COMPLEX HYPERBOLIC FENCHEL-NIELSEN COORDINATES

JOHN R. PARKER AND IOANNIS D. PLATIS

Abstract. Let Σ be a closed, orientable surface of genus g. It is known that the $\mathrm{SU}(2,1)$ representation variety of $\pi_1(\Sigma)$ has $2g-3$ components of (real) dimension $16g-16$ and two components of dimension $8g-6$. Of special interest are the totally loxodromic, faithful (that is quasi-Fuchsian) representations. In this paper we give global real analytic coordinates on a subset of the representation variety that contains the quasi-Fuchsian representations. These coordinates are a natural generalisation of Fenchel-Nielsen coordinates on the Teichmüller space of Σ and complex Fenchel-Nielsen coordinates on the (classical) quasi-Fuchsian space of Σ.

1. Introduction

In their famous manuscript, recently published as [5], Fenchel and Nielsen gave global coordinates for the Teichmüller space of a closed surface Σ of genus $g \geq 2$. These coordinates are defined as follows; see also Wolpert [26], [27]. First, let γ_j for $j = 1, \ldots, 3g-3$ be a maximal collection of disjoint, simple, closed curves on Σ that are neither homotopic to each other nor homotopically trivial. We call such a collection a *curve system*; it is also called a partition by some authors. The complement of such a curve system is a collection of $2g-2$ three-holed spheres, or pairs of pants. If Σ has a hyperbolic metric then, without loss of generality, we may choose each γ_j in our curve system to be the geodesic in its homotopy class. The hyperbolic metric on each three-holed sphere is completely determined by the hyperbolic length $l_j > 0$ of each of its boundary geodesics. Each γ_j is in the boundary of exactly two three-holed spheres (including the case where it corresponds to two boundary curves of the same three-holed sphere). There is a twist parameter $k_j \in \mathbb{R}$ that determines how these three-holed spheres are attached to one another. This is defined as follows. On each three-holed sphere with its hyperbolic metric, take disjoint orthogonal geodesic arcs between each pair of boundary geodesics. On each geodesic γ_j, the feet of these perpendiculars on the same side are diametrically opposite. The twist parameter k_j measures the hyperbolic distance along γ_j between the feet of the perpendiculars on opposite sides. As we have just defined it, the parameter k_j lies between $\pm l_j/2$. Performing a Dehn twist about γ_j adds $\pm l_j$ to k_j, the sign depending on the direction of twist. Thus we can make the twist parameter a well defined real number with reference to an initial homotopy class. The theorem of Fenchel and Nielsen states that each $(6g-6)$-tuple

$$(l_1, \ldots, l_{3g-3}, k_1, \ldots, k_{3g-3}) \in \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$$

determines a unique hyperbolic metric on Σ and each hyperbolic metric arises in this way.

We will take the point of view that Teichmüller space is the collection of discrete, faithful, purely loxodromic representations of $\pi_1(\Sigma)$ to $\mathrm{SL}(2,\mathbb{R})$, up to conjugation. In this case the discreteness of the representation follows from the fact that it is totally loxodromic, but we include discreteness as a hypothesis for emphasis. Wolpert gives a careful description of the Fenchel-Nielsen coordinates for Riemann surfaces in [26]. Given such a representation, the Fenchel-Nielsen coordinates may be computed directly from the matrices; see [14] for an explicit way to do this. The length parameters

1991 *Mathematics Subject Classification.* 32G05,32M05.

JRP was partly supported by a CNRS position held at the University of Paris VI. IDP was supported by a Marie Curie Intra-European fellowship (Contract No.MEIF-CT-2003-500074) within the 6th Community Framework Programme.

l_j may read off from the traces of the corresponding matrices and the twist parameters k_j from cross-ratios of certain combinations of fixed points. Note that all these quantities are conjugation invariant. We also remark that it is not possible to determine the representation up to conjugacy by merely using $6g-6$ trace parameters, in fact one needs $6g-5$; see [16] and [19].

In [12] and [20] Kourouniotis and Tan defined complex Fenchel-Nielsen coordinates on the quasi-Fuchsian space of Σ, in which both the length parameters and the twist parameters become complex. The group elements corresponding to γ_j in the curve system are now loxodromic with, in general, non-real trace. Thus the imaginary part of the length parameter represents the holonomy angle when moving around γ_j. Likewise, the imaginary part of the twist parameter becomes the parameter of a bending deformation about γ_j; see also [18] for more details of this correspondence and how to relate these parameters to traces of matrices. The main difference from the situation with real Fenchel-Nielsen coordinates is that, while distinct quasi-Fuchsian representations determine distinct complex Fenchel-Nielsen coordinates, it is not at all clear which set of coordinates give rise to discrete representations, and hence to a quasi-Fuchsian structure. In fact the boundary of the set of realisable coordinates is fractal.

Another generalisation of Fenchel-Nielsen coordinates is given by Goldman in [8], where he considers the space of convex real projective structures on a compact surface. There he constructs $16g-16$ real parameters. Goldman uses two real parameters generalising the length of each γ_j and two real parameters generalising the twist parameters. Which gives $12g-12$ in total. For the remaining $4g-4$ parameters, Goldman shows that one must associate an additional two real parameters to each three-holed sphere.

The purpose of this paper is to define analogous Fenchel-Nielsen coordinates for complex hyperbolic quasi-Fuchsian representations of surface groups, that is discrete, faithful, totally loxodromic representations; see [17]. (Once again a totally loxodromic representation is automatically discrete.) In this setting the representation space, and hence the quasi-Fuchsian space, is more complicated. There is a natural invariant of representations of surface groups to $\mathrm{SU}(2,1)$, called the *Toledo invariant*; see [21]. The Toledo invariant is an even integer lying in the interval $[\chi,-\chi]$, where $\chi=\chi(\Sigma)$ is the Euler characteristic of Σ; see [9]. Moreover, the Toledo invariant distinguishes the components of the $\mathrm{SU}(2,1)$ representations variety; see [28]. Each component contains discrete, faithful, totally loxodromic representations; see [9]. A representation preserves a complex line if and only if its Toledo invariant equals $\pm\chi$; see [21]. The corresponding two components comprise reducible representations and they are the direct product of Teichmüller space (within the complex line) and representations of $\pi_1(\Sigma)$ to $\mathrm{U}(1)$ (rotations around the complex line). The representation is reducible and the corresponding components have dimension $8g-6$; see Theorem 6(d) of Goldman [7]. The remaining components correspond to itteducible representations and so, using Weil's formula [22] their dimension is $16g-16$; see also Lemma 1 of [7].

The definition of the Toledo invariant uses an equivariant embedding of the universal cover of the surface Σ (that is the hyperbolic plane) into complex hyperbolic space. We do not explicitly use this surface. However, we will have this embedding in the backs of our minds when we use phrases like 'attaching groups along peripheral elements' and 'closing a handle'. These phrases are carried over from plane hyperbolic geometry and do not make direct sense in four dimensions, although we could make them precise by using equivariant embeddings of the surfaces in question.

Suppose that we are given a curve system $\gamma_1,\ldots,\gamma_{3g-3}$ on a closed surface Σ of genus $g\geq 2$, as described above. We consider representations $\pi_1(\Sigma)$ to $\mathrm{SU}(2,1)$ for which the $3g-3$ group elements representing the γ_j are all loxodromic with distinct fixed points. It is clear that this is a proper subset of the representation variety; but this subset contains all (discrete) faithful, totally loxodromic representations. That is, it contains the complex hyperbolic quasi-Fuchsian space.

In fact, we restrict our attention to a particular type of curve systems. Namely, we suppose that there are g of the curves γ_j that correspond to two boundary components of the same three-holed sphere. We call such a curve system *simple*. See Figure 2.1 for an example of a simple curve system. This restriction makes our computations easier and should not be necessary in general.

Our goal is to describe $16g-16$ real parameters that distinguish non-conjugate irreducible representations and $8g-6$ real parameters that distinguish non-conjugate representations that preserve a complex line. As with the complex Fenchel-Nielsen coordinates described by Kourouniotis and Tan it is not clear which coordinates correspond to discrete representations. However, our coordinates determine the group up to conjugacy and distinguish between non-conjugate representations.

The major innovation in this paper is the use of cross-ratios in addition to complex length and twist-bend parameters. Following Korányi and Reimann [11], there are 24 complex cross-ratios associated to the different permutations of four ordered points. Certain permutations of the four points preserve these cross ratios or send them to their complex conjugate, to their reciprocal or to their conjugate reciprocal; see either page 225 of [6] or else [25]. After taking account of these symmetries, one is left with three complex cross-ratios. Falbel [2] shows that these satisfy two real equations and so lie on a real four-dimensional variety in $\mathbb{C}^3$. This variety is Falbel's *cross-ratio variety* which we denote by $\mathfrak{X}$. Moreover, following Falbel, these three cross-ratios determine the four ordered points up to $\mathrm{SU}(2,1)$ equivalence.

In the case where our representation does not preserve a complex line, we assign parameters as follows. To each of the $2g-2$ three-holed spheres we associate two complex traces and a point on $\mathfrak{X}$. This gives eight real parameters. These $16g-16$ real parameters are subject to $3g-3$ complex constraints that are compatibility conditions for gluing the three-holed spheres together. This reduces the number of independent parameters to $10g-10$. There are then $3g-3$ complex twist-bend parameters, one associated to each gluing operation. This gives $16g-16$ real parameters in total; see Theorem 2.1. This parameter count is the same as Goldman's [8], but his real parameters are not combined into complex numbers.

Representations that preserve a complex line are reducible. A result analogous to Theorem 2.1 may be deduced by splitting the representation to one in $\mathrm{SU}(1,1)$ and one in $\mathrm{U}(1)$. The first corresponds to a point in Teichmüller space and is determined by $6g-6$ real parameters (for example Fenchel-Nielsen coordinates). The second is abelian and is completely determined by $2g$ real parameters, for example the arguments of the generators. In Theorem 2.2 we show that certain of our parameters are real in this case and the parameters analogous to those indicated in Theorem 2.1 give $8g-6$ real parameters that completely determine $\rho:\pi_1(\Sigma)\longrightarrow\Gamma<\mathrm{S}\big(\mathrm{U}(1)\times\mathrm{U}(1,1)\big)$ up to conjugation.

The paper is organised as follows. We give the statements of the main results in Section 2. After covering the necessary background material in Section 3 we discuss loxodromic isometries in some detail, Section 4. Following this we discuss the properties of Korányi-Reimann cross-ratios and Cartan angular invariants in Section 5. In Section 6 we show how to associate a point on $\mathfrak{X}$ to a pair of loxodromic maps A and B and we investigate the relationship between cross-ratios and traces of elements of $\langle A,B\rangle$. We are then able to begin to discuss Fenchel-Nielsen coordinates. We begin with coordinates for three-holed spheres, Section 7, and then go on to discuss in Section 8 the twist-bend parameters that describe the ways to glue three-holed spheres to form four-holed spheres or one-holed tori. This completes the list of ingredients necessary for Section 2. Additionally, in Section 7.3 we investigate what happens if we only use traces (that is complex lengths) to parametrise three-holed spheres. We show that it is not sufficient to use four traces, but we must use five traces subject to two real equations.

A large fraction of this paper is devoted to both showing that other possible coordinates do not work (Section 7.3) and also treating the special case where the group preserves a complex line (Sections 2.2, 5.4, 6.3 and 8.4). Readers who do not want to go into this material may by-pass it as follows. A good overview can be obtained by reading the outline in Section 2.1; the background material in Sections 3, 4, 5.1 and 5.2 and then Sections 6.1, 7.1, 8.1, 8.2 and 8.3. However, this

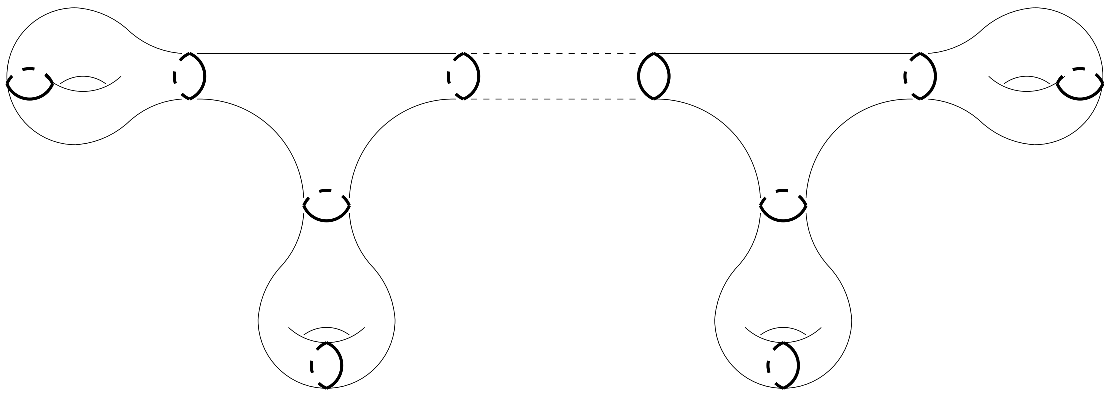

FIGURE 2.1. *An example of a simple curve system.*

reading scheme omits certain crucial results, for example Proposition 5.10, which could be assumed from Falbel's work [2].

Part of this work was carried out while both authors were visiting the University of Paris VI, where we had many useful discussions. We would particularly like to thank Elisha Falbel for pointing out a subtle, but serious, error in an earlier version of this manuscript. We would also like to thank the referee for giving us useful suggestions for improving the paper.

2. Complex hyperbolic Fenchel-Nielsen coordinates

2.1. **Representations that do not preserve a complex line.** We now summarise our construction of Fenchel-Nielsen coordinates for complex hyperbolic quasi-Fuchsian surface groups. For the details the reader should see subsequent sections and we give precise references as we go along.

As mentioned in the introduction we only consider curve systems with the property that each handle is closed from inside the same three-holed sphere, and we call such a curve system *simple*; see Figure 2.1 for an example of a simple curve system. Given a simple curve system on a closed Riemann surface of genus g we consider representations of the fundamental group so that each curve in our system is represented by a loxodromic map; see Section 4 for more details about loxodromic isometries. The restriction that our curve system is simple should not be necessary in general. In the classical case the effect of a change of curve system has been investigated by Okai [15]. This could probably be extended to the complex hyperbolic setting, but we will not pursue it here.

The good thing about simple curve systems is that the surface may be built up using the following recursive process. Begin with a single three-holed sphere. Attach a second three-holed sphere along a boundary curve. In order to do this, two of the boundary curves, one from each three-holed sphere, must be compatible. The result is a four-holed sphere. Keep adding pairs of three-holed spheres so that at each stage the boundary curves are grouped in pairs and each pair belong to the same three-holed sphere. Eventually one ends up with $2g-2$ three-holed spheres attached together to form a $2g$-holed sphere. These $2g$ holes naturally come in pairs, each pair belonging to the same three-holed sphere. For each such pair we close the handle. The result is a surface of genus g that is naturally made up of $2g-2$ three-holed spheres attached along $3g-3$ curves γ_j and it is these curves that make up our curve system, which by construction is simple. At each stage we have required that the boundary components that are attached are compatible, both when adding new three-holed spheres and when closing handles.

This way of using three-holed spheres to build up our surface with a simple curve system is very well adapted to the Fenchel-Nielsen coordinates we shall construct. In this section we will examine how this works for representations of $\pi_1(\Sigma)$ that do not preserve a complex line. Each three-holed sphere corresponds to a $(0,3)$ group and this is described up to conjugation by eight real parameters (locally by four complex parameters): namely two complex traces and a point on a cross-ratio variety; see Theorem 7.1. When attaching two $(0,3)$ groups together to form a $(0,4)$ group we require that two of the peripheral elements are compatible (that is one is conjugate to the inverse of the other); see Section 8.1 for a discussion of compatibility. This gives one fewer complex degree of freedom. However, there is one complex parameter associated to the attaching process, namely the Fenchel-Nielsen twist-bend. Thus there are still sixteen real parameters describing an attached pair of $(0,3)$ groups (that is eight for each $(0,3)$ group); see Theorem 8.4. Continuing in the same way, each $(0,3)$ group we attach is described by eight real parameters, two of which are constrained by the compatibility condition. But there is one complex degree of freedom in the attaching process. Thus once we have attached all $2g-2$ of our $(0,3)$ groups we will have $8\cdot(2g-2)=16g-16$ real parameters. In order to close the g handles we need to impose the compatibility condition on each of the g pairs of boundary curves. These g complex constraints reduce our number of real parameters to $14g-16$. But there are g complex twist-bend parameters, one for each handle we close; see Theorem 8.6. This gives a grand total of $16g-16$ real parameters. This is the number we require.

We call the resulting coordinates *complex hyperbolic Fenchel-Nielsen coordinates* for the group $\Gamma=\rho\big(\pi_1(\Sigma)\big)$. Specifically, these coordinates are the $3g-3$ complex twist-bend parameters; the $4g-4$ complex traces and $2g-2$ points on the cross-ratio variety $\mathfrak{X}$, all subject to $3g-3$ complex constraints. It remains to check that these are independent and that they completely determine our representations up to conjugacy. Our main theorem is the following:

Theorem 2.1. *Let Σ be a surface of genus g with a simple curve system $\gamma_1,\ldots,\gamma_{3g-3}$. Let $\rho:\pi_1(\Sigma)\longrightarrow\Gamma<\mathrm{SU}(2,1)$ be a representation of the fundamental group of Σ with the property that $\rho(\gamma_j)=A_j$ is loxodromic for each $j=1,\ldots,3g-3$. Suppose that Γ does not preserve a complex line. Then the Fenchel-Nielsen coordinates of ρ are independent and two representations have the same Fenchel-Nielsen coordinates of and only if they are conjugate in* $\mathrm{SU}(2,1)$.

Proof. This theorem will follow from the results we prove below. In Theorem 7.1 we show that the representations of each $(0,3)$ group may be parametrised by the trace of two peripheral curves and a point of the corresponding cross-ratio variety. For each $(0,3)$ group we may choose the two peripheral curves in three ways. Making a different choice corresponds to an analytic change of coordinates; see Theorem 7.2. This gives a total of $4g-4$ traces and $2g-2$ points in the cross-ratio variety (which has four real dimensions). The compatibility conditions when gluing impose $3g-3$ complex conditions on these parameters. There are $3g-3$ twist-bend parameters κ_j, each in $\mathbb{C}$ with $-\pi<\Im(\kappa_j)\leq\pi$. The only relations between the parameters in adjacent $(0,3)$ groups are the compatibility conditions. There are no relations between parameters in non-adjacent $(0,3)$ groups. Thus, all other parameters are independent.

Now suppose we have two representations with the same coordinates. The coordinates of each three-holed sphere are the same in both representations and so they are pairwise conjugate; see Theorem 7.1. But when gluing across each curve in the system the resulting $(0,4)$ group or $(1,1)$ group is determined up to conjugation; see Theorems 8.4 and 8.6. Thus the whole group is determined up to conjugation.

Conversely, suppose we have two representations that are conjugate. By definition, the traces $\mathrm{tr}(A_j)$ are the same. This is also true of the parameters $\mathbb{X}_l$ provided we have chosen cross-ratios of corresponding points. If not, then one cross-ratio, together with three length coordinates, determines all other cross-ratios for that particular $(0,3)$ group by a real analytic change of coordinates; see Proposition 7.5. Finally, we know that the twist-bend parameters are the same.

This proves the result. □

2.2. **Representations preserving a complex line.** The two components of the representation variety with extreme Toledo invariant are made up of groups that preserve a complex line. These representations are reducible and the components have real dimension $8g-6$. Specifically, the components are a direct product of Teichmüller space, of dimension $6g-6$, and $2g$ copies of $\mathrm{U}(1)$. In this section we describe what happens to our Fenchel-Nielsen coordinates in this case.

Let $A_j = \rho(\gamma_j)$ be the group elements representing the simple closed curves γ_j in our simple curve system. These $3g-3$ curves fall into two classes. First, there are $2g-3$ curves used to attach distinct three-holed spheres and, secondly, there are g curves used to close handles. In Proposition 6.8 we show that, if γ_j is $2g-3$ curves used to attach distinct three-holed spheres, then $\mathrm{tr}(A_j)$ is real. Furthermore, there can be no bending across such curves; see the discussion in Section 8.2. Hence, each of these $2g-3$ complex twist-bend parameters κ_j is forced to be a real twist parameter k_j (which is just the classical Fenchel-Nielsen twist).

Additionally, the cross-ratios are all real and satisfy certain equations; see Proposition 5.13. Moreover, arguing as in Proposition 7.6, we may express this cross-ratio in terms of the traces. In fact, using the notation of Proposition 7.6, in this case we have

$$\mathbb{X}_1(A,B) = \frac{(\mathrm{tr}(AB) - \tau(\lambda - \overline{\mu}))}{(e^{\lambda} - e^{-\overline{\lambda}})(e^{\mu} - e^{-\overline{\mu}})}.$$

Thus in this case there are no independent cross-ratio parameters.

Thus we have proved that when $\rho\big(\pi_1(\Sigma)\big)$ preserves a complex line the Fenchel-Nielsen coordinates from Theorem 2.1 have degenerated as follows. First there are $2g$ complex parameters, namely the complex length and twist-bend parameters λ_j and κ_j for $j = 1, \ldots, g$ associated to curves γ_j that are used to close a handle. Then there are $4g-6$ real parameters, namely the length and twist parameters l_j and k_j for $j = g+1, \ldots, 3g-3$ associated to the other curves in the system. We call these the *Fenchel-Nielsen coordinates* for ρ.

In fact l_j and k_j for $j = 1, \ldots, 3g-3$ (where $l_j = \Re(\lambda_j)$ and $k_j = \Re(\kappa_j)$ for $j = 1, \ldots, g$) are just the classical Fenchel-Nielsen coordinates on the Teichmüller space of Σ. The other $2g$ parameters correspond to rotations around the complex line fixed by Γ. They may be thought of as a (necessarily abelian) representation of $\pi_1(\Sigma)$ into $\mathrm{U}(1)$. The stabiliser of a complex line is isomorphic to $\mathrm{S}\big(\mathrm{U}(1)\times\mathrm{U}(1,1)\big)$, the first factor corresponding to rotation around the complex line and the second to isometries of the hyperbolic metric on the complex line. These representations are clearly independent. Thus we have proved:

Theorem 2.2. *Let Σ be a surface of genus g with a simple curve system $\gamma_1, \ldots, \gamma_{3g-3}$. Let $\rho : \pi_1(\Sigma) \longrightarrow \Gamma < \mathrm{SU}(2,1)$ be a representation of the fundamental group of Σ preserving a complex line and with the property that $\rho(\gamma_j) = A_j$ is loxodromic for each $j = 1, \ldots, 3g-3$. Then, the Fenchel-Nielsen coordinates of ρ are independent and two representations have the same Fenchel-Nielsen coordinates of and only if they are conjugate in* $\mathrm{SU}(2,1)$.

3. Preliminaries

3.1. **Complex Hyperbolic Space.** Let $\mathbb{C}^{2,1}$ be the vector space $\mathbb{C}^3$ with the Hermitian form of signature $(2,1)$ given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* J \mathbf{z} = z_1\overline{w}_3 + z_2\overline{w}_2 + z_2\overline{w}_1$$

with matrix

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We consider the following subspaces of $\mathbb{C}^{2,1}$:

$$V_- = \Big\{ \mathbf{z} \in \mathbb{C}^{2,1} \;:\; \langle \mathbf{z},\, \mathbf{z} \rangle < 0 \Big\},$$
$$V_0 = \Big\{ \mathbf{z} \in \mathbb{C}^{2,1} - \{\mathbf{0}\} \;:\; \langle \mathbf{z},\, \mathbf{z} \rangle = 0 \Big\}.$$

Let $\mathbb{P} : \mathbb{C}^{2,1} - \{0\} \longrightarrow \mathbb{C}P^2$ be the canonical projection onto complex projective space. Then *complex hyperbolic space* $\mathbf{H}^2_{\mathbb{C}}$ is defined to be $\mathbb{P}V_-$ and its boundary $\partial\mathbf{H}^2_{\mathbb{C}}$ is $\mathbb{P}V_0$. Specifically, $\mathbb{C}^{2,1} - \{\mathbf{0}\}$ may be covered with three charts H_1, H_2, H_3 where H_j comprises those points in $\mathbb{C}^{2,1} - \{\mathbf{0}\}$ for which $z_j \neq 0$. It is clear that V_- is contained in H_3. The canonical projection from H_3 to $\mathbb{C}^2$ is given by $\mathbb{P}(\mathbf{z}) = (z_1/z_3,\, z_2/z_3) = z$. Therefore we can write $\mathbf{H}^2_{\mathbb{C}} = \mathbb{P}(V_-)$ as

$$\mathbf{H}^2_{\mathbb{C}} = \big\{ (z_1,\, z_2) \in \mathbb{C}^2 \;:\; 2\Re(z_1) + |z_2|^2 < 0 \big\}.$$

There are distinguished points in V_0 which we denote by $\mathbf{o}$ and $\boldsymbol{\infty}$:

$$\mathbf{o} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{\infty} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then $V_0 - \{\boldsymbol{\infty}\}$ is contained in H_3 and $V_0 - \{\mathbf{o}\}$ (in particular $\boldsymbol{\infty}$) is contained in H_1. Let $\mathbb{P}\mathbf{o} = o$ and $\mathbb{P}\boldsymbol{\infty} = \infty$. Then we can write $\partial\mathbf{H}^2_{\mathbb{C}} = \mathbb{P}(V_0)$ as

$$\partial\mathbf{H}^2_{\mathbb{C}} - \{\infty\} = \big\{ (z_1,\, z_2) \in \mathbb{C}^2 \;:\; 2\Re(z_1) + |z_2|^2 = 0 \big\}.$$

In particular $o = (0,0) \in \mathbb{C}^2$. In this manner, $\mathbf{H}^2_{\mathbb{C}}$ is the Siegel domain in $\mathbb{C}^2$; see [6].

Conversely, given a point z of $\mathbb{C}^2 = \mathbb{P}(H_3) \subset \mathbb{C}P^2$ we may lift $z = (z_1, z_2)$ to a point $\mathbf{z}$ in $H_3 \subset \mathbb{C}^{2,1}$, called the *standard lift* of z, by writing $\mathbf{z}$ in non-homogeneous coordinates as

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix}.$$

The *Bergman metric* on $\mathbf{H}^2_{\mathbb{C}}$ is defined by the distance function ρ given by the formula

$$\cosh^2\left(\frac{\rho(z,w)}{2}\right) = \frac{\langle \mathbf{z}, \mathbf{w} \rangle \, \langle \mathbf{w}, \mathbf{z} \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle \, \langle \mathbf{w}, \mathbf{w} \rangle} = \frac{\big|\langle \mathbf{z}, \mathbf{w} \rangle\big|^2}{|\mathbf{z}|^2 |\mathbf{w}|^2}$$

where $\mathbf{z}$ and $\mathbf{w}$ in V_- are the standard lifts of z and w in $\mathbf{H}^2_{\mathbb{C}}$ and $|\mathbf{z}| = \sqrt{-\langle \mathbf{z}, \mathbf{z} \rangle}$. Alternatively,

$$ds^2 = -\frac{4}{\langle \mathbf{z}, \mathbf{z} \rangle^2} \det \begin{bmatrix} \langle \mathbf{z}, \mathbf{z} \rangle & \langle d\mathbf{z}, \mathbf{z} \rangle \\ \langle \mathbf{z}, d\mathbf{z} \rangle & \langle d\mathbf{z}, d\mathbf{z} \rangle \end{bmatrix}.$$

The holomorphic sectional curvature of $\mathbf{H}^2_{\mathbb{C}}$ equals -1 and its real sectional curvature is pinched between -1 and $-1/4$.

There are no totally geodesic, real hypersurfaces of $\mathbf{H}^2_{\mathbb{C}}$, but there are are two kinds of totally geodesic 2-dimensional subspaces of complex hyperbolic space, (see Section 3.1.11 of [6]). Namely:

(i) complex lines L, which have constant curvature -1, and
(ii) totally real Lagrangian planes R, which have constant curvature $-1/4$.

Both of these subspaces are isometrically embedded copies of the hyperbolic plane.

3.2. **Isometries.** Let $\mathrm{U}(2,1)$ be the group of unitary matrices for the Hermitian form $\langle\cdot,\cdot\rangle$. Each such matrix A satisfies the relation $A^{-1} = JA^*J$ where A^* is the Hermitian transpose of A.

The full group of holomorphic isometries of complex hyperbolic space is the *projective unitary group* $\mathrm{PU}(2,1) = \mathrm{U}(2,1)/\mathrm{U}(1)$, where $\mathrm{U}(1) = \{e^{i\theta}I, \theta \in [0, 2\pi)\}$ and I is the 3×3 identity matrix. For our purposes we shall consider instead the group $\mathrm{SU}(2,1)$ of matrices which are unitary with respect to $\langle\cdot,\cdot\rangle$, and have determinant 1. Therefore $\mathrm{PU}(2,1) = \mathrm{SU}(2,1)/\{I, \omega I, \omega^2 I\}$, where ω is a non real cube root of unity, and so $\mathrm{SU}(2,1)$ is a 3-fold covering of $\mathrm{PU}(2,1)$. This is the direct analogue of the fact that $\mathrm{SL}(2,\mathbb{C})$ is the double cover of $\mathrm{PSL}(2,\mathbb{C})$.

Every complex line L is the image under some $A \in \mathrm{SU}(2,1)$ of the complex line where the second coordinate is zero. The subgroup of $\mathrm{SU}(2,1)$ stabilising this particular complex line is thus (conjugate to) the group of block diagonal matrices $\mathrm{S}\big(\mathrm{U}(1) \times \mathrm{U}(1,1)\big) < \mathrm{SU}(2,1)$. Similarly, every Lagrangian plane is the image under some element of $\mathrm{SU}(2,1)$ of the Lagrangian plane $R_{\mathbb{R}}$ where both coordinates are real. This is preserved by the subgroup of $\mathrm{SU}(2,1)$ comprising matrices with real entries, that is $\mathrm{SO}(2,1) < \mathrm{SU}(2,1)$.

Holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$ are classified as follows:

(i) An isometry is *loxodromic* if it fixes exactly two points of $\partial\mathbf{H}^2_{\mathbb{C}}$, one of which is attracting and the other repelling.

(ii) An isometry is *parabolic* if it fixes exactly one point of $\partial\mathbf{H}^2_{\mathbb{C}}$.

(iii) An isometry is *elliptic* if it fixes at least one point of $\mathbf{H}^2_{\mathbb{C}}$.

4. Loxodromic Isometries

4.1. **Eigenvalues and eigenvectors of loxodromic matrices.** Let $A \in \mathrm{SU}(2,1)$ be a matrix representing a loxodromic isometry. By definition A has an attracting fixed point. From the matrix point of view, this means that A has an eigenvalue e^{λ} with $|e^{\lambda}| = e^{\Re(\lambda)} > 1$. In other words $\Re(\lambda) > 0$. Since elements of $\mathrm{SU}(2,1)$ preserve the Hermitian form, it is not hard to show that if e^{λ} is an eigenvalue of A then so is $e^{-\overline{\lambda}}$ (Lemma 6.2.5 of [6]) and since $\det(A) = 1$, its third eigenvalue must be $e^{\overline{\lambda}-\lambda}$. We may also assume that $\Im(\lambda) \in (-\pi, \pi]$ and, in this way, $\lambda \in S$ where S is the region defined by:

$$S = \big\{\lambda \in \mathbb{C} \,:\, \Re(\lambda) > 0,\ \Im(\lambda) \in (-\pi, \pi]\big\}. \tag{4.1}$$

Let $a_A \in \partial\mathbf{H}^2_{\mathbb{C}}$ be the attractive fixed point of A. Then any lift $\mathbf{a}_A$ of a_A to V_0 is an eigenvector of A and the corresponding eigenvalue is e^{λ} with $\lambda \in S$. Likewise, if $r_A \in \partial\mathbf{H}^2_{\mathbb{C}}$ is the repelling fixed point of A, then any lift $\mathbf{r}_A$ of r_A to V_0 is an eigenvector of A with eigenvalue $e^{-\overline{\lambda}}$. The fixed points a_A and r_A span a complex line L_A in $\mathbf{H}^2_{\mathbb{C}}$, called the *complex axis* of A. The geodesic joining r_A and a_A is called the *real axis* of A. The eigenvector $\mathbf{n}_A$ of A corresponding to $e^{\overline{\lambda}-\lambda}$ is a polar vector to the complex axis of A.

For any $\lambda \in \mathbb{C}$ with $-\pi < \Im(\lambda) \leq \pi$ define $E(\lambda)$ by

$$E(\lambda) = \begin{bmatrix} e^{\lambda} & 0 & 0 \\ 0 & e^{\overline{\lambda}-\lambda} & 0 \\ 0 & 0 & e^{-\overline{\lambda}} \end{bmatrix} \tag{4.2}$$

It is easy to check that $E(\lambda)$ is in $\mathrm{SU}(2,1)$ for all λ. If $\lambda \in S$ then $E = E(\lambda)$ is a loxodromic map with attractive eigenvalue e^{λ} and fixed points $a_E = \infty$, $r_E = o$. If $\Re(\lambda) = 0$ then $E(\lambda)$ is elliptic (or the identity) and fixes the complex line spanned by o and ∞. If $\Re(\lambda) < 0$ then $-\overline{\lambda} \in S$ and $E(\lambda)$ is a loxodromic map with attractive eigenvalue $e^{-\overline{\lambda}}$ and fixed points $a_E = o$, $r_E = \infty$.

Let A be a general loxodromic map with attracting eigenvalue e^{λ} for $\lambda \in S$. Since $\mathrm{SU}(2,1)$ acts 2-transitively on $\partial\mathbf{H}^2_{\mathbb{C}}$ then there exists a $Q \in \mathrm{SU}(2,1)$ whose columns are projectively $\mathbf{a}_A$, $\mathbf{n}_A$, $\mathbf{r}_A$. Moreover, $\mathbf{a}_A = Q(\infty)$ and $\mathbf{r}_A = Q(\mathbf{o})$. Thus we may write: $A = QE(\lambda)Q^{-1}$, where $E(\lambda)$ is given by (4.2).

If A lies in $\mathrm{SO}(2,1)$ and corresponds to a loxodromic isometry of the hyperbolic plane then λ is real and so $\mathrm{tr}(A) = 2\cosh(\lambda) + 1$ is real and greater than 3. If $\Im(\lambda) = \pi$ then A corresponds to a hyperbolic glide reflection on $\mathbf{H}^2_{\mathbb{R}}$ and $\mathrm{tr}(A) = -2\cosh(\Re(\lambda)) + 1 < -1$.

4.2. **The trace function for a loxodromic matrix.** Let A be a loxodromic matrix and let e^{λ} be its attracting eigenvalue, where $\lambda \in S$. As indicated in Section 4.1 the other eigenvalues are $e^{-\overline{\lambda}}$ and $e^{\overline{\lambda}-\lambda}$ and so the trace of A is given by the following function of λ which we denote by $\tau(\lambda)$:

$$\mathrm{tr}(A) = \tau(\lambda) = e^{\lambda} + e^{\overline{\lambda}-\lambda} + e^{-\overline{\lambda}}. \tag{4.3}$$

This generalises the well known formula $\mathrm{tr}(A) = e^{\lambda} + e^{-\lambda}$ for $\mathrm{SL}(2,\mathbb{C})$. However our function $\tau(\lambda)$ is not holomorphic. It is easy to see that $\tau(\lambda)$ has the following properties:

(i) τ is a real analytic function of λ.
(ii) $\tau(-\overline{\lambda}) = \tau(\lambda)$.
(iii) $\tau(\lambda + 2\pi i) = \tau(\lambda)$.

The latter two properties prevent τ from being one-to-one in the whole of $\mathbb{C}$. We therefore restrict our attention to those λ lying in the strip S defined by (4.1). We now determine the image in $\mathbb{C}$ of S under τ. In order to do so, following Goldman §6.2.3 of [6], we define the function $f : \mathbb{C} \longrightarrow \mathbb{R}$ by

$$f(\tau) = |\tau|^4 - 8\Re(\tau^3) + 18|\tau|^2 - 27. \tag{4.4}$$

In Theorem 6.2.4 (2) of [6], Goldman proves that the matrix $A \in \mathrm{SU}(2,1)$ is loxodromic if and only if $f\big(\mathrm{tr}(A)\big) > 0$. Therefore we define the region T of $\mathbb{C}$ by

$$T = \big\{\tau \in \mathbb{C} \,:\, f(\tau) > 0\big\}. \tag{4.5}$$

This region is the exterior of a closed curve in $\mathbb{C}$ called a deltoid. We can now prove

Lemma 4.1. *The function $\tau(\lambda) = e^{\lambda} + e^{\overline{\lambda}-\lambda} + e^{-\overline{\lambda}}$ is a real analytic diffeomorphism from S onto T.*

Proof. Writing $\lambda = l + i\theta$, we calculate the Jacobian of $\tau(\lambda)$:

$$\begin{aligned} |J_\tau(\lambda)| &= \left|\frac{\partial\tau}{\partial\lambda}\right|^2 - \left|\frac{\partial\tau}{\partial\overline{\lambda}}\right|^2 \\ &= \left|e^{\lambda} - e^{\overline{\lambda}-\lambda}\right|^2 - \left|e^{\overline{\lambda}-\lambda} - e^{-\overline{\lambda}}\right|^2 \\ &= 2\sinh(2l) - 4\sinh(l)\cos(3\theta) \\ &= 4\sinh(l)\big(\cosh(l) - \cos(3\theta)\big), \end{aligned}$$

which is clearly different from 0 whenever $l \neq 0$. Hence τ is a local diffeomorphism on S.

We now show that τ is injective on S. Suppose that $\lambda = l + i\theta$ and $\lambda' = l' + i\theta'$ are two points of S with $\tau(\lambda) = \tau(\lambda')$. By equating real and imaginary parts we have

$$\begin{aligned} 2\cosh(l)\cos(\theta) + \cos(2\theta) &= 2\cosh(l')\cos(\theta') + \cos(2\theta'), \\ 2\cosh(l)\sin(\theta) - \sin(2\theta) &= 2\cosh(l')\sin(\theta') - \sin(2\theta'). \end{aligned}$$

Eliminating l' and using the addition rule for $\sin(\alpha+\beta)$ gives

$$2\cosh(l)\sin(\theta'-\theta)=\sin(3\theta')-\sin(2\theta+\theta')=2\cos(2\theta'+\theta)\sin(\theta'-\theta).$$

Since $\cosh(l)>1\geq\cos(2\theta'+\theta)$ we see that $\sin(\theta'-\theta)=0$. Hence $\theta'=\theta+k\pi$. Plugging this into the expression for τ we see that

$$2\cosh(l)e^{i\theta}+e^{-2i\theta}=(-1)^k 2\cosh(l')e^{i\theta}+e^{-2i\theta}.$$

Cancelling $e^{-2i\theta}$ from each side and comparing signs, we see that k is even and so $e^{i\theta}=e^{i\theta'}$. Hence we also have $\cosh(l)=\cosh(l')$. Since λ and λ' both lie in S we see that $\lambda=\lambda'$ as required. Hence τ is an injective, local diffeomorphism and so is a global diffeomorphism onto its image.

We now show that the image of S under τ is T. If $f(\tau)$ is Goldman's function given by (4.4), a brief calculation shows that

$$f\bigl(e^{\lambda}+e^{\overline{\lambda}-\lambda}+e^{-\overline{\lambda}}\bigr)=16\sinh^2(l)\bigl(\cosh(l)-\cos(3\theta)\bigr)^2=\bigl|J_\tau(\lambda)\bigr|^2>0$$

and so $\tau(S)\subset T$. Conversely, if $\tau\in T$ then τ is the trace of a loxodromic map by Goldman's theorem and we may take e^{λ} to be its eigenvalue of largest modulus. By construction $\lambda\in S$ and so $T\subset\tau(S)$. □

Even though it is not holomorphic, the function $\tau(\lambda)$ does enjoy a stronger property than merely being real analytic. Namely, in Proposition 4.2 we show that $\tau(\lambda)$ it is quasiconformal, and hence this is also true of its inverse $\lambda(\tau)$. This result and its proof are very short and are are only included for interest. We will not use them in the rest of the paper. Further information about quasiconformality may be found in Lehto and Virtanen [13]. For any $\epsilon>0$ define S_ϵ by

$$S_\epsilon=\bigl\{\lambda\in S\,:\,\Re(\lambda)\geq\epsilon\bigr\}. \tag{4.6}$$

Proposition 4.2. *For each $\epsilon>0$ the function $\tau(\lambda)$ is $e^{-\epsilon}$-quasiconformal on S_ϵ.*

Proof. The Beltrami differential $\mu_\tau(\lambda)$ is well defined on S and given by

$$\mu_\tau(\lambda)=\frac{\partial\tau/\partial\overline{\lambda}}{\partial\tau/\partial\lambda}=\frac{e^{\overline{\lambda}-\lambda}-e^{-\overline{\lambda}}}{e^{\lambda}-e^{\overline{\lambda}-\lambda}}=e^{-\lambda}\frac{e^{\overline{\lambda}}-e^{\lambda-\overline{\lambda}}}{e^{\lambda}-e^{\overline{\lambda}-\lambda}}.$$

Therefore $\bigl|\mu_\tau(\lambda)\bigr|=|e^{-\lambda}|<e^{-\epsilon}$ on S_ϵ. □

5. Cross-ratios and angular invariants

5.1. **The Korányi-Reimann cross-ratio.** Cross-ratios were generalised to complex hyperbolic space by Korányi and Reimann [11]. Following their notation, we suppose that z_1, z_2, z_3, z_4 are four distinct points of $\partial\mathbf{H}^2_{\mathbb{C}}$. Let $\mathbf{z}_1$, $\mathbf{z}_2$, $\mathbf{z}_3$ and $\mathbf{z}_2$ be corresponding lifts in $V_0\subset\mathbb{C}^{2,1}$. Their *complex cross-ratio* is defined to be

$$\mathbb{X}=[z_1,z_2,z_3,z_4]=\frac{\langle\mathbf{z}_3,\mathbf{z}_1\rangle\langle\mathbf{z}_4,\mathbf{z}_2\rangle}{\langle\mathbf{z}_4,\mathbf{z}_1\rangle\langle\mathbf{z}_3,\mathbf{z}_2\rangle}.$$

Since the z_i are distinct we see that $\mathbb{X}$ is finite and non-zero. We note that $\mathbb{X}$ is invariant under $\mathrm{SU}(2,1)$ and independent of the chosen lifts. More properties of the complex cross-ratio may be found in Section 7.2 of [6]. We highlight the following properties, which are Theorem 7.2.1 and Property 7 on page 226 of [6].

Proposition 5.1. *Let $\mathbb{X}=[z_1,z_2,z_3,z_4]$ be the complex cross-ratio of the distinct points z_1, z_2, z_3, $z_4\in\partial\mathbf{H}^2_{\mathbb{C}}$. Then*

(i) $\mathbb{X}<0$ *if and only if all z_i lie on a complex line and z_1,z_2 separate z_3,z_4;*

(ii) $\mathbb{X} > 0$ *if and only if* z_3, z_4 *lie in the same orbit of the stabiliser of* z_1, z_2*;*
(iii) $\mathbb{X} > 0$ *if and only if there is an antiholomorphic involution swapping* z_1, z_2 *and swapping* z_3, z_4.

We remark that Proposition 5.1 (iii) corrects a mistake in Theorem 7.2.1 of [6] (this error was pointed out to us by Pierre Will). We now give a proof.

Proof. [Proposition 5.1 (iii)] Suppose that such an antiholomorphic involution ι exists. Then, using Properties 2 and 5 on page 225 of [6] we have:

$$\begin{aligned}[z_1, z_2, z_2, z_4] &= \overline{[\iota(z_1), \iota(z_2), \iota(z_3), \iota(z_4)]} \\ &= \overline{[z_2, z_1, z_4, z_3]} \\ &= \overline{[z_1, z_2, z_3, z_4]}.\end{aligned}$$

Hence $[z_1, z_2, z_3, z_4]$ is real. (It is non-zero since the z_j are distinct.)

Suppose that $[z_1, z_2, z_3, z_4] < 0$. Then, using Proposition 5.1 (i), all the points z_i lie on a complex line L and z_1, z_2 separate z_3, z_4. Another way of saying this is that the geodesics γ_{12} and γ_{34} with endpoints z_1, z_2 and z_3, z_4 respectively intersect in a point z of L. There is a holomorphic isometry I_z in $\mathrm{SU}(2,1)$ fixing z and interchanging z_1, z_2 and z_3, z_4. Therefore $I_z\iota$ is an antiholomorphic isometry fixing z_1, z_2, z_3 and z_4. Thus these points lie on a Lagrangian plane. This is a contradiction, since four distinct boundary points cannot lie on both a complex line and a Lagrangian plane. Hence if ι exists then $[z_1, z_2, z_3, z_4]$ is real and positive.

Conversely, suppose that $[z_1, z_2, z_3, z_4]$ is real and positive. Using Proposition 5.1 (ii) we see that there exists $A \in \mathrm{SU}(2,1)$ so that $A(z_1) = z_1$, $A(z_2) = z_2$ and $A(z_3) = z_4$. Using the construction of Falbel and Zocca [4], there is a decomposition $A = \iota_1\iota_2$ as a product of two antiholomorphic involutions ι_1 and ι_2, each of which interchanges z_1 and z_2. Moreover, we are free to choose ι_2 among all involutions interchanging z_1 and z_2 and this determines ι_1. We choose ι_2 to be the involution fixing z_3, that is $\iota_2(z_1) = z_2$ and $\iota_2(z_3) = z_3$. Using $A = \iota_1\iota_2$ gives $\iota_1(z_3) = \iota_1\iota_2(z_3) = z_4$. Hence ι_1 interchanges z_3 and z_4. Since it also interchanges z_1 and z_2, it is the involution we require.

Alternatively, one can follow Goldman's proof after observing that $[z_1, z_2, z_3, z_4] = \Pi(z_3)/\Pi(z_4)$ must be positive if $\Pi(z_3)$ and $\Pi(z_4)$ lie on a hypercycle. □

5.2. **The cross-ratio variety.** By choosing different orderings of our four points we may define other cross-ratios. There are some symmetries associated to certain permutations, see Property 5 on page 225 of [6]. After taking these into account, there are only three cross-ratios that remain. Given distinct points $z_1, \ldots, z_4 \in \partial\mathbf{H}^2_{\mathbb{C}}$, we define

$$\mathbb{X}_1 = [z_1, z_2, z_3, z_4], \quad \mathbb{X}_2 = [z_1, z_3, z_2, z_4], \quad \mathbb{X}_3 = [z_2, z_3, z_1, z_4]. \tag{5.1}$$

In [2] Falbel has given a general setting for cross-ratios that includes both Korányi-Reimann cross-ratios and the standard real hyperbolic cross-ratio. We use a different normalisation to his. Our three cross-ratios satisfy two real equations, which we now derive. In Falbel's normalisation, the analogous relations are given in Proposition 2.3 of [2]. In his general setting there are six cross-ratios that lie on a complex algebraic variety in $\mathbb{C}^6$. Our cross-ratios correspond to the fixed locus of an antiholomorphic involution on this variety.

Proposition 5.2. *Let* z_1*,* z_2*,* z_3*,* z_4 *be any four distinct points in* $\partial\mathbf{H}^2_{\mathbb{C}}$*. Let* $\mathbb{X}_1$*,* $\mathbb{X}_2$ *and* $\mathbb{X}_3$ *be defined by (5.1). Then*

$$|\mathbb{X}_2| = |\mathbb{X}_1|\,|\mathbb{X}_3|, \tag{5.2}$$

$$2|\mathbb{X}_1|^2\Re(\mathbb{X}_3) = |\mathbb{X}_1|^2 + |\mathbb{X}_2|^2 + 1 - 2\Re(\mathbb{X}_1 + \mathbb{X}_2). \tag{5.3}$$

Proof. Since $\mathrm{SU}(2,1)$ acts 2-transitively on $\partial\mathbf{H}^2_{\mathbb{C}}$ we may suppose that $z_2=\infty$ and $z_3=o$. Let $\mathbf{z}_1$ and $\mathbf{z}_4$ be lifts of z_1 and z_4 chosen so that $\langle\mathbf{z}_1,\mathbf{z}_4\rangle=1$. We write them in coordinates as:

$$\mathbf{z}_1=\begin{bmatrix}\xi_1\\ \eta_1\\ \zeta_1\end{bmatrix},\quad \mathbf{z}_2=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad \mathbf{z}_3=\begin{bmatrix}0\\0\\1\end{bmatrix},\quad \mathbf{z}_4=\begin{bmatrix}\xi_4\\ \eta_4\\ \zeta_4\end{bmatrix}. \tag{5.4}$$

Then we have

$$0 = \langle\mathbf{z}_1,\mathbf{z}_1\rangle = \xi_1\overline{\zeta}_1+\zeta_1\overline{\xi}_1+|\eta_1|^2, \tag{5.5}$$

$$1 = \langle\mathbf{z}_4,\mathbf{z}_1\rangle = \xi_4\overline{\zeta}_1+\zeta_4\overline{\xi}_1+\eta_4\overline{\eta}_1, \tag{5.6}$$

$$0 = \langle\mathbf{z}_4,\mathbf{z}_4\rangle = \xi_4\overline{\zeta}_4+\zeta_4\overline{\xi}_4+|\eta_4|^2. \tag{5.7}$$

From the definitions of the cross-ratios, we have

$$\begin{aligned}
\mathbb{X}_1 &= [z_1,z_2,z_3,z_4] = \frac{\langle\mathbf{z}_3,\mathbf{z}_1\rangle\langle\mathbf{z}_4,\mathbf{z}_2\rangle}{\langle\mathbf{z}_4,\mathbf{z}_1\rangle\langle\mathbf{z}_3,\mathbf{z}_2\rangle} = \zeta_4\overline{\xi}_1,\\
\mathbb{X}_2 &= [z_1,z_3,z_2,z_4] = \frac{\langle\mathbf{z}_2,\mathbf{z}_1\rangle\langle\mathbf{z}_4,\mathbf{z}_3\rangle}{\langle\mathbf{z}_4,\mathbf{z}_1\rangle\langle\mathbf{z}_2,\mathbf{z}_3\rangle} = \xi_4\overline{\zeta}_1,\\
\mathbb{X}_3 &= [z_2,z_3,z_1,z_4] = \frac{\langle\mathbf{z}_1,\mathbf{z}_2\rangle\langle\mathbf{z}_4,\mathbf{z}_3\rangle}{\langle\mathbf{z}_4,\mathbf{z}_2\rangle\langle\mathbf{z}_1,\mathbf{z}_3\rangle} = \frac{\xi_4\zeta_1}{\zeta_4\xi_1}.
\end{aligned}$$

We immediately see that $|\mathbb{X}_3|=|\mathbb{X}_2|/|\mathbb{X}_1|$. Using equations (5.5), (5.6) and (5.7) we have:

$$\begin{aligned}
|\mathbb{X}_1|^2|\mathbb{X}_3-1|^2 &= |\zeta_4\xi_1-\xi_4\zeta_1|^2\\
&= |\zeta_4\xi_1|^2+|\xi_4\zeta_1|^2+\zeta_4\overline{\xi}_4\big(\zeta_1\overline{\xi}_1+|\eta_1|^2\big)+\xi_4\overline{\zeta}_4\big(\xi_1\overline{\zeta}_1+|\eta_1|^2\big)\\
&= |\zeta_4\overline{\xi}_1+\xi_4\overline{\zeta}_1|^2-|\eta_1\eta_4|^2\\
&= |\mathbb{X}_1+\mathbb{X}_2|^2-|1-\mathbb{X}_1-\mathbb{X}_2|^2.
\end{aligned}$$

Rearranging this gives the identity we want. □

Since $-|\mathbb{X}_3|\le\Re(\mathbb{X}_3)\le|\mathbb{X}_3|$ an immediate consequence of the identities (5.2) and (5.3) is:

Corollary 5.3. *Let $\mathbb{X}_1$ and $\mathbb{X}_2$ be defined by (5.1). Then*

$$\big(|\mathbb{X}_1|-|\mathbb{X}_2|\big)^2\le 2\Re(\mathbb{X}_1+\mathbb{X}_2)-1\le\big(|\mathbb{X}_1|+|\mathbb{X}_2|\big)^2.$$

In particular, $2\Re(\mathbb{X}_1+\mathbb{X}_2)\ge 1$.

Corollary 5.4. *Let $\mathbb{X}_1$, $\mathbb{X}_2$ and $\mathbb{X}_3$ be defined by (5.1). Then $\mathbb{X}_1+\mathbb{X}_2=1$ if and only if either $\mathbb{X}_3=-\mathbb{X}_2/\mathbb{X}_1$ or $\mathbb{X}_3=-\overline{\mathbb{X}}_2/\overline{\mathbb{X}}_1$.*

Proof. We can rearrange (5.3) as:

$$2|\mathbb{X}_1|^2\Re(\mathbb{X}_3+\mathbb{X}_2/\mathbb{X}_1)=|\mathbb{X}_1+\mathbb{X}_2-1|^2.$$

Therefore $\mathbb{X}_1+\mathbb{X}_2=1$ if and only if $\Re(\mathbb{X}_3)=\Re(-\mathbb{X}_2/\mathbb{X}_1)$. Since $|\mathbb{X}_3|=|\mathbb{X}_2|/|\mathbb{X}_1|$ this is true if and only if $\mathbb{X}_3=-\mathbb{X}_2/\mathbb{X}_1$ or $\mathbb{X}_3=-\overline{\mathbb{X}}_2/\overline{\mathbb{X}}_1$. □

We now show that any three complex numbers satisfying the identities of Proposition 5.2 are the cross-ratios of four points. Again, this follows Falbel, Proposition 2.6 of [2].

Proposition 5.5. *Let x_1, x_2 and x_3 be three complex numbers satisfying*

$$|x_2| = |x_1|\,|x_3| \quad \text{and} \quad 2|x_1|^2\Re(x_3) = |x_1|^2 + |x_2|^2 + 1 - 2\Re(x_1 + x_2).$$

Then there exist points z_1, z_2, z_3, z_4 in $\partial\mathbf{H}^2_{\mathbb{C}}$ so that

$$\mathbb{X}_1 = [z_1, z_2, z_3, z_4] = x_1, \quad \mathbb{X}_2 = [z_1, z_3, z_2, z_4] = x_2, \quad \mathbb{X}_3 = [z_2, z_3, z_1, z_4] = x_3.$$

Proof. Suppose that $z_2 = \infty$ and $z_3 = o$. Then, making a consistent choice of square roots of x_1, x_2, x_3 and $1 - x_1 - x_2$, define z_1 and z_4 by:

$$\mathbf{z}_1 = \begin{bmatrix} -\overline{x}_1^{1/2} \\ \left(2\Re(x_1^{1/2}\overline{x}_2^{1/2}e^{i\delta})\right)^{1/2}e^{-i\eta} \\ \overline{x}_2^{1/2}e^{i\delta} \end{bmatrix}, \quad \mathbf{z}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{z}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{z}_4 = \begin{bmatrix} x_2^{1/2}e^{i\delta} \\ \left(2\Re(x_1^{1/2}\overline{x}_2^{1/2}e^{-i\delta})\right)^{1/2}e^{i\eta} \\ -x_1^{1/2} \end{bmatrix}$$

where 2δ is the argument of x_3 and 2η is the argument of $1 - x_1 - x_2$ provided $1 \neq x_1 + x_2$. Arguing as in Corollary 5.4, if $x_1 + x_2 = 1$ then either $x_3 = -x_2/x_1$ or $x_3 = -\overline{x}_2/\overline{x}_1$. This implies that $\Re(x_1^{1/2}\overline{x}_2^{1/2}e^{i\delta}) = 0$ or $\Re(x_1^{1/2}\overline{x}_2^{1/2}e^{-i\delta}) = 0$ respectively. Hence when $x_1 + x_2 = 1$ the middle entry of either $\mathbf{z}_1$ or $\mathbf{z}_4$ (or both) is zero, and so we are free to choose η to be any angle.

One may easily check that $\langle \mathbf{z}_j, \mathbf{z}_j \rangle = 0$ for $j = 1, 2, 3, 4$ and also

$$\langle \mathbf{z}_3, \mathbf{z}_2 \rangle = 1, \quad \langle \mathbf{z}_3, \mathbf{z}_1 \rangle = \langle \mathbf{z}_4, \mathbf{z}_2 \rangle = -x_1^{1/2}, \quad \langle \mathbf{z}_2, \mathbf{z}_1 \rangle = x_2^{1/2}e^{-i\delta}, \quad \langle \mathbf{z}_4, \mathbf{z}_3 \rangle = x_2^{1/2}e^{i\delta}.$$

Also, since $|x_1|\,|x_2|\cos(2\delta) = |x_1|^2\Re(x_3)$, we have:

$$\begin{aligned} 2\Re(x_1^{1/2}\overline{x}_2^{1/2}e^{i\delta})2\Re(x_1^{1/2}\overline{x}_2^{1/2}e^{-i\delta}) &= x_1\overline{x}_2 + 2|x_1|\,|x_2|\cos(2\delta) + x_2\overline{x}_1 \\ &= x_1\overline{x}_2 + |x_1|^2 + |x_2|^2 + 1 - 2\Re(x_1 + x_2) + x_2\overline{x}_1 \\ &= |1 - x_1 - x_2|^2. \end{aligned}$$

Therefore

$$\begin{aligned} \langle \mathbf{z}_4, \mathbf{z}_1 \rangle &= x_2 + \left(2\Re(x_1^{1/2}\overline{x}_2^{1/2}e^{i\delta})2\Re(x_1^{1/2}\overline{x}_2^{1/2}e^{-i\delta})\right)^{1/2}e^{2i\eta} + x_1 \\ &= x_2 + |1 - x_1 - x_2|e^{2i\eta} + x_1 = 1, \end{aligned}$$

where we have used $|1 - x_1 - x_2|e^{2i\eta} = 1 - x_1 - x_2$. Thus

$$\begin{aligned} [z_1, z_2, z_3, z_4] &= \frac{\langle \mathbf{z}_3, \mathbf{z}_1 \rangle \langle \mathbf{z}_4, \mathbf{z}_2 \rangle}{\langle \mathbf{z}_4, \mathbf{z}_1 \rangle \langle \mathbf{z}_3, \mathbf{z}_2 \rangle} = x_1, \\ [z_1, z_3, z_2, z_4] &= \frac{\langle \mathbf{z}_2, \mathbf{z}_1 \rangle \langle \mathbf{z}_4, \mathbf{z}_3 \rangle}{\langle \mathbf{z}_4, \mathbf{z}_1 \rangle \langle \mathbf{z}_2, \mathbf{z}_3 \rangle} = x_2 \\ [z_2, z_3, z_1, z_4] &= \frac{\langle \mathbf{z}_1, \mathbf{z}_2 \rangle \langle \mathbf{z}_4, \mathbf{z}_3 \rangle}{\langle \mathbf{z}_4, \mathbf{z}_2 \rangle \langle \mathbf{z}_1, \mathbf{z}_3 \rangle} = \frac{|x_2|e^{2i\delta}}{|x_1|} = x_3. \end{aligned}$$

□

Therefore any triple of complex numbers (x_1, x_2, x_3) is the triple of cross-ratios $(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3)$ of an ordered quadruple of points z_1, z_2, z_3, z_4 in $\partial\mathbf{H}^2_{\mathbb{C}}$ if and only if they satisfy the two real identities from Proposition 5.5. In other words, $(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3)$ lie in a four dimensional real algebraic variety in $\mathbb{C}^3$. We call this variety the *cross-ratio variety* and we denote it by $\mathfrak{X}$. From Falbel's point of view, this variety is the moduli space of CR tetrahedra [2] and he has used it to model the figure eight knot complement [3]. From our point of view it is the moduli space of ordered pairs of oriented geodesics, that is the axes of A and B.

Notice that we may express $|\mathbb{X}_3|$ and $\Re(\mathbb{X}_3)$ as real analytic functions of $\mathbb{X}_1$ and $\mathbb{X}_2$. Therefore we may determine $\Im(\mathbb{X}_3)$ from $\mathbb{X}_1$ and $\mathbb{X}_2$ up to an ambiguity of sign. Thus there is an involution on $\mathfrak{X}$ obtained by sending $(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3)$ to $(\mathbb{X}_1, \mathbb{X}_2, \overline{\mathbb{X}}_3)$. This involution is not given by a permutation of the points (see [25] for all the maps given by permutations) and its geometric action on the collection of quadruples of four points seems to be very mysterious. Away from the fixed point set of this involution, that is the locus where $\mathbb{X}_3$ is real, the complex numbers $\mathbb{X}_1$, $\mathbb{X}_2$ give local complex coordinates on $\mathfrak{X}$.

Similarly, we may use the identities from Proposition 5.5 to write $|\mathbb{X}_2|$ and $\Re(\mathbb{X}_2)$ as real analytic functions of $\mathbb{X}_1$ and $\mathbb{X}_3$. There is again a sign ambiguity when solving for $\Im(\mathbb{X}_2)$ and so the complex numbers $\mathbb{X}_1$ and $\mathbb{X}_3$ give local coordinates away from the locus where $\mathbb{X}_2$ is real. Finally, a similar argument shows that the complex numbers $\mathbb{X}_2$ and $\mathbb{X}_3$ give local coordinates away from the locus where $\mathbb{X}_1$ is real. In Section 5.4 we show that all three of $\mathbb{X}_1$, $\mathbb{X}_2$ and $\mathbb{X}_3$ are real if and only if the four points either lie in the same complex line or on the same Lagrangian plane. Hence $\mathfrak{X}$ has local complex coordinates away from this set.

5.3. **Cartan's angular invariant.** Let z_1, z_2, z_3 be three distinct points of $\partial \mathbf{H}^2_{\mathbb{C}}$ with lifts $\mathbf{z}_1$, $\mathbf{z}_2$ and $\mathbf{z}_3$. *Cartan's angular invariant* [1] is defined as follows:

$$\mathbb{A}\,(z_1, z_2, z_3) = \arg\big(-\langle \mathbf{z}_1, \mathbf{z}_2\rangle\langle \mathbf{z}_2, \mathbf{z}_3\rangle\langle \mathbf{z}_3, \mathbf{z}_1\rangle\big).$$

The angular invariant is independent of the chosen lifts $\mathbf{z}_j$ of the points z_j. It is clear that applying an element of $\mathrm{SU}(2,1)$ to our triple of points does not change the Cartan invariant. The converse is also true; the following result is Theorem 7.1.1 of [6]:

Proposition 5.6. *Let z_1, z_2, z_3 and z_1', z_2', z_3' be triples of distinct points of $\partial \mathbf{H}^2_{\mathbb{C}}$. Then $\mathbb{A}(z_1, z_2, z_3) = \mathbb{A}(z_1', z_2', z_3')$ if and only if there exists an $A \in \mathrm{SU}(2,1)$ so that $A(z_j) = z_j'$ for $j = 1$, 2, 3. Moreover, A is unique unless the three points lie on a complex line.*

The properties of $\mathbb{A}$ may be found in Section 7.1 of [6]. We shall make use of the following, which are Corollary 7.1.3 and Theorem 7.1.4 on pages 213-4.

Proposition 5.7. *Let z_1, z_2, z_3 be three distinct points of $\partial \mathbf{H}^2_{\mathbb{C}}$ and let $\mathbb{A} = \mathbb{A}(z_1, z_2, z_3)$ be their angular invariant. Then,*

(i) $\mathbb{A} \in [-\pi/2, \pi/2]$;
(ii) $\mathbb{A} = \pm\pi/2$ *if and only if z_1, z_2 and z_3 all lie on a complex line;*
(iii) $\mathbb{A} = 0$ *if and only if z_1, z_2 and z_3 all lie on a Lagrangian plane.*

We can relate cross-ratios and angular invariants as follows:

Proposition 5.8. *Let $z_1, \ldots, z_4$ be distinct points of $\partial \mathbf{H}^2_{\mathbb{C}}$ and let $\mathbb{X}_1$, $\mathbb{X}_2$, $\mathbb{X}_3$ denote the cross-ratios defined by (5.1). Let $\mathbb{A}_1 = \mathbb{A}(z_4, z_3, z_2)$ and $\mathbb{A}_2 = (z_3, z_2, z_1)$. Then*

$$\mathbb{A}_1 + \mathbb{A}_2 = \arg(\overline{\mathbb{X}}_1 \mathbb{X}_2), \tag{5.8}$$

$$\mathbb{A}_1 - \mathbb{A}_2 = \arg(\mathbb{X}_3). \tag{5.9}$$

Proof. We have

$$\overline{\mathbb{X}}_1\mathbb{X}_2 = \frac{\langle \mathbf{z}_1, \mathbf{z}_3\rangle\langle \mathbf{z}_2, \mathbf{z}_4\rangle}{\langle \mathbf{z}_1, \mathbf{z}_4\rangle\langle \mathbf{z}_2, \mathbf{z}_3\rangle} \cdot \frac{\langle \mathbf{z}_2, \mathbf{z}_1\rangle\langle \mathbf{z}_4, \mathbf{z}_3\rangle}{\langle \mathbf{z}_4, \mathbf{z}_1\rangle\langle \mathbf{z}_2, \mathbf{z}_3\rangle} = \frac{\langle \mathbf{z}_4, \mathbf{z}_3\rangle\langle \mathbf{z}_3, \mathbf{z}_2\rangle\langle \mathbf{z}_2, \mathbf{z}_4\rangle \cdot \langle \mathbf{z}_1, \mathbf{z}_3\rangle\langle \mathbf{z}_3, \mathbf{z}_2\rangle\langle \mathbf{z}_2, \mathbf{z}_1\rangle}{\big|\langle \mathbf{z}_2, \mathbf{z}_3\rangle\big|^4\big|\langle \mathbf{z}_4, \mathbf{z}_1\rangle\big|^2}.$$

This clearly has argument $\mathbb{A}_1 + \mathbb{A}_2$. Likewise

$$\mathbb{X}_3 = \frac{\langle \mathbf{z}_1, \mathbf{z}_2\rangle\langle \mathbf{z}_4, \mathbf{z}_3\rangle}{\langle \mathbf{z}_4, \mathbf{z}_2\rangle\langle \mathbf{z}_1, \mathbf{z}_3\rangle} = \frac{\langle \mathbf{z}_4, \mathbf{z}_3\rangle\langle \mathbf{z}_3, \mathbf{z}_2\rangle\langle \mathbf{z}_2, \mathbf{z}_4\rangle\big|\langle \mathbf{z}_1, \mathbf{z}_2\rangle\big|^2}{\langle \mathbf{z}_3, \mathbf{z}_2\rangle\langle \mathbf{z}_2, \mathbf{z}_1\rangle\langle \mathbf{z}_1, \mathbf{z}_3\rangle\big|\langle \mathbf{z}_2, \mathbf{z}_4\rangle\big|^2},$$

which has argument $\mathbb{A}_1 - \mathbb{A}_2$. □

The following result, which should be compare to Corollary 5.4, follows immediately:

Corollary 5.9. *Let $\mathbb{X}_i$ be given by (5.1). Then $\mathbb{X}_3 = -\mathbb{X}_2/\mathbb{X}_1$ if and only if z_1, z_2 and z_3 lie on the same complex line. Similarly, $\mathbb{X}_3 = -\overline{\mathbb{X}}_2/\overline{\mathbb{X}}_1$ if and only if z_2, z_3 and z_4 lie on a complex line.*

Proof. First, $\mathbb{X}_3 = -\mathbb{X}_2/\mathbb{X}_1$ if and only if $\arg(\mathbb{X}_3) = \arg(\overline{\mathbb{X}}_1\mathbb{X}_2) \pm \pi$. From (5.8) and (5.9) this is true if and only if $\mathbb{A}_2 = \pm\pi/2$. The result follows from Proposition 5.7 (ii). A similar argument shows $\mathbb{X}_3 = -\overline{\mathbb{X}}_2/\overline{\mathbb{X}}_1$ if and only if $\mathbb{A}_1 = \pm\pi/2$. □

We can use Proposition 5.8 to prove the following crucial result; see also [2].

Proposition 5.10. *Let $z_1, \ldots, z_4$ be distinct points of $\partial\mathbf{H}^2_{\mathbb{C}}$ with cross ratios $\mathbb{X}_1$, $\mathbb{X}_2$, $\mathbb{X}_3$ given by (5.1). Let $z_1', \ldots, z_4'$ be another set be distinct points of $\partial\mathbf{H}^2_{\mathbb{C}}$ with corresponding cross-ratios $\mathbb{X}_1'$, $\mathbb{X}_2'$ and $\mathbb{X}_3'$. If $\mathbb{X}_i' = \mathbb{X}_i$ for $i = 1, 2, 3$ then there exists $A \in \mathrm{SU}(2,1)$ so that $A(z_j) = z_j'$ for $j = 1, 2, 3, 4$.*

Proof. As in the proof of Proposition 5.2, applying elements of $\mathrm{SU}(2,1)$ if necessary, we suppose that $z_2 = z_2' = \infty$ and $z_3 = z_3' = o$. We write lifts of the other points as

$$\mathbf{z}_1 = \begin{bmatrix}\xi_1\\ \eta_1\\ \zeta_1\end{bmatrix}, \quad \mathbf{z}_4 = \begin{bmatrix}\xi_4\\ \eta_4\\ \zeta_4\end{bmatrix}, \quad \mathbf{z}_1' = \begin{bmatrix}\xi_1'\\ \eta_1'\\ \zeta_1'\end{bmatrix}, \quad \mathbf{z}_4' = \begin{bmatrix}\xi_4'\\ \eta_4'\\ \zeta_4'\end{bmatrix}.$$

We may suppose that the lifts of these points are chosen so that both $1 = \langle \mathbf{z}_4, \mathbf{z}_1\rangle = \zeta_4\overline{\xi}_1 + \eta_4\overline{\eta}_1 + \xi_4\overline{\zeta}_1$ and $1 = \langle \mathbf{z}_4', \mathbf{z}_1'\rangle = \zeta_4'\overline{\xi}_1' + \eta_4'\overline{\eta}_1' + \xi_4'\overline{\zeta}_1'$. Then our condition on the cross-ratios is

$$\zeta_4\overline{\xi}_1 = \zeta_4'\overline{\xi}_1', \quad \xi_4\overline{\zeta}_1 = \xi_4'\overline{\zeta}_1', \quad \frac{\zeta_1\xi_4}{\zeta_4\xi_1} = \frac{\zeta_1'\xi_4'}{\zeta_4'\xi_1'}.$$

Hence we also have $\eta_4\overline{\eta}_1 = \eta_4'\overline{\eta}_1'$.

As above, denote the angular invariants of the points by $\mathbb{A}_1 = \mathbb{A}(z_4, z_3, z_2)$, $\mathbb{A}_2 = (z_3, z_2, z_1)$, $\mathbb{A}_1' = \mathbb{A}(z_4', z_3', z_2')$ and $\mathbb{A}_2' = (z_3', z_2', z_1')$. Using Proposition 5.8 we see that $\mathbb{A}_1 + \mathbb{A}_2 = \mathbb{A}_1' + \mathbb{A}_2'$ and $\mathbb{A}_1 - \mathbb{A}_2 = \mathbb{A}_1' - \mathbb{A}_2'$. Hence $\mathbb{A}_1 = \mathbb{A}_1'$ and $\mathbb{A}_2 = \mathbb{A}_2'$. From Proposition 5.6 we see that there exists $A \in \mathrm{SU}(2,1)$ sending z_3, z_2, z_1 to $z_3' = z_3$, $z_2' = z_2$, z_1' respectively.

We now show that A sends z_4 to z_4', which will prove the result. Because A fixes $z_2 = \infty$ and $z_3 = 0$ it must be diagonal and so, from (4.2), has the form $E(\alpha)$ given in (4.2) for some $\alpha \in \mathbb{C}$ with $-\pi < \Im(\alpha) \leq \pi$. Hence (multiplying $\mathbf{z}_1'$ by a unit modulus complex number if necessary) we have $\xi_1' = e^{\alpha}\xi_1$, $\eta_1' = e^{\overline{\alpha}-\alpha}\eta_1$ and $\zeta_1' = e^{-\overline{\alpha}}\zeta_1$. Therefore

$$\xi_4' = \frac{\xi_4'\overline{\zeta}_1'}{\overline{\zeta}_1'} = \frac{\xi_4\overline{\zeta}_1}{e^{-\alpha}\zeta_1} = e^{\alpha}\xi_4, \quad \eta_4' = \frac{\eta_4'\overline{\eta}_1'}{\overline{\eta}_1'} = \frac{\eta_4\overline{\eta}_1}{e^{\alpha-\overline{\alpha}}\overline{\eta}_1} = e^{\overline{\alpha}-\alpha}\eta_4, \quad \zeta_4' = \frac{\zeta_4'\overline{\xi}_1'}{\overline{\xi}_1'} = \frac{\zeta_4\overline{\xi}_1}{e^{\overline{\alpha}}\overline{\xi}_1} = e^{-\overline{\alpha}}\zeta_4.$$

Hence $A = E(\alpha)$ also sends z_4 to z_4'. □

We remark that this result is false if we only know that two of the cross-ratios the same. Suppose we have two quadruples of points $z_1, \ldots, z_4$ and $z_1', \ldots, z_4'$ with cross ratios $\mathbb{X}_i$ and $\mathbb{X}_i'$ respectively for $i = 1, 2, 3$. If we only know that $\mathbb{X}_1 = \mathbb{X}_1'$ and $\mathbb{X}_2 = \mathbb{X}_2'$ then either $\mathbb{X}_3 = \mathbb{X}_3'$ or $\mathbb{X}_3 = \overline{\mathbb{X}}_3'$. In the latter case we have $\mathbb{A}_1 = \mathbb{A}_2'$ and $\mathbb{A}_2 = \mathbb{A}_1'$, where the angular invariants $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_1'$ and $\mathbb{A}_2'$ are as defined in the proof above. When $\mathbb{X}_3$ is not real we know that $\mathbb{A}_1 \neq \mathbb{A}_2$ from (5.9). Thus $\mathbb{A}_1 \neq \mathbb{A}_1'$ and therefore there is no element of $\mathrm{SU}(2,1)$ sending z_j to z_j' for $j = 2, 3, 4$.

Composing the above result with complex conjugation gives

Corollary 5.11. *Let $z_1, \ldots, z_4$ be distinct points of $\partial\mathbf{H}^2_{\mathbb{C}}$ with cross ratios $\mathbb{X}_1$, $\mathbb{X}_2$, $\mathbb{X}_3$ given by (5.1). Let $z'_1, \ldots, z'_4$ be another set be distinct points of $\partial\mathbf{H}^2_{\mathbb{C}}$ with corresponding cross-ratios $\mathbb{X}'_1$, $\mathbb{X}'_2$ and $\mathbb{X}'_3$. If $\mathbb{X}'_i = \overline{\mathbb{X}}_i$ for $i = 1, 2, 3$ then there exists an anti-holomorphic complex hyperbolic isometry ι so that $\iota(z_j) = z'_j$ for $j = 1, 2, 3, 4$.*

5.4. **When all the cross-ratios are real.** In this section we consider the special case where all three cross-ratios are real. Putting this into equation (5.2) implies that $\mathbb{X}_3 = \pm\mathbb{X}_2/\mathbb{X}_1$. We show that these two cases correspond to our four points either lying in a complex line or a Lagrangian plane; compare [25]. Moreover, there are six components to the locus where all three cross-ratios are real: three each for the cases where the points lie on complex line or a Lagrangian plane. The three cases are determined by the relative separation properties of the points.

Proposition 5.12. *Suppose that $\mathbb{X}_1$, $\mathbb{X}_2$ and $\mathbb{X}_3$ are all real.*

(i) *If $\mathbb{X}_3 = -\mathbb{X}_2/\mathbb{X}_1$ then the points z_j all lie on a complex line.*
(ii) *If $\mathbb{X}_3 = \mathbb{X}_2/\mathbb{X}_1$ then the points z_j all lie on a Lagrangian plane.*

Proof. If $\mathbb{X}_3 = -\mathbb{X}_2/\mathbb{X}_1$ then at least one of them is negative and the result follows from Proposition 5.1 (i).

If $\mathbb{X}_3 = \mathbb{X}_2/\mathbb{X}_1$ then either all three of them are positive or two of them are negative. In the latter case the separation conditions of Proposition 5.1 (i) lead to a contradiction. Thus they are all positive. From Proposition 5.1 (iii) there are antiholomorphic involutions ι_1, ι_2 and ι_3 so that

$$\iota_1(z_1) = z_2, \quad \iota_1(z_3) = z_4; \quad \iota_2(z_1) = z_3, \quad \iota_2(z_2) = z_4; \quad \iota_3(z_2) = z_3, \quad \iota_3(z_1) = z_4.$$

One immediately checks that $\iota_3\iota_2\iota_1$ fixes each of z_1, z_2, z_3, z_4. Therefore the four points are all fixed by the same antiholomorphic isometry, and so must be in the same Lagrangian plane; see Lemma 7.1.6 (i) of [6]. □

We now prove the converse to Proposition 5.12. We begin with the case where the points lie on a complex line.

Proposition 5.13. *Suppose that z_1, z_2, z_3 and z_4 all lie on the same complex line. Then $\mathbb{X}_1$, $\mathbb{X}_2$ and $\mathbb{X}_3$ are each real and satisfy $\mathbb{X}_3 = -\mathbb{X}_2/\mathbb{X}_1$.*

Proof. From Corollary 5.9 we see that both $\mathbb{X}_3 = -\mathbb{X}_2/\mathbb{X}_1$ and $\mathbb{X}_3 = -\overline{\mathbb{X}}_2/\overline{\mathbb{X}}_1$. Thus $\mathbb{X}_3$ is real. Using Corollary 5.4 we also have $\mathbb{X}_1 + \mathbb{X}_2 = 1$. Since the ratio and sum of $\mathbb{X}_1$ and $\mathbb{X}_2$ are both real then they must also be real. This proves the result. □

Proposition 5.14. *Suppose that all of the fixed points of A and B are contained in same Lagrangian plane. Then $\mathbb{X}_1$, $\mathbb{X}_2$, $\mathbb{X}_3$ are each real, positive and satisfy $\mathbb{X}_3 = \mathbb{X}_2/\mathbb{X}_1$.*

Proof. Let ι be the antiholomorphic involution fixing the Lagrangian plane. Then applying ι to the points z_j we see that $\mathbb{X}_i = \overline{\mathbb{X}}_i$, for $i = 1, 2, 3$. Hence all the cross-ratios are real. Using Proposition 5.7 (iii) we have $\mathbb{A}_1 = \mathbb{A}_2 = 0$. Thus from Proposition 5.8, we have $\arg(\mathbb{X}_3) = \arg(\overline{\mathbb{X}}_1\mathbb{X}_2) = 0$. Hence $\mathbb{X}_3$ and $\mathbb{X}_2/\mathbb{X}_1$ are both real and positive and hence are equal. Finally, putting this into (5.3) and rearranging gives

$$2\mathbb{X}_1 + 2\mathbb{X}_2 = 1 + (\mathbb{X}_1 - \mathbb{X}_2)^2 > 0.$$

Since $\mathbb{X}_2/\mathbb{X}_1 > 0$ this implies $\mathbb{X}_1$ and $\mathbb{X}_2$ are both positive. □

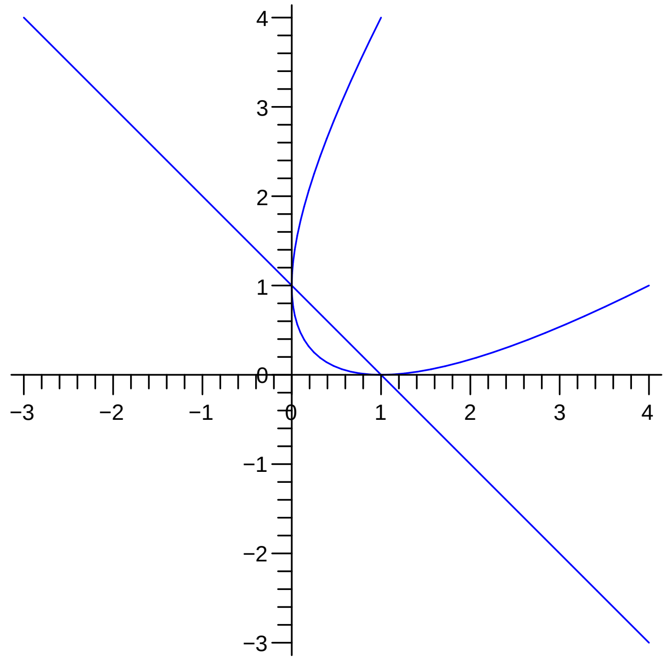

FIGURE 5.1. *The line* $\mathbb{X}_1 + \mathbb{X}_2 = 1$ *where the points lie on a complex line and the parabola* $\mathbb{X}_1^2 + \mathbb{X}_2^2 + 1 - 2\mathbb{X}_1 - 2\mathbb{X}_2 - 2\mathbb{X}_1\mathbb{X}_2 = 0$ *where they lie on a Lagrangian plane.*

If the points z_j lie on a complex line then $\mathbb{X}_3 = -\mathbb{X}_2/\mathbb{X}_1$ and so either one or all three of the $\mathbb{X}_i$ must be negative. Moreover, using Corollary 5.4, we have $\mathbb{X}_1 + \mathbb{X}_2 = 1$ and so that all three of them cannot be negative. Thus two of the $\mathbb{X}_i$ are positive and the third is negative. Furthermore, by using Proposition 5.1 (i), the one that is negative is determined by the separation properties of the points z_j. This gives three components to the cross-ratio variety associated to quadruples of points on a complex line. In Figure 5.1 we draw this locus in the $(\mathbb{X}_1, \mathbb{X}_2)$ plane. The three components are obtained from the line $\mathbb{X}_1 + \mathbb{X}_2 = 1$ by removing the points $(1, 0)$ and $(0, 1)$.

Similarly, if the points lie on a Lagrangian plane then the $\mathbb{X}_i$ are each real, positive and satisfy $\mathbb{X}_3 = \mathbb{X}_2/\mathbb{X}_1$. In this case, we can rearrange (5.3) to give

$$\begin{aligned} 0 &= \mathbb{X}_1^2 + \mathbb{X}_2^2 + 1 - 2\mathbb{X}_1 - 2\mathbb{X}_2 - 2\mathbb{X}_1\mathbb{X}_2 \\ &= \left(\mathbb{X}_1^{1/2} + \mathbb{X}_2^{1/2} + 1\right)\left(\mathbb{X}_1^{1/2} + \mathbb{X}_2^{1/2} - 1\right)\left(\mathbb{X}_1^{1/2} - \mathbb{X}_2^{1/2} + 1\right)\left(\mathbb{X}_1^{1/2} - \mathbb{X}_2^{1/2} - 1\right) \end{aligned}$$

for some consistent choice of square roots of the positive numbers $\mathbb{X}_1$ and $\mathbb{X}_2$. By making a suitable normalisation, it is not hard to show which of these brackets is zero from the separation properties of the points, and so we deduce that there are again three components:

Corollary 5.15. *Suppose that the four points* z_j *lie on a Lagrangian plane. Then the positive square roots of* $\mathbb{X}_1$ *and* $\mathbb{X}_2$ *satisfy:*

(i) $\mathbb{X}_1^{1/2} + \mathbb{X}_2^{1/2} = 1$ *if* z_1 *and* z_4 *separate* z_2 *and* z_3*;*
(ii) $\mathbb{X}_1^{1/2} - \mathbb{X}_2^{1/2} = 1$ *if* z_1 *and* z_3 *separate* z_2 *and* z_4*;*
(iii) $-\mathbb{X}_1^{1/2} + \mathbb{X}_2^{1/2} = 1$ *if* z_1 *and* z_2 *separate* z_3 *and* z_4*.*

In Figure 5.1 we also draw this locus in the $(\mathbb{X}_1, \mathbb{X}_2)$ plane. The three components are obtained by removing the points $(1, 0)$ and $(0, 1)$ from the parabola $\mathbb{X}_1^2 + \mathbb{X}_2^2 + 1 - 2\mathbb{X}_1 - 2\mathbb{X}_2 - 2\mathbb{X}_1\mathbb{X}_2 = 0$. (Compare this with Figure 2 of [10] where the same locus is plotted in the $(\mathbb{X}_1^{1/2}, \mathbb{X}_2^{1/2})$ plane.)

6. Cross-ratios and pairs of loxodromic maps

6.1. **Associating cross-ratios to pairs of loxodromic transformations.** Let A and B be loxodromic transformations with attracting fixed points a_A, a_B and repelling fixed points r_A, r_B respectively. Suppose that these fixed points correspond to attractive eigenvectors $\mathbf{a}_A$, $\mathbf{a}_B$ and repulsive eigenvectors $\mathbf{r}_A$, $\mathbf{r}_B$ respectively. For the rest of this paper we only consider the case where neither r_B nor a_B equals either r_A or a_A, that is the real axes of A and B are distinct and do not share an end-point. Cross-ratios associated to pairs of loxodromic maps were used in [10] to generalise Jørgensen's inequality to complex hyperbolic space. Some of the properties of the cross-ratios we use in this section are generalisations of properties used there.

Following (5.1), we define the first, second and third cross-ratios of the loxodromic maps A and B to be

$$\mathbb{X}_1(A,B) = [a_B, a_A, r_A, r_B] = \frac{\langle \mathbf{r}_A, \mathbf{a}_B\rangle\langle \mathbf{r}_B, \mathbf{a}_A\rangle}{\langle \mathbf{r}_B, \mathbf{a}_B\rangle\langle \mathbf{r}_A, \mathbf{a}_A\rangle}, \tag{6.1}$$

$$\mathbb{X}_2(A,B) = [a_B, r_A, a_A, r_B] = \frac{\langle \mathbf{a}_A, \mathbf{a}_B\rangle\langle \mathbf{r}_B, \mathbf{r}_A\rangle}{\langle \mathbf{r}_B, \mathbf{a}_B\rangle\langle \mathbf{a}_A, \mathbf{r}_A\rangle}, \tag{6.2}$$

$$\mathbb{X}_3(A,B) = [a_A, r_A, a_B, r_B] = \frac{\langle \mathbf{a}_B, \mathbf{a}_A\rangle\langle \mathbf{r}_B, \mathbf{r}_A\rangle}{\langle \mathbf{r}_B, \mathbf{a}_A\rangle\langle \mathbf{a}_B, \mathbf{r}_A\rangle}. \tag{6.3}$$

Since the fixed points were assumed to be distinct, none of these cross-ratios is either zero or infinity. These three numbers satisfy the identities of of Proposition 5.2. Therefore they define a point on the cross-ratio variety $\mathfrak{X}$ associated to these four points. We call this the *cross-ratio variety* of the pair of loxodromic maps A and B and we call it $\mathfrak{X}(A,B)$. Using Property 5 on page 225 of [6] we immediately obtain.

Proposition 6.1. *The following hold:*

$$\begin{array}{lll}
\mathbb{X}_1(B,A) = \mathbb{X}_1(A,B), & \mathbb{X}_2(B,A) = \overline{\mathbb{X}_2(A,B)}, & \mathbb{X}_3(B,A) = \overline{\mathbb{X}_3(A,B)};\\
\mathbb{X}_1(A^{-1},B) = \mathbb{X}_2(A,B), & \mathbb{X}_2(A^{-1},B) = \mathbb{X}_1(A,B), & \mathbb{X}_3(A^{-1},B) = 1/\mathbb{X}_3(A,B);\\
\mathbb{X}_1(A,B^{-1}) = \overline{\mathbb{X}_2(A,B)}, & \mathbb{X}_2(A,B^{-1}) = \overline{\mathbb{X}_1(A,B)}, & \mathbb{X}_3(A,B^{-1}) = 1/\mathbb{X}_3(A,B);\\
\mathbb{X}_1(A^{-1},B^{-1}) = \overline{\mathbb{X}_1(A,B)}, & \mathbb{X}_2(A^{-1},B^{-1}) = \overline{\mathbb{X}_2(A,B)}, & \mathbb{X}_3(A^{-1},B^{-1}) = \mathbb{X}_3(A,B).
\end{array}$$

Therefore either swapping A and B or else replacing either or both of A and B with their inverse defines an automorphism of $\mathfrak{X}(A,B)$.

6.2. **Traces and cross-ratios.** In this section we investigate the relationship between the cross-ratios $\mathbb{X}_i(A,B)$ and traces of elements of the group $\langle A,B\rangle$. We shall use this when discussing change of coordinates on a three-holed sphere in Section 7.2 and also trace coordinates in Section 7.3. In what follows we make use of the following normalisation; see [10]. Our main results are independent of this normalisation, but it will be useful for calculations. We normalise so that A fixes o and ∞, that is it as the form (4.2):

$$A = E(\lambda) = \begin{bmatrix} e^{\lambda} & 0 & 0 \\ 0 & e^{\overline{\lambda}-\lambda} & 0 \\ 0 & 0 & e^{-\overline{\lambda}} \end{bmatrix} \tag{6.4}$$

where $\lambda \in S$. As in Section 4.1 we can write

$$B = QE(\mu)Q^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \begin{bmatrix} e^{\mu} & 0 & 0 \\ 0 & e^{\overline{\mu}-\mu} & 0 \\ 0 & 0 & e^{-\overline{\mu}} \end{bmatrix} \begin{bmatrix} \overline{j} & \overline{f} & \overline{c} \\ \overline{h} & \overline{e} & \overline{b} \\ \overline{g} & \overline{d} & \overline{a} \end{bmatrix} \tag{6.5}$$

where $\mu \in S$ and $Q \in \mathrm{SU}(2,1)$.

Lemma 6.2. *If A and B are as given in (6.4) and (6.5) then $\mathbb{X}_1(A,B)=j\overline{a}$, $\mathbb{X}_2(A,B)=c\overline{g}$ and $\mathbb{X}_3(A,B)=cg/aj$.*

Proof. We have

$$\mathbf{a}_A=\boldsymbol{\infty}=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad \mathbf{r}_A=\mathbf{o}=\begin{bmatrix}0\\0\\1\end{bmatrix},\quad \mathbf{a}_B=Q(\boldsymbol{\infty})=\begin{bmatrix}a\\d\\g\end{bmatrix},\quad \mathbf{r}_B=Q(\mathbf{o})=\begin{bmatrix}c\\f\\j\end{bmatrix}.$$

Therefore

$$\begin{aligned}
\mathbb{X}_1(A,B) &= \big[Q(\infty),\infty,o,Q(o)\big] = \frac{\langle\mathbf{o},Q(\boldsymbol{\infty})\rangle\langle Q(\mathbf{o}),\boldsymbol{\infty}\rangle}{\langle Q(\mathbf{o}),Q(\boldsymbol{\infty})\rangle\langle\mathbf{o},\boldsymbol{\infty}\rangle} = j\overline{a},\\
\mathbb{X}_2(A,B) &= \big[Q(\infty),o,\infty,Q(o)\big] = \frac{\langle\boldsymbol{\infty},Q(\boldsymbol{\infty})\rangle\langle Q(\mathbf{o}),\mathbf{o}\rangle}{\langle Q(\mathbf{o}),Q(\boldsymbol{\infty})\rangle\langle\boldsymbol{\infty},\mathbf{o}\rangle} = c\overline{g},\\
\mathbb{X}_3(A,B) &= \big[\infty,o,Q(\infty),Q(o)\big] = \frac{\langle Q(\boldsymbol{\infty}),\boldsymbol{\infty}\rangle\langle Q(\mathbf{o}),\mathbf{o}\rangle}{\langle Q(\mathbf{o}),\boldsymbol{\infty}\rangle\langle Q(\boldsymbol{\infty}),\mathbf{o}\rangle} = \frac{cg}{aj}.
\end{aligned}$$

□

We define $\sigma(\mu)=e^{\mu}-e^{\overline{\mu}-\mu}$. Note that $\sigma(-\mu)=-e^{-\overline{\mu}}\sigma(\mu)$ and $\sigma(\overline{\mu})=\overline{\sigma(\mu)}$.

Lemma 6.3. *If B is by given by (6.5) then, writing $\sigma(\mu)=e^{\mu}-e^{\overline{\mu}-\mu}$ we have*

$$B=\begin{bmatrix} e^{\overline{\mu}-\mu}+a\overline{j}\sigma(\mu)+c\overline{g}\sigma(-\overline{\mu}) & a\overline{f}\sigma(\mu)+c\overline{d}\sigma(-\overline{\mu}) & a\overline{c}\sigma(\mu)+c\overline{a}\sigma(-\overline{\mu})\\ d\overline{j}\sigma(\mu)+f\overline{g}\sigma(-\overline{\mu}) & e^{\overline{\mu}-\mu}+d\overline{f}\sigma(\mu)+f\overline{d}\sigma(-\overline{\mu}) & d\overline{c}\sigma(\mu)+f\overline{a}\sigma(-\overline{\mu})\\ g\overline{j}\sigma(\mu)+j\overline{g}\sigma(-\overline{\mu}) & g\overline{f}\sigma(\mu)+j\overline{d}\sigma(-\overline{\mu}) & e^{\overline{\mu}-\mu}+g\overline{c}\sigma(\mu)+j\overline{a}\sigma(-\overline{\mu})\end{bmatrix}.$$

Proof. This is proved by performing the matrix multiplication and then substituting identities that come from $QQ^{-1}=I$. For example, the top left hand entry is

$$a\overline{j}e^{\mu}+b\overline{h}e^{\overline{\mu}-\mu}+c\overline{g}e^{-\overline{\mu}}=a\overline{j}e^{\mu}+(1-a\overline{j}-c\overline{g})e^{\overline{\mu}-\mu}+c\overline{g}e^{-\overline{\mu}}=e^{\overline{\mu}-\mu}+a\overline{j}\sigma(\mu)+c\overline{g}\sigma(-\overline{\mu}),$$

where we have used the identity $1=a\overline{j}+b\overline{h}+c\overline{g}$ which comes from the top left hand entry of $QQ^{-1}=I$. □

Proposition 6.4. *If A and B are given by (6.4) and (6.5) then the traces of their product and their commutator are given by*

$$\begin{aligned}
\mathrm{tr}(AB) &= (e^{\lambda}+e^{-\overline{\lambda}})e^{\overline{\mu}-\mu}+e^{\overline{\lambda}-\lambda}(e^{\mu}+e^{-\overline{\mu}})-e^{\overline{\lambda}-\lambda}e^{\overline{\mu}-\mu}\\
&\quad +\mathbb{X}_1\,\sigma(-\overline{\lambda})\sigma(-\overline{\mu})+\overline{\mathbb{X}}_1\,\sigma(\lambda)\sigma(\mu)+\mathbb{X}_2\,\sigma(\lambda)\sigma(-\overline{\mu})+\overline{\mathbb{X}}_2\,\sigma(-\overline{\lambda})\sigma(\mu)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{tr}[A,B] &= 3-2\Re\Big(\mathbb{X}_1\,\sigma(\overline{\lambda})\sigma(-\overline{\lambda})\sigma(\overline{\mu})\sigma(-\overline{\mu})+\mathbb{X}_2\,\sigma(\lambda)\sigma(-\lambda)\sigma(\overline{\mu})\sigma(-\overline{\mu})\Big)\\
&\quad +\Big(1-2\Re(\mathbb{X}_1+\mathbb{X}_2)\Big)\Big(\big|\sigma(\lambda)\sigma(\mu)\big|^2+\big|\sigma(-\lambda)\sigma(-\mu)\big|^2\Big)\\
&\quad +\Big|\mathbb{X}_1\,\sigma(\overline{\lambda})\sigma(\overline{\mu})+\overline{\mathbb{X}}_1\,\sigma(-\lambda)\sigma(-\mu)+\mathbb{X}_2\,\sigma(-\lambda)\sigma(\overline{\mu})+\overline{\mathbb{X}}_2\,\sigma(\overline{\lambda})\sigma(-\mu)\Big|^2\\
&\quad +\Big(|\mathbb{X}_2|^2-|\mathbb{X}_1|^2\mathbb{X}_3\Big)\Big(\big|\sigma(\lambda)\big|^2-\big|\sigma(-\lambda)\big|^2\Big)\Big(\big|\sigma(\mu)\big|^2-\big|\sigma(-\mu)\big|^2\Big),
\end{aligned}$$

where $\sigma(\lambda)=e^{\lambda}-e^{\overline{\lambda}-\lambda}$.

Proof. We may conjugate so that

$$A = E(\lambda), \quad B = QE(\mu)Q^{-1}.$$

Then substituting $j\overline{a} = \mathbb{X}_1$, $c\overline{g} = \mathbb{X}_2$ and $f\overline{d} = 1 - j\overline{a} - c\overline{g} = 1 - \mathbb{X}_1 - \mathbb{X}_2$ into Lemma 6.3, a short calculation yields

$$\begin{aligned} \mathrm{tr}(AB) &= e^{\lambda}\Big(e^{\overline{\mu}-\mu} + \overline{\mathbb{X}}_1\,\sigma(\mu) + \mathbb{X}_2\,\sigma(-\overline{\mu})\Big) \\ &+ e^{\overline{\lambda}-\lambda}\Big(e^{\overline{\mu}-\mu} + (1-\overline{\mathbb{X}}_1-\overline{\mathbb{X}}_2)\,\sigma(\mu) + (1-\mathbb{X}_1-\mathbb{X}_2)\,\sigma(-\overline{\mu})\Big) \\ &+ e^{-\overline{\lambda}}\Big(e^{\overline{\mu}-\mu} + \overline{\mathbb{X}}_2\,\sigma(\mu) + \mathbb{X}_1\,\sigma(-\overline{\mu})\Big). \end{aligned}$$

Rearranging this expression and using $\sigma(\lambda) = e^{\lambda} - e^{\overline{\lambda}-\lambda}$ gives the first part of the result. A similar but lengthier calculation, which also uses (5.2) and (5.3), gives the second. □

Using $\sigma(\lambda) = -e^{\overline{\lambda}}\sigma(-\lambda)$ and $\lambda,\,\mu \in S$, we have:

Corollary 6.5. *Let A and B be loxodromic maps with $\mathrm{tr}(A) = \tau(\lambda)$ and $\mathrm{tr}(B) = \tau(\mu)$. Let $\mathbb{X}_i = \mathbb{X}_i(A,B)$ be their cross-ratios. Then*

$$\mathbb{X}_3 = \frac{F(\lambda,\mu,\mathbb{X}_1,\mathbb{X}_2) - \mathrm{tr}[A,B]}{|\mathbb{X}_1|^2\big(|e^{\lambda}|^2-1\big)\big|\sigma(-\lambda)\big|^2\big(|e^{\mu}|^2-1\big)\big|\sigma(-\mu)\big|^2},$$

where $F(\lambda,\mu,\mathbb{X}_1,\mathbb{X}_2)$ is a real valued, real analytic function of λ, μ, $\mathbb{X}_1$ and $\mathbb{X}_2$.

6.3. **Representations that preserve a complex line.** We now consider representations that preserve a complex line. We show that certain traces are also forced to be real, and so the associated complex length parameter λ_j will be a real length parameter $l_j \in \mathbb{R}_+$.

Lemma 6.6. *Let A and B be elements of $\mathrm{SU}(2,1)$ that both preserve the same complex line with A loxodromic. Then $[A,B] = ABA^{-1}B^{-1}$ has real trace.*

Proof. The imaginary part of the trace arises from the representation into the group U(1) of rotations around the complex line. This representation is necessarily abelian and so the commutator is represented by the identity.

Alternatively, we now show the result directly. We may suppose that A and $BA^{-1}B^{-1}$ have the forms (6.4) and (6.5) with $\mu = -\lambda$. Since they preserve a complex line we know that their cross-ratios are real and sum to 1. Putting this information into Proposition 7.3 (with $BA^{-1}B^{-1}$ in place of B) and simplifying, gives $\mathrm{tr}(ABA^{-1}B^{-1}) = 3 + 2\big(\cosh(\lambda+\overline{\lambda}) - 1\big)\mathbb{X}_2$. □

Lemma 6.7. *Let A and B be loxodromic elements of $\mathrm{SU}(2,1)$ preserving a complex line and both having real trace. Then $\mathrm{tr}(AB)$ is real.*

Proof. This again uses Proposition 7.3 but is even easier as λ, μ, $\mathbb{X}_1$ and $\mathbb{X}_2$ are all real. □

These seemingly innocent lemmas have a far reaching consequence for the traces associated curves in our curve system.

Proposition 6.8. *Let γ_j be a simple curve system on Σ. Suppose that $\rho : \pi_1(\Sigma) \longrightarrow \mathrm{SU}(2,1)$ preserves a complex line. Let $\rho(\gamma_j) = A_j$ for $j = 1, \ldots, 3g-3$. Suppose that γ_j is in the boundary of distinct three-holed spheres (that is γ_j is not associated to a handle). Then $\mathrm{tr}(A_j)$ is real.*

Proof. Consider the g three-holed spheres that are used to close a handle. Each of these corresponds to a $(1,1)$ group and the boundary component is a commutator $[A,B]$. Thus it is represented by a loxodromic map with real trace, Lemma 6.6. If $g=2$ we are done. Suppose $g\geq 3$. Consider the remaining $g-2$ three-holed spheres that are not used to form a handle. These are glued together to form a g-holed sphere, and each of the g group elements representing a hole is a commutator. Of these $g-2$ three-holed spheres, there is at least one with two boundary loops represented by commutators, and hence which have real trace. The third peripheral element of this three-holed sphere is the product of the inverses of the other two peripheral elements. Hence, using Lemma 6.7, it too has real trace. Now consider the remaining $g-3$ three-holed spheres. These are attached together to form a $(g-1)$-holed sphere, where all $g-1$ holes are represented by group elements with real trace. Thus we may repeat the above argument and, by induction, we see that in each of the $g-2$ three-holed spheres that is not used to form a handle, each peripheral element has real trace. This proves the result. □

7. Fenchel-Nielsen coordinates of three-holed spheres

7.1. **Parameters associated to a three-holed sphere.** The first step in defining Fenchel-Nielsen coordinates is to parametrise complex hyperbolic groups representing three-holed spheres. In the classical case, one parametrises each three-holed sphere by the three lengths l_j of the geodesic boundary curves γ_j. From the group theory perspective, a three-holed sphere corresponds to a subgroup generated by two loxodromic transformations A and B in $\mathrm{SL}(2,\mathbb{R})$ whose product is also loxodromic. (One needs to restrict to the case where the axes of A, B and AB are disjoint and do not separate each other.) Such a group is called a $(0,3)$ group, that is it corresponds to a surface of genus 0 with 3 boundary components. The three boundary curves γ_1, γ_2 and γ_3 are represented by A, B and $B^{-1}A^{-1}$ which are called the *peripheral elements* of $\langle A,B\rangle$. Actually, the boundary components correspond to the conjugacy classes of A, B and $B^{-1}A^{-1}$. Going once around each boundary curve in turn gives a trivial loop and this corresponds to the relation $AB(B^{-1}A^{-1})=I$ and explains why we have used $B^{-1}A^{-1}$ for the third boundary curve. The three length parameters l_1, l_2 and l_3 may be read off from the traces of A, B and $B^{-1}A^{-1}$. Plane hyperbolic geometry tells us that these three numbers are independent and completely determine the three-holed sphere, or equivalently A and B, up to conjugation.

We want to mimic this construction in the complex hyperbolic setting. Once again a $(0,3)$ group is a group generated by two loxodromic elements A and B whose product AB is also loxodromic. The three boundary curves are again represented by (the conjugacy classes of) A, B and $B^{-1}A^{-1}$. We choose to focus on the representation theory viewpoint and so we want to parametrise conjugacy classes of groups generated by two loxodromic transformations A and B. Unfortunately, three complex numbers are not enough to parametrise such groups and so the obvious analogy with the classical case breaks down. In fact one needs to use eight real parameters.

The parameters we use to describe $(0,3)$ groups $\langle A,\,B\rangle$, and so also to parametrise the associated three-holed spheres, are the traces $\mathrm{tr}(A)$, $\mathrm{tr}(B)$, which each lie in the domain $T\subset\mathbb{C}$ given by (4.5), together with a point on the cross-ratio variety $\mathfrak{X}(A,B)$. We call these parameters the *Fenchel-Nielsen coordinates* of the $(0,3)$ group $\langle A,\,B\rangle$. Away from the locus where $\mathbb{X}_3(A,B)$ is real, the cross-ratios $\mathbb{X}_1(A,B)$ and $\mathbb{X}_2(A,B)$ give local complex coordinates on $\mathfrak{X}(A,B)$, but, as remarked above, these are not global coordinates. The goal of this section is to show that these parameters determine the $(0,3)$ group up to conjugation. The collection of pairs of loxodromic isometries with distinct fixed points have also been parametrised by Will [24] using their traces and a particular normalisation of their fixed points. One may write his fixed points in terms of our cross-ratios and *vice versa*.

The following Theorem establishes that complex hyperbolic Fenchel-Nielsen coordinates for $(0,3)$ groups are unique up to conjugation.

Theorem 7.1. *The $(0,3)$ group $\langle A,\, B\rangle$ is determined up to conjugation in $\mathrm{SU}(2,1)$ by its Fenchel-Nielsen coordinates: $\mathrm{tr}(A)$, $\mathrm{tr}(B)$ and a point on $\mathfrak{X}(A,B)$.*

Proof. Suppose that A, B, A' and B' are loxodromic transformations for which $\mathrm{tr}(A) = \mathrm{tr}(A')$, $\mathrm{tr}(B) = \mathrm{tr}(B')$ and $\mathbb{X}_i(A,B) = \mathbb{X}_i(A',B')$ for $i = 1, 2, 3$. Since the cross-ratios are equal, Proposition 5.10 implies that there exists a $C \in \mathrm{SU}(2,1)$ so that $a_{A'} = C(a_A)$, $r_{A'} = C(r_A)$, $a_{B'} = C(a_B)$ and $r_{B'} = C(r_B)$. Therefore A' and CAC^{-1} have the same fixed points. Since they also have the same trace, we must have $A' = CAC^{-1}$. Similarly, B' and CBC^{-1} are equal as they have the same fixed points and the same trace. Thus $\langle A',B'\rangle = \langle CAC^{-1},CBC^{-1}\rangle = C\langle A,B\rangle C^{-1}$ as claimed. □

7.2. **Change of coordinates on the same three-holed sphere.** There is a natural three-fold symmetry associated to a three-holed sphere which is respected by the classical Fenchel-Nielsen coordinates, both for Teichmüller space and for quasi-Fuchsian space. It is glaringly obvious that our parameters fail to respect this symmetry. Namely, they use the group elements corresponding to two of the boundary components and completely ignore the third. This is not an ideal situation. In this section we partially rectify this problem by showing that passing from the Fenchel-Nielsen coordinates determined by one pair of boundary components to those coordinates determined by another pair is a real analytic change of variables. Let $\langle A,\, B\rangle$ be a $(0,3)$ group with peripheral curves represented by A, B and $B^{-1}A^{-1}$. Our goal will be to show that the Fenchel-Nielsen coordinates associated to the pair A, $B^{-1}A^{-1}$ may be expressed as a real analytic function of the Fenchel-Nielsen coordinates associated to A, B. Using this result and Proposition 6.1, we could do the same for the peripheral curves B, $B^{-1}A^{-1}$. We leave the details for the reader.

Another way of symmetrising our Fenchel-Nielsen coordinates for a three-holed sphere would be to take the three traces $\mathrm{tr}(A)$, $\mathrm{tr}(B)$ and $\mathrm{tr}(B^{-1}A^{-1})$ together with a point on each of the cross-ratio varieties $\mathfrak{X}(A,B)$, $\mathfrak{X}(A,B^{-1}A^{-1})$ and $\mathfrak{X}(B,B^{-1}A^{-1})$ subject to suitable (real analytic) relations. These relations could be deduced from the results below and we will not pursue this idea.

Theorem 7.2. *Let A, B and $B^{-1}A^{-1}$ be loxodromic elements of $\mathrm{SU}(2,1)$. Then $\mathrm{tr}(B^{-1}A^{-1})$, $\mathbb{X}_1(A,B^{-1}A^{-1})$, $\mathbb{X}_2(A,B^{-1}A^{-1})$ and $\mathbb{X}_3(A,B^{-1}A^{-1})$ may be expressed as real analytic functions of $\mathrm{tr}(A)$, $\mathrm{tr}(B)$, $\mathbb{X}_1(A,B)$, $\mathbb{X}_2(A,B)$ and $\mathbb{X}_3(A,B)$.*

Let e^{λ} and e^{μ}, for λ, $\mu \in S$, be the attractive eigenvalues of the loxodromic maps A and B, respectively. Then, using Lemma 4.1, we can write λ and μ as a real analytic functions of $\mathrm{tr}(A)$ and $\mathrm{tr}(B)$ respectively. Thus, to prove Theorem 7.2 it suffices to show that $\mathrm{tr}(B^{-1}A^{-1})$, $\mathbb{X}_1(A,B^{-1}A^{-1})$, $\mathbb{X}_2(A,B^{-1}A^{-1})$ and $\mathbb{X}_3(A,B^{-1}A^{-1})$ may be expressed as real analytic functions of λ, μ, $\mathbb{X}_1(A,B)$, $\mathbb{X}_2(A,B)$ and $\mathbb{X}_3(A,B)$.

The following result is an immediate consequence of Proposition 6.4. Note that $\mathrm{tr}(B^{-1}A^{-1})$ may be obtained from from $\mathrm{tr}(AB)$ by replacing λ and μ with $-\lambda$ and $-\mu$ respectively.

Proposition 7.3. *Let A and B be loxodromic maps in $\mathrm{SU}(2,1)$ with attracting eigenvalues e^{λ}, e^{μ} where λ, $\mu \in S$ and let also $\mathbb{X}_1 = \mathbb{X}_1(A,B)$ and $\mathbb{X}_2 = \mathbb{X}_2(A,B)$ be their first two cross-ratios. Then $\mathrm{tr}(B^{-1}A^{-1})$ may be expressed as a real analytic function of λ, μ, $\mathbb{X}_1$ and $\mathbb{X}_2$.*

We now show the same thing for the cross ratios $\mathbb{X}_i(A,B^{-1}A^{-1})$.

Proposition 7.4. *Let A and B be loxodromic maps in* $\mathrm{SU}(2,1)$ *with attracting eigenvalues e^{λ}, e^{μ} where λ, $\mu\in S$ and let also $\mathbb{X}_1=\mathbb{X}_1(A,B)$, $\mathbb{X}_2=\mathbb{X}_2(A,B)$ and $\mathbb{X}_3=\mathbb{X}_3(A,B)$ be their cross-ratios. Then $\mathbb{X}_j(A,B^{-1}A^{-1})$ may be expressed as a real analytic function of λ, μ, $\mathbb{X}_1$, $\mathbb{X}_3$ and $\mathbb{X}_3$.*

Proof. We may conjugate so that

$$A=E(\lambda),\quad B=QE(\mu)Q^{-1},\quad B^{-1}A^{-1}=RE(\nu)R^{-1}.$$

Then multiplying through, equating the upper left hand entries of $B^{-1}A^{-1}$ and those of AB and using standard properties of the entries of Q and R we obtain

$$\begin{aligned} e^{\overline{\nu}-\nu}+\overline{\mathbb{X}}_1(A,B^{-1}A^{-1})\,\sigma(\nu)+\mathbb{X}_2(A,B^{-1}A^{-1})\,\sigma(-\overline{\nu}) &= e^{-\lambda}\Big(e^{\mu-\overline{\mu}}+\overline{\mathbb{X}}_1\,\sigma(-\mu)+\mathbb{X}_2\,\sigma(\overline{\mu})\Big),\\ e^{\nu-\overline{\nu}}+\overline{\mathbb{X}}_1(A,B^{-1}A^{-1})\,\sigma(-\nu)+\mathbb{X}_2(A,B^{-1}A^{-1})\,\sigma(\overline{\nu}) &= e^{\lambda}\Big(e^{\overline{\mu}-\mu}+\overline{\mathbb{X}}_1\,\sigma(\mu)+\mathbb{X}_2\,\sigma(-\overline{\mu})\Big). \end{aligned}$$

Since $\sigma(\nu)=-e^{\overline{\nu}}\sigma(-\nu)$ and $|e^{\nu}|\neq 1$, we may eliminate either $\overline{\mathbb{X}}_1(A,B^{-1}A^{-1})$ or $\mathbb{X}_2(A,B^{-1}A^{-1})$ from these equations. Thus we can write each of $\mathbb{X}_1(A,B^{-1}A^{-1})$ and $\mathbb{X}_2(A,B^{-1}A^{-1})$ as a real analytic function of λ, μ, ν, $\mathbb{X}_1$ and $\mathbb{X}_2$.

Using Corollary 6.5 as well as $|e^{\lambda}|\neq 1$ and $|e^{\nu}|\neq 1$, we can write $\mathbb{X}_3(A,B^{-1}A^{-1})$ as a real analytic function of λ, ν, $\mathbb{X}_1(A,B^{-1}A^{-1})$, $\mathbb{X}_2(A,B^{-1}A^{-1})$ and $\mathrm{tr}[A,B^{-1}A^{-1}]$. As is well known, $[A,B^{-1}A^{-1}]=A(AB)^{-1}A^{-1}(AB)=B^{-1}(BAB^{-1}A^{-1})B=B^{-1}[A,B]^{-1}B$. Hence, we have $\mathrm{tr}[A,B^{-1}A^{-1}]=\overline{\mathrm{tr}[A,B]}$. Using the second part of Proposition 6.4 we can write $\mathrm{tr}[A,B]$ as a real analytic function of λ, μ, $\mathbb{X}_1$, $\mathbb{X}_2$ and $\mathbb{X}_3$. Substituting this in the previous expression, we can write $\mathbb{X}_3(A,B^{-1}A^{-1})$ as a real analytic function of λ, μ, ν, $\mathbb{X}_1$, $\mathbb{X}_2$ and $\mathbb{X}_3$.

Hence we can express $\mathbb{X}_1(A,B^{-1}A^{-1})$, $\mathbb{X}_2(A,B^{-1}A^{-1})$ and $\mathbb{X}_3(A,B^{-1}A^{-1})$ as real analytic functions of λ, μ, ν, $\mathbb{X}_1$, $\mathbb{X}_2$ and $\mathbb{X}_3$. Using Proposition 7.3 we see that $\mathrm{tr}(B^{-1}A^{-1})$, and hence ν, may be expressed as a real analytic function of λ, μ, $\mathbb{X}_1$ and $\mathbb{X}_2$. Substituting for ν in the expressions above gives the result. □

Thus we have proved Theorem 7.2. In the application to surface groups we shall consider the following situation which is not quite covered by the preceding results. We shall want to specify the traces of the peripheral elements A, B and $B^{-1}A^{-1}$ and find all possible Fenchel-Nielsen coordinates. We now show that we can find $\mathbb{X}_1(A,B)$ in terms of $\mathrm{tr}(A)$, $\mathrm{tr}(B)$, $\mathrm{tr}(B^{-1}A^{-1})$ and $\mathbb{X}_2(A,B)$. Hence we can find $\big|\mathbb{X}_3(A,B)\big|$ and $\Re\big(\mathbb{X}_3(A,B)\big)$ and so we can determine two possible points on $\mathfrak{X}(A,B)$; or one if $\Im\big(\mathbb{X}_3(A,B)\big)=0$.

Proposition 7.5. *Let A and B be loxodromic maps in* $\mathrm{SU}(2,1)$ *with attracting eigenvalues e^{λ}, e^{μ} where λ, $\mu\in S$ and let also $\mathbb{X}_1=\mathbb{X}_1(A,B)$ and $\mathbb{X}_2=\mathbb{X}_2(A,B)$ be their first two cross-ratios. Suppose that $B^{-1}A^{-1}$ is loxodromic with attracting eigenvalue e^{ν} for $\nu\in S$. Then $\mathbb{X}_1$ may be expressed as a real analytic function of λ, μ, ν and $\mathbb{X}_2$.*

Proof. Since $\mathrm{tr}(B^{-1}A^{-1})=e^{\nu}+e^{\overline{\nu}-\nu}+e^{-\overline{\nu}}$ is linear in $\mathbb{X}_1$, $\overline{\mathbb{X}}_1$, $\mathbb{X}_2$ and $\overline{\mathbb{X}}_2$ with coefficients that are analytic functions of λ and μ, we can conjugate and eliminate $\overline{\mathbb{X}}_1$ and so express $\mathbb{X}_1$ as a real analytic function of λ, μ, ν and $\mathbb{X}_2$. This function is well defined provided

$$\mathbb{X}_1\,\sigma(-\overline{\lambda})\sigma(-\overline{\mu})+\overline{\mathbb{X}}_1\,\sigma(\lambda)\sigma(\mu),$$

viewed as a function of $\mathbb{X}_1$ and $\overline{\mathbb{X}}_1$, is not a multiple of its complex conjugate. This is true provided

$$\big|\sigma(-\lambda)\big|\,\big|\sigma(-\mu)\big|\neq\big|\sigma(\lambda)\big|\,\big|\sigma(\mu)\big|=|e^{\lambda}|\,\big|\sigma(-\lambda)\big|\,|e^{\mu}|\,\big|\sigma(-\mu)\big|.$$

in other words, provided $|e^{\lambda}|\,|e^{\mu}|\neq 1$. Since λ and μ lie in S this condition is satisfied. This gives the result. □

We remark that the same argument enables us to express $\mathbb{X}_2$ as a real analytic function of λ, μ, ν and $\mathbb{X}_1$ provided $|e^\lambda| \neq |e^\nu|$. This is not always the case for λ, $\mu \in S$. In particular, when we close a handle below we will have $\mu = \overline{\lambda}$.

7.3. **Trace coordinates.** In this section we discuss the number of trace parameters that are needed to parametrise $(0,3)$ groups. We do not use this when constructing Fenchel-Nielsen coordinates, but we include it for completeness. We first show that the cross-ratios $\mathbb{X}_1(A,B)$ and $\mathbb{X}_2(A,B)$ may be expressed uniquely in terms of $\mathrm{tr}(A)$, $\mathrm{tr}(B)$, $\mathrm{tr}(AB)$ and $\mathrm{tr}(A^{-1}B)$.

Proposition 7.6. *Let A and B be loxodromic maps in $\mathrm{SU}(2,1)$ with attracting eigenvalues e^λ, e^μ where $\lambda, \mu \in S$. Then $\mathbb{X}_1 = \mathbb{X}_1(A,B)$ and $\mathbb{X}_2 = \mathbb{X}_2(A,B)$ may be expressed as a real analytic function of λ, μ, $\mathrm{tr}(AB)$ and $\mathrm{tr}(A^{-1}B)$.*

Proof. By Proposition 6.4 we have

$$\begin{aligned}\mathrm{tr}(AB) &= (e^\lambda + e^{-\overline{\lambda}})e^{\overline{\mu}-\mu} + e^{\overline{\lambda}-\lambda}(e^\mu + e^{-\overline{\mu}}) - e^{\overline{\lambda}-\lambda}e^{\overline{\mu}-\mu} \\ &\quad + \Big(\mathbb{X}_1\,\sigma(-\overline{\mu}) + \overline{\mathbb{X}}_2\,\sigma(\mu)\Big)\sigma(-\overline{\lambda}) + \Big(\overline{\mathbb{X}}_1\,\sigma(\mu) + \mathbb{X}_2\,\sigma(-\overline{\mu})\Big)\sigma(\lambda).\end{aligned}$$

From the fact that the attractive eigenvalue of A^{-1} is $e^{\overline{\lambda}}$ we similarly have

$$\begin{aligned}\mathrm{tr}(A^{-1}B) &= (e^{-\lambda} + e^{\overline{\lambda}})e^{\overline{\mu}-\mu} + e^{\lambda-\overline{\lambda}}(e^\mu + e^{-\overline{\mu}}) - e^{\lambda-\overline{\lambda}}e^{\overline{\mu}-\mu} \\ &\quad + \Big(\mathbb{X}_1\,\sigma(-\overline{\mu}) + \overline{\mathbb{X}}_2\,\sigma(\mu)\Big)\sigma(\overline{\lambda}) + \Big(\overline{\mathbb{X}}_1\,\sigma(\mu) + \mathbb{X}_2\,\sigma(-\overline{\mu})\Big)\sigma(-\lambda).\end{aligned}$$

Using elementary linear algebra we may solve these two equations for $\mathbb{X}_1\,\sigma(-\overline{\mu}) + \overline{\mathbb{X}}_2\,\sigma(\mu)$ and $\overline{\mathbb{X}}_1\,\sigma(\mu) + \mathbb{X}_2\,\sigma(-\overline{\mu})$. This uses $\big|\sigma(\lambda)\big| = |e^\lambda|\,\big|\sigma(-\lambda)\big|$ and $|e^\lambda| \neq 1$. Then complex conjugating one of the resulting equations we may solve for $\mathbb{X}_1$ and $\mathbb{X}_2$. This uses $\big|\sigma(\mu)\big| = |e^\mu|\,\big|\sigma(-\mu)\big|$ and $|e^\mu| \neq 1$. □

We can use the previous result to eliminate the cross-ratios and only deal with traces. However, as we show here this does not determine $\langle A, B\rangle$ up to conjugacy in $\mathrm{SU}(2,1)$. This is because the sign of $\Im\big(\mathbb{X}_3(A,B)\big)$ is not determined by the traces of A, B, AB and AB^{-1}.

Proposition 7.7. *Suppose that $\Gamma = \langle A,\, B\rangle$ and $\Gamma' = \langle A',\, B'\rangle$ are $(0,3)$ groups with $\mathrm{tr}(A) = \mathrm{tr}(A')$, $\mathrm{tr}(B) = \mathrm{tr}(B')$, $\mathrm{tr}(AB) = \mathrm{tr}(A'B')$ and $\mathrm{tr}(A^{-1}B) = \mathrm{tr}({A'}^{-1}B')$. Then either there exists a holomorphic isometry C in $\mathrm{SU}(2,1)$ so that $A' = CAC^{-1}$ and $B' = CBC^{-1}$ or else there is an anti-holomorphic isometry ι so that $A' = \iota A^{-1}\iota^{-1}$ and $B' = \iota B^{-1}\iota^{-1}$. In particular the groups Γ and Γ' are conjugate by an isometry, which may not be holomorphic.*

Proof. Write $\mathbb{X}_i = \mathbb{X}_i(A,B)$ and $\mathbb{X}_i' = \mathbb{X}_i(A',B')$. From Proposition 7.6 we see that our hypotheses on the traces imply that $\mathbb{X}_1' = \mathbb{X}_1$ and $\mathbb{X}_2' = \mathbb{X}_2$. From equations (5.2) and (5.3) we see that either $\mathbb{X}_3' = \mathbb{X}_3$ or else $\mathbb{X}_3' = \overline{\mathbb{X}}_3$.

In the former case, by Proposition 5.10, there exists a holomorphic isometry C with $a_{A'} = C(a_A)$, $r_{A'} = C(r_A)$, $a_{B'} = C(a_B)$ and $r_{B'} = C(r_B)$. Since A' and CAC^{-1} have the same traces and fixed points they must be equal. Likewise, B' and CBC^{-1} are equal.

Now consider the latter case, namely $\mathbb{X}_1' = \mathbb{X}_1$, $\mathbb{X}_2' = \mathbb{X}_2$ and $\mathbb{X}_3' = \overline{\mathbb{X}}_3$. Using Proposition 6.1 we have

$$\begin{aligned}\mathbb{X}_1(A',B') &= \mathbb{X}_1(A,B) = \overline{\mathbb{X}_1(A^{-1},B^{-1})}, \\ \mathbb{X}_2(A',B') &= \mathbb{X}_2(A,B) = \overline{\mathbb{X}_2(A^{-1},B^{-1})}, \\ \mathbb{X}_3(A',B') &= \overline{\mathbb{X}_3(A,B)} = \overline{\mathbb{X}_3(A^{-1},B^{-1})}.\end{aligned}$$

Therefore, from Corollary 5.11, there is an antiholomorphic isometry ι sending the attractive and repulsive fixed points of A^{-1} and B^{-1} to those of A' and B'. In other words, $a_{A'} = \iota(r_A)$, $r_{A'} = \iota(a_A)$, $a_{B'} = \iota(r_B)$ and $r_{B'} = \iota(a_B)$. Since ι is antiholomorphic, we have

$$\mathrm{tr}(A') = \mathrm{tr}(A) = \overline{\mathrm{tr}(A^{-1})} = \mathrm{tr}(\iota A^{-1}\iota^{-1}).$$

Because the fixed points and traces of A' and $\iota A^{-1}\iota^{-1}$ are the same, they are equal. Likewise, $B' = \iota B^{-1}\iota^{-1}$. This proves the result. □

There is a strong contrast between the previous result and the classical case, where these four traces determine the group up to conjugacy by a holomorphic (that is orientation preserving) isometry. Thus one must be very careful when using trace parameters to determine $\mathrm{SU}(2,1)$ conjugacy classes. However, we can conclude that five traces are sufficient. Of course, these five traces satisfy two real equations, which may be deduced from Proposition 6.4. This is in the spirit of the theorem of Okumura [16] and Schmutz [19].

Proposition 7.8. *Let $\langle A, B\rangle$ and $\langle A', B'\rangle$ be two $(0,3)$ groups. If $\mathrm{tr}(A) = \mathrm{tr}(A')$, $\mathrm{tr}(B) = \mathrm{tr}(B')$, $\mathrm{tr}(AB) = \mathrm{tr}(A'B')$, $\mathrm{tr}(A^{-1}B) = \mathrm{tr}(A'^{-1}B')$ and $\mathrm{tr}[A,B] = \mathrm{tr}[A',B']$ then $\langle A, B\rangle$ and $\langle A', B'\rangle$ are conjugate in $\mathrm{SU}(2,1)$.*

Proof. Using Proposition 7.6 we may show that $\mathrm{tr}(A) = \mathrm{tr}(A')$, $\mathrm{tr}(B) = \mathrm{tr}(B')$, $\mathrm{tr}(AB) = \mathrm{tr}(A'B')$ and $\mathrm{tr}(A^{-1}B) = \mathrm{tr}(A'^{-1}B')$ imply that both $\mathbb{X}_1(A,B) = \mathbb{X}_1(A',B')$ and $\mathbb{X}_2(A,B) = \mathbb{X}_2(A',B')$. If we also have $\mathrm{tr}[A,B] = \mathrm{tr}[A',B']$ then these facts and Corollary 6.5 imply $\mathbb{X}_3(A,B) = \mathbb{X}_3(A',B')$. Hence the groups correspond to the same point on the cross-ratio variety. In other words, they have the same Fenchel-Nielsen coordinates and so, from Theorem 7.1, they are conjugate. □

This should be compared to the discussion on page 102 of [23], where Wen shows that in $\mathrm{SL}(3,\mathbb{C})$ one may express the traces of any element of $\langle A, B\rangle$ as a polynomial in the traces of A, A^{-1}, B, B^{-1}, AB, $B^{-1}A^{-1}$, $A^{-1}B$, $B^{-1}A$ and ABA^2B^2. Moreover, the last of these traces satisfies a quadratic polynomial whose coefficients are polynomials in the other eight. In other words there are two possibilities for this trace. In our setting $\mathrm{tr}(A^{-1}) = \overline{\mathrm{tr}(A)}$ and so Wen's first eight variables may be replaced with just four. Moreover, in order to determine the group up to conjugation we just need a choice of sign for $\Im(\mathbb{X}_3)$. This translates into two possible values for $\mathrm{tr}[A,B]$. See also [25] for a more detailed discussion of this material.

8. Twist-bend parameters

8.1. **The complex hyperbolic Fenchel-Nielsen twist-bend.** Suppose that we are given two three-holed spheres with the property that two of the boundary components, one on each three-holed sphere, are compatible (in a sense to be made precise below). We now discuss how to parametrise the possible ways to attach the two three-holed spheres to form a four-holed sphere. An analogous situation is that of a single three-holed sphere where two of the boundary components are compatible and we want to discuss how to parametrise ways to attach these boundary components to form a one holed torus group. We will discuss the details of these two cases in separate sections below, but the general principles in each case are the same.

First we must explain what we meant by the word 'compatible' in the previous paragraph. To be precise suppose that $\langle A, B\rangle$ and $\langle C, D\rangle$ are two $(0,3)$ groups (which may be conjugate). We say that the boundary components associated to A and D are *compatible* if and only if $D = A^{-1}$; compare Wolpert [27]. Why do we need an inverse? If we were dealing with the case where $\langle A, B\rangle$ and $\langle C, D\rangle$ are Fuchsian groups then saying that $D = A^{-1}$ means that A and D have the same (oriented) axis but that the universal covers of the two three-holed spheres are subsets of the hyperbolic plane on opposite sides of the axis (perhaps by adopting the convention that, when viewed from inside the surface the orientation on the boundary curves is always to the right). We can make sense of this idea in the complex hyperbolic setting by equivariantly embedding the universal covers of our three-holed spheres into $\mathbf{H}^2_{\mathbb{C}}$ so that the boundary curves are mapped onto the axes of A, B, $B^{-1}A^{-1}$ and their conjugates. We leave the details of this to the reader.

A complex hyperbolic Fenchel-Nielsen twist-bend consists of taking these two three-holed spheres in $\mathbf{H}^2_{\mathbb{C}}$ that are glued together along the axis of $A = D^{-1}$ and, while fixing the surface corresponding to $\langle A, B\rangle$, moving the surface corresponding to $\langle C, D\rangle$ by a hyperbolic translation along the axis of A (the twist) and a rotation around the complex axis of A (the bend). In other words, we take a map K that commutes with $A = D^{-1}$ and we conjugate $\langle C, D\rangle$ by K. A twist by a hyperbolic distance $k \in \mathbb{R}$ corresponds to K being purely hyperbolic with trace $2\cosh(k/2) + 1 = \tau(k/2)$ and a bend through an angle $\beta \in (-\pi, \pi]$ corresponds to K being a boundary elliptic with trace $2e^{i\beta/3} + e^{-2i\beta/3} = \tau(\beta/3)$. Putting this together, we see that if $\mathrm{tr}(K) = \tau(\kappa) = \tau(-\overline{\kappa})$ then K corresponds to a twist through distance $\pm\Re(2\kappa)$ and a bend through angle $\Im(3\kappa) = \Im(-3\overline{\kappa})$. We remark that the ambiguity in the sign of the twist is also present when passing from twists to traces in the classical construction as well; see [18]. We call κ defined in this way the *twist-bend parameter*. In the above description we started with a given way of attaching $\langle A, B\rangle$ to $\langle C, D\rangle$ to form the group $\langle A, B, C\rangle$ and then performed the twist-bend relative to this initial choice of group. We remark that the twist-bend is not an absolute invariant but must always be chosen relative to some starting group $\langle A, B, C\rangle$. We are free to fix this group once and for all at the beginning.

The issue of the direction of twist is subtle and it can be very easy to introduce ambiguities. So we now make very clear what we are doing. Conjugating if necessary, we assume that $a_A = \infty$ and $r_A = o$. This means that $A = E(\lambda)$ for some $\lambda \in S$. Let $\kappa \in \mathbb{C}$ with $-\pi < \Im(\kappa) \le \pi$. Then we define $K = E(\kappa)$, that is:

$$K = E(\kappa) = \begin{bmatrix} e^{\kappa} & 0 & 0 \\ 0 & e^{\overline{\kappa}-\kappa} & 0 \\ 0 & 0 & e^{-\overline{\kappa}} \end{bmatrix}.$$

If $\Re(\kappa) > 0$ then $\kappa \in S$ and K is loxodromic. Its attractive fixed point is $a_K = a_A$ and its repulsive fixed point is $r_K = r_A$. Thus the twist goes in the same direction as A (from r_A to a_A). If $\Re(\kappa) = 0$ then K is boundary elliptic and κ corresponds to a pure bend, that is there is no twist. If $\Re(\kappa) < 0$ then $-\overline{\kappa} \in S$ and again K is loxodromic. This time $a_K = r_A$ and $r_K = a_A$ and the twist goes in the opposite direction to A. We say that the twist-bend parameter κ is *oriented consistently* with A if when we write $A = QE(\lambda)Q^{-1}$ the matrix K is given by $QE(\kappa)Q^{-1}$. Note that with respect to $\langle C, D\rangle$ we must twist $\langle A, B\rangle$ by K^{-1}. The orientation of K^{-1} with respect to $D = A^{-1}$ is the same as that of K with respect to A. In other words, $-\kappa$ is oriented consistently with D.

A crucial special case is when either $\langle A, B\rangle$ or $\langle C, D\rangle$ preserves a complex line. In this case there are no bends. In order to see that, we observe that if $\langle A, B\rangle$ preserves a complex line then it must be L_A, the complex axis of A. Moreover, if K is a boundary elliptic commuting with A then it too fixes L_A. Hence K commutes with both A and B and so leaves $\langle A, B\rangle$ unchanged. Hence there is no bending in this case. This phenomenon contributes to the reduction in the number of parameters for representations whose Toledo invariant is $\pm\chi(\Sigma)$, that is surface groups that preserve complex lines.

We need to find a conjugation invariant way of measuring the twist-bend parameter κ. We do this using the cross-ratios of the fixed points $a_A = r_{KDK^{-1}}$, $r_A = a_{KDK^{-1}}$, a_B and $r_{KCK^{-1}} = K(r_C)$. We define

$$\widetilde{\mathbb{X}}_1(\kappa) = [a_B, a_A, r_A, K(r_C)], \quad \widetilde{\mathbb{X}}_2(\kappa) = [a_B, r_A, a_A, K(r_C)]. \tag{8.1}$$

Note that if $a_B = K(r_C)$ then both of these cross-ratios are infinite. We remark that the angular invariants $\mathbb{A}(a_A, r_A, a_B)$ and $\mathbb{A}\big(a_A, r_A, K(r_C)\big)$ are independent of κ. For the latter one, we see this by observing that $\mathbb{A}\big(K(a_A), K(r_A), K(r_C)\big) = \mathbb{A}(a_A, r_A, r_C)$, Then using Proposition 5.8 we see that there are in fact only two degrees of freedom in $\widetilde{\mathbb{X}}_1(\kappa)$ and $\widetilde{\mathbb{X}}_2(\kappa)$, as we should expect.

Proposition 8.1. *Let A, B and C be loxodromic transformations. Let a_A, r_A, a_B, r_B, a_C, r_C be the fixed points of A, B and C respectively. Suppose that neither a_B nor r_C lies on L_A, the complex axis of A. Let κ and κ' be twist-bend parameters that are oriented consistently with A. If*

$$\widetilde{\mathbb{X}}_1(\kappa) = \widetilde{\mathbb{X}}_1(\kappa') \quad \textit{and} \quad \widetilde{\mathbb{X}}_2(\kappa) = \widetilde{\mathbb{X}}_2(\kappa')$$

(which are possibly infinity) then $\kappa = \kappa'$.

Proof. Without loss of generality, suppose that $a_A = \infty$ and $r_A = o$. That is $A = E(\lambda)$, $K = E(\kappa)$ and $K' = E(\kappa')$ where $\lambda \in S$ and κ, κ' are any complex numbers with $\Im(\kappa), \Im(\kappa') \in (-\pi, \pi]$. Write lifts of a_B, r_B, a_C and r_C as

$$\mathbf{a}_B = \begin{bmatrix} a \\ d \\ g \end{bmatrix}, \quad \mathbf{r}_B = \begin{bmatrix} c \\ f \\ j \end{bmatrix}, \quad \mathbf{a}_C = \begin{bmatrix} a' \\ d' \\ g' \end{bmatrix}, \quad \mathbf{r}_C = \begin{bmatrix} c' \\ f' \\ j' \end{bmatrix}.$$

Then

$$\begin{aligned} \widetilde{\mathbb{X}}_1(\kappa) &= \frac{\langle K\mathbf{r}_C, \boldsymbol{\infty}\rangle\langle \mathbf{o}, \mathbf{a}_B\rangle}{\langle K\mathbf{r}_C, \mathbf{a}_B\rangle\langle \mathbf{o}, \boldsymbol{\infty}\rangle} = \frac{e^{-\overline{\kappa}} j'\overline{a}}{e^{-\overline{\kappa}} j'\overline{a} + e^{\overline{\kappa}-\kappa} f'\overline{d} + e^{\kappa} c'\overline{g}}, \\ \widetilde{\mathbb{X}}_2(\kappa) &= \frac{\langle K\mathbf{r}_C, \mathbf{o}\rangle\langle \boldsymbol{\infty}, \mathbf{a}_B\rangle}{\langle K\mathbf{r}_C, \mathbf{a}_B\rangle\langle \boldsymbol{\infty}, \mathbf{o}\rangle} = \frac{e^{\kappa} c'\overline{g}}{e^{-\overline{\kappa}} j'\overline{a} + e^{\overline{\kappa}-\kappa} f'\overline{d} + e^{\kappa} c'\overline{g}}. \end{aligned}$$

Since we know that a_B and r_C are distinct from o and ∞ we automatically see that neither $\widetilde{\mathbb{X}}_1(\kappa)$ nor $\widetilde{\mathbb{X}}_2(\kappa)$ is zero. Since neither a_B nor r_C lies on L_A, using Corollaries 5.9 and 5.4, we see that $\widetilde{\mathbb{X}}_1(\kappa) + \widetilde{\mathbb{X}}_2(\kappa) \neq 1$.

If $\widetilde{\mathbb{X}}_1(\kappa)$ is infinite then $a_B = K(r_C)$. But $\widetilde{\mathbb{X}}_1(\kappa')$ must also be infinite and so $a_B = K'(r_C)$. Thus $K^{-1}K' = E(\kappa' - \kappa)$ fixes r_C. Thus either $\kappa' = \kappa$ or else r_C lies in L_A, a contradiction.

Suppose that $\widetilde{\mathbb{X}}_1(\kappa) = \widetilde{\mathbb{X}}_1(\kappa')$ is finite (and non-zero). Since $\widetilde{\mathbb{X}}_1(\kappa) + \widetilde{\mathbb{X}}_2(\kappa) = \widetilde{\mathbb{X}}_1(\kappa') + \widetilde{\mathbb{X}}_2(\kappa') \neq 1$, we have

$$e^{2\overline{\kappa}-\kappa}\frac{f'\overline{d}}{j'\overline{a}} = \frac{1 - \widetilde{\mathbb{X}}_1(\kappa) - \widetilde{\mathbb{X}}_2(\kappa)}{\widetilde{\mathbb{X}}_1(\kappa)} = \frac{1 - \widetilde{\mathbb{X}}_1(\kappa') - \widetilde{\mathbb{X}}_2(\kappa')}{\widetilde{\mathbb{X}}_1(\kappa')} = e^{2\overline{\kappa}'-\kappa'}\frac{f'\overline{d}}{j'\overline{a}}.$$

Thus $e^{2\overline{\kappa}'-\kappa'} = e^{2\overline{\kappa}-\kappa}$ and so $\kappa = \kappa'$. □

Corollary 8.2. *Let A, B and C be loxodromic transformations. Suppose that neither $\langle A, B\rangle$ nor $\langle A, C\rangle$ preserves a complex line. Let κ and κ' be twist-bend parameters that are oriented consistently with A and let K and K' be the corresponding elements of* $\mathrm{SU}(2,1)$ *that commute with A. Then $\langle A,\, B,\, KCK^{-1}\rangle = \langle A,\, B,\, K'CK'^{-1}\rangle$ if and only if $\kappa = \kappa'$.*

Proof. Clearly if $\kappa = \kappa'$ then $K = K'$ and $\langle A,\, B,\, KCK^{-1}\rangle = \langle A,\, B,\, K'CK'^{-1}\rangle$.

Conversely, let a_A, r_A, a_B, r_B, a_C and r_C denote the fixed points of A, B, C. Suppose that neither a_B nor r_C lies on L_A. Since $\langle A,\, B,\, KCK^{-1}\rangle = \langle A,\, B,\, K'CK'^{-1}\rangle$ we have

$$[a_B, a_A, r_A, K(r_C)] = [a_B, a_A, r_A, K'(r_C)] \quad \text{and} \quad [a_B, r_A, a_A, K(r_C)] = [a_B, r_A, a_A, K'(r_C)].$$

From Proposition 8.1 we have $\kappa = \kappa'$.

If a_B lies on L_A then, since $\langle A, B\rangle$ does not preserve L_A, we see that r_B does not lie on L_A. Thus repeating the above argument with B^{-1} in place of B gives the result. Likewise if r_C lies on L_A then we replace C with C^{-1}. □

8.2. **Attaching pairs of three-holed spheres.** We define a $(0,4)$ subgroup of $\mathrm{SU}(2,1)$ to be a group with four loxodromic generators satisfying the single relation that their product is the identity. These four loxodromic maps correspond to the boundary curves and (their conjugacy classes) are called *peripheral*. Note that the $(0,4)$ group is freely generated by any three of these loxodromic maps.

Let $\langle A, B\rangle$ and $\langle C, D\rangle$ be two $(0,3)$-groups with $A = D^{-1}$. We now show how to construct a $(0,4)$ group from $\langle A, B\rangle$ and $\langle C, D\rangle$. Algebraically this is done by taking the free product of $\langle A, B\rangle$ and $\langle C, D\rangle$ with amalgamation along the common cyclic subgroup $\langle A\rangle = \langle D\rangle$. We are free to conjugate $\langle C, D\rangle$ by any $K \in \mathrm{SU}(2,1)$ that commutes with $A = D^{-1}$ and doing so yields a new $(0,4)$ group depending on K. As explained above, varying this K corresponds to a Fenchel-Nielsen twist deformation.

Lemma 8.3. *Suppose that $\Gamma_1 = \langle A, B\rangle$ and $\Gamma_2 = \langle C, D\rangle$ are two $(0,3)$ groups with peripheral elements A, B, $B^{-1}A^{-1}$ and C, D and $D^{-1}C^{-1}$ respectively. Moreover suppose that $A = D^{-1}$. Let K be any element of $\mathrm{SU}(2,1)$ that commutes with $A = D^{-1}$. Then the group $\langle A, B, KCK^{-1}\rangle$ is a $(0,4)$-group with peripheral elements B, $B^{-1}A^{-1}$, KCK^{-1} and $KD^{-1}C^{-1}K^{-1}$.*

Proof. The loxodromic transformations

$$B, \quad B^{-1}A^{-1}, \quad KD^{-1}C^{-1}K^{-1}, \quad KCK^{-1}$$

generate a $(0,4)$ group since we see that their product is

$$(B)(B^{-1}A^{-1})(KD^{-1}C^{-1}K^{-1})(KCK^{-1}) = A^{-1}KD^{-1}K^{-1}.$$

which is the identity since $D = A^{-1}$ and $KAK^{-1} = A$. □

Note that in Lemma 8.3 the generator D is redundant and so we just speak of the $(0,4)$ group $\langle A, B, KCK^{-1}\rangle$ obtained from the $(0,3)$ groups $\langle A, B\rangle$ and $K\langle C, A^{-1}\rangle K^{-1}$ by gluing along A with twist-bend parameter κ corresponding to K, relative to some specified group. We then associate to $\langle A, B, KCK^{-1}\rangle$ the four complex numbers

$$\mathrm{tr}(A), \quad \mathrm{tr}(B), \quad \mathrm{tr}(C), \quad \kappa$$

together with a point on each of the cross-ratio varieties $\mathfrak{X}(A, B)$ and $\mathfrak{X}(A, C)$. We call these sixteen real parameters the *Fenchel-Nielsen coordinates* of $\langle A, B, KCK^{-1}\rangle$. As remarked above, if either of the $(0,3)$ groups $\langle A, B\rangle$ or $\langle C, A^{-1}\rangle$ preserves a complex line (that is L_A equals L_B or L_C) then the twist-bend parameter must be real.

Theorem 8.4. *Suppose that $\langle A, B\rangle$ and $\langle C, A^{-1}\rangle$ are two $(0,3)$ groups neither of which preserves a complex line. Let κ be a twist-bend parameter oriented consistently with A and let $\langle A, B, KCK^{-1}\rangle$ be the corresponding $(0,4)$ group. Then $\langle A, B, KCK^{-1}\rangle$ is uniquely determined up to conjugation in $\mathrm{SU}(2,1)$ by its Fenchel-Nielsen coordinates: a point on each of the cross-ratio varieties $\mathfrak{X}(A, B)$ and $\mathfrak{X}(A, C)$ together with the four complex numbers*

$$\mathrm{tr}(A), \quad \mathrm{tr}(B), \quad \mathrm{tr}(C), \quad \kappa$$

Proof. Suppose that $\langle A, B, KCK^{-1}\rangle$ and $\langle A', B', K'C'K'^{-1}\rangle$ are two $(0,4)$ groups with

$$\mathrm{tr}(A) = \mathrm{tr}(A'), \quad \mathrm{tr}(B) = \mathrm{tr}(B'), \quad \mathrm{tr}(C) = \mathrm{tr}(C'), \quad \kappa = \kappa'$$

and

$$\begin{array}{lll} \mathbb{X}_1(A,B) = \mathbb{X}_1(A',B'), & \mathbb{X}_2(A,B) = \mathbb{X}_2(A',B'), & \mathbb{X}_3(A,B) = \mathbb{X}_3(A',B'), \\ \mathbb{X}_1(A,C) = \mathbb{X}_1(A',C'), & \mathbb{X}_2(A,C) = \mathbb{X}_2(A',C'), & \mathbb{X}_3(A,C) = \mathbb{X}_3(A',C'). \end{array}$$

Since the Fenchel-Nielsen coordinates of $\langle A, B\rangle$ and $\langle A', B'\rangle$ are the same, then using Theorem 7.1, they are conjugate. Conjugating if necessary, we suppose that $A = A'$ and $B = B'$. Similarly $\langle C, A^{-1}\rangle$ and $\langle C', A'^{-1}\rangle$ are conjugate. The twist-bend parameters must be defined relative to the same initial group which we take to be $\langle A, B, C\rangle$. Then by construction, since $\kappa = \kappa'$ it is clear that $\langle A, B, KCK^{-1}\rangle$ and $\langle A', B', K'C'K'^{-1}\rangle$ are conjugate.

Conversely suppose that $\langle A, B, KCK^{-1}\rangle$ and $\langle A', B', K'C'K'^{-1}\rangle$ are conjugate. Then clearly $\mathrm{tr}(A) = \mathrm{tr}(A')$, $\mathrm{tr}(B) = \mathrm{tr}(B')$ and $\mathrm{tr}(C) = \mathrm{tr}(C')$. Also because cross-ratios are $\mathrm{SU}(2,1)$ invariant we also have $\mathbb{X}_i(A,B) = \mathbb{X}_i(A',B')$ and $\mathbb{X}_i(A,C) = \mathbb{X}_i(A',C')$ for $i = 1, 2, 3$. Thus it remains to show that $\kappa = \kappa'$. Again using the invariance of cross-ratios, we have

$$[a_B, a_A, r_A, K(r_C)] = [a_B, a_A, r_A, K'(r_C)] \quad \text{and} \quad [a_B, r_A, a_A, K(r_C)] = [a_B, r_A, a_A, K'(r_C)].$$

In other words, $\widetilde{\mathbb{X}}_1(\kappa) = \widetilde{\mathbb{X}}_1(\kappa')$ and $\widetilde{\mathbb{X}}_2(\kappa) = \widetilde{\mathbb{X}}_2(\kappa')$. Using Proposition 8.1 we see that $\kappa = \kappa'$. □

8.3. **Closing a handle.** Most of the results of this section run in parallel with the corresponding results in the previous section. We will be interested in the case of one-holed tori in complex hyperbolic space obtained by attaching two of the boundary components of a single three-holed sphere. We call the process of attaching these two ends *closing a handle*. For this to work, one of the peripheral elements of the corresponding $(0,3)$ group must be conjugate to the inverse of another, so that they are compatible. Suppose that these two (conjugacy classes of) peripheral elements are A and $BA^{-1}B^{-1}$, which we suppose are loxodromic. Clearly the $(0,3)$ group is freely generated by A and $BA^{-1}B^{-1}$ and, by hypothesis the third peripheral element $BAB^{-1}A^{-1} = [B, A]$ must also be loxodromic. A $(1,1)$ subgroup Γ of $\mathrm{SU}(2,1)$ is a group that corresponds to a one-holed torus. That is, this group has three generators A, B, C where C is the commutator of A and B, that is, $C = [B, A] = BAB^{-1}A^{-1}$ (and so $A(BA^{-1}B^{-1})C$ is the identity). In particular, Γ is freely generated by A, B. From the group theory point of view, closing a handle is the same as taking the HNN-extension of the $(0,3)$ group $\langle A, BA^{-1}B^{-1}\rangle$ by adjoining the element B to form a $(1,1)$ group. Hence our $(1,1)$ group is $\langle A, B\rangle$ and its peripheral element is $BAB^{-1}A^{-1}$, which is not affected by our attaching operation.

Clearly when we close a handle (that is when we take the HNN-extension) the map B is not unique. If K is any element of $\mathrm{SU}(2,1)$ that commutes with A then $(BK)A^{-1}(BK)^{-1} = BA^{-1}B^{-1}$. So we could just as well have taken our $(1,1)$ group to be $\langle A, BK\rangle$. Varying K corresponds to a Fenchel-Nielsen twist-bend as above. If $A = QE(\lambda)Q^{-1}$ for $\lambda \in S$ we define the twist-bend parameter κ by $K = QE(\kappa)Q^{-1}$ just as before, and we say that κ is oriented consistently with A. Again κ is any complex number with $-\pi < \Im(\kappa) \leq \pi$ and the real part of κ corresponds to a twist and its imaginary part to a bend. Also, just as before, κ is only defined relative to a fixed reference group.

Proposition 8.5. *Let $\langle A, BA^{-1}B^{-1}\rangle$ be a $(0,3)$ group. Let B be a fixed choice of an element in $\mathrm{SU}(2,1)$ conjugating A^{-1} to $BA^{-1}B^{-1}$. Let κ and κ' be twist-bend parameters oriented consistently with A. Then $\langle A, BK\rangle$ is conjugate to $\langle A, BK'\rangle$ if and only if $\kappa = \kappa'$.*

Proof. Clearly if $\kappa=\kappa'$ then $BK=BK'$ and the groups are equal.

Conversely, suppose that $\langle A, BK\rangle$ is conjugate to $\langle A, BK'\rangle$ are conjugate. Then the conjugating element D must commute with A, and so fixes a_A and r_A. Since $BA^{-1}B^{-1}$ is specified we must have

$$BA^{-1}B^{-1}=(BK')A^{-1}(BK')^{-1}=(DBKD^{-1})A^{-1}(DBKD^{-1})^{-1}=D(BA^{-1}B^{-1})D^{-1}.$$

Thus D also commutes with $BA^{-1}B^{-1}$ and so fixes $a_{BA^{-1}B^{-1}}=B(r_A)$ and $r_{BA^{-1}B^{-1}}=B(a_A)$. As these fixed points are distinct, the only possibilities are that either D is the identity or else a_A, r_A, $B(r_A)$ and $B(a_A)$ all lie on a complex line fixed by D. In the latter case D commutes with B as well as A (and hence with K and K'). Thus in either case $BK'=DBKD^{-1}=BK$. In other words, $K=K'$ and so $\kappa=\kappa'$ as required. □

Suppose that $\langle A, BK\rangle$ is a $(1,1)$ group obtained by closing the handle in the $(0,3)$ group $\langle A, BA^{-1}B^{-1}\rangle$ with twist-bend parameter κ. Then we associate to $\langle A, BK\rangle$ a point on the cross-ratio variety $\mathfrak{X}(A, BA^{-1}B^{-1})$ and the two complex numbers $\mathrm{tr}(A)$ and κ. We call these parameters the *Fenchel-Nielsen coordinates* of $\langle A, BK\rangle$: Our main theorem in this section is

Theorem 8.6. *The $(1,1)$ group $\langle A, BK\rangle$ is determined up to conjugation in $\mathrm{SU}(2,1)$ by its Fenchel-Nielsen coordinates: a point on the cross-ratio variety $\mathfrak{X}(A, BA^{-1}B^{-1})$ and the complex numbers $\mathrm{tr}(A)$ and κ.*

Proof. Suppose that $\langle A, BK\rangle$ and $\langle A', B'K'\rangle$ are two $(1,1)$ groups with the same Fenchel-Nielsen coordinates. In particular, $\mathrm{tr}(A)=\mathrm{tr}(A')$ and so

$$\mathrm{tr}(BA^{-1}B^{-1})=\overline{\mathrm{tr}(A)}=\overline{\mathrm{tr}(A')}=\mathrm{tr}(B'A'^{-1}B'^{-1}).$$

Moreover, $\mathbb{X}_i(A, BA^{-1}B^{-1})=\mathbb{X}_i(A', B'A'^{-1}B'^{-1})$ for $i=1$, 2, 3, so we see that the $(0,3)$ groups $\langle A, BA^{-1}B^{-1}\rangle$ and $\langle A', B'A'^{-1}B'^{-1}\rangle$ have the same Fenchel-Nielsen coordinates and so, using Theorem 7.1, are conjugate. Thus we may suppose that $A=A'$ and $BA^{-1}B^{-1}=B'A'^{-1}B'^{-1}$. Using Proposition 8.5 we see that, since $\kappa=\kappa'$ (with reference to the same conjugating element $B=B'$), then $\langle A, BK\rangle$ and $\langle A', B'K'\rangle$ are conjugate.

Conversely, suppose that $\langle A, BK\rangle$ and $\langle A', B'K'\rangle$ are conjugate. It is clear that $\mathrm{tr}(A)=\mathrm{tr}(A')$, $\mathbb{X}_i(A, BA^{-1}B^{-1})=\mathbb{X}_i(A', B'A'^{-1}B'^{-1})$ for $i=1$, 2, 3. Conjugating if necessary, we may suppose that $A=A'$ and $B=B'$ (the latter being a fixed choice of conjugating element with reference to which κ and κ' are defined). Again using Proposition 8.5, we see that $\kappa=\kappa'$. □

8.4. **Twist-bends for groups preserving a complex line.** We now consider what happens when we attach two $(0,3)$ groups $\langle A, B\rangle$ or $\langle A, C\rangle$ (at least) one of which preserves a complex line. In this case, as indicated above, there can be no bending around this complex line and so the twist-bend parameter degenerates into a real twist parameter which we call k. Once again k is only defined relative to a specific reference group. More precisely, we cannot distinguish between different bending angles (rather like the origin in polar coordinates). Thus, in the irreducible case, a group where one of the $(0,3)$ groups preserves a complex line can be the limit of groups which do not preserve a complex line and which have any bending angles. In the reducible case, all the bending angles between distinct $(0,3)$ groups are undetermined at each point and we take them all to be identically zero.

Proposition 8.7. *Let A, B and C be loxodromic transformations. Let a_A, r_A, a_B, r_B, a_C, r_C be the fixed points of A, B and C respectively. Suppose that either a_B and r_B or a_C and r_C lie on L_A,* *the complex axis of A. Let k and k' be (real) twist parameters oriented consistently with A and let K and K' be the corresponding matrices in* $\mathrm{SU}(2,1)$ *that commute with A. If*

$$\widetilde{\mathbb{X}}_1(k) = \widetilde{\mathbb{X}}_1(k') \quad \text{and} \quad \widetilde{\mathbb{X}}_2(k) = \widetilde{\mathbb{X}}_2(k')$$

(which are possibly infinity) then $k = k'$.

Proof. This is similar to the proof of Proposition 8.1. Again we suppose that $A = E(\lambda)$, $K = E(k)$ and $K' = E(k')$. In this case Corollaries 5.9 and 5.4 imply $\widetilde{\mathbb{X}}_1(k) + \widetilde{\mathbb{X}}_2(k) = \widetilde{\mathbb{X}}_1(k') + \widetilde{\mathbb{X}}_2(k') = 1$ With the notation used in the proof of Proposition 8.1, we have:

$$\begin{aligned}
\widetilde{\mathbb{X}}_1(k) &= \frac{\langle K\mathbf{r}_C, \infty\rangle\langle \mathbf{o}, \mathbf{a}_B\rangle}{\langle K\mathbf{r}_C, \mathbf{a}_B\rangle\langle \mathbf{o}, \infty\rangle} = \frac{e^{-k}j'\overline{a}}{e^{-k}j'\overline{a} + e^{k}c'\overline{g}}, \qquad \widetilde{\mathbb{X}}_1(k') = \frac{e^{-k'}j'\overline{a}}{e^{-k'}j'\overline{a} + e^{k'}c'\overline{g}},\\
\widetilde{\mathbb{X}}_2(k) &= \frac{\langle K\mathbf{r}_C, \mathbf{o}\rangle\langle \infty, \mathbf{a}_B\rangle}{\langle K\mathbf{r}_C, \mathbf{a}_B\rangle\langle \infty, \mathbf{o}\rangle} = \frac{e^{k}c'\overline{g}}{e^{-k}j'\overline{a} + e^{k}c'\overline{g}}, \qquad \widetilde{\mathbb{X}}_2(k') = \frac{e^{k'}c'\overline{g}}{e^{-k'}j'\overline{a} + e^{k'}c'\overline{g}},
\end{aligned}$$

Thus

$$e^{2k'}\frac{c'\overline{g}}{j'\overline{a}} = \frac{\widetilde{\mathbb{X}}_2(k')}{\widetilde{\mathbb{X}}_1(k')} = \frac{\widetilde{\mathbb{X}}_2(k)}{\widetilde{\mathbb{X}}_1(k)} = e^{2k}\frac{c'\overline{g}}{j'\overline{a}}.$$

Hence $k' = k$ as claimed. □

Corollary 8.8. *Let $A = QE(\lambda)Q^{-1}$, B and C be loxodromic transformations. Suppose that one or both of $\langle A, B\rangle$ or $\langle C, A^{-1}\rangle$ preserves a complex line. Let k and k' be twist parameters oriented consistently with A and let K and K' be the corresponding matrices in* $\mathrm{SU}(2,1)$ *that commute with A. Then $\langle A,\, B,\, KCK^{-1}\rangle = \langle A,\, B,\, K'CK'^{-1}\rangle$ if and only if $k = k'$.*

We could mimic the proof of Theorem 8.4 and show that if either $\langle A, B\rangle$ or $\langle C, A^{-1}\rangle$ preserves a complex line then $(0,4)$ group $\langle A, B, KCK^{-1}\rangle$ is uniquely determined by its Fenchel-Nielsen parameters. The main difference is that some of the parameters that were complex will now be real. For example, if $\langle A, B\rangle$ preserves a complex line then $\mathbb{X}_1(A,B)$ and $\mathbb{X}_2(A,B)$ are both real and satisfy $\mathbb{X}_3(A,B) = -\mathbb{X}_2(A,B)/\mathbb{X}_1(A,B)$ and $\mathbb{X}_1(A,B) + \mathbb{X}_2(A,B) = 1$. Moreover $\kappa = k$ is real. The details in the case where the whole surface group (and so each $(0,3)$ group) preserves a complex line are given in Section 2.2.

Finally, we remark that, unlike the $(0,4)$ case, even if the $(0,3)$ group $\langle A, BA^{-1}B^{-1}\rangle$ preserves a complex line, if we close the handle to obtain $\langle A, BK\rangle$, the twist-bend parameter κ associated to K is still complex. The point is that we are not conjugating by K and so we can see the effect of twists around L_A. This is even the case when the whole surface group preserves a complex line. In other words, there is still a two parameter family of ways to close a handle in $(0,3)$ groups that preserve a complex line.

References

[1] É. Cartan; *Sur le groupe de la géométrie hypersphérique*. Comm. Math. Helv. **4** (1932) 158–171.

[2] E. Falbel; *Geometric structures associated to triangulations as fixed point sets of involutions*. Topology Appl **154** (2007) 1041–1052.

[3] E. Falbel; *A spherical CR structure on the complement of the figure eight knot with discrete holonomy*. Preprint 2005.

[4] E. Falbel & V. Zocca; *A Poincaré's polyhedron theorem for complex hyperbolic geometry*. J. reine angew. Math., **516** (1999) 133–158.

[5] W. Fenchel & J. Nielsen; Discontinuous groups of isometries in the hyperbolic plane. *Edited by Asmus L. Schmidt*, de Gruyter (2003).

[6] W.M. Goldman; Complex Hyperbolic Geometry. Oxford Mathematical Monographs. Oxford University Press (1999).

[7] W.M. Goldman; *Representations of fundamental groups of surfaces*. In: Geometry and Topology, ed J. Alexander & J. Harer, *Lecture Notes in Maths.*, **1167**, Springer, 1985, 95–117.

[8] W.M. Goldman; *Convex real projective structures on compact surfaces*. J. Diff. Geom. **31** (1990) 791–845.

[9] W.M. Goldman, M. Kapovich & B. Leeb; *Complex hyperbolic manifolds homotopy equivalent to a Riemann surface*. Communications in Analysis and Geometry **9** (2001) 61–95.

[10] Y. Jiang, S. Kamiya & J.R. Parker; *Jørgensen's inequality for complex hyperbolic space*. Geometriae Dedicata **97** (2003) 55–80.

[11] A. Korányi & H.M. Reimann; *The complex cross-ratio on the Heisenberg group*. L'Enseign. Math. **33** (1987) 291–300.

[12] C. Kourouniotis; *Complex length coordinates for quasi-Fuchsian groups*. Mathematika **41** (1994) 173–188.

[13] O. Lehto & K.I. Virtanen; Quasiconformal Mappings in the Plane. *Grundlehren der mathematischen Wissenschaften* **126**, Springer, 1973.

[14] B. Maskit; *Matrices for Fenchel-Nielsen coordinates*. Ann. Acad. Sci. Fenn. Math. **26** (2001) 267–304.

[15] T. Okai; *Effects of a change of pants decompositions on their Fenchel-Nielsen coordinates*. Kobe J. Maths. **10** (1993) 215–223.

[16] Y. Okumura; *Global real analytic length parameters for Teichmüller spaces*. Hiroshima Math. J. **26** (1996) 165-179.

[17] J.R. Parker & I.D. Platis; *Open sets of maximal dimension in complex hyperbolic quasi-Fuchsian space*. J. Diff. Geom **73** (2006) 319–350.

[18] J.R. Parker & C. Series; *Bending formulae for convex hull boundaries*. J. d'Analyse Math. **67** (1995) 165–198.

[19] P. Schmutz; *Die Parametrisierung des Teichmüllerraumes durch geodätische Längenfunktionen*. Comment. Math. Helv. **68** (1993) 278–288.

[20] S.P. Tan; *Complex Fenchel-Nielsen coordinates for quasi-Fuchsian structures*. Internat. J. Math. **5** (1994) 239–251.

[21] D. Toledo; *Representations of surface groups on complex hyperbolic space*. Journal of Differential Geometry **29** (1989) 125–133.

[22] A. Weil; *Remarks on the cohomology of groups*. Annals of Maths. **80** (1964) 149–157.

[23] Z.-X. Wen; *Relations polynomiales entre les traces de produits de matrices* C. R. Acad. Sci. Paris **318** (1994) 99–104.

[24] P. Will; *Lagrangian decomposability of some two-generator subgroups of* $\mathrm{PU}(2,1)$. C. R. Acad. Sci. Paris **340** (2005) 353–358.

[25] P. Will; *Groupes libres, groupes triangulaires et tore épointé dans* $\mathrm{PU}(2,1)$. Ph.D. thesis University of Paris VI.

[26] S. Wolpert; *The length spectra as moduli for compact Riemann surfaces*. Annals of Maths. **109** (1979), 323–351.

[27] S. Wolpert; *The Fenchel–Nielsen deformation*. Annals of Maths. **115** (1982), 501–528.

[28] E.Z. Xia; *The moduli of flat* $\mathrm{PU}(2,1)$ *structures on Riemann surfaces*. Pacific Journal of Mathematics **195** (2000) 231–256.

Department of Mathematical Sciences, University of Durham, Durham, United Kingdom.
E-mail address: j.r.parker@dur.ac.uk, i.d.platis@dur.ac.uk