 0$ for all $j' > j$ then*

$$(n-1)(2g^*(K) - 1) \geq -j.$$

As a corollary we obtain another proof of Milnor's conjecture on the slice genus of torus knots.

Corollary 1.3. *The slice genus (see Definition 1.1) $g^*(K_{p,q})$ of a (p,q)-torus knot agrees with the genus $g(K_{p,q})$ of the knot and is given by the formula:*

$$g^*(K_{p,q}) = (p-1)(q-1)/2.$$

The topological result which allows us to avoid the algebraic complexity of Reidemeister move III, is best stated as a "normal form" theorem for closed surfaces. In [5] , Kawauchi, Shibuya, and Suzuki propose the following definition of a *normal form* for presentations of closed surfaces in 4-space. They then show analytically that any surface has such a normal form presentation.

Definition 1.4. *Normal form for a closed surface in 4-space.*

Given a smooth embedding of the genus g surface $i : \Sigma_g \hookrightarrow \mathbb{R}^4$, then a normal form for i is a height function $h : \mathbb{R}^4 \to \mathbb{R}$ which restricts to a Morse function $f = h \circ i$ on the surface Σ_g with the following properties:

1. *The index 0 critical points of f are contained in $f^{-1}(3)$.*
2. *The index 2 critical points of f are contained in $f^{-1}(-3)$.*
3. *The index 1 critical points of f are contained in $f^{-1}(\{2, 1, -1, -2\})$. In terms of handles, $f^{-1}(\{2, -1\})$ contains only fusion 1-handles (handles connecting two link components), $f^{-1}(\{1, -2\})$ contains only fission 1-handles (handles splitting one link component into two).*

4. *$f^{-1}(1.5)$ and $f^{-1}(-1.5)$ are connected and $f^{-1}(0)$ has $g+1$ components.*

Theorem 1.5. *(See [5]) For every smooth embedding of a surface $\Sigma_g \hookrightarrow \mathbb{R}^4$, there exists a height function with the properties given in Definition 1.4.*

In terms of elementary cobordisms, an index i critical point corresponds to the addition of an i-handle, so Theorem 1.5 contains information about achievable orderings of handle additions. It also gives an achievable ordering on fusion 1-handles (handles joining two components into one) versus fission 1-handles (handles splitting one component into two). Theorem 1.6, on the other hand, contains information about achievable orderings of Reidemeister moves in a presentation of an embedded surface as a sequence of elementary cobordisms. We use Theorem 1.5 to achieve the 1-handle orderings of Theorem 1.6.

Theorem 1.6. *Suppose $\Sigma_g : L_0 \to L_1$ is a connected genus g knot cobordism between the two links L_0 and L_1. Suppose further that D_i is a diagram of L_i for $i = 0, 1$. Puncturing Σ_g k times gives a cobordism $\overline{\Sigma_g} : L_0 \to \overline{L_1}$ where $\overline{L_1}$ is the link composed of the disjoint union of L_1 with the k-component unlink.*

For some k there exists a presentation of $\overline{\Sigma_g}$ as a sequence of elementary cobordisms in which the elementary cobordisms come in the following order:

1. *The presentation begins with the diagram D_0.*

2. *Then all the 0-handles of the presentation.*

3. *Then a sequence of Reidemeister I and II moves, each of which increases the number of crossings.*

4. *Then a sequence of fusion 1-handles, ending in a 1-component knot diagram.*

5. *Then g fission 1-handles.*

6. *Then g fusion 1-handles.*

7. *Then a sequence of Reidemeister I and II moves and fission 1-handles ending in a diagram $\overline{D_1}$ of $\overline{L_1}$, which is the disjoint union (as an immersed 1-manifold in $\mathbb{R}^2$) of D_1, 0-crossing diagrams of the unknot, and diagrams as in Figure 36.*

In the final stages of the preparation of this paper a preprint by Wu [13] appeared on the arXiv, which contains, among other results, a proof of Theorem 1.2. The approaches of the two papers are distinguished in part by our paper's use of the topological result Theorem 1.6.

1.5 Overview of this paper

Section 2 contains definitions and theorems whose statements echo those to be found in [2] and [7] but adapted to our choice of potential. One purpose of the section is to show that the results of [2] hold for more general choices of potential than those considered in that paper. As a consequence of the results in this section it will follow that for D a diagram of an l-component link, $HKh_w(D)$ consists of n^l copies of $\mathbb{C}$. We give explicit chain representatives of these n^l generators of the homology. When $l = 1$, $HKh_w(D)$ is supported in homological degree 0.

Although conventional practice dictates that one should deal with invariance under Reidemeister moves before other considerations, issues of logical dependence encourage us to postpone this until Section 4.

Section 3 deals with Morse moves for knot cobordisms as in Figure 2. To link diagrams D, D' differing by a Morse move we associate an isomorphism $HKh_w(D) \to HKh_w(D')$, induced by a chain map $CKh_w(D) \to CKh_w(D')$, that is filtered of quantum degree $1 - n$ in the case of the 0-handle and 2-handle Morse moves and of degree $n - 1$ in the case of the 1-handle Morse move. We see that the maps we define have good properties in terms of preserving the generators defined in Section 2.

Section 4 is also computational. The purpose of this section is to show that for link diagrams D and D' differing by a Reidemeister move I or II there is a map $HKh_w(D) \to HKh_w(D')$, filtered of quantum-degree 0 and induced by a chain map $CKh_w(D) \to CKh_w(D')$, that preserves the generators of $HKh_w(D)$. If we could show this also for Reidemeister move III, a consequence would be that the graded groups associated to the quantum filtration $HKh_w^{i,j}(D) = \mathcal{F}^j HKh_w^i / \mathcal{F}^{j-1} HKh_w^i$ would be invariants of the link represented by the diagram D.

Section 5 contains the proof of Theorem 1.6. Section 6 shows how Theorem 1.6 and the computations of Sections 3 and 4 give us Theorem 1.2 and hence another proof of Milnor's conjecture Corollary 1.3.

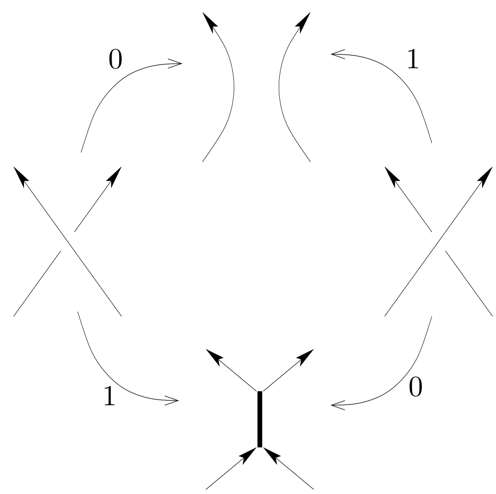

Figure 3: This diagram explains how to form resolutions and how to associate each one with a vertex of the cube $[0,1]^n$. Note that all but one of the resolutions will have one or more *thick edges*.

1.6 Acknowledgement

The author wishes to thank Peter Kronheimer for many useful conversations during the writing of this paper.

2 Construction of perturbed $sl(n)$ Khovanov-Rozansky homology

In this section we construct and find chain representatives for a basis of $HKh_w(D)$.

2.1 The Khovanov cube

Here we outline the construction of HKh_w. This is similar to the construction of the original Khovanov-Rozansky homology HKh_n. Indeed, replacing any occurence of the word “filtered” by the word “graded” in this Subsection will give an outline of the construction of HKh_n.

To define HKh_w of an oriented link diagram D with m crossings we start by forming the 2^m possible *resolutions* of D and decorating each vertex of the cube $[0,1]^m$ with one of these resolutions. We form the resolutions as shown in Figure 3.

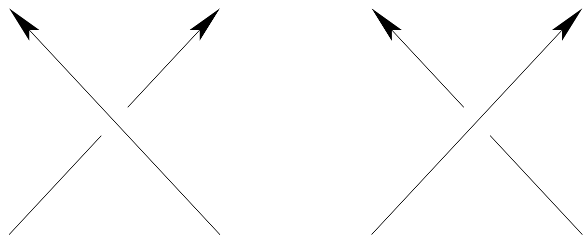

Figure 4: These are the signs associated to each crossing of a diagram D. The total signed number of crossings is called the *writhe* $W(D)$.

Choosing an (arbitrary) ordering of the crossings allows us to associate each resolution Γ of D with one of the corners of the cube. For a corner v call the associated resolution Γ_v. In Subsection 2.4 we shall see how to associate to Γ_v a filtered vector space $H(\Gamma_v)$. We shall often refer to the filtration as the *quantum* filtration.

If two corners of the cube v, v' are connected by an edge e then we see that their associated resolutions Γ_v and $\Gamma'_{v'}$ differ only in the resolution of a single crossing of D. Suppose that v is at the 0-coordinate of the edge and v' is at the 1-coordinate of the edge. In the following subsections we shall define a map $\Phi_e : H(\Gamma_v) \to H(\Gamma_{v'})$ of quantum filtered degree 1. If e_1, e_2, e_3, e_4 are edges bounding a face of the cube then $\Phi_{e_1}\Phi_{e_2} = \Phi_{e_3}\Phi_{e_4}$ if the edges are ordered so the composition makes sense.

For a vertex v write the sum of its coordinates as $s(v)$ (this will be an integer between 0 and m). We write W for the *writhe* of the knot diagram D: the signed number of crossings of D as in Figure 4. We define the chain groups of a chain complex by

$$C^k = \sum_{s(v)=k+(1/2)(n+W)} H(\Gamma_v)\{(n-1)W - k\},$$

where by $\{-\}$ we mean a shift in the quantum filtration.

The differentials of the chain complex are defined by

$$d_k = \sum \pm\Phi_e : C^k \to C^{k+1}$$

where the sum is taken over all edges for which it makes sense. The $\pm$ signs are chosen so that each face of the cube has exactly 1 or 3 of its edges decorated with a minus sign; the commutivity of the maps associated to the edges of any face will then ensure that $d_{k+1}d_k = 0$. The shift in the quantum filtrations of the $H(\Gamma_v)$'s ensures that the differentials are filtered of degree 0.

Taking homology we obtain a vector space graded in the homological direction and filtered in the quantum direction. This is our perturbed homology theory $\mathcal{F}^j HKh^i_w(D)$.

2.2 Introduction to matrix factorizations

The definition of the $sl(n)$ Khovanov homology, standard or perturbed, makes use of the notion of a *matrix factorization*. Here we first define *ungraded* matrix factorizations, in the next Subsection we see how to associate *filtered* matrix factorizations to trivalent graphs.

A polynomial $p(x) \in \mathbb{C}[x_1, ..., x_n] = R$ may not admit a non-trivial factorization into polynomials. However if $M^0 = M^1$ is a free module over R then we may be able to find R-module maps f_0, f_1

$$M^0 \xrightarrow{f_0} M^1 \xrightarrow{f_1} M^0$$

such that $f_0 f_1 = f_1 f_0 = p(x)$. Thus we would have factored $p(x)$ into a product of R-module maps: this is called a *matrix factorization* of $p(x)$.

If M and $\tilde{M}$ are both matrix factorizations of $p(x) \in \mathbb{C}[x_1, ..., x_n] = R$ then a *map of matrix factorizations* $G : M \to \tilde{M}$ is a pair of R-module maps

$$G_0 : M^0 \to \tilde{M}^0, \, G_1 : M^1 \to \tilde{M}^1$$

satisfying $\tilde{f}_0 G_0 = G_1 f_0$ and $\tilde{f}_1 G_1 = G_0 f_1$. Note that this echoes the definition of a map of chain complexes. Similarly, considering a matrix factorization as being something akin to a 2-periodic chain complex, we can define the notion of a tensor product:

Suppose M and $\tilde{M}$ are matrix factorizations of $p(x), q(x) \in R$ respectively. Then we define the matrix factorization $M \otimes \tilde{M}$ of $p(x) + q(x)$ by $(M \otimes \tilde{M})^0 = (M^0 \otimes \tilde{M}^0) \oplus (M^1 \otimes \tilde{M}^1)$, $(M \otimes \tilde{M})^1 = (M^1 \otimes \tilde{M}^0) \oplus (M^0 \otimes \tilde{M}^1)$ with maps

$$\begin{pmatrix} f_0 & \tilde{g}_1 \\ -\tilde{g}_0 & f_1 \end{pmatrix} : (M \otimes \tilde{M})^0 \to (M \otimes \tilde{M})^1,$$

$$\begin{pmatrix} f_1 & -\tilde{g}_1 \\ \tilde{g}_0 & f_0 \end{pmatrix} : (M \otimes \tilde{M})^1 \to (M \otimes \tilde{M})^0.$$

Since matrix factorizations have the look of 2-periodic complexes (but with d^2 now being a polynomial, not necessarily zero), we can define the

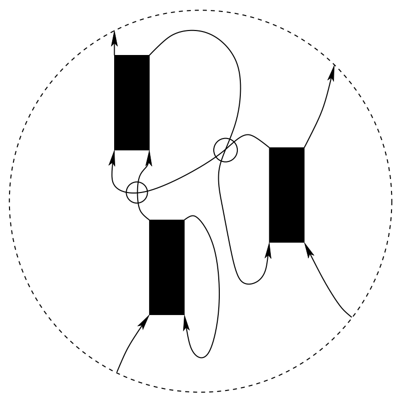

Figure 5: An example of a trivalent graph. The circles show where two edges do not intersect or give rise to an overcrossing or undercrossing.

obvious notions of homotopic maps of matrix factorizations and homotopy equivalent matrix factorizations. We will be working in the category of matrix factorizations and homotopy classes of maps.

2.3 Trivalent graphs

Khovanov and Rozansky [7] describe a way of associating a matrix factorization to any finite trivalent graph with thick edges and labelled boundary components. An example of what we mean by a trivalent graph with thick edges is given in Figure 5, here we have indicated the boundary as a dotted circle which shall hereafter be omitted from our diagrams of trivalent graphs.

Note that the thin edges are oriented and at a thick edge the orientations of the incident thin edges look like Figure 6. Each trivalent vertex has one thick edge and two thin edges incident to it. Only thin edges are allowed to end on the boundary of the graph. We allow closed thin loops.

Also note that we have included what appear to be crossings (which are circled) of thin edges. But a trivalent graph need not come with an embedding into $\mathbb{R}^2$ or even $\mathbb{R}^3$, and we are thinking of these thin edges as neither intersecting nor as giving rise to an "overcrossing" or an "undercrossing".

Remark 2.1. *These trivalent graphs should be thought of as the $sl(n)$ analogues of Kauffman's smoothings of a knot diagram giving rise to the state sum model of the Jones polynomial [4]. Given a knot diagram D of a knot K, the $sl(n)$ quantum polynomial $P_n(K)$ can be expressed as a signed sum*

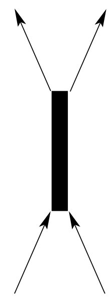

Figure 6: The neighbourhood of a thick edge.

of polynomials $P_n(\Gamma)$ where Γ ranges over all resolutions of D, normalized by the writhe of D. Murakami, Ohtsuki, and Yamada [10] studied these polynomials $P_n(\Gamma)$ and gave a complete set of relations for computing them.

To associate a matrix factorization to such a trivalent graph will first involve choosing a *potential*: a polynomial $w(x) \in \mathbb{C}[x]$ of degree $n+1$. Khovanov's and Rozansky's original theory took $w = x^{n+1}$ and Gornik used the perturbation $w = x^{n+1} + (n+1)\beta^n x$ for $\beta \in \mathbb{C} - 0$. In the course of this paper we use a general polynomial w of degree $n+1$, satisfying the condition that $\partial_x w = dw/dx$ factors as a product of distinct linear factors.

Every variable x_i that appears in the polynomial rings used to define matrix factorizations is taken to have quantum degree 2 (we often use the word *quantum* so as to distinguish from *homological*). We will be using *filtered* matrix factorizations, which means each module M is filtered $\ldots \subseteq \mathcal{F}^i M \subseteq \mathcal{F}^{i+1} M \subseteq \ldots$. The filtration on M (which will always be a free module over R) shall be induced by the grading on R with possibly an overall shift. To denote a shift of quantum degree d in the filtration grading of a module M we use the notation $M\{d\}$. The two differentials in the matrix factorization should have filtered degree $n+1$ (note multiplication by $w(x)$ is filtered of degree $2(n+1)$). Maps of matrix factorizations, unless stated otherwise, are of filtered degree 0, homotopies are of filtered degree $-(n+1)$.

We define two fundamental factorizations: that of an oriented thin line segment (Figure 7) and that of a neighbourhood of a thick edge (Figure 8). Then we can obtain the factorization $C(\Gamma)$ coming from any trivalent graph Γ by tensoring together these fundamental factorizations, identifying boundary variables where we have joined the fundamental factorizations.

Definition 2.2. *The fundamental factorization in Figure 7 is the factorization of $w(x_1) - w(x_2)$ over $R = \mathbb{C}[x_1, x_2]$ given by*

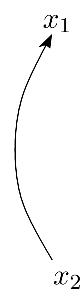

Figure 7: An oriented line segment with labelled boundary points.

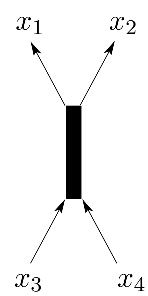

Figure 8: A thick edge with labelled boundary points.

$$M^0 \xrightarrow{\pi_{x_1x_2}} M^1 \xrightarrow{x_1-x_2} M^0$$

where $M^0 = R$ and $M^1 = R\{1-n\}$ are rank 1 modules over R and $\pi_{x_1x_2} = (w(x_1) - w(x_2))/(x_1 - x_2)$. Remember the pair of curly brackets denotes a grading shift so that each of our differentials is of filtered degree $n+1$.

We now define the factorization in Figure 8 over the ring $R = \mathbb{C}[x_1, x_2, x_3, x_4]$. This is a factorization of the polynomial $w(x_1) + w(x_2) - w(x_3) - w(x_4)$.

Consider the unique polynomial p in two variables such that $p(x + y, xy) = w(x) + w(y)$.

Then

$$\begin{aligned}
&w(x_1) + w(x_2) - w(x_3) - w(x_4) \\
&\quad = p(x_1 + x_2, x_1x_2) - p(x_3 + x_4, x_3x_4) \\
&\quad = p(x_1 + x_2, x_1x_2) - p(x_3 + x_4, x_1x_2) + p(x_3 + x_4, x_1x_2) - p(x_3 + x_4, x_3x_4) \\
&\quad = (x_1 + x_2 - x_3 - x_4)u_1 + (x_1x_2 - x_3x_4)u_2
\end{aligned}$$

for some polynomials u_1, u_2.

Definition 2.3. *The second fundamental factorization is the tensor product over R of the two factorizations*

$$R\{-1\} \xrightarrow{u_1} R\{-n\} \xrightarrow{x_1+x_2-x_3-x_4} R\{-1\},$$

$$R \xrightarrow{u_2} R\{3-n\} \xrightarrow{x_1x_2-x_3x_4} R.$$

Note that the differentials of the resulting factorization have filtered degree $n+1$.

Hence the factorization $C(\Gamma)$ (a tensor product of fundamental factorizations), associated to a general trivalent graph Γ, also has differentials of filtered degree $n+1$. Let $R(\Gamma) = Q[X_e | e \in \text{edge endpoints}]$, then the graph Γ defines a matrix factorization $C(\Gamma)$ over $R(\Gamma)$

$$M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} M^0$$

where M^0 and M^1 are free modules over $R(\Gamma)$ and

$$d^1d^0 = d^0d^1 = \sum_{\substack{\text{edge endpoints e} \\ \text{pointing out of } \Gamma}} w(X_e) - \sum_{\substack{\text{edge endpoints e} \\ \text{pointing into } \Gamma}} w(X_e). \tag{1}$$

2.4 The generators of $HKh_w(D)$

We have required of our potential w that $\partial_x w$ is a product of distinct linear factors. This condition is the main ingredient in many of the results in [2], the proofs of which carry across very easily to our potential. We shall content ourselves, most often, with stating these results as they apply in our case without proof, but in a few cases we shall give a proof where things are more difficult for our potential than for that of Gornik's or where we feel it is important for the rest of the paper to understand the proof.

Assume now that Γ is a closed trivalent graph so that $C(\Gamma)$ (the matrix factorization associated to Γ) is actually a 2-periodic complex (see Equation 1), call the homology of this 2-complex $H(\Gamma)$. Since $C(\Gamma)$ is filtered, $H(\Gamma)$ is also filtered $\ldots \subseteq \mathcal{F}^i H(\Gamma) \subseteq \mathcal{F}^{i+1} H(\Gamma) \subseteq \ldots$. Let $R(\Gamma)$ be the polynomial ring over $\mathbb{C}$ generated by variables X_e as e runs over the thin edges of Γ (note: not just the thin edges with boundary as in the previous section). Certainly the homology $H(\Gamma)$ is a module over $R(\Gamma)$, but in fact it is also a module over the ring $\bar{R}(\Gamma)$ obtained from $R(\Gamma)$ by quotienting out any polynomial in the X_e's appearing in the definition of the fundamental factorizations making up $C(\Gamma)$, (since these polynomials define 0-homotopic endomorphisms of $C(\Gamma)$).

Proposition 2.4. *(See [2] Proposition 2.4) The algebra $\bar{R}(\Gamma)$ is generated by elements X_e (by abuse of notation X_e means the image of $X_e \in R(\Gamma)$ in $\bar{R}(\Gamma)$) where e runs over the thin edges of Γ, and each X_e satisfies*

$$\partial_x w(X_e) = 0.$$

□

Definition 2.5. *Writing Σ_n for the roots of $\partial_x w$, a state ϕ of Γ is an assignment*

$$\phi : e(\Gamma) \to \Sigma_n$$

of elements of Σ_n to the thin edges $e(\Gamma)$ of Γ.

Proposition 2.6. *(See [2] Proposition 2.5) Given a state ϕ, define $Q_\phi \in \bar{R}(\Gamma)$ by*

$$Q_\phi = \prod_{e\in e(\Gamma)} \left(\prod_{\alpha\in\Sigma_n\setminus\phi(e)} \frac{1}{\phi(e)-\alpha} \right) \frac{1}{n+1} \frac{\partial_x w(X_e)}{X_e - \phi(e)},$$

where $\partial_x w(X_e)/(X_e - \phi(e))$ is the result of substituting X_e for x in the polynomial $\partial_x w(x)/(x - \phi(e))$.

Then we have

$$Q_{\phi_1} Q_{\phi_2} = \begin{cases} Q_{\phi_1}, & \phi_1 = \phi_2 \\ 0, & \phi_1 \neq \phi_2 \end{cases}$$

$$\sum_\phi Q_\phi = 1.$$

Proof. It suffices to check that the relations hold in the ring

$$\bigotimes_{e\in e(\Gamma)} \mathbb{C}[X_e]/\partial_x w(X_e)$$

since $\bar{R}(\Gamma)$ is a quotient of this ring. □

Given an embedding of Γ into $\mathbb{R}^2$ we can define the notion of *admissability* of states. In the construction of the Khovanov-Rozansky homology of a knot, we start with a diagram of the knot and all Γ's appearing in the construction will come with an embedding into $\mathbb{R}^2$ induced by the knot diagram.

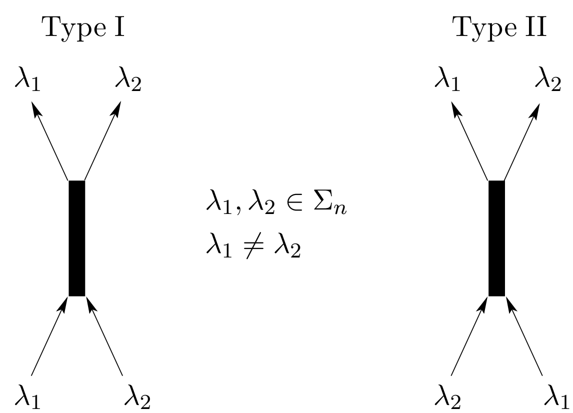

Figure 9: Type I and II admissable states. Each thin edge is decorated with an element of Σ_n, subject to satisfying the conditions shown at each thick edge.

Definition 2.7. *A state ϕ is called* admissable *if the assignment $\phi : \Sigma_n \to e(\Gamma)$ looks like either of the possibilities in Figure 9 in a neighbourhood of each thick edge. We call the set of all states $S'(\Gamma)$ and the set of admissable states $S(\Gamma)$.*

In the following theorem we refer to the polynomial $P_n(\Gamma)$ of a trivalent graph Γ. For the purposes of this paper, Remark 2.1 is sufficient information for the reader. For further information we refer readers to [7, 10].

Theorem 2.8. *(See [2] Theorem 3) For non-admissable states ϕ we have*

$$Q_\phi = 0.$$

For admissable states ϕ we have

$$0 \neq \mathbb{C}Q_\phi = \bar{R}(\Gamma)Q_\phi \text{ and so } \dim_{\mathbb{C}}\bar{\mathrm{R}}(\Gamma)\mathrm{Q}_\phi = 1.$$

We have a decomposition as a $\mathbb{C}$-algebra

$$\bar{R}(\Gamma) = \oplus_{\phi \in S(\Gamma)} \mathbb{C}Q_\phi$$

and

$$dim_{\mathbb{C}} \bar{R}(\Gamma) = P_n(\Gamma)|_{q=1}.$$

$\square$

Proposition 2.9. *Let $k \in \mathbb{Z}$. Let $Kh^k(\Gamma)$ be the quantum degree k piece of the classical $sl(n)$ Khovanov-Rozansky homology of the graph Γ. Then there is an isomorphism of vector spaces*

$$\Phi : Kh^k(\Gamma) \to \mathcal{F}^k H(\Gamma)/\mathcal{F}^{k-1}H(\Gamma).$$

Corollary 2.10. *Since the graded dimension of $Kh(\Gamma)$ is just $P_n(\Gamma)$ (the $sl(n)$ polynomial of Γ), the filtered dimension of $H(\Gamma)$ is $P_n(\Gamma)$. The number of admissable states is $P_n(\Gamma)(1)$ so that the dimension of $H(\Gamma)$ as a complex vector space agrees with the number of admissable states.* □

Here since the translation of Gornik's proof is not completely straightforward, we give some details.

Proof. (of Proposition 2.9) Khovanov and Rozansky have shown that the classical 2-periodic complex $CKh(\Gamma)$ has cohomology only in one of the two homological gradings. Suppose without loss of generality that it lies in grading 1.

$CKh(\Gamma)$ looks like

$$CKh^0(\Gamma) \xrightarrow{d_0} CKh^1(\Gamma) \xrightarrow{d_1} CKh^0(\Gamma)$$

where d_0 and d_1 are *graded* of degree $n+1$. The perturbed 2-periodic complex $C(\Gamma)$ looks like

$$C^0(\Gamma) \xrightarrow{d_0'} C^1(\Gamma) \xrightarrow{d_1'} C^0(\Gamma)$$

where d_0' and d_1' are *filtered* of degree $n+1$. In fact we can decompose d_i' as

$$d_i' = d_i^0 + d_i^1 + d_i^2 + \dots$$

where d_i^j is graded of degree $n+1-2j$ and $d_i^0 = d_i$ for $i = 1, 2$. To define the Φ mentioned in the Proposition we give first a lift of Φ

$$\phi : (\ker d_1)^k \to \mathcal{F}^k H(\Gamma)/\mathcal{F}^{k-1}H(\Gamma)$$

$$\phi : \alpha \mapsto \alpha + \alpha^1 + \alpha^2 + \dots$$

where $(\ker d_1)^k$ is the k^{th} graded piece of $\ker d_1$, and α^i has degree $k - 2i$. We define each α^i inductively. We wish to define each α^i so that

$$\sum_{l=0}^{k} d_1^i(\alpha^{j-i}) = 0 \;\forall j. \tag{2}$$

Writing $\alpha = \alpha^0$ gives us the root case. Suppose that we know (2) for $k \leq K$; we wish to define α^{K+1} so that we have (2) for $k \leq K+1$.

If we can show that the following holds:

$$d_0^0\left(\sum_{i=0}^{K} d_1^{i+1}(\alpha^{K-i})\right) = 0 \tag{3}$$

then we will be done since we know that $\ker d_0^0 = \mathrm{im} d_1^0$ so we can find an α^{K+1} satisfying

$$d_1^0(-\alpha^{K+1}) = \sum_{i=0}^{K} d_1^{i+1}(\alpha^{K-i}).$$

Now,

$$\begin{aligned}
&d_0^0\left(\sum_{i=0}^{K} d_1^{i+1}(\alpha^{K-i})\right)\\
&= \sum_{i=0}^{K} d_0^0 d_1^{i+1}(\alpha^{K-i}) = -\sum_{i}^{K}\sum_{j=0}^{i} d_0^{j+1} d_1^{i-j}(\alpha^{K-i})\\
&= -\sum_{j=0}^{K}\sum_{i=j}^{K} d_0^{j+1} d_1^{i-j}(\alpha^{K-i}) = -\sum_{j=0}^{K} d_0^{j+1}\sum_{i=0}^{K-j} d_1^{i}(\alpha^{(K-j)-i}) = 0.
\end{aligned}$$

It is easy to check that the map ϕ so defined gives a well-defined isomorphism Φ. □

Proposition 2.11.

$$H(\Gamma) = \bigoplus_{\phi\in S(\Gamma)} Q_\phi H(\Gamma)$$

$$\dim_{\mathbb{C}} \mathrm{Q}_\phi \mathrm{H}(\Gamma) = 1$$

Proof. For dimensional reasons, it is enough to show that for any $\phi \in S(\Gamma)$ we can find a non-zero element of $Q_\phi H(\Gamma)$. Such a non-zero element is described explicitly below.

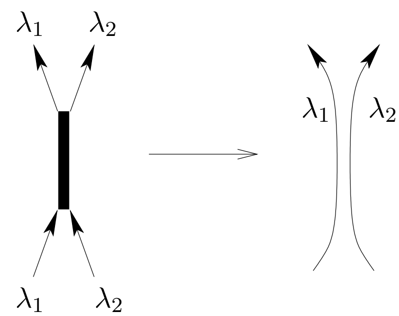

Figure 10: The rule for forming a new graph Γ' when starting with a type I decoration ϕ of Γ.

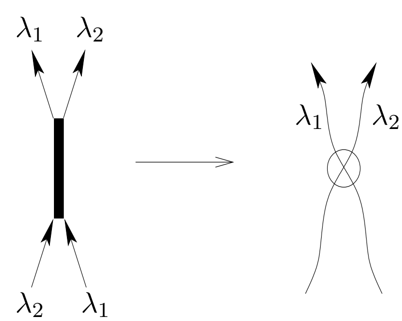

Figure 11: The rule for forming a new graph Γ' when starting with a type II decoration ϕ of Γ.

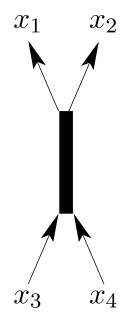

Figure 12: The graph Γ_1, to which is associated the fundamental matrix factorization $C(\Gamma_1)$, as defined in Subsection 2.4.

Given $\phi \in S(\Gamma)$ form a new graph Γ', with each component decorated with an element $\lambda \in \Sigma$ by following the local rule in Figures 10 and 11 at all thick edges. We remind the reader that a trivalent graph describing a matrix need not come with an embedding into $\mathbb{R}^2$. The circled crossings in Γ' should not be thought of as undercrossings or overcrossings but just as the two strands not intersecting.

The state ϕ gives a corresponding state ϕ' of Γ'. Now since Γ' is a union of circles, $H(\Gamma') = R(\Gamma')$ naturally so we can pick unambiguously a $Q_{\phi'} \in H(\Gamma)$.

We apply χ_0 and η_0 repeatedly to $H(\Gamma')$ until we arrive in $H(\Gamma)$. (χ_0 is defined in the remainder of this section and η_0 is defined in Appendix 7.2). We recycle old notation that we will not use again and define Q_ϕ to be the image of $Q_{\phi'}$ under these map. Gornik shows that this element is non-zero. In fact, applying the obvious combination of χ_1 and η_1 maps repeatedly to $H(\Gamma)$ takes Q_ϕ back to a non-zero multiple of $Q_{\phi'} \in H(\Gamma')$, and hence $Q_\phi \neq 0$. $\square$

To define $HKh_w(D)$ for a link diagram D we have to give two maps of matrix factorizations

$$\chi_0 : C(\Gamma_0) \to C(\Gamma_1) \quad \chi_1 : C(\Gamma_1) \to C(\Gamma_0).$$

From these maps we build up the differentials in the chain complex of matrix factorization $CKh_w(D)$. Given a knot diagram D, the Khovanov cube with closed trivalent graphs (giving matrix factorizations of 0 i.e. 2-periodic complexes) at the vertices and maps of matrix factorizations on each edge is given by taking the tensor product of

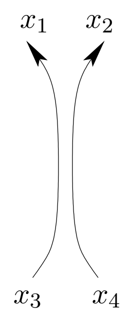

Figure 13: The graph Γ_0, to which is associated the fundamental matrix factorization $C(\Gamma_0)$, as defined in Subsection 2.4.

$$0 \to C(\Gamma_0)\{1-n\} \xrightarrow{\chi_0} C(\Gamma_1)\{-n\} \to 0$$

for every positive crossing (the left of Figure 4) and

$$0 \to C(\Gamma_1)\{n\} \xrightarrow{\chi_1} C(\Gamma_0)\{n-1\} \to 0$$

for every negative crossing (the right of Figure 4). This definition clearly agrees with that of Subsection 1.5.

We give an explicit description (R is the ring $\mathbb{C}[x_1, x_2, x_3, x_4]$). The factorization $C(\Gamma_0)$ is

$$\left(\begin{array}{c} R \\ R\{2-2n\} \end{array} \right) \xrightarrow{P_0} \left(\begin{array}{c} R\{1-n\} \\ R\{1-n\} \end{array} \right) \xrightarrow{P_1} \left(\begin{array}{c} R \\ R\{2-2n\} \end{array} \right)$$

$$P_0 = \left(\begin{array}{cc} \pi_{13} & x_2 - x_4 \\ \pi_{24} & x_3 - x_1 \end{array} \right) \quad P_1 = \left(\begin{array}{cc} x_1 - x_3 & x_2 - x_4 \\ \pi_{24} & -\pi_{13} \end{array} \right)$$

and $C(\Gamma_1)$ is the factorization

$$\left(\begin{array}{c} R\{-1\} \\ R\{3-2n\} \end{array} \right) \xrightarrow{Q_0} \left(\begin{array}{c} R\{-n\} \\ R\{2-n\} \end{array} \right) \xrightarrow{Q_1} \left(\begin{array}{c} R\{-1\} \\ R\{3-2n\} \end{array} \right)$$

$$Q_0 = \left(\begin{array}{cc} u_1 & x_1x_2 - x_3x_4 \\ u_2 & x_4 + x_3 - x_1 - x_2 \end{array} \right) \quad Q_1 = \left(\begin{array}{cc} x_1 + x_2 - x_3 - x_4 & x_1x_2 - x_3x_4 \\ u_2 & -u_1 \end{array} \right)$$

where u_1 and u_2 are the polynomials given in Definition 2.3.

The map $\chi_0 : C(\Gamma_0) \to C(\Gamma_1)$ is defined by the pair of maps:

$$U_0 = \begin{pmatrix} x_3 - x_2 & 0 \\ a & 1 \end{pmatrix} \quad U_1 = \begin{pmatrix} x_3 & -x_2 \\ -1 & 1 \end{pmatrix},$$

where $a = (u_1 + x_3 u_2 - \pi_{24})/(x_1 - x_3)$.

The map $\chi_1 : C(\Gamma_1) \to C(\Gamma_0)$ is defined by the pair of maps:

$$V_0 = \begin{pmatrix} 1 & 0 \\ -a & x_3 - x_2 \end{pmatrix} \quad V_1 = \begin{pmatrix} 1 & x_2 \\ 1 & x_3 \end{pmatrix}.$$

Note that $\chi_0\chi_1$ and $\chi_1\chi_0$ are both homotopic to multiplication by $x_2 - x_3$ (we can just take the zero homotopies in both cases).

Proposition 2.12. *(See [2]) The homology groups $HKh_w(D)$ of an l-component link diagram D for our potential w, have total dimension*

$$\dim_{\mathbb{C}} \oplus_{i,j} HKh_w^{i,j}(D) = n^l.$$

There is a canonical basis of generators, one generator for each assignment

$$\psi : components(D) \to \Sigma_n.$$

Each such ψ defines in an obvious way a type II admissable state of a resolution Γ_ψ. The generator corresponding to ψ lives in the chain group summand $H(\Gamma_\psi)$, and comes from the state ψ as described in the proof of Proposition 2.11. □

It is these generators that will play the same roles in this paper as Lee's generators [9] played in Rasmussen's slice genus result [12].

3 Morse moves

A link cobordism can be written as a finite sequence of link diagrams, where successive diagrams differ either by a Reidemeister move or a Morse move. The Morse moves correspond to adding 0-, 1-, and 2-handles.

In this section we will assign to each Morse move a filtered chain map between the complexes of the diagrams that it connects. Then we shall compute what each Morse move does to the canonical generators of the homology of a link diagram.

3.1 0-handle move

A 0-handle move is the creation of a simple loop as in Figure 2.

The 2-periodic complex associated to a circle is

$$A\langle 1\rangle = (0 \to A \to 0)$$

where $A = (\mathbb{C}[x]/\partial_x w(x))\{1-n\}$ (the curly brackets indicate a shift in the quantum filtration).

The unit map $i : \mathbb{C} \to A$ has filtered degree $1-n$.

Definition 3.1. *To the* 0*-handle move above we associate the map of complexes*

$$1 \otimes i : CKh_w(D) \to CKh_w(D \sqcup S^1).$$

The canonical generators of the cohomology of $A\langle 1\rangle$ are the chain elements

$$q_\beta = \frac{1}{n+1}\left(\prod_{\alpha\in\Sigma_n\setminus\beta}\frac{1}{\beta-\alpha}\right)\frac{\partial_x w(x)}{x-\beta},$$

and they satisfy

$$1 = \sum_{\beta\in\Sigma_n} q_\beta.$$

So our map $1\otimes i$ takes a canonical generator $g \in CKh_w(D)$ to the sum of canonical generators $\sum_{\beta\in\Sigma_n} g\otimes q_\beta$.

3.2 2-handle move

A 2-handle move is the removal of a simple closed loop as in Figure 2.

The trace map $\epsilon : A \to \mathbb{C}$ is defined by

$$\epsilon(x^i) = \begin{cases} 1, & i = n-1 \\ 0, & i < n-1 \end{cases}$$

Definition 3.2. *To the* 2*-handle move above we associate the map of complexes*

$$1 \otimes \epsilon : CKh_w(D \sqcup S^1) \to CKh_w(D)$$

and this has filtered degree $1-n$*.*

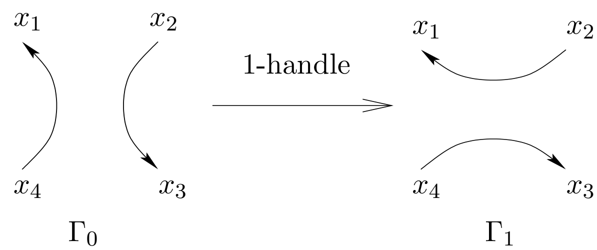

Figure 14: 1-handle addition. The boundary points have been labelled with variables which generate the ring R.

Since the coefficient of x^{n-1} in q_β is just $\prod_{\alpha\in\Sigma_n\setminus\beta}\frac{1}{\beta-\alpha}$ our map takes a canonical generator $g\otimes q_\beta \in CKh_w(D\sqcup S^1)$ to

$$\left(\prod_{\alpha\in\Sigma_n\setminus\beta}\frac{1}{\beta-\alpha}\right)g.$$

3.3 1-handle move

The 1-handle move is the addition of a saddle as in Figure 14.

Here $C(\Gamma_0)$ is the factorization

$$\begin{pmatrix} R \\ R\{2-2n\}\end{pmatrix} \xrightarrow{P_0} \begin{pmatrix} R\{1-n\} \\ R\{1-n\}\end{pmatrix} \xrightarrow{P_1} \begin{pmatrix} R \\ R\{2-2n\}\end{pmatrix}$$

$$P_0 = \begin{pmatrix} \pi_{14} & -(x_3-x_2) \\ \pi_{23} & x_1-x_4\end{pmatrix} \quad P_1 = \begin{pmatrix} x_1-x_4 & x_3-x_2 \\ -\pi_{23} & \pi_{14}\end{pmatrix}.$$

And $C(\Gamma_1)\langle 1\rangle$ is the factorization

$$\begin{pmatrix} R\{1-n\} \\ R\{1-n\}\end{pmatrix} \xrightarrow{Q_0} \begin{pmatrix} R \\ R\{2-2n\}\end{pmatrix} \xrightarrow{Q_1} \begin{pmatrix} R\{1-n\} \\ R\{1-n\}\end{pmatrix}$$

$$Q_0 = \begin{pmatrix} x_1-x_2 & x_3-x_4 \\ -\pi_{34} & \pi_{12}\end{pmatrix} \quad Q_1 = \begin{pmatrix} \pi_{12} & -(x_3-x_4) \\ \pi_{34} & x_1-x_2\end{pmatrix}.$$

Ignoring the grading shift, the filtered $\mathbb{C}[x_1,x_2,x_3,x_4]$-module of filtered matrix factorization maps $C(\Gamma_0)\to C(\Gamma_1)\langle 1\rangle$ is isomorphic to the quotient $\mathbb{C}[x_1,x_2,x_3,x_4]/(x_1=x_2=x_3=x_4, \partial_x w(x_1)=0)$.

Definition 3.3. *To the* 1*-handle move of Figure 14 we associate the map of complexes* $F : CKh_w(\Gamma_0) \to CKh_w(\Gamma_1)\langle 1\rangle$ *given by the pair of matrices:*

$$F_0 = \begin{pmatrix} e_{123} & -1 \\ -e_{134} & -1 \end{pmatrix} \quad F_1 = \begin{pmatrix} -1 & 1 \\ e_{123} & e_{134} \end{pmatrix},$$

where

$$e_{ijk} = \frac{\pi_{ik} - \pi_{jk}}{x_i - x_j}.$$

The map F *is filtered of degree* $n-1$*.*

Suppose we have two link diagrams D_1 and D_2 (of links L_1 L_2) such that D_2 is obtained from D_1 by the addition of a 1-handle. We will compute the map induced by F

$$HKh_w(D_1) \to HKh_w(D_2)$$

in terms of the generators of $HKh_w(D_1)$ and $HKh_w(D_2)$ that were constructed in Subsection 2.4.

Recall that for each choice of assignment $\phi : \{components\, of\ L_1\} \to \Sigma_n$ we got a corresponding resolution Γ_ϕ of D_1 and a state $\phi : e(\Gamma_\phi) \to \Sigma_n$ giving us an element $Q_\phi \in H(\Gamma_\phi) \subseteq CKh_w(D_1)$ which is a chain representative of one of the basis elements of $HKh_w(D_1)$.

To construct Q_ϕ we followed the recipe which converts the resolution Γ_ϕ into a disjoint union of circles Γ'_ϕ and pushes forward an element (again determined by ϕ) of $H(\Gamma'_\phi)$ to $H(\Gamma_\phi)$ via repeated use of η_0. Since F commutes with η_0, it is enough for us to determine the map induced by F acting on the homology of disjoint circles.

There are thus two possibilities: either the 1-handle move joins two components of the link as in Figure 15 or it splits a single component into two as in Figure 16.

3.3.1 First case

Performing the endpoint indentifications in Figure 15, we have

$$C(\Gamma'_0) = \begin{pmatrix} R \\ R\{2-2n\} \end{pmatrix} \xrightarrow{P'_0} \begin{pmatrix} R\{1-n\} \\ R\{1-n\} \end{pmatrix} \xrightarrow{P'_1} \begin{pmatrix} R \\ R\{2-2n\} \end{pmatrix}$$

$$P'_0 = \begin{pmatrix} \partial_x w(x_1) & 0 \\ \partial_x w(x_2) & 0 \end{pmatrix} \quad P'_1 = \begin{pmatrix} 0 & 0 \\ -\partial_x w(x_2) & \partial_x w(x_1) \end{pmatrix}$$

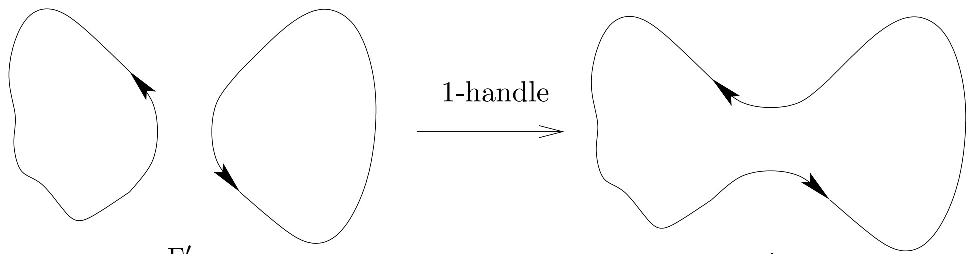

Figure 15: 1-handle addition, the first case where two components are joined into one.

where $R = \mathbb{C}[x_1, x_2]$ thought of as an infinite dimensional $\mathbb{C}$-module. And

$$C(\Gamma'_1)\langle 1\rangle = \left(\begin{array}{c} R\{1-n\} \\ R\{1-n\} \end{array}\right) \xrightarrow{Q'_0} \left(\begin{array}{c} R \\ R\{2-2n\} \end{array}\right) \xrightarrow{Q'_1} \left(\begin{array}{c} R\{1-n\} \\ R\{1-n\} \end{array}\right)$$

$$Q'_0 = \left(\begin{array}{cc} (x_1 - x_2) & -(x_1 - x_2) \\ -\pi_{12} & \pi_{12} \end{array}\right) \quad Q'_1 = \left(\begin{array}{cc} \pi_{12} & (x_1 - x_2) \\ \pi_{12} & (x_1 - x_2) \end{array}\right).$$

The induced map F' is given by

$$F'_0 = \left(\begin{array}{cc} e' & -1 \\ -f' & -1 \end{array}\right) \quad F'_1 = \left(\begin{array}{cc} -1 & 1 \\ e' & f' \end{array}\right),$$

where

$$e' = \frac{\pi_{12} - \partial_x w(x_2)}{x_1 - x_2} \quad f' = \frac{\partial_x w(x_1) - \pi_{12}}{x_1 - x_2}.$$

We observe that $H(\Gamma'_0) = ker(P'_0)/im(P'_1)$. The canonical generators indexed by pairs of roots $\alpha, \beta \in \Sigma_n$ have representatives in the 0th chain group of $C(\Gamma_0)$:

$$\left(\begin{array}{c} 0 \\ q^1_\alpha q^2_\beta \end{array}\right) \in \left(\begin{array}{c} R \\ R\{2-2n\} \end{array}\right).$$

Also, $H(\Gamma'_1\langle 1\rangle) = ker(Q'_0)/im(Q'_1)$, and the canonical generators indexed by roots $\alpha \in \Sigma_n$ have representatives in the 0th chain group of $C(\Gamma_1)$:

$$\left(\begin{array}{c} q^1_\alpha \\ q^1_\alpha \end{array}\right) \in \left(\begin{array}{c} R\{1-n\} \\ R\{1-n\} \end{array}\right)$$

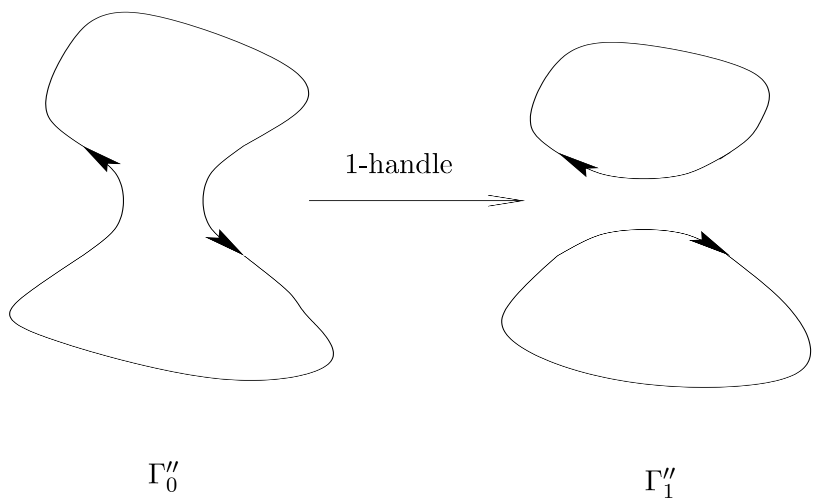

Figure 16: 1-handle addition, second case where one component is split into two.

where

$$q_\alpha^i = \frac{1}{n+1}\left(\prod_{\beta\in\Sigma_n\setminus\alpha}\frac{1}{\alpha-\beta}\right)\frac{\partial_x w(x_i)}{x_i-\alpha} \text{ for i} = 1,2.$$

Now $H(\Gamma_1'\langle 1\rangle)$ is isomorphic to $\mathbb{C}[x_1,x_2]/(x_1=x_2,\partial_x w(x_1)=0)$, and in this module

$$q_\alpha^1 q_\beta^2 = \begin{cases} 0, & \alpha\neq\beta \\ q_\alpha^1, & \alpha=\beta \end{cases}.$$

So it is easy to see what Ψ' does to the generators of $H(\Gamma_0')$:

$$\begin{aligned}\begin{pmatrix} e' & -1 \\ -f' & -1 \end{pmatrix}\begin{pmatrix} 0 \\ q_\alpha^1 q_\beta^2 \end{pmatrix} &= \begin{pmatrix} -q_\alpha^1 q_\beta^2 \\ -q_\alpha^1 q_\beta^2 \end{pmatrix} \\ &= \begin{pmatrix} -q_\alpha^1 \\ -q_\alpha^1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 0 \end{pmatrix}\end{aligned}$$

depending as $\alpha=\beta$ or $\alpha\neq\beta$.

3.3.2 Second case

Under the identifications in Figure 16, we have

$$C(\Gamma_0'') = \begin{pmatrix} R \\ R\{2-2n\} \end{pmatrix} \xrightarrow{P_0''} \begin{pmatrix} R\{1-n\} \\ R\{1-n\} \end{pmatrix} \xrightarrow{P_1''} \begin{pmatrix} R \\ R\{2-2n\} \end{pmatrix}$$

$$P_0'' = \begin{pmatrix} \pi_{13} & -(x_3 - x_1) \\ \pi_{13} & x_1 - x_3 \end{pmatrix} \quad P_1'' = \begin{pmatrix} x_1 - x_3 & x_3 - x_1 \\ -\pi_{13} & \pi_{13} \end{pmatrix}$$

where $R = \mathbb{C}[x_1, x_3]$.

$$C(\Gamma_1'')\langle 1\rangle = \begin{pmatrix} R\{1-n\} \\ R\{1-n\} \end{pmatrix} \xrightarrow{Q_0''} \begin{pmatrix} R \\ R\{2-2n\} \end{pmatrix} \xrightarrow{Q_1''} \begin{pmatrix} R\{1-n\} \\ R\{1-n\} \end{pmatrix}$$

$$Q_0'' = \begin{pmatrix} 0 & 0 \\ -\partial_x w(x_3) & \partial_x w(x_1) \end{pmatrix} \quad Q_1'' = \begin{pmatrix} \partial_x w(x_1) & 0 \\ \partial_x w(x_3) & 0 \end{pmatrix}.$$

The induced map F'' is given by

$$F_0'' = \begin{pmatrix} e'' & -1 \\ -f'' & -1 \end{pmatrix} \quad F_1'' = \begin{pmatrix} -1 & 1 \\ e'' & f'' \end{pmatrix}$$

where

$$e'' = \frac{\partial_x w(x_1) - \pi_{13}}{x_1 - x_3} \quad f'' = \frac{\pi_{13} - \partial_x w(x_3)}{x_1 - x_3}.$$

We observe that $H(\Gamma_0'') = ker(P_1'')/im(P_0'')$. The canonical generators indexed by roots $\alpha \in \Sigma_n$ have representatives in the 0th chain group of $C(\Gamma_0'')$

$$\begin{pmatrix} q_\alpha^1 \\ q_\alpha^1 \end{pmatrix} \in \begin{pmatrix} R\{1-n\} \\ R\{1-n\} \end{pmatrix}.$$

Also $H(\Gamma_1''\langle 1\rangle) = ker(Q_1'')/im(Q_0'')$, and the canonical generators indexed by roots $\alpha, \beta \in \Sigma_n$ have representatives in the 0th chain group of $C(\Gamma_1'')\langle 1\rangle$

$$\begin{pmatrix} 0 \\ q_\alpha^1 q_\beta^3 \end{pmatrix} \in \begin{pmatrix} R \\ R\{2-2n\} \end{pmatrix},$$

where

$$q_\alpha^i = \frac{1}{n+1} \left(\prod_{\beta \in \Sigma_n \backslash \alpha} \frac{1}{\alpha - \beta} \right) \frac{\partial_x w(x_i)}{x_i - \alpha} \text{ for i} = 1, 3.$$

Now $H(\Gamma_1''\langle 1\rangle)$ is isomorphic to $\mathbb{C}[x_1, x_3]/(\partial_x w(x_1), \partial_x w(x_3))$ and in this module

$$\left(\frac{\partial_x w(x_1) - \partial_x w(x_3)}{x_1 - x_3}\right)\left(\frac{\partial_x w(x_1)}{x_1 - \alpha}\right) = \left(\frac{\partial_x w(x_1)}{x_1 - \alpha}\right)\left(\frac{\partial_x w(x_3)}{x_3 - \alpha}\right),$$

since

$$\frac{\partial_x w(x_1) - \partial_x w(x_3)}{x_1 - x_3} - \frac{\partial_x w(x_3)}{x_3 - \alpha}$$

has a factor of $(x_1 - \alpha)$.

And so it is easy to see what F'' does to the generators:

$$\begin{aligned}
\begin{pmatrix} -1 & 1 \\ e'' & f'' \end{pmatrix}\begin{pmatrix} q_\alpha^1 \\ q_\alpha^1 \end{pmatrix} &= \begin{pmatrix} 0 \\ (e'' + f'')q_\alpha^1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \frac{\partial_x w(x_1) - \partial_x w(x_3)}{x_1 - x_3} q_\alpha^1 \end{pmatrix} \\
&= \left(\prod_{\beta \in \Sigma_n \setminus \alpha} (\alpha - \beta)\right)\begin{pmatrix} 0 \\ q_\alpha^1 q_\alpha^3 \end{pmatrix}.
\end{aligned}$$

and this a non-zero multiple of the canonical generator

$$\begin{pmatrix} 0 \\ q_\alpha^1 q_\alpha^3 \end{pmatrix} \in \begin{pmatrix} R \\ R\{2 - 2n\} \end{pmatrix},$$

in $H(\Gamma_1''\langle 1\rangle)$.

4 Reidemeister moves I and II

In this section we prove invariance of $HKh_w(D)$ under changing the (closed) link diagram D by an oriented Reidemeister I or II move.

The diagrams on either side of an oriented Reidemeister move each give a chain complex of matrix factorizations. We wish to define two chain maps (one in each direction) between these two chain complexes. Each chain map shall be quantum-graded of degree 0. These chain maps will then induce degree 0 chain maps on the chain complexes associated to *closed* link diagrams differing locally by the oriented Reidemester move. We have a description of chain representatives for the generators of the homology HKh_w of each link diagram and we show that the maps preserve the generators up to multiplication by a non-zero number. Since the chain maps were of degree 0 this tells us that the graded vector space associated to the quantum filtration of HKh_w is invariant under changing the input diagram by the Reidemeister move.

Every matrix in this section describes a map between free modules over polynomial rings. Each of these modules, as per the definition of CKh_w, is possibly subject to a shift in quantum filtration. In what follows, we suppress mention of these shifts. For those readers wishing to check the validity of the following computations, this omission should not encumber them with any great difficulties, while explicitly describing the shifts would thicken somewhat our exposition. In this section we write $\partial_i w$ for $\partial_x w(x_i)$.

4.1 Reidemeister move I

There are two variants of Reidemeister move I, which can be seen in Figures 17 and 20. We shall first tackle the invariance of HKh_w under Reidemeister move I.1 by explicit algebraic computation. The invariance of HKh_w under Reidemeister move I.2 proceeds by a topological argument: we decompose the move as a sequence of other elementary cobordisms for which we have already made algebraic computations.

4.1.1 Reidemeister move I.1

The first case of the Reidemeister I move is shown in Figure 17. Either side of the move corresponds to a chain complex of matrix factorizations. Our first task is to define chain maps of degree 0 between the left chain complex and the right complex. Then we wish to see that these chain maps induce maps on the homologies of two closed link diagrams, differing locally by Reidemeister move I.1, which preserve the generators of the homology.

The set-up is shown in Figure 18. The chain complex C from the left of Figure 17 is above, and the chain complex C' from the right is below. To give the two chain maps $C \to C'$ and $C' \to C$ we need to define the maps of matrix factorizations $f : M \to N$ and $g : N \to M$. First we shall write down the three factorizations M, N, and P.

Let $R = \mathbb{C}[x_2, x_3]$ and $S = \mathbb{C}[x_1, x_2, x_3]$ considered as a module over R. The factorizations M, N, and P are

$$M : R\{1-n\} \overset{x_2-x_3}{\to} R \overset{\pi_{23}}{\to} R\{1-n\}$$

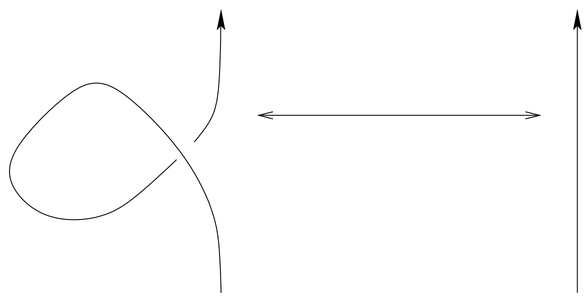

Figure 17: Reidemeister move I.1.

$$N : \left(\begin{array}{c} S\{2-2n\} \\ S\{2-2n\} \end{array} \right) \xrightarrow{A^0} \left(\begin{array}{c} S\{1-n\} \\ S\{3-3n\} \end{array} \right) \xrightarrow{A^1} \left(\begin{array}{c} S\{2-2n\} \\ S\{2-2n\} \end{array} \right)$$

$$A^0 = \left(\begin{array}{cc} 0 & x_2 - x_3 \\ -\pi_{23} & \partial_1 w \end{array} \right) \quad A^1 = \left(\begin{array}{cc} \partial_1 w & -(x_2 - x_3) \\ \pi_{23} & 0 \end{array} \right)$$

$$P : \left(\begin{array}{c} S\{-2n\} \\ S\{2-2n\} \end{array} \right) \xrightarrow{B^0} \left(\begin{array}{c} S\{-1-n\} \\ S\{3-3n\} \end{array} \right) \xrightarrow{B^1} \left(\begin{array}{c} S\{-2n\} \\ S\{2-2n\} \end{array} \right)$$

$$B^0 = \left(\begin{array}{cc} x_2 - x_3 & x_1(x_2 - x_3) \\ -u_2 & u_1 \end{array} \right) \quad B^1 = \left(\begin{array}{cc} u_1 & -x_1(x_2 - x_3) \\ u_2 & x_2 - x_3 \end{array} \right)$$

where

$$\pi_{23} = \frac{w(x_2) - w(x_3)}{x_2 - x_3}, \ \partial_1 w = w'(x_1),$$

$$u_1 = \frac{p(x_1 + x_2, x_1 x_2) - p(x_1 + x_3, x_1 x_2)}{x_2 - x_3},$$

$$u_2 = \frac{p(x_1 + x_3, x_1 x_2) - p(x_1 + x_3, x_1 x_3)}{x_1(x_2 - x_3)}.$$

As usual, p denotes the unique two-variable polynomial such that $p(x + y, xy) = w(x) + w(y)$.

The eagle-eyed reader may spot that there has been a shift in the usual $\mathbb{Z}/2$-grading of the matrix factorizations N and P. This is due to the introduction of an extra component in the oriented resolutions of the diagram defining these factorizations. We shall not mention this kind of shift in the

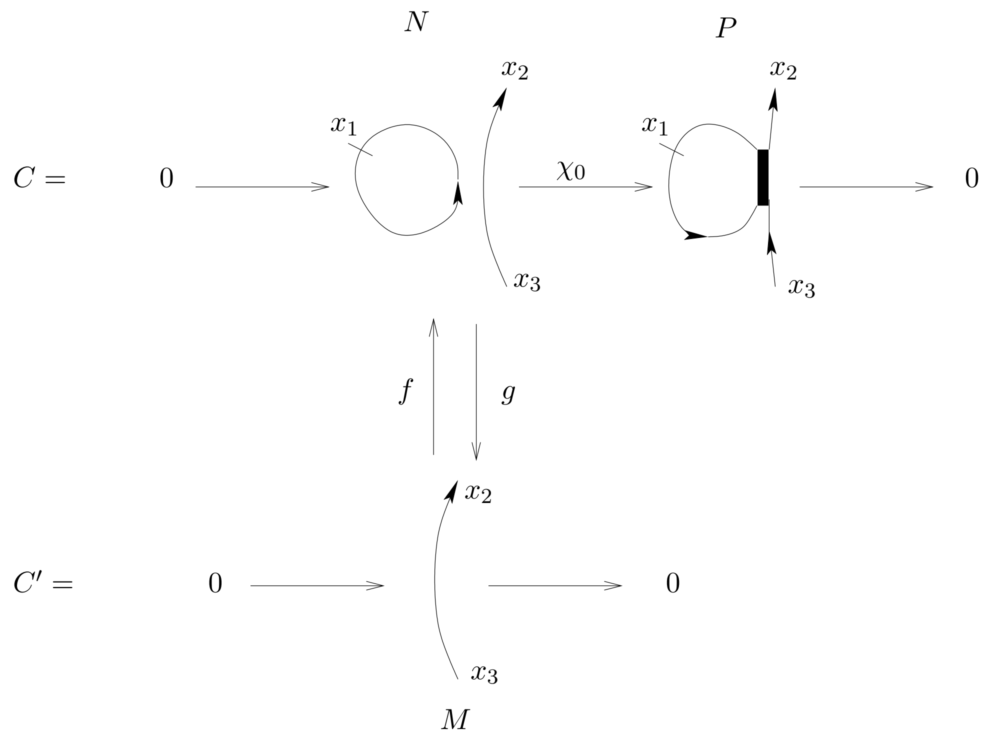

Figure 18: Reidemeister move I.1 chain maps. The upper row is the chain complex C of matrix factorizations corresponding to the left-hand side of Figure 17, the lower row is the chain complex C' of matrix factorizations corresponding to the right-hand side of Figure 17. M, N, and P are the matrix factorizations corresponding to the thick-edged trivalent graphs shown. To define chain maps between C and C', it is enough to give the maps of matrix factorizations f and g and then to check that there is enough commutativity to give chain maps. The label x_1 shows where two arc endpoints have been joined, and the endpoint labels identified.

remainder of this paper where it occurs, but content ourselves most often with giving the factorizations explicitly on which we are focussed.

The map χ_0 is given by the pair of matrices

$$\begin{pmatrix} x_1 & -x_2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} x_1 - x_2 & 0 \\ -a & 1 \end{pmatrix}.$$

Finally, we are in a position to define the first of our chain maps, whose only non-zero component is $f : M \to N$.

The map of matrix factorizations $f : M \to N$ comes as the pair of matrices

$$\begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\alpha \end{pmatrix}$$

where $\alpha = (p(x_1 + x_2, x_1x_2) - p(x_1 + x_2, x_1x_3))/(x_1(x_2 - x_3))$.

Lemma 4.1. *The map $f : M \to N$ gives a degree-0 map of chain complexes $F : C' \to C$.*

Proof. It is straightforward to check that f is a map of matrix factorizations, and it is clearly filtered of degree 0. However, it is less straightforward to check that f defines a map of chain complexes F; to do so we need to see that $\chi_0 f$ is homotopic to the zero map of chain complexes. The map $\chi_0 f$ is given by the pair of matrices:

$$\begin{pmatrix} x_1\alpha \\ -\alpha \end{pmatrix}, \begin{pmatrix} 0 \\ -\alpha \end{pmatrix}.$$

We define a homotopy $H^0 : M_0 \to P_1 \quad H^1 : M_1 \to P_0$:

$$H^0 = \begin{pmatrix} -1 \\ \frac{u_2-\alpha}{x_2-x_3} \end{pmatrix} \quad H^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(a glance at the definitions of α and u_2 assures us that the second entry of H^0 is a polynomial).

To see that H is a homotopy between $\chi_0 f : M \to P$ and $0 : M \to P$ we need to compute that

$$\begin{pmatrix} x_1\alpha \\ -\alpha \end{pmatrix} = H^1(\pi_{23}) + B^1H^0 \quad \begin{pmatrix} 0 \\ -\alpha \end{pmatrix} = H^0(x_2 - x_3) + B^0H^1$$

which is left as an exercise. $\square$

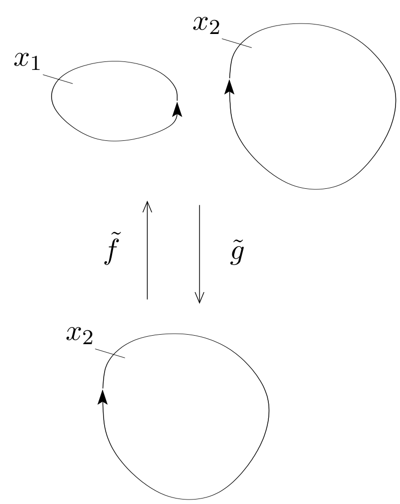

Figure 19: This diagram shows the maps $\tilde{f}$ and $\tilde{g}$ induced by f and g respectively. To check that the chain maps we have defined preserve chain representatives of the generators of the homology HKh_w of two closed link diagrams differing by a Reidemeister I.1 move, we need to compute where $\tilde{f}$ and $\tilde{g}$ take certain generators of the homologies H of the upper two circles and the lower circle.

We have now exhibited a degree 0 chain map $F : C' \to C$. To complete the proof of one direction of the Reidemeister I.1 move, it remains to check that F preserves the generators of the homology of a closed link diagram.

Proposition 4.2. *Write D, D' for closed link diagrams which are the same except as they differ locally as the left and right parts of Figure 17 respectively. There is an obvious correspondence between decorations of components of D and of D' with elements of Σ_n. Then a standard basis element of $HKh_w(D')$ coming from a given decoration of the components of D', gets taken by F to (a non-zero multiple of) the basis element of $HKh_w(D)$ which is obtained by the corresponding decoration of D.*

Proof. Because of the way that the generators are defined (as the image of various elements of the homology of some disjoint circles under η_0 maps), it is enough to check that the generators are preserved by the map $\tilde{f}$ that f induces when x_2 is identified with x_3 as in Figure 19.

We write $p = p(s,t)$ where $s = x_1 + x_2$ and $t = x_1 x_2$. Looking above at the definition of the polynomial α that appears in the entries of the matrix f, we see that result of substituting x_2 for x_3 in α is

$$\begin{aligned}
\alpha_{3\to 2} &= \frac{\partial}{\partial t} p(s,t) = \frac{\partial}{\partial t}(w_1 + w_2) \\
&= \frac{\partial x_1}{\partial t}\partial_1 w + \frac{\partial x_2}{\partial t}\partial_2 w \\
&= (x_2 - x_1)^{-1}\partial_1 w - (x_2 - x_1)^{-1}\partial_2 w \\
&= -\frac{\partial_1 w - \partial_2 w}{x_1 - x_2}
\end{aligned}$$

Taking homology of the graphs in Figure 19, $\tilde{f}$ becomes multiplication by $\alpha_{3\to 2}$

$$\tilde{f} = \alpha_{3\to 2} : \mathbb{C}[x_2]/(\partial_2 w) \to \mathbb{C}[x_1, x_2]/(\partial_1 w, \partial_2 w).$$

A general basis element, as constructed in Section 2, of the homology of the lower circle in Figure 19 is written (up to non-zero multiple) as $\partial_2 w/(x_2 - \psi)$ where ψ is a root of ∂w. Under $\tilde{f}$ this gets mapped to

$$\begin{aligned}
\alpha_{3\to 2}\frac{\partial_2 w}{x_2 - \psi} &= -\left(\frac{\partial_1 w - \partial_2 w}{x_1 - x_2}\right)\left(\frac{\partial_2 w}{x_2 - \psi}\right) \\
&= -\left(\frac{\partial_1 w}{x_1 - \psi}\right)\left(\frac{\partial_2 w}{x_2 - \psi}\right)
\end{aligned}$$

since

$$\frac{\partial_1 w - \partial_2 w}{x_1 - x_2} - \frac{\partial_1 w}{x_1 - \psi}$$

has a factor of $(x_2 - \psi)$ and $\partial_2 w = 0$ in the image of $\tilde{f}$. $\square$

Next we need to define the map of matrix factorizations $g : N \to M$ to give us a degree-0 chain map $G : C \to C'$.

As a $\mathbb{C}[x_2, x_3]$-module, $\mathbb{C}[x_1, x_2, x_3]$ is isomorphic to the following direct sum

$$\mathbb{C}[x_1, x_2, x_3] = \bigoplus_{i=0}^{n-2} x_1^i \mathbb{C}[x_2, x_3] \oplus \bigoplus_{i \geq 0} x_1^{i-1}(\partial_1 w - c)\mathbb{C}[x_2, x_3],$$

where we have written c for the constant term in the polynomial ∂w. Let β be the $\mathbb{C}[x_2, x_3]$-module map

$$\beta : \mathbb{C}[x_1, x_2, x_3] \to \mathbb{C}[x_2, x_3]$$

$$1, x_1, x_1^2, \ldots, x_1^{n-2} \mapsto 0, x_1^{i-1}(\partial_1 w - c) \mapsto (x_2 - x_3)^i \ \forall i \geq 0.$$

And let $1_{1\to 2,3}$ be the map

$$1_{1\to 2,3} : \mathbb{C}[x_1, x_2, x_3] \to \mathbb{C}[x_2, x_3]$$

$$x_1^i \mapsto (x_2 - x_3)^i \ \forall i \geq 0.$$

We define the map of matrix factorizations g by the pair of matrices

$$\begin{pmatrix} \beta & 0 \end{pmatrix}, \begin{pmatrix} 1_{1\to 2,3} & -\beta \end{pmatrix}.$$

Lemma 4.3. *The map $g : N \to M$ gives a degree-0 map of chain complexes $G : C \to C'$.*

Proof. We can see that so long as g is a map of matrix factorizations of degree-0, it will define a chain map $G : C \to C'$ since the only commuting square that we need to worry about will be automatically 0 in both directions. It is straightforward to check that g is a map of matrix factorizations. □

Next, as we did for $\tilde{f}$ in the proof of Proposition 4.2, we need to compute what the induced map $\tilde{g}$ does on the homologies of the closed-up graphs in Figure 19.

Proposition 4.4. *We use the same D and D' as in Proposition 4.2. Then a standard basis element of $HKh_w(D)$ coming from a given decoration of the components of D gets taken by G to (a non-zero multiple of) the basis element of $HKh_w(D')$ which is obtained by the corresponding decoration of D'.*

Proof. Looking at the definition of β we see that $\tilde{g}$ will take

$$\left(\frac{\partial_1 w}{x_1 - \psi}\right)\left(\frac{\partial_2 w}{x_2 - \psi}\right) \in \mathbb{C}[x_1, x_2]/(\partial_1 w, \partial_2 w)$$

to

$$\frac{\partial_2 w}{x_2 - \psi} \in \mathbb{C}[x_2]/(\partial_2 w).$$

□

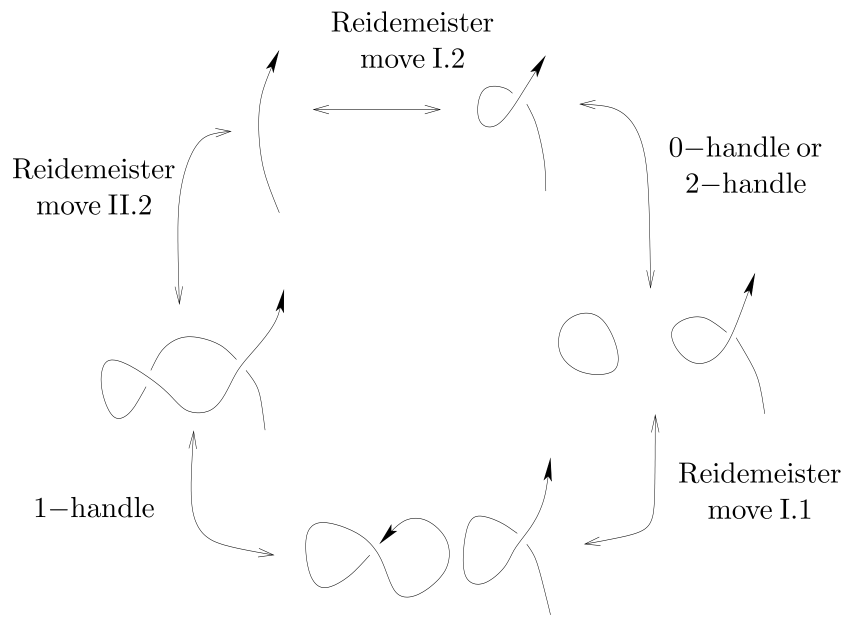

Figure 20: Here we show how to decompose either direction of a Reidemeister move I.2 as a sequence of other elementary cobordisms.

4.1.2 Reidemeister move I.2

Figure 20 decomposes the Reidemeister I.2 move into other elementary cobordisms. From other parts of this paper it follows that this gives degree-0 chain maps (by composition of chain maps corresponding to the elementary cobordisms used) between $CKh_w(D)$ and $CKh_w(D')$ where D and D' are closed link diagrams differing locally by the Reidemeister I.2 move. It also follows that these chain maps will preserve generators of $HKh_w(D)$ and $HKh_w(D)$ up to non-zero multiples.

4.2 Reidemeister move II

Since our link comes with an orientation, there are two cases to compute, the first is in Figure 21. Either side of Figure 21 corresponds to a chain complex of matrix factorizations. We shall define chain maps in both directions between them and check that these induce chain maps between the homologies HKH_w of two closed link diagrams differing locally by the move which preserve the generators of HKh_w.

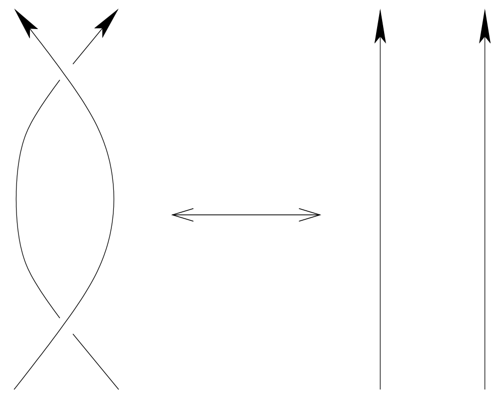

Figure 21: Reidemeister move II.1.

4.2.1 Reidemeister move II.1

We start with a note about ordering of bases. The trivalent graphs in Figure 12 and Figure 13 are associated to factorizations defined explicitly in Section 2. We write either of these factorizations as

$$\begin{pmatrix} M_{00} \\ M_{11} \end{pmatrix} \to \begin{pmatrix} M_{10} \\ M_{01} \end{pmatrix} \to \begin{pmatrix} M_{00} \\ M_{11} \end{pmatrix}$$

where each M_{ij} is a free module of rank 1 over the ring $\mathbb{C}[x_1, x_2, x_3, x_4]$. The basis that we use for the tensor product of two of these factorizations, as appears in the lower half of Figure 22 for example, is

$$\begin{pmatrix} M_{0000} \\ M_{0011} \\ M_{0101} \\ M_{0110} \\ M_{1001} \\ M_{1010} \\ M_{1100} \\ M_{1111} \end{pmatrix} \to \begin{pmatrix} M_{0001} \\ M_{0010} \\ M_{0100} \\ M_{1000} \\ M_{1110} \\ M_{1101} \\ M_{1011} \\ M_{0111} \end{pmatrix} \to \begin{pmatrix} M_{0000} \\ M_{0011} \\ M_{0101} \\ M_{0110} \\ M_{1001} \\ M_{1010} \\ M_{1100} \\ M_{1111} \end{pmatrix}$$

where $M_{ijkl} = M_{ij} \otimes M_{kl}$ and M_{ij} is a summand of the higher matrix factorization, M_{kl} a summand of the lower.

Each M_{ijkl} is $\mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]$ considered as a module over the ring $\mathbb{C}[x_1, x_2, x_3, x_4]$. In what follows we have many 8×8 matrices. Each matrix entry should be understood as a $\mathbb{C}[x_1, x_2, x_3, x_4]$-module map $M_{ijkl} \to M_{i'j'k'l'}$. We will often abuse notation, for example x_5 appearing as a matrix entry will mean the "multiply by x_5" map, even though x_5 technically lies in the module and not the ring. Also, we suppress the notation $C(\Gamma)$ to stand for the matrix factorization associated to the trivalent graph Γ, and instead just denote this factorization as Γ.

The complex of matrix factorizations C coming from the left-hand side of Reidemeister move II.1 is in the lower half of Figure 22, and the complex coming from the right-hand side is above it. In this diagram, as in the others in this section, where a point would normally be labelled with a variable x_i we have suppressed the x and just labelled with i. The single-headed arrows between the trivalent graphs are components of the chain differentials.

In Figure 22 there are also some double-headed arrows, each double-headed arrow is two maps of matrix factorizations (one in either direction). These maps define two chain maps $C \to C'$ and $C' \to C$. It is our first task in this section to define these maps of matrix factorizations so that we indeed *do* have two chain maps.

We also aim to define these maps so that the induced chain maps between two complexes coming from closed link diagrams differing by a single Reidemeister II.1 move preserve the generators of the homology HKh_w.

Two of the double-headed arrows are automatically the zero map in both directions. The double-headed arrow labelled *id* is homotopy equivalences of matrix factorizations as described in Lemma 7.1 in the Appendix. For the remaining double-headed arrow, we shall write down two explicit maps of matrix factorizations $A : N \to \Gamma_0$ and $B : \Gamma_0 \to N$.

First we consider the chain map $C \to C'$. In Figure 23 we have isolated the relevant part of Figure 22 and factored the map $A : N \to \Gamma_0$ into the composition of two maps $A = \chi_1 \circ \psi$. We have a chain map upwards if we can define the map $\psi : N \to \Gamma_1$ so that Figure 23 is anti-commutative.

Using the basis conventions discussed earlier we define ψ to be

$$\psi_0 = -\Pi \circ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\psi_1 = -\Pi \circ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where Π is the map which on each module summand $\mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]$ looks like

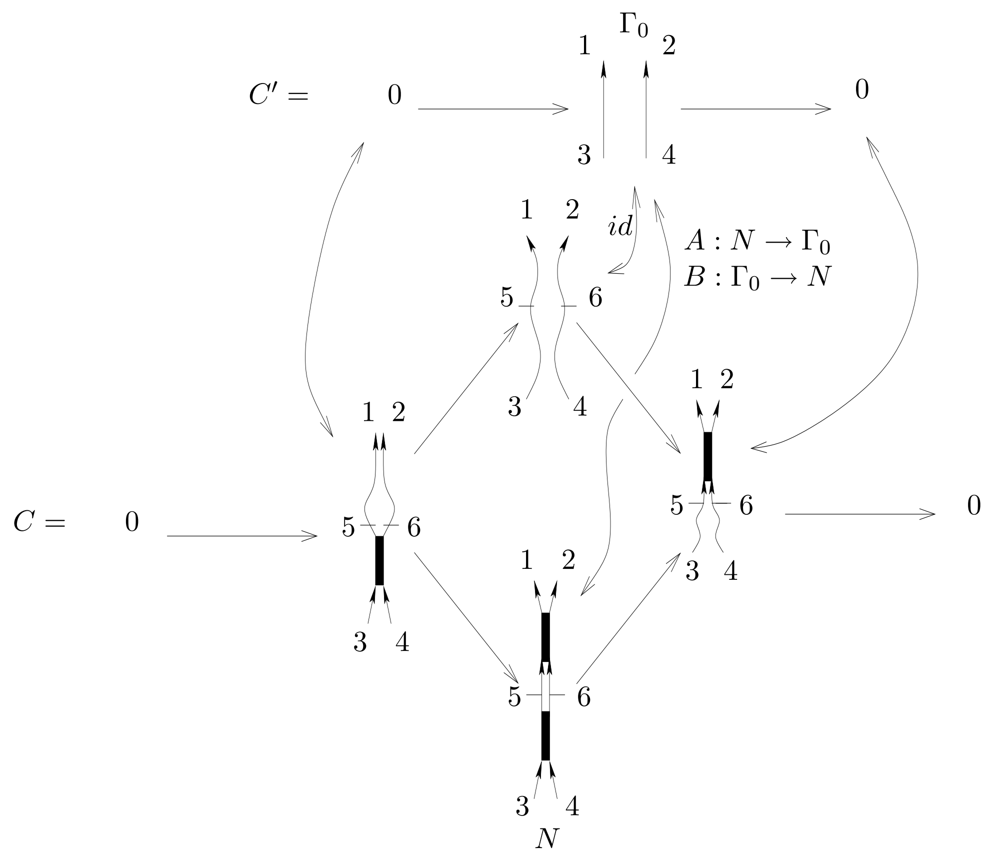

Figure 22: Reidemeister II.1 chain maps. The uppermost row is the chain complex C' of matrix factorizations coming from the right-hand side of Figure 21. The rest of the diagram is the chain complex C of matrix factorizations coming from the left-hand side of Figure 21. All of the chain differentials go from left to right, and components of the differentials are denoted by single-headed arrows. Each double-headed arrow represents two maps of matrix factorizations, one in each direction. Some of these maps are necessarily the 0 map since they end or originate at the 0 factorization. The double-headed arrow labelled id is the identity map in the homotopy category of matrix factorizations, and we define below the two maps comprising the final double-headed arrow, $A : N \to \Gamma_0$ and $B : \Gamma_0 \to N$. This gives us chain maps $C \to C'$ and $C' \to C$.

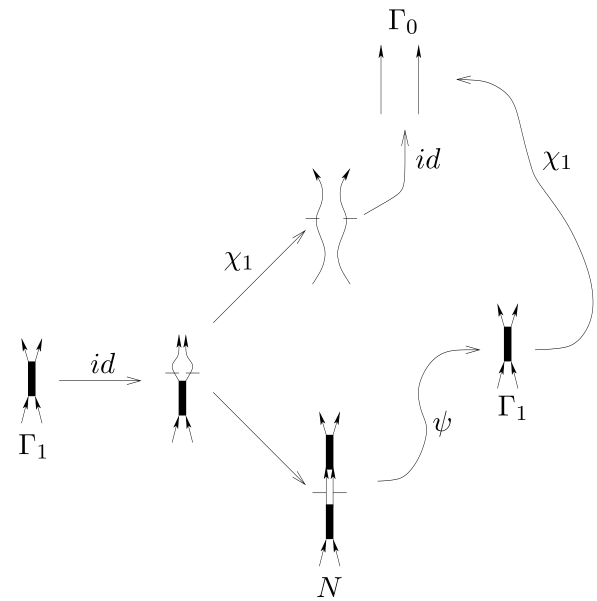

Figure 23: Factoring $A : N \to \Gamma_0$. Here we isolate the only two components of the upwards chain map $C \to C'$ which are possibly non-zero. One of these we are defining to be the identity (labelled id), and the other we decompose into two maps of matrix factorizations $A = \chi_1 \circ \psi : N \to \Gamma_0$. We define $\psi : N \to \Gamma_1$ in the body of the text. We have included an extra Γ_1 factorization and a homotopy equivalence between this and the leftmost non-zero factorization of the complex C.

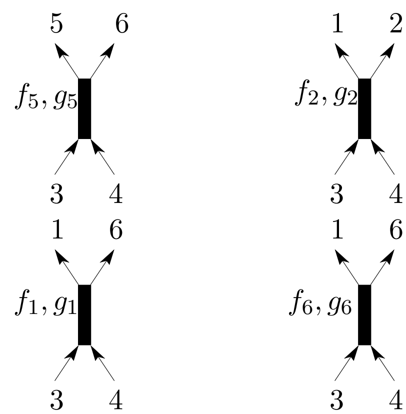

Figure 24: For the purposes of notation, we here show four thick-edged trivalent graphs which differ only in the labels allocated to the endpoints. We write the matrix factorization that each is associated to as the pair of matrices f, g with subscripts as shown.

$$\begin{aligned}
\mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6] &\to \mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]/(x_5 + x_6 - x_3 - x_4, x_5x_6 - x_3x_4) \\
&= \mathbb{C}[x_1, x_2, x_3, x_4]1 \oplus \mathbb{C}[x_1, x_2, x_3, x_4]x_5 \\
&\to \mathbb{C}[x_1, x_2, x_3, x_4]
\end{aligned}$$

(the last map is projection onto the *second* module summand).

Lemma 4.5. *Using this definition of $\psi : N \to \Gamma_1$, we have defined a chain map $C \to C'$.*

Proof. The map ψ is easily checked to be a map of matrix factorizations. We want to see that we have defined a chain map by seeing that Figure 23 anti-commutes.

We compute the map in Figure 23 which runs from the leftmost Γ_1 factorization to the Γ_1 factorization which is the target of ψ (this is a composition of three maps of matrix factorizations). We will denote this map by the pair of matrices

$$\begin{pmatrix} b^{01} & b^{02} \\ b^{03} & b^{04} \end{pmatrix}, \begin{pmatrix} b^{11} & b^{12} \\ b^{13} & b^{14} \end{pmatrix}.$$

We write $f : M^0 \to M^1$ and $g : M^1 \to M^0$ with suitable subscripts to stand for the matrix factorizations in Figure 24.

Let

$$f_{15} = (f_1 - f_5)/(x_1 - x_5),\ g_{15} = (g_1 - g_5)/(x_1 - x_5),$$

$$f_{26} = (f_2 - f_6)/(x_2 - x_6),\ g_{26} = (g_2 - g_6)/(x_2 - x_6)$$

Then we have

$$\begin{pmatrix} b^{01} & b^{02} \\ b^{03} & b^{04} \end{pmatrix} = \psi_0 \circ \begin{pmatrix} x_5 - x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_5 - x_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -x_2 & 0 & x_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_2 & 0 & x_5 & 0 & 0 \\ -a_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -a_1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\circ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} id_{R^2} \\ f_{26} \\ -g_{15} \circ f_{26} \\ f_{15} \end{pmatrix}$$

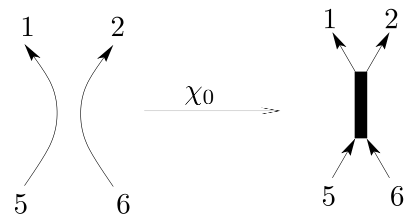

Figure 25: For the purposes of notation, we show the endpoint variables of the χ_0 map mentioned in the text.

$$\begin{pmatrix} b^{11} & b^{12} \\ b^{13} & b^{14} \end{pmatrix} = \psi_1 \circ \begin{pmatrix} x_5 - x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_5 - x_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_2 & x_5 & 0 & 0 & 0 & 0 \\ 0 & -a_1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -a_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_5 & -x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\circ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} g_{15} \\ -f_{15} \circ g_{26} \\ g_{26} \\ id_{R^2} \end{pmatrix}$$

where a_1 is the usual polynomial appearing in the χ_0 map in Figure 25, and id_{R^2} stands for the identity map on $R \oplus R$. In our previous notation (with now $x_5 + x_6$ and $x_5 x_6$ replacing $x_3 + x_4$ and $x_3 x_4$ respectively in the definitions of u_1 and u_2) we have

$$a_1 = \frac{u_1 + x_5 u_2 - \pi_{26}}{x_1 - x_5} = -\frac{u_1 + x_2 u_2 - \pi_{15}}{x_2 - x_6}.$$

Multiplying out the matrices:

$$\begin{pmatrix} b^{01} & b^{02} \\ b^{03} & b^{04} \end{pmatrix} = -\Pi \circ \begin{pmatrix} x_5 - x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} id_{R^2} \\ f_{26} \\ -g_{15}f_{26} \\ f_{15} \end{pmatrix}$$

$$= -\Pi \circ \left(\begin{pmatrix} x_5 - x_2 & 0 \\ -a_1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} g_{15}f_{26} \right).$$

Let us write

$$f_2 = \begin{pmatrix} v_{12} & x_3x_4 - x_1x_2 \\ v_{22} & x_1 + x_2 - x_3 - x_4 \end{pmatrix} \quad f_6 = \begin{pmatrix} v_{16} & x_3x_4 - x_1x_2 \\ v_{26} & x_1 + x_2 - x_3 - x_4 \end{pmatrix}$$

so

$$f_{26} = \begin{pmatrix} \frac{v_{12}-v_{16}}{x_2-x_6} & -x_1 \\ \frac{v_{22}-v_{26}}{x_2-x_6} & 1 \end{pmatrix}.$$

Now the first row of g_{15} is

$$g_{15} = \begin{pmatrix} 1 & x_6 \\ . & . \end{pmatrix}$$

so that we see immediately $b^{01} = -1, b^{02} = 0, b^{04} = -1$.

It remains to compute b^{03}:

$$\begin{aligned} b^{03} &= -\Pi \frac{1}{x_2 - x_6}[(u_1 + x_2u_2 - \pi_{15}) - (v_{12} - v_{16}) - (x_6(v_{22} - v_{26}))] \\ &= -\Pi \left(\frac{1}{x_2 - x_6}(u_1 + x_2u_2 - \pi_{15} - v_{12} + v_{16} - x_2v_{22} + x_6v_{26}) + v_{22} \right) \\ &= -\Pi \left(\frac{1}{x_2 - x_6}(\pi_{15} + v_{16} + x_6v_{26}) + v_{22} \right) \end{aligned}$$

(the last equality follows from the action of Π).

Now v_{22} certainly gets killed by Π and

$$\frac{-\pi_{15} + v_{16} + x_6v_{26}}{x_2 - x_6}$$

is a polynomial with no term involving x_2 in the numerator and hence also gets killed by Π. Thus we have shown that $b^{03} = 0$.

Now we work on $b^{11}, b^{12}, b^{13}, b^{14}$:

$$\begin{aligned}\begin{pmatrix} b^{11} & b^{12} \\ b^{13} & b^{14} \end{pmatrix} &= -\Pi \circ \begin{pmatrix} x_5 & 0 & 0 & 0 & -x_2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} g_{15} \\ -f_{15}g_{26} \\ g_{26} \\ id_{R^2} \end{pmatrix} \\ &= -\Pi \circ \left(\begin{pmatrix} x_5 & 0 \\ -1 & 0 \end{pmatrix} g_{15} + \begin{pmatrix} -x_2 & 0 \\ 1 & 0 \end{pmatrix} g_{26} \right)\end{aligned}$$

We compute the first rows of g_{15} and g_{26}:

$$g_{15} = \begin{pmatrix} 1 & x_6 \\ . & . \end{pmatrix} \quad g_{26} = \begin{pmatrix} 1 & x_1 \\ . & . \end{pmatrix}.$$

Hence

$$\begin{pmatrix} b^{11} & b^{12} \\ b^{13} & b^{14} \end{pmatrix} = -\Pi \circ \begin{pmatrix} x_5 - x_2 & x_5x_6 - x_1x_2 \\ -1+1 & -x_6 + x_1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now it is clear that Figure 23 anti-commutes, and hence we have defined a chain map $C \to C'$. □

Proposition 4.6. *The chain map $C \to C'$ that we have defined preserves the generators of the homology. Specifically, suppose we are given two (closed) link diagrams D and D' which differ locally as the right and left sides respectively of Figure 22. Then a basis element of $HKh_w(D)$ coming from a decoration of the the components of D with elements of Σ_n, gets taken by the map $HKH_w(D) \to HKh_w(D')$ induced by $C \to C'$ to a non-zero multiple of the basis element of $HKh_w(D')$ coming from the corresponding decoration of D'.*

Proof. The first step to define a basis element involves decorating the link components with roots of ∂w. We then resolve the diagram by adding a thick edge at each crossing where the roots disagree and adding the oriented resolution at each crossing where the roots agree.

If the decoration of the two strands of the Reidemeister II move is by the same root then we have nothing to show since the relevant component of the chain map is homotopic to the identity.

Suppose now that we have a decoration ϕ in which the two strands are decorated with different roots. We use the result Lemma 7.2 and refer to Figure 26. Suppose that the strand of D on the left of Figure 21 is

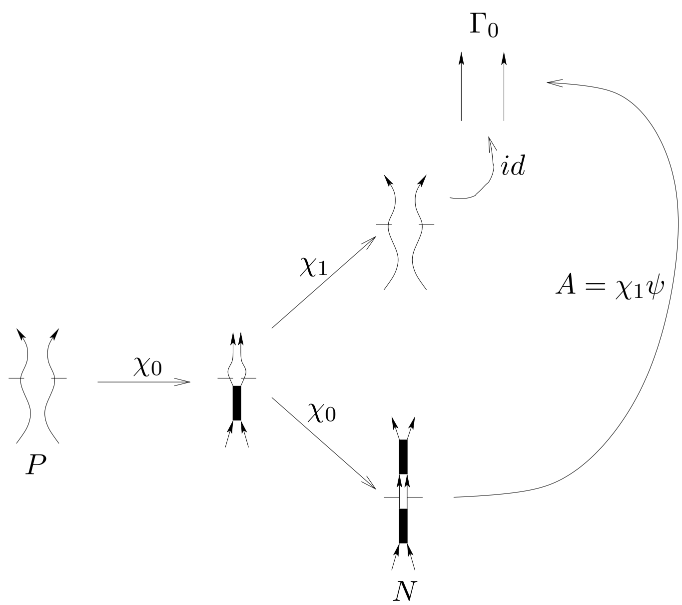

Figure 26: We use this diagram in our discussion of the preservation of generators of HKh_w under the Reidemeister II.1 move.

decorated by the root $\alpha_1 \in \Sigma_n$ and the strand on the right is decorated by the root $\alpha_2 \in \Sigma_n$.

Let $Q_{\phi,N} \in H(\Delta_N)$ be the standard chain representative of the basis element of $HKh(D)$ corresponding to the decoration ϕ. We write Δ_N for the resolution of D, determined by ϕ, that looks locally like N in Figure 26. To prove the proposition we need to show that $A(Q_{\phi,N}) = Q_{\phi,\Gamma_0}$ up to non-zero multiple where $Q_{\phi,\Gamma_0} \in H(\Delta_{\Gamma_0})$ represents the basis element of $HKh(D')$ corresponding to the decoration ϕ. We write Δ_{Γ_0} for the resolution of D, determined by ϕ, that looks locally like Γ_0 in Figure 26.

Recall that to produce $Q_{\phi,N}$ we start with a resolution of D which has the homology of a set of disjoint circles and then push forward a basis element (specified by ϕ) of the homology of that resolution by η_0 maps. Lemma 7.2 of Appendix 7.2 tells us that, at the site of the Reidemeister II.1 move, pushing forward by two η_0 maps gives us the same result, up to sign, as pushing forward by two χ_0 maps.

In other words, we can create $Q_{\phi,N}$ by pushing forward the corresponding basis element $Q_{\phi,P} \in H(\Delta_P)$ of the homology of the diagram looking locally like resolution P in Figure 26 by the two χ_0 maps indicated. Since we have shown that this is an anti-commutative diagram, the image of $Q_{\phi,N}$ up to

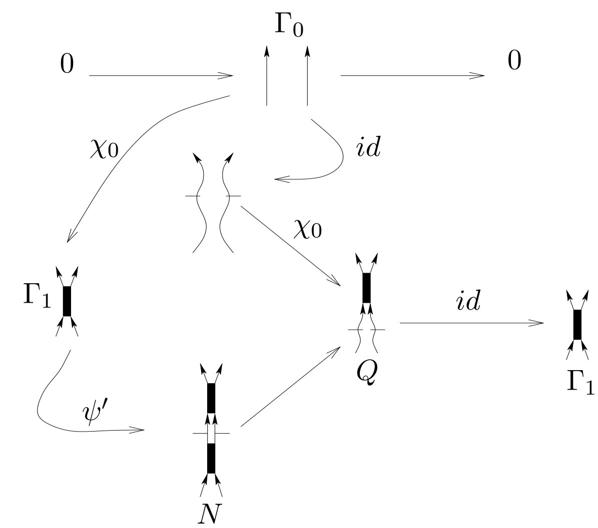

Figure 27: Factoring the downwards chain map. As we did for the upwards chain map, we isolate the only two possible components of the upwards chain map $C' \to C$ which are possibly non-zero. One of these we are defining to be the identity (labelled id), and the other we decompose into two maps of matrix factorizations $B = \psi' \circ \chi_0$. We define $\psi' : \Gamma_1 \to N$ in the body of the text.

sign in $H(\Delta_{\phi,\Gamma_0})$ under the chain map $C \to C'$ is the same up to sign as $id\chi_1\chi_0(Q_{\phi,P})$. Earlier, we saw that $\chi_1\chi_0$ is the same up to homotopy as the map induced by multiplication by $x_6 - x_3$. So

$$\chi_1\chi_0(Q_{\phi,P}) = (x_6 - x_3)Q_{\phi,P} = (\alpha_1 - \alpha_2)Q_{\phi,P} \in H(\Delta_P)$$

and so the anti-commutativity of Figure 26 implies that our generator is preserved up to non-zero multiple. $\square$

Now we shall consider the chain map $C' \to C$. In Figure 27 we have isolated the relevant part of the diagram and factored the map $B : \Gamma_0 \to N$ into the compositon of two maps $B = \psi' \circ \chi_0$. The arrows marked id are the identity in the homotopy category of matrix factorizations. We shall have a chain map if we can give the map ψ' so that the diagram anti-commutes.

Let F and G be the 2×2 matrices of the factorization in Figure 8. The entries of F and G are polynomials in $x_1+x_2, x_3+x_4, x_1x_2, x_3x_4$, and we shall write F_{5+6} and G_{5+6} for the matrices obtained from F and G by replacing $x_3 + x_4$ with $x_5 + x_6$ and F^{56}_{5+6} and G^{56}_{5+6} for the matrices obtained from F and G by replacing $x_3 + x_4$ and x_3x_4 with $x_5 + x_6$ and x_5x_6 respectively.

Writing $P(x+y,xy)=w(x)+w(y)$ we define

$$u_1(x,y,z)=\frac{P(x,z)-P(y,z)}{x-y}.$$

Using the basis-ordering conventions given at the start of this section we define ψ' to be

$$\psi'_0 \quad = \quad -\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} id_{R^2} \\ X_1 \\ X_2 \\ X_3+X_4 \end{pmatrix}$$

$$\psi'_1 \quad = \quad -\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} id_{R^2} \\ Y_1 \\ Y_2 \\ Y_3+Y_4 \end{pmatrix}$$

where

$$\begin{aligned}
X_1 &= \frac{F-F_{5+6}}{x_3+x_4-x_5-x_6}, \quad X_2=\frac{F_{5+6}-F^{56}_{5+6}}{x_3x_4-x_5x_6}, \\
X_3 &= \frac{u_1(x_5+x_6,x_3+x_4,x_3x_4)-u_1(x_5+x_6,x_3+x_4,x_5x_6)}{x_3x_4-x_5x_6} id_{R^2}, \\
X_4 &= \frac{G_{5+6}-G^{56}_{5+6}}{x_3x_4-x_5x_6}\frac{F-F_{5+6}}{x_3+x_4-x_5-x_6}, \\
Y_1 &= \frac{G-G_{5+6}}{x_3+x_4-x_5-x_6}, \quad Y_2=\frac{G_{5+6}-G^{56}_{5+6}}{x_3x_4-x_5x_6}, \\
Y_3 &= -\frac{u_1(x_5+x_6,x_3+x_4,x_3x_4)-u_1(x_5+x_6,x_3+x_4,x_5x_6)}{x_3x_4-x_5x_6} id_{R^2}, \\
Y_4 &= \frac{F_{5+6}-F^{56}_{5+6}}{x_3x_4-x_5x_6}\frac{G-G_{5+6}}{x_3+x_4-x_5-x_6}.
\end{aligned}$$

Lemma 4.7. *Using this definition of $\psi' : \Gamma_1 \to N$, we have defined a chain map $C' \to C$.*

Proof. It is a simple matter to check ψ' is a well-defined map of matrix factorizations. We now check that we have a chain map by seeing that Figure 27 anti-commutes.

If we let Π be the map which on each module summand looks like

$$\begin{aligned} \Pi &: \mathbb{C}[x_1,x_2,x_3,x_4,x_5,x_6] \to \mathbb{C}[x_1,x_2,x_3,x_4,x_5,x_6]/(x_5-x_3,x_6-x_4) \\ &= \mathbb{C}[x_1,x_2,x_3,x_4], \end{aligned}$$

then the components of the map in Figure 27 from the leftmost Γ_1 to the rightmost Γ_1 are

$$\begin{aligned} \Pi &\circ \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \end{pmatrix} \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ -a_2&a_3&0&0&0&0&0&0 \\ 0&0&x_3&1&0&0&0&0 \\ 0&0&x_6&1&0&0&0&0 \\ 0&0&0&0&x_3&1&0&0 \\ 0&0&0&0&x_6&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&-a_2&a_3 \end{pmatrix} \\ &\circ \begin{pmatrix} -1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&-1&0 \\ 0&0&0&0&0&-1&0&0 \\ 0&0&0&-1&0&0&0&0 \\ 0&0&0&0&-1&0&0&0 \\ 0&0&-1&0&0&0&0&0 \\ 0&-1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&-1 \end{pmatrix} \begin{pmatrix} id_{R^2} \\ X_1 \\ X_2 \\ X_3+X_4 \end{pmatrix} \\ &= \begin{pmatrix} -1&0 \\ 0&-1 \end{pmatrix} \end{aligned}$$

and

$$
\Pi \circ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_3 & 0 & 0 \\ 0 & 0 & 0 & -a_2 & 0 & 0 & a_3 & 0 \\ 0 & 0 & -a_2 & 0 & 0 & 0 & 0 & a_3 \end{pmatrix}
$$

$$
\circ \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} id_{R^2} \\ Y_1 \\ Y_2 \\ Y_3 + Y_4 \end{pmatrix}
$$

$$
= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
$$

Hence we see, just as before, that we have indeed defined a chain map. □

Proposition 4.8. *The chain map $C' \to C$ that we have defined preserves the generators of the homology. Specifically, using the notation of Lemma 4.6, a basis element of $HKh_w(D')$ gets taken by the map $HKh_w(D') \to HKh_w(D)$ induced by $C' \to C$ to a non-zero multiple of the corresponding basis element of $HKh_w(D)$.*

Proof. As in Proposition 4.6, the only case that is not immediate is that when the roots of ∂w decorating the two strands of the Reidemeister move are distinct. Let us suppose that we are given such a decoration ϕ where the strand on the left of Figure 21 is decorated by the root α_1 and the strand on the right is decorated by the root α_2.

We use the same Δ notation as in Proposition 4.6. Let $Q_{\phi,\Gamma_0} \in H(\Delta_{\Gamma_0})$ be the standard chain representative of the basis element of $HKh(D')$ corresponding to the resolution ϕ. To prove the proposition we need to show that $B(Q_{\phi,\Gamma_0}) = Q_{\phi,N} \in H(\Delta_N)$ where $Q_{\phi,N}$ is the chain representative of the basis element of $HKh(D')$ corresponding to the decoration ϕ.

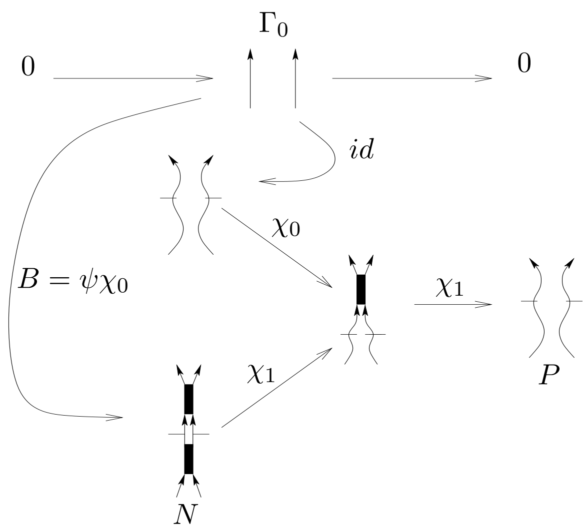

Figure 28: We use this diagram in our discussion of the preservation of generators of HKh_w under the Reidemeister II.1 move.

The proof of Proposition 2.11 tells us that applying two η_1 maps to $H(\Delta_N)$ locally at N is an injective map, and gives us a recipe to construct a non-zero multiple of the image of $Q_{\phi,N}$. Taken together with Lemma 7.2, we see that $\chi_1\chi_1 : H(\Delta_N) \to H(\Delta_P)$ is injective and takes $Q_{\phi,N}$ to a non-zero multiple of $Q_{\phi,P}$. Now,

$$\begin{aligned} \chi_1\chi_0 id(Q_{\phi,\Gamma_0}) &= \chi_1\chi_0(Q_{\phi,P}) = (x_2 - x_5)Q_{\phi,P} \\ &= (\alpha_2 - \alpha_1)Q_{\phi,P} = Q_{\phi,P} \end{aligned}$$

up to non-zero multiple. Hence the anti-commutativity of Figure 4.6 implies that our generator is preserved up to non-zero multiple. □

4.2.2 Reidemeister move II.2

Proposition 4.9. *If D and D' are two closed link diagrams differing locally by the Reidemeister II.2 move (as shown in Figure 29), then there exist degree 0 chain maps $CKh_w(D) \to CKh_w(D')$ and $CKh_w(D') \to CKh_w(D)$ which preserve, up to non-zero multiple, chain representatives of the canonical bases of $HKh_w(D)$ and $HKh_w(D')$.*

Proof. Figure 29 decomposes the Reidemeister II.2 move into other elementary cobordisms. It follows that this gives degree 0 chain maps (by com-

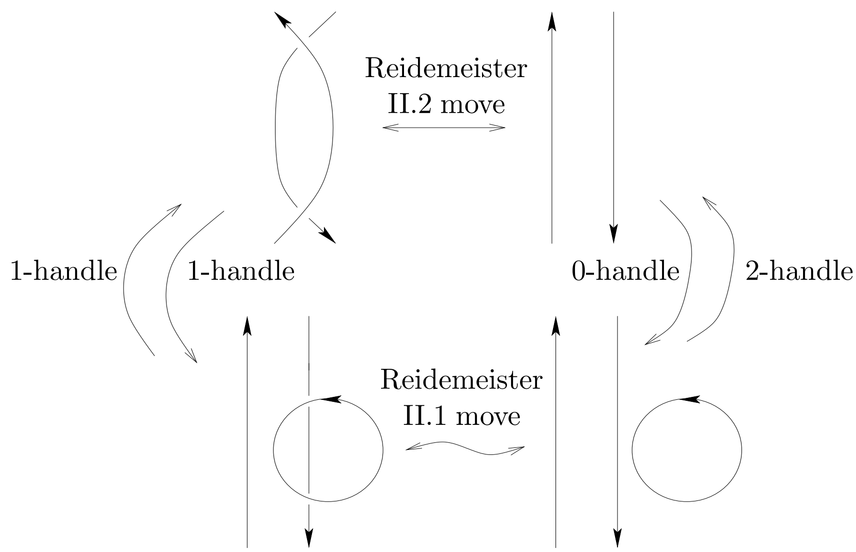

Figure 29: Decomposition of the Reidemeister II.2 move as a product of three other elementary cobordisms. This enables us to avoid making similar algebraic calculations for the Reidemeister II.2 move as we did for the II.1 move.

position of chain maps corresponding to the elementary cobordisms used) between $CKh_w(D)$ and $CKh_w(D')$ where D and D' are closed link diagrams differing locally by the Reidemeister II.2 move. These chain maps preserve generators of $HKh_w(D)$ and $HKh_w(D')$ up to non-zero multiples. $\square$

5 Presentations of surfaces in 4-space

In this section we give a proof of Theorem 1.6, but before doing so we discuss a way to visualize how a general link cobordism can be described by a finite sequence of elementary cobordisms.

5.1 General remarks

Consider a link cobordism $\Sigma : L_0 \to L_1$ between the two links L_0 and L_1.

$$i : \Sigma \hookrightarrow \mathbb{R}^3 \times [0,1]$$

$$\partial\Sigma = L_0 \sqcup L_1,\ L_i \hookrightarrow \mathbb{R}^3 \times \{i\} \text{ for i} = 0, 1.$$

We give one of the coordinates of $\mathbb{R}^3$ a label: $\mathbb{R}^3 = \mathbb{R}_h \times \mathbb{R}^2$. By a small perturbation of the embedding i, we can ensure both that the composition of i with the projection collapsing $\mathbb{R}_h$, $f : \Sigma \to \mathbb{R}^2 \times [0,1]$ is generic, and that the further composition with the projection to the interval $\Sigma \to [0,1]$ is a Morse function on Σ and also a Morse function when restricted to each singular stratum of $f : \Sigma \to \mathbb{R}^2 \times [0,1]$.

The dimension$= 0$ stratum consists of triple points and cusps of f, and the dimension$= 1$ stratum also includes the double points of f. The singular set of f is a singular compact 1-manifold with a finite number of boundary points which occur either on the boundary of $\mathbb{R}^2 \times [0,1]$ or at cusps, and with a finite number of singularities which are locally homeomorphic to the subset $xy(x-y) = 0$ of the (x, y) plane, and occur at triple points.

If we think of $[0,1]$ as being a time coordinate t, then we have a finite number of times t (say $0 < t_1 < t_2 < \ldots < t_d < 1$) at which

$$f(\Sigma) \cap \mathbb{R}^2 \times \{t\} \subset \mathbb{R}^2 \times \{t\}$$

fails to be a link diagram (note that we can determine which branch of a double point is the overcrossing and which the undercrossing, by looking at their relative projections to the 'height' coordinate $\mathbb{R}_h$). These are the times at which there is either a point of the dimension$= 0$ stratum of f, a critical point of the Morse function on the dimension$= 1$ stratum of f, or a critical point of the Morse function on Σ (remember that both Morse functions are just the projections to $[0,1]$). If $t_j < s < t < t_{j+1}$ then we have

$$(f(\Sigma) \cap \mathbb{R}^2 \times \{s\} \subset \mathbb{R}^2 \times \{s\}) = (f(\Sigma) \cap \mathbb{R}^2 \times \{t\} \subset \mathbb{R}^2 \times \{t\})$$

diffeomorphically as link diagrams, by the stratified Morse theory version of the usual Morse flow argument.

At each t_j there is a singular link diagram $f(\Sigma) \cap \mathbb{R}^2 \times \{t_j\} \subset \mathbb{R}^2 \times \{t_j\}$, corresponding to altering the diagram at time $t_j - \epsilon$ to the diagram at $t_j + \epsilon$ for small $\epsilon > 0$ by an elementary cobordism as follows:

- If the Morse function on Σ has a local maximum, this corresponds to adding a 0-handle.
- If the Morse function on Σ has a local minimum, this corresponds to adding a 2-handle.
- If the Morse function on Σ has a saddle point, this corresponds to adding a 1-handle.

- If the Morse function on the dimension= 1 stratum of f has a local maximum or minimum, this corresponds to a Reidemeister II move.
- If there is a cusp, this corresponds to a Reidemeister I move.
- If there is a triple point of f, this corresponds to a Reidemeister III move.

5.2 Delaying Reidemeister III

The main point of Theorem 1.6 is to delay the Reidemeister III moves in a movie presentation of a knot cobordism to the end of the presentation when they can take place within a 'simple' knot diagram. The Khovanov-Rozansky homology of this simple knot diagram is very amenable to computation, and this enables us to derive the slice genus lower bound without needing to make complicated calculations involving the Reidemeister III move taking place within a completely general knot diagram.

In the formalism of the previous section, delaying Reidemeister III moves means moving triple points of $f : \Sigma \rightarrow \mathbb{R}^2 \times [0, 1]$ down with respect to the time coordinate $t \in [0, 1]$ (we are thinking of 0 as being 'above' 1).

The obvious way to try and do this is to pull down the triple point within a small cylindrical neighbourhood $D^2 \times [0, 1]$ of the point. Generically this small cylindrical neighbourhood will intersect $f(\Sigma)$ in the three discs which intersect in the triple point, and in a finite number of disjoint discs. When the triple point is pulled down, we are forced to introduce pairs of canceling 1- and 2-handles in each disc below the triple point to avoid introducing more triple points. We explain this in more detail later, but for the general idea refer to Figures 30, 31.

There is, however, a problem with this simple approach. In order to apply Theorem 1.6 in our proof of Theorem 1.2, the requirement that the sequence of elementary cobordisms includes an intermediary 1-component knot diagram is essential. Hence we find it necessary to keep some control over the 1-handles as well, and this gives rise to the approach of the next section.

5.3 Proof of topological results

We now are in a position to prove Theorem 1.6.

Proof. Assume that $i : \Sigma_g \hookrightarrow \mathbb{R}^3 \times [0, 1]$ is a connected knot cobordism between links L_0 and L_1 as in the hypotheses of Theorem 1.6. We use the

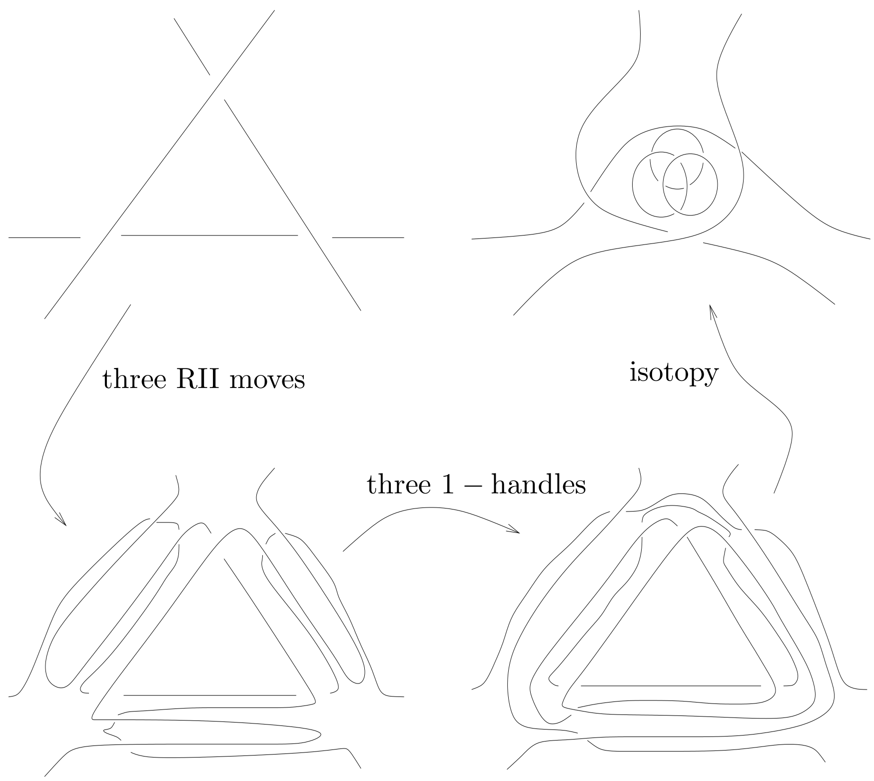

Figure 30: Here we show the effect of pulling a triple point of the immersion $f : {\Sigma_g}' \to \mathbb{R}^2 \times [0,1]$ down in the t direction. In terms of elementary cobordisms, the RIII move is replaced by three RII moves and three 1-handles. This introduces an extra three boundary components to ${\Sigma_g}'$, one in each sheet in which the triple point of f' is included.

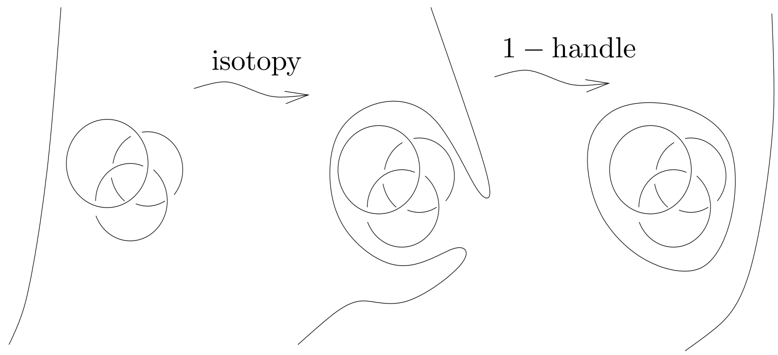

Figure 31: When we pull down a triple point of the immersion $f' : {\Sigma_g}' \to \mathbb{R}^2 \times [0,1]$, we also introduce 1-handles in sheets of the immersion f' that pass beneath the triple point. In this way we avoid introducing more triple points. Here we show this in terms of a movie picture.

notation $\mathbb{R}^3 \times [0,1] = \mathbb{R}_h \times \mathbb{R}^2 \times [0,1]_t$ as we discussed in Subsection 5.1, and we make the assumption that the projection to the time interval $[0,1]$ is a stratified Morse function.

Suppose a 0- or a 2-handle of i occurs at a point $(h,p,q,t) \in \mathbb{R}_h \times \mathbb{R}^2 \times [0,1]$. Generically $\{(h,p,q)\} \times [0,1]$ will not intersect $i(\Sigma_g)$, so by isotoping 0-handles 'up' and 2-handles 'down' (we think of 0 as lying above 1) in the time t direction and then rescaling $[0,1]$, we may assume that the 0-handles of i occur when $t = 1/6$ and the 2-handles when $t = 5/6$. Furthermore we may assume by Theorem 1.5 [5], that the 1-handles of i occur at time $t = 1/2$ and that they come with an ordering which satisfies the conditions on fusion and fission in the statement of Theorem 1.6.

We shall write Σ_g^t for $i^{-1}(\mathbb{R}_h \times \mathbb{R}^2 \times [0,t])$, by i^t we shall mean the restriction $i^t = i|_{\Sigma_g^t} \to \mathbb{R}_h \times \mathbb{R}^2 \times [0,t]$, and by f^t we shall mean the $\mathbb{R}_h$-collapsing projection $f^t : \Sigma_g^t \to \mathbb{R}^2 \times [0,t]$.

Suppose that i has h_1 saddle points. We let I_j be the part of the ascending manifold corresponding to the j^{th} saddle point which is contained in $\mathbb{R}_h \times \mathbb{R}^2 \times [1/3, 2/3]$. This is illustrated in Figure 33.

We shall write $\overline{I_j}$ and $\overline{\overline{I_j}}$ for the projections of I_j to $\mathbb{R}_h \times \mathbb{R}^2$ and to $\mathbb{R}^2$ respectively. By rescaling t if necessary, we may assume that

$$(\overline{I_j} \times [1/3, 2/3]) \cap i(\Sigma_g) = I_j$$

for all j.

Let h_0 be the number of 0-handles of i. Since only 0-handles occur when

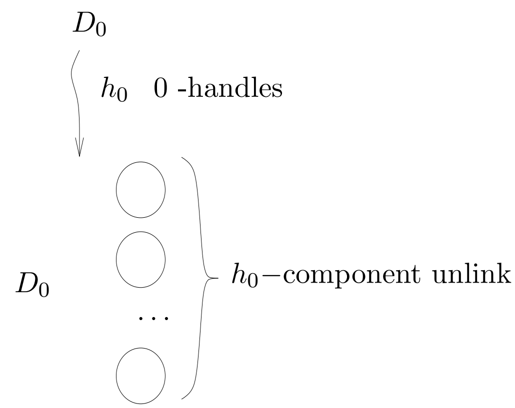

Figure 32: Here we show the movie presentation corresponding to the projection $f^{1/3} : \Sigma_g^{1/3} \to \mathbb{R}^2 \times [0, 1/3]$. The projection $f^{1/3}$ has only index$= 0$ critical points, and no critical points of positive codimensional strata.

$0 < t < 1/3$, $i^{1/3}$ is an embedding of $L_0 \times [0, 1/3]$ (with fibre $L_0 \subset \mathbb{R}^3$ at each value of t), along with h_0 trivially embedded discs D^2. Hence, by possibly composing with a time-dependent diffeomorphism of $\mathbb{R}^3$, we can assume that the $\mathbb{R}_h$-collapsing projection $f^{1/3}$ has no dimension$= 0$ stratum and no singular points of the dimension$= 1$ stratum, with fibre D_0 over $t = 0$ and fibre $D_0 \sqcup \coprod^{h_0} U$ over $t = 1/3$, where U is the 0-crossing diagram of the unknot. We illustrate the corresponding movie presentation in Figure 32.

Let $N\overline{I_j}$ be a small tubular neighbourhood $\overline{I_j} \times D^2$ of $\overline{I_j}$, chosen small enough that $(N\overline{I_j} \times [1/3, 2/3]) \cap i(\Sigma_g)$ consists of a neighbourhood of I_j in $i(\Sigma_g)$. We form the space

$$(\mathbb{R}_h \times \mathbb{R}^2 \times [0, 1/3]) \cup \left(\coprod_j N\overline{I_j} \times (1/3, 2/3] \right),$$

and call the smoothing of this space X.

Note that there are no critical points of the Morse function on Σ_g which are mapped by i to $(\mathbb{R}_h \times \mathbb{R}^2 \times [0, 2/3]) \setminus X$ and that points of $(i(\Sigma_g) \cap \partial X) \setminus (\mathbb{R}_h \times \mathbb{R}^2 \times \{0\})$ are in one-to-one correspondence with points of $i(\Sigma_g) \cap (\mathbb{R}_h \times \mathbb{R}^2 \times \{2/3\})$ since there is a unique flow-line in

$$i(\Sigma_g) \cap ((\mathbb{R}_h \times \mathbb{R}^2 \times [1/3, 2/3]) \setminus X)$$

connecting them. Indeed, by a standard Morse flow argument, $i(\Sigma_g) \cap X \subset X$ is diffeomorphic as a pair to $i^{2/3}(\Sigma_g^{2/3}) \subset \mathbb{R}_h \times \mathbb{R}^2 \times [0, 2/3]$.

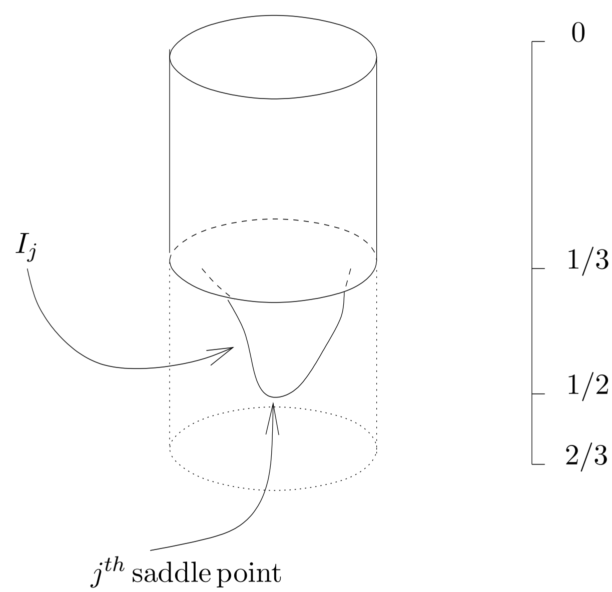

Figure 33: The j^{th} saddle point of the embedding i occurs at a time $t = 1/2$. Here we draw the part I_j of the corresponding ascending manifold contained in $\mathbb{R}_h \times \mathbb{R}^2 \times [1/3, 2/3]$. For the horizontal discs, one should imagine 3-dimensional space $\mathbb{R}_h \times \mathbb{R}^2$.

Furthermore, since $(N\overline{I_j} \times [1/3, 2/3]) \cap i(\Sigma_g)$ consists of a neighbourhood of I_j in $i(\Sigma_g)$, the pair $i(\Sigma_g) \cap X \subset X$ is determined up to ambient isotopy by the $\overline{I_j} \subset (\mathbb{R}_h \times \mathbb{R}^2)$ and a framing of the $\overline{I_j}$ which lifts to the surface framing of the I_j.

Now the $\overline{\overline{I_j}}$ (the projections of the I_j to $\mathbb{R}^2$) are generically immersed arcs in $\mathbb{R}^2$ with endpoints on the diagram $D_0 \sqcup \coprod^{h_0} U$, and elsewhere transverse to this diagram. For any framing of the $\overline{I_j}$, there exists a cobordism (realising this framing) $(\Sigma_g^{2/3} \setminus \Sigma_g^{1/3}) \hookrightarrow (\mathbb{R}_h \times \mathbb{R}^2 \times [1/3, 2/3])$ whose projection to $\mathbb{R}^2$ is supported on $D_0 \sqcup \coprod^{h_0} U$ and on a small neighbourhood of the $\overline{\overline{I_j}}$. This is illustrated in Figure 34, where it is observed that the corresponding movie presentation of this cobordism consists of only Reidemeister I and II moves which introduce crossings and of 1-handle Morse moves. We have shown that we can create the cobordism $i^{2/3}$ up to ambient isotopy by composing a cobordism of this type with the cobordism $i^{1/3}$.

In other words we have found a new decomposition of the ambient space as $\mathbb{R}_h \times \mathbb{R}^2 \times [0, 2/3]$ (we use the same notation for the new decomposition as for the old) such that the projection $f^{2/3} : \Sigma_g^{2/3} \to \mathbb{R}^2 \times [0, 2/3]$ has no triple points and no minima of the dimension= 1 stratum. As before, all the 1-handles occur at time $t = 1/2$ and it is possible to perturb f in a neighbourhood of $t = 1/2$ such that the conditions on the orders of fusion and fission 1-handles in Theorem 1.6 hold.

Suppose now that there are h_2 2-handles of i. Then by construction

$$i(\Sigma_g) \cap (\mathbb{R}_h \times \mathbb{R}^2 \times \{2/3\}) \subset \mathbb{R}^3$$

is the link which consists of the disjoint union of K_1 with the h_2-component unlink. Furthermore,

$$f(\Sigma_g) \cap (\mathbb{R}^2 \times \{2/3\}) \subset (\mathbb{R}^2 \times \{2/3\})$$

is a diagram of this link. Since the link diagram $D_1 \sqcup \coprod^{h_2} U$ as illustrated in Figure 35 is also a diagram of this link, there exists a sequence of Reidemeister moves taking one to the other. In other words we can extend the immersion $f^{2/3} : \Sigma_g^{2/3} \to \mathbb{R}^2 \times [0, 2/3]$ to an immersion $f' : \Sigma'_g \to \mathbb{R}^2 \times [0, 1]$ where ${\Sigma_g}'$ is Σ_g punctured h_2 times, the projection of ${\Sigma_g}'$ to $[0, 1]$ contains no critical points with values in $[2/3, 1]$, and

$$f'({\Sigma_g}') \cap (\mathbb{R}^2 \times \{1\}) \subset (\mathbb{R}^2 \times \{1\})$$

is the link diagram $D_1 \sqcup \coprod^{h_2} U$.

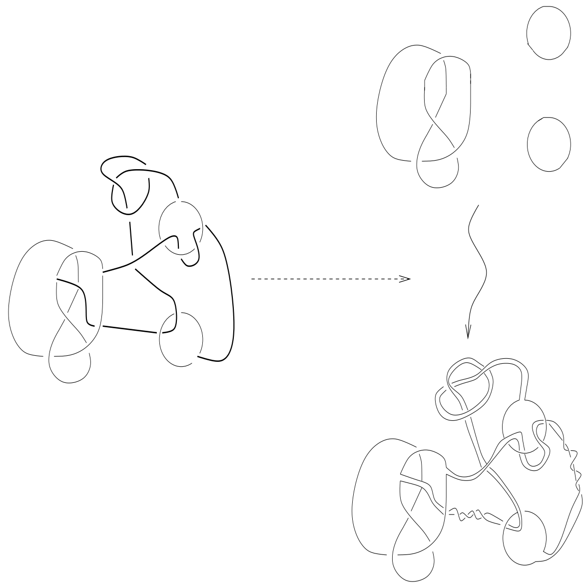

Figure 34: On the left of this figure is $D_0 \sqcup \coprod^{h_0} U$, (for example's sake we have taken $h_0 = 2$ and D_0 to be a diagram of a figure-8 knot), along with the $\overline{\overline{I_j}}$ (again for example's sake we have drawn two of these). Such a diagram along with a choice of (surface) framing for each $\overline{I_j}$ determines the cobordism $i^{2/3}$. On the right are the start and end diagrams of a corresponding movie presentation which is supported on $D_0 \sqcup \coprod^{h_0} U$ and on a neighbourhood of the $\overline{\overline{I_j}}$. It is clear how to compose Reidemeister I and II moves and two 1-handle moves to achieve such a presentation. The twisting of the ribbons (achieved with Reidemeister I moves) should be chosen to give the required surface framing of each $\overline{I_j}$.

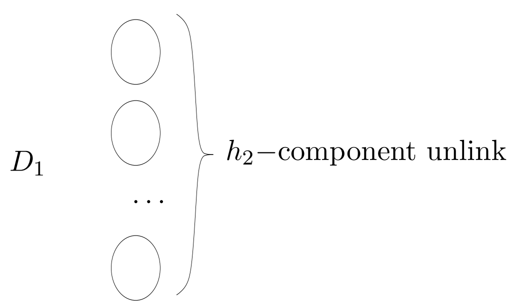

Figure 35: This is what we mean by the diagram $D_1 \sqcup \coprod^{h_2} U$.

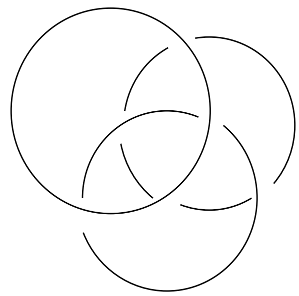

Figure 36: This is a diagram of a 3-component unlink. One of these occurs on $\mathbb{R}^2 \times \{1\}$ for each triple point of f' that we remove by pulling it down to the boundary of $\mathbb{R}^2 \times [0,1]$.

The lift of f' to a link cobordism $i' : {\Sigma_g}' \hookrightarrow (\mathbb{R}_h \times \mathbb{R}^2 \times [0,1])$ is the cobordism i punctured h_2 times. It is a cobordism between L_0 and $L_1 \cup \coprod^{h_2} W$ where W stands for the unknot.

Finally it remains to pull down the triple points of f' which occur in $\mathbb{R}^2 \times [2/3, 1]$. If the triple point occurs at (p, q, t) this means altering the surface in a small tubular neighbourhood of $\{(p, q)\} \times [2/3, 1]$. We puncture each of the three sheets where the triple point occurs, and pull down the punctures in the small tubular neighbourhood until they reach the boundary $\mathbb{R}^2 \times \{1\}$. Doing this introduces a 1-handle in each sheet, as well as a critical point (hence a Reidemeister II move) at three points of the dimension$= 1$ stratum of f'. We illustrate this in Figure 30.

Suppose f' has T triple points. Generically, below each triple point of f' there will be a number of sheets of $f'({\Sigma_g}')$ - call the total number of such sheets S. We pull down each triple point and also puncture and pull down the sheets below it to form a new immersion $F : {\Sigma_g}'' \to \mathbb{R}^2 \times [0,1]$. We puncture and pull down the sheets below each triple point to avoid introducing any new intersections when we pull down the triple points - see Figure 31. Here, ${\Sigma_g}''$ is Σ_g punctured $k = h_2 + 3T + S$ times and the diagram

$$F({\Sigma_g}'') \cap (\mathbb{R}^2 \times \{1\}) \subset (\mathbb{R}^2 \times \{1\})$$

is a diagram which consists of the disjoint union of D_1 with T copies of the diagram in Figure 36 and S copies of the 0-crossing knot diagram U. We illustrate and explain further in Figures 31, 37. We note that in taking the disjoint union we allow the diagram components to nest, so long as they remain disjoint from each other as subsets of $\mathbb{R}^2$. The Khovanov-Rozansky complex does not see the nesting of diagram components.

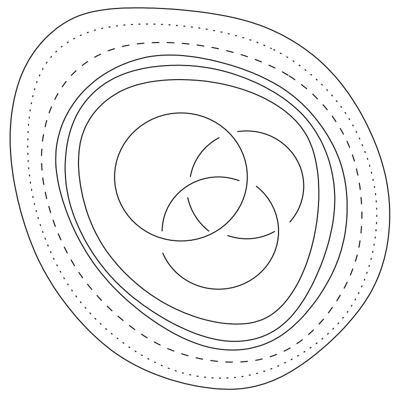

Figure 37: We introduce a 1-handle in every sheet of $f' : {\Sigma_g}' \to \mathbb{R}^2 \times [0,1]$ passing beneath a triple point. This means puncturing ${\Sigma_g}'$ a number of times (three times for each triple point that we pull down, and once for each sheet of f' directly below a triple point). Here we show how these punctures appear when pulled down to $\mathbb{R}^2 \times \{1\}$, giving a link diagram. It is a 6-crossing diagram of a 3-component unlink, nested inside a number of 0-crossing knot diagrams, one for each sheet below the triple point.

The movie presentation corresponding to the immersion F satisfies the requirements of Theorem 1.6 and so we are done. $\square$

6 Derivation of slice genus bound

In this section we deduce Theorem 1.2 on the lower bound for the slice genus of a knot coming from HKh_w, and Milnor's conjecture on the value of the slice genus of a torus knot Corollary 1.3.

6.1 Heuristics

Given a knot K that bounds a connected surface in the 4-ball, we can puncture the surface once to get a connected knot cobordism $\Sigma_g : K \to W$ between K and the unknot W.

Suppose there is a presentation of Σ_g as a sequence of elementary cobordisms, not including the Reidemeister III move, between a diagram D of K and the 0-crossing diagram U of the unknot. By work of previous sections, this gives us an induced map on the perturbed homologies

$$\mathcal{F}^j HKh^0_w(D) \to \mathcal{F}^{j-(n-1)\chi(\Sigma_g)} HKh^0_w(U).$$

Since we know that this map is an isomorphism as a plain (unfiltered) vector space homomorphism and we know the homology of U, we can deduce a slice genus lower bound for K from the associated graded vector space to the filtered vector space $\mathcal{F}^j HKh^0_w(D)$.

However, it is not always possible to find such a composition of elementary cobordisms that avoids the Reidemeister III move. But, as we have shown in Section 5, it is possible to puncture Σ_g a number of times to get a cobordism ${\Sigma_g}'$ between K and an unlink, such that there is a presentation of ${\Sigma_g}'$ that includes no Reidemeister III moves, that starts with D, and that ends in a 'simple' diagram Z of the unlink (see Figure 37 for the worst case scenario).

Under the action of this presentation on homology, we shall see that it is possible to compute directly the filtration grading in $HKh^0_w(Z)$ of the image of a generator of $HKh^0_w(D)$ under the induced map

$$\mathcal{F}^j HKh^0_w(D) \to \mathcal{F}^{j-(n-1)\chi({\Sigma_g}')} HKh^0_w(Z).$$

This is because we have been careful not to allow Z to be a very complicated diagram of the unlink.

In this way we recover the same slice genus bound on K as we would expect to achieve using a much more laborious algebraic computation of the invariance of HKh_w under the Reidemeister III move.

6.2 Computation

Let $\Sigma : K \to U$ be a connected knot cobordism of genus g between the 1-component knot K and the unknot U, and suppose we are given a diagram D of K. We know that by puncturing Σ a number of times to get the cobordism $\Sigma' : K \to E$ (where E is the e-component unlink), we can find a presentation S' of Σ' satisfying the conditions of Theorem 1.6. The presentation S' ends with the diagram Z of E where each component of Z (as a subset of $\mathbb{R}^2$) comes as a 0-crossing unknot or as a diagram of a 3-component unlink as appears in the left of Figure 38.

The presentation S' has been constructed so that it contains no Reidemeister III moves. By the work of previous sections, associated to S' is a map

$$HKh_w(S') : \mathcal{F}^j HKh^i_w(D) \to \mathcal{F}^{j-(n-1)\chi(\Sigma')} HKh_w(Z).$$

Proposition 6.1. *If $\alpha_\xi \in CKh_w(D)$ is the chain representative of the generator $[\alpha_\xi] \in HKh_w(D)$ associated to decorating K with the root ξ of ∂w then*

$$HKh_w(S')([\alpha_\xi]) = [\alpha_\xi] \in HKh_w(Z)$$

where we write $[\alpha_\xi]$ also for the element of $HKh_w(Z)$ associated to decorating each component of Z with ξ.

Proof. This follows from the results proved in Sections 3 and 4, where we considers what happens to chain representatives of basis elements of $HKh_w(D)$ upon performing a Reidemeister I or II move or a handle addition to D. It is important here also to remember Theorem 1.6, which ensures that there is an intermediate diagram D' of a connected knot in the presentation S'. This will mean that $[\alpha_\xi] \in HKh_w(D)$ gets mapped to $[\alpha_\xi] \in HKh_w(Z)$, since every 0-handle of S' gets joined to D by 1-handles before other 1-handles split any 0-handles into more pieces. $\square$

Recall that there is a filtration

$$\ldots \subseteq \mathcal{F}^{j-1}HKh^i \subseteq \mathcal{F}^j HKh^i \subseteq \ldots$$

and an associated graded vector space

$$HKh_w^{i,j} = \mathcal{F}^j HKh^i / \mathcal{F}^{j-1} HKh^i.$$

Necessarily $[\alpha_\xi] \in HKh^0(Z)$, we would like now to know in which degree of the quantum grading of $HKh_w(Z)$ the element $[\alpha_\xi]$ lies.

Lemma 6.2.

$$[\alpha_\xi] \in HKh^{0,e(n-1)}(Z).$$

Proof. First note that if $Z = U$ it is automatic that $[\alpha_\xi] \in HKh^{0,n-1}(U)$. Write B for the unlink that appears in the left of in Figure 38. Since the homology of a diagram composed of disjoint pieces is the tensor product of the homologies of the pieces, we are done if we can show that $[\alpha_\xi] \in HKh_w^{0,3n-3}(B)$.

Figure 38 shows B and the oriented resolution B^o of B. The chain representative α_ξ of $[\alpha_\xi]$ lies in $H(B^o)$ which is a summand of the 0th homological degree chain group $CKh_w^0(B)$.

The canonical chain representative $\alpha_\xi \in H(B^o)$ is, up to non-zero scalar multiplication, the module element

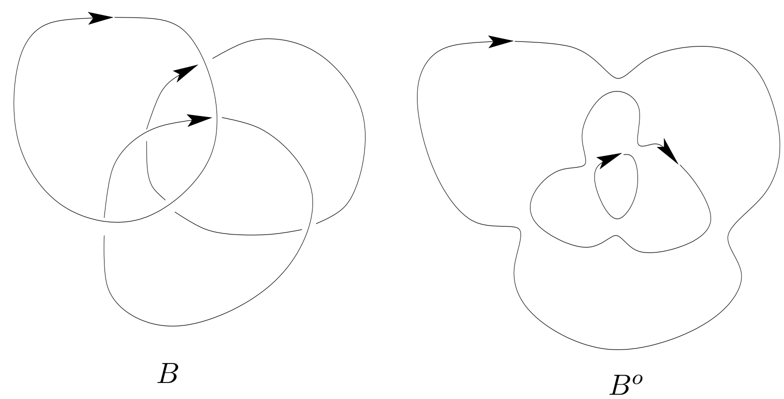

Figure 38: The diagram B of the 3-component unlink possibly occurs a number of times as a component of Z. The diagram B^o is the oriented resolution of B.

$$\prod_{i=1,2,3} \frac{\partial_i w}{x_i - \xi} \in (\mathbb{C}[x_1, x_2, x_3]/(\partial_1 w, \partial_2 w, \partial_3 w))\{3 - 3n\}$$

which is of top filtration grading $3n - 3$. *A priori*, of course, it does not follow that $[\alpha_\xi] \in HKh_w^{0,3n-3}(B)$, since we have not yet seen that α_ξ is not homologous to an element of $CKh_w(B)$ which is of a lower filtered degree. We shall show this now.

The diagram B is a diagram of the unlink, so for dimensional reasons the spectral sequence converging from $E_2 = HKh_n^{i,j}(B)$ to $E_\infty = HKh_w^{i,j}(B)$ collapses immediately ($E_2 = E_\infty$).

The reduction $\tilde{\alpha}_\xi \in \mathcal{F}^{3n-3}H(B^o)/\mathcal{F}^{3n-4}H(B^o)$ is a cycle in the page $E_1 = CKh_n^{i,j}$. If we can show that $\tilde{\alpha}_\xi$ is not a boundary with respect to the E_1 differential (which is just the standard Khovanov-Rozansky $sl(n)$ differential) then, since $E_2 = E_\infty$, $\tilde{\alpha}_\xi$ will represent a non-zero class on the E_∞ page. The grading of this class (which is necessarily $j = 3n - 3$) will be the grading in which $[\alpha_\xi]$ lies in $HKh_w^{0,j}(B)$.

So, in order to show that $[\alpha_\xi] \in HKh_w^{0,3n-3}(B)$, we need to see that $\tilde{\alpha}_\xi$ represents a non-zero class in $HKh_n(B)$. Khovanov and Rozansky [7] have provided quantum-degree 0 chain homotopy equivalences between the chain complexes CKh_n corresponding to tangle diagrams differing by a single Reidemeister move. We shall change B by certain of these Reidemeister moves to arrive at the 0-crossing 3-component unlink. We shall see that the chain maps of Khovanov-Rozansky's chain homotopy equivalences will

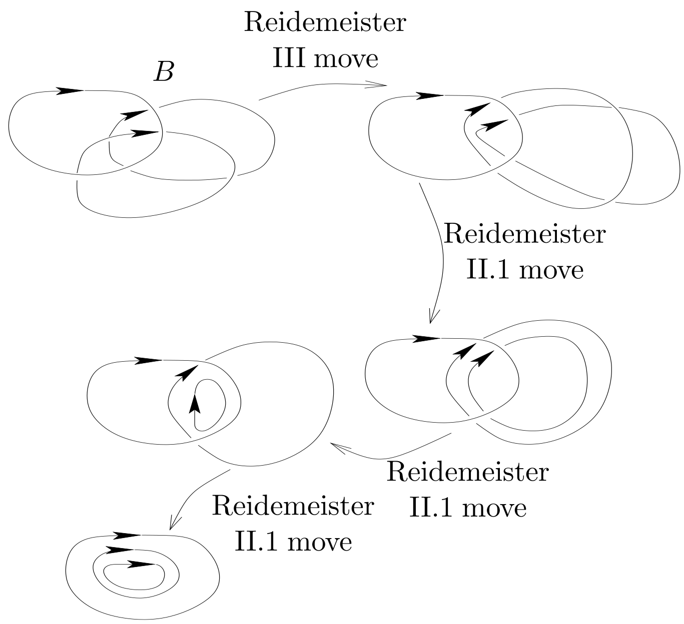

Figure 39: Uncrossing the unlink. We show the sequence of moves we use in the proof of Lemma 6.2 when converting the 3-component unlink B into the 3-component 0-crossing diagram of the unlink.

act on $\tilde{\alpha}_\xi$, mapping it to a cycle of the chain complex corresponding to the 0-crossing 3-component unlink. This cycle represents a non-zero class in the homology, so $\tilde{\alpha}_\xi$ represents a non-zero class. This will complete our argument that $[\alpha_\xi] \in HKh_w^{0,3n-3}$.

In Figure 39 we have shown the sequence of Reidemeister moves that we use to convert B to the 0-crossing 3-component unlink. Note that all the Reidemeister II moves are II.1 moves, that the Reidemeister III move involves three positive crossings, and that the number of circles in the oriented resolution never changes.

The cycle $\tilde{\alpha}_\xi$ in CKh_n lies in the 0th chain group summand $H(B^o)$. Up to non-zero scalar multiplication:

$$\tilde{\alpha}_\xi = x_1^{n-1} x_2^{n-1} x_3^{n-1} \in (\mathbb{C}[x_1, x_2, x_3]/x_1^n x_2^n x_3^n)\{3 - 3n\} = H(B^o).$$

Unpacking the proofs a little in [7], we see that (for the Reidemeister II.1 move which reduces the number of crossings and for the Reidemeister III move with all positive crossings) the maps induced from the chain group summands corresponding to the oriented resolutions are just the identity map of matrix factorizations. (The fact that, in Figure 22, the double-headed arrow labeled *id* represents the identity map, is a reflection of this fact).

This means that the element $\tilde{\alpha}_\xi$ always gets mapped to the top-degree element in the chain group summand that is the homology of the oriented resolution. In the final diagram of the 0-crossing 3-component unlink, the homology of the oriented resolution is the *only* chain group summand. So the image of $\tilde{\alpha}_\xi$ represents a non-zero element in homology and, since the chain maps are all chain homotopy equivalences, $\tilde{\alpha}_\xi$ also represents a non-zero element in homology. $\square$

We now have enough to deduce our result on the slice genus Theorem 1.2.

Proof of Theorem 1.2. Consider the map

$$HKh_w(S') : \mathcal{F}^{j+(n-1)(\chi(\Sigma)-e+1)} HKh^0_w(D) \to \mathcal{F}^j HKh^0_w(Z).$$

By Proposition 6.1 and Lemma 6.2, we know that $HKh_w(S')([\alpha_\xi])$ is a non-zero element of $\mathcal{F}^{e(n-1)} HKh^0_w - \mathcal{F}^{e(n-1)-1} HKh^0_w$. Hence $HKh^0_w(D) - \mathcal{F}^{(n-1)(\chi(\Sigma)+1)-1} HKh^0_w(D)$ is non-empty. So there exists some $j \geq (n-1)(\chi(\Sigma)+1) = (1-n)(2g(\Sigma)-1)$ for which $HKh^{0,j}_w(D) \neq 0$. Theorem 1.2 follows. $\square$

We deduce Milnor's conjecture on the slice genus of torus knots as Corollary 1.3 to Theorem 1.2. We start by discussing *positive* knots.

Suppose that a knot K has a diagram D in which all the crossings are positive (i.e. they look like the crossing on the left of Figure 4). Suppose that D has k crossings and l circles in its oriented resolution. We compute the top grading of $HKh_w(D)$.

Since all the crossings are positive, the homology of the oriented resolution is the leftmost chain group in the Khovanov cube (also the chain group of homological degree 0). Hence the homology $HKh_w(D)$ will just be the kernel of the differential coming from this chain group. In particular the filtration grading of chain representatives of the homology will be the same taken in the chain group CKh^0_w as taken in the homology HKh^0_w since there is no boundary group by which to quotient.

Each basis element $[\alpha_\xi]$ of $HKh_w(D)$ lies in $HKh^{0,(1-n)(k-l)}_w(D)$. In particular, $HKh^{0,(1-n)(k-l)}_w(D) \neq 0$. Now, $(1-n)(k-l)$ is the highest filtration degree of the 0th chain group, so Theorem 1.2 says that

$$(n-1)(2g^*(K)-1) \geq (n-1)(k-l).$$

There exists an explicit description of a Seifert surface for K which consists of l disks (filling the circles of the oriented resolution of D), connected by k bands (where the crossings of D are). The Euler characteristic of this surface with boundary is $l-k$, so the surface is of genus $(k-l+1)/2$. Pushing this Seifert surface slightly into the 4-ball yields a slice surface for K of the same genus. Since we have seen that

$$2g^*(K) \geq k-1+1$$

it follows that $2g^*(K) = k-l+1$.

Proof of Corollary 1.3. Performing this computation in the case of the standard diagram of a torus knot yields Milnor's conjecture on the slice genus of torus knots. □

7 Appendix

Here we give two basic results used (sometimes implicitly) throughout this paper.

7.1 Removal of Marks

Below, we omit mention of the various quantum grading shifts that occur, although readers friendly with matrix factorizations can easily assure themselves that the filtration grading of each map is as expected.

To first explain Figure 40: the matrix factorization N (resp. N') is obtained from Q (resp. Q') by tensoring with the matrix factorization P. Also, the matrix factorization M (resp. M') is the same as Q (resp. Q') with the formal replacement of the variable y by x.

Lemma 7.1. *Removal of marks*

We prove that the matrix factorizations M (resp. M') and N (resp. N') (in which we intend the circles to contain the same arbitrary trivalent graph with thick edges) in Figure 40 are equal in the homotopy category of matrix factorizations.

Furthermore, suppose M', N', Q' are factorizations, maybe different from M, N, Q but with the same boundary labels. If we then have a map of matrix factorizations $\alpha_x : M \to M'$ (and, replacing x by y, $\alpha_y : Q \to Q'$), inducing a map of matrix factorizations $A : N \to N'$, then

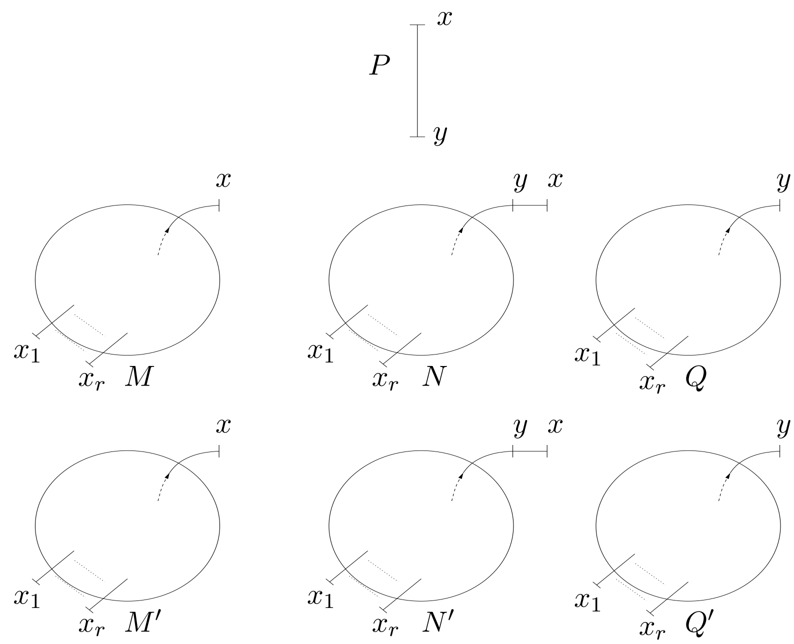

Figure 40: Removal of marks

$$\alpha_x = L \circ A \circ K$$

where $L : N \to M$ and $K : M \to N$ are the homotopy equivalences constructed.

Proof. We exhibit filtered degree-0 maps $M \to N$, $N \to M$ which we show are homotopy equivalences.

The matrix factorizations M and N are both defined over the same ground ring $R = \mathbb{C}[x_1, x_2, ..., x_r, x]$. We write M as

$$M^0 \xrightarrow{f_x} M^1 \xrightarrow{g_x} M^0.$$

The matrix factorization Q is the same but with any occurence of x relabelled as y. We shall consequently write Q as

$$Q^0 \xrightarrow{f_y} Q^1 \xrightarrow{g_y} Q^0$$

where the subscripts are to remind us that the difference between the factorizations M and Q is just the interchanging of x with y.

Then if P is the factorization

$$P^0 \stackrel{\pi_{xy}}{\to} P^1 \stackrel{x-y}{\to} P^0$$

where P^0 and P^1 are free rank-1 modules over $\mathbb{C}[x,y]$, we have that $N^0 = (Q^0 \otimes P^0) \oplus (Q^1 \otimes P^1)$, $N^1 = (Q^1 \otimes P^0) \oplus (Q^0 \otimes P^1)$ and N can be written

$$N^0 \stackrel{\begin{pmatrix} f_y & -(x-y) \\ \pi_{xy} & g_y \end{pmatrix}}{\longrightarrow} N^1 \stackrel{\begin{pmatrix} g_y & x-y \\ -\pi_{xy} & f_y \end{pmatrix}}{\longrightarrow} N^0$$

in which we are implicitly thinking of N as a factorization over the polynomial ring $\mathbb{C}[x_1, x_2, ..., x_r, x, y]$ with degenerate potential.

Now we give the homotopy equivalence between M and N. We define maps of matrix factorizations $K : M \to N$ and $L : N \to M$ by the following $\mathbb{C}[x_1, x_2, ..., x_r, x]$-module maps:

$$K_0 = \begin{pmatrix} 1 \\ -\frac{f_x - f_y}{x-y} \end{pmatrix} \quad K_1 = \begin{pmatrix} 1 \\ \frac{g_x - g_y}{x-y} \end{pmatrix}$$

$$L_0 = \begin{pmatrix} P_{y \mapsto x} & 0 \end{pmatrix} \quad L_1 = \begin{pmatrix} P_{y \mapsto x} & 0 \end{pmatrix}$$

where by $P_{y \mapsto x}$ we mean the map which looks like the ring map

$$\mathbb{C}[x_1, x_2, ..., x_r, x, y] \to \mathbb{C}[x_1, x_2, ..., x_r, x, y]/(x-y) = \mathbb{C}[x_1, x_2, ..., x_r, x]$$

on each module summand $\mathbb{C}[x_1, x_2, ..., x_r, x, y]$.

Note that $L \circ K : M \to M$ is already the identity map.

We define an homotopy $H_0 : N^0 \to N^1, H_1 : N^1 \to N^0$ as follows:

$$H_0 = \begin{pmatrix} 0 & 0 \\ \frac{1 - P_{y \mapsto x}}{x-y} & 0 \end{pmatrix} \quad H_1 = \begin{pmatrix} 0 & 0 \\ \frac{P_{y \mapsto x} - 1}{x-y} & 0 \end{pmatrix}.$$

It is now an exercise to see that

$$id_{N^0} - K_0 \circ L_0 = H_1 \circ \begin{pmatrix} f_y & -(x-y) \\ \pi_{xy} & g_y \end{pmatrix} + \begin{pmatrix} g_y & x-y \\ -\pi_{xy} & f_y \end{pmatrix} \circ H_0$$

and

$$id_{N^1} - K_1 \circ L_1 = \begin{pmatrix} f_y & -(x-y) \\ \pi_{xy} & g_y \end{pmatrix} \circ H_1 + H_0 \circ \begin{pmatrix} g_y & x-y \\ -\pi_{xy} & f_y \end{pmatrix}.$$

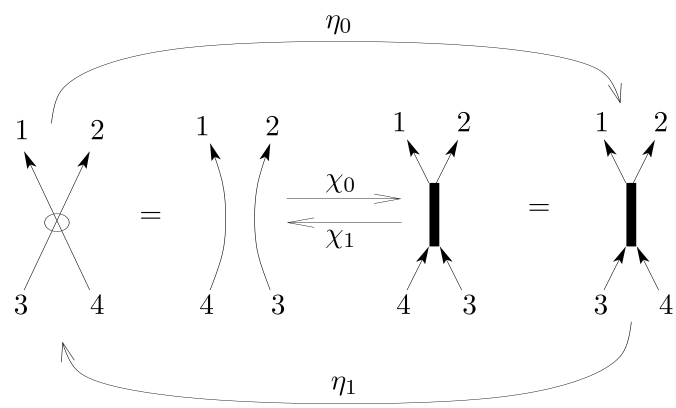

Figure 41: Definition of the η maps. By this diagram we mean the reader to understand that the maps η_0 and η_1 are defined as conjugation of the maps χ_0 and χ_1 respectively. This is not a commutative diagram.

The second part of the theorem amounts to the observation that if $\alpha_{y,0}$ and $\alpha_{y,1}$ are the components of $\alpha_y : Q \to Q'$ then the induced map $A : N \to N'$ has components

$$A_0 = \begin{pmatrix} \alpha_{y,0} & 0 \\ 0 & \alpha_{y,1} \end{pmatrix} \quad A_1 = \begin{pmatrix} \alpha_{y,1} & 0 \\ 0 & \alpha_{y,0} \end{pmatrix}$$

and then it is immediate that

$$\alpha_{x,0} = \begin{pmatrix} P_{y\to x} & 0 \end{pmatrix} \begin{pmatrix} \alpha_{y,0} & 0 \\ 0 & \alpha_{y,1} \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{f_x - f_y}{x-y} \end{pmatrix}$$

and

$$\alpha_{x,1} = \begin{pmatrix} P_{y\to x} & 0 \end{pmatrix} \begin{pmatrix} \alpha_{y,1} & 0 \\ 0 & \alpha_{y,0} \end{pmatrix} \begin{pmatrix} 1 \\ \frac{g_x - g_y}{x-y} \end{pmatrix}.$$

$\square$

7.2 The η map

The η_0 and η_1 maps from Section 2 are defined as in Figure 41.

The equalities in Figure 41 are equalities as bare matrix factorizations. It is useful for us to know how this η map would be different if it were defined as conjugation of the χ map by swapping x_1 and x_2 instead of x_3 and x_4. In particular, at the end of our proofs of Propositions 4.6 and 4.8,

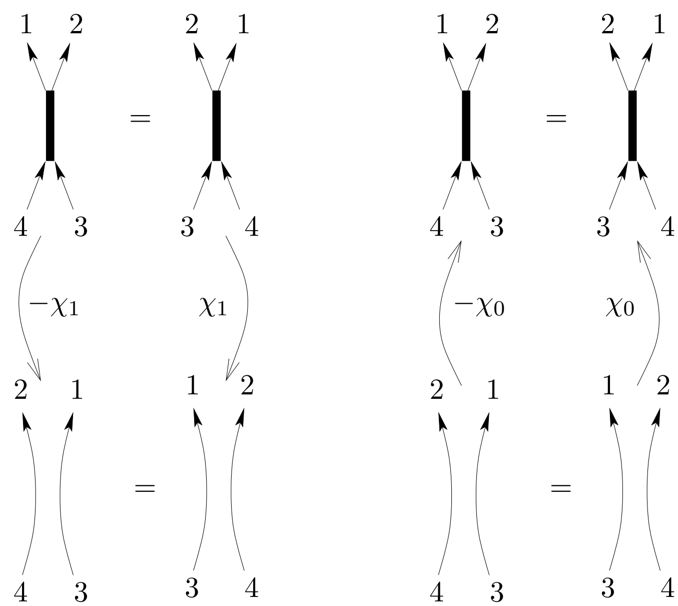

Figure 42: A commutative diagram.

we implicitly use the fact that the η map would merely change up to sign. We state this as Lemma 7.2, the proof of which is just to write down the relevant matrices and multiply them out:

Lemma 7.2. *Figure 42 is commutative.* □

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