Date 20 May 2009

Title of technical note:

COMPARISON BETWEEN COUPLED AND UNCOUPLED CONSOLIDATION ANALYSIS OF A RIGID SPHERE IN A POROUS ELASTIC INFINITE SPACE

Ashraf S. Osman
Lecturer,
School of Engineering,
Durham University,
South Road,
Durham DH1 3LE, UK
Tel: +44 191 334 2388;
Fax: +44 191 334 2390;
E-mail: ashraf.osman@durham.ac.uk

ABSTRACT

Linear consolidation analyses are usually treated either by means of Terzaghi-Rendulic uncoupled theory or Biot's consolidation theory. In this note, the problem of consolidation displacements around an axially loaded sphere was considered. It is demonstrated that both the uncoupled analysis and the coupled analysis give the same governing equation for pore fluid pressure dissipation with time. A simplified procedure for deriving transient strain components is illustrated. A general solution for time-dependent displacements is obtained using uncoupled consolidation analysis. Close agreement is evident between the new approximate uncoupled analysis solution and the existing coupled analysis solution with a maximum error of less than 0.5%.

Keywords: Elastic analysis; Poroelasticity; Spheres; Deformation.

INTRODUCTION

Consolidation is the process by which the pore fluid pressure is dissipated through a porous skeleton, following a change in the state of stress. Theories of consolidation fall into two main categories:

i. Uncoupled theory where it is assumed that the total stress remains constant everywhere throughout the consolidation process and the strains are caused only by the change of pore fluid pressure. This theory is attributed to Terzaghi (1923) and Rendulic (1936).

ii. Coupled Biot theory: in which the continuing interaction between skeleton and pore-fluid is included in the formulation. This leads, in general, to more complex equations for the solution (Biot, 1941). The partial differential equations governing the displacements **u** and pore fluid pressure p, for both a mechanically and hydraulically isotropic porous skeleton and an incompressible pore fluid, take the forms:

$$G\nabla^2\mathbf{u} + (\lambda + G)\nabla(\nabla.\mathbf{u}) = \nabla p \qquad (1)$$

$$\frac{k}{\gamma_f}\nabla^2 p = \frac{\partial}{\partial t}(\nabla.\mathbf{u}) \qquad (2)$$

where ∇ is the gradient operator; ∇^2 is Laplace's operator; λ and G are Lamé's constants for the porous elastic skeleton; k is the hydraulic conductivity; and γ_f is the unit weight of the fluid.

Sills (1975) considered the conditions under which the coupled consolidation analysis and the uncoupled analysis gives similar variation of pore fluid pressure with time. By introducing a scalar function ϕ, the displacements can be expressed as $\mathbf{u} = \nabla\phi$. Substituting in equation (1) and integrating gives:

$$2G\left[\frac{1-v}{1-2v}\right](\nabla.\mathbf{u}) = \mathrm{p} + \mathrm{K(t)} \qquad (3)$$

where v is Poisson's ratio and K(t) is some function of time.

Substituting (3) in (2) gives:

$$\frac{\partial \mathrm{p}}{\partial \mathrm{t}} + \frac{\partial \mathrm{K(t)}}{\partial \mathrm{t}} = \frac{2\mathrm{k}G(1-v)}{\gamma_\mathrm{f}(1-2v)}\nabla^2\mathrm{p} \qquad (4)$$

Sills (1975) concluded that for the cases where the flow occurs in one-direction when K(t) is constant, the variation of pore pressure with time calculated from Biot's coupled analysis becomes identical to that calculated using Terzaghi's uncoupled analysis.

Using uncoupled analysis could lead to a relatively simpler calculation procedure compared with coupled analysis. However, any complete consolidation analysis requires not only establishing the relation between the pore fluid pressure and the consolidation time, but also the variation of displacements and stresses with time. The stresses, strains, and displacements can be divided into transient and long-term components. The long-term components do not vary with time and can be calculated from the elastic solution. The transient components are time dependent and vary with the change in pore fluid pressure. In uncoupled analysis of one-dimensional consolidation problems, it is a relatively straightforward procedure to derive the transient components of displacements and strains from the change in pore fluid pressure. In two and three-dimensional consolidation problems, calculating displacements using uncoupled analysis becomes more complicated since information is needed regarding the changes of each transient component with the change in pore fluid pressure.

In this technical note we will present a simplified procedure for calculating time-dependent displacement using uncoupled consolidation analysis in a three-dimensional problem. This represents an improvement in the current practical approach of using uncoupled analysis only with the assumption that displacements occur in one-direction (as is the case in shallow foundation problems (Skempton and Bjerrum, 1957)).

We will consider the problem of consolidation around an axially-loaded rigid sphere in the interior of porous elastic space (Figure 1). This problem has practical implication in civil engineering especially with the increasing use of ball penetrometers in in-situ testing of soil (Randolph et al. 2005). It is also relevant to consolidation around base of under-reamed piles.

The potential applications of the solution presented in this note are not restricted to geomechanics; it has the potential to be applied to other fluid-saturated porous media. For example, it could be applied to the embedded rigid inclusion problems in biomechanical materials.

We will start by deriving a linear elastic solution for a rigid sphere embedded in an elastic infinite space. Next, we will present an uncoupled consolidation solution for the displacements. Finally, we will compare this uncoupled solution with the existing coupled consolidation solution of de Josselin de Jong (1955).

In this note, the reference stresses and the pore fluid pressure before loading the sphere is taken to be zero. The tensile stresses and strains are taken to be positive and the total stress $\boldsymbol{\sigma}$ is taken to be given by:

$$\boldsymbol{\sigma} = -p\mathbf{I} + \lambda(\nabla.\mathbf{u})\mathbf{I} + 2G\boldsymbol{\varepsilon} = -p\mathbf{I} + \boldsymbol{\sigma}' \tag{5}$$

where $\boldsymbol{\varepsilon}$ is the strain, $\boldsymbol{\sigma}'$ is the effective stress and $\mathbf{I}$ is the unity matrix.

LINEAR ELASTIC SOLUTION FOR STRESSES AND DISPLACEMENTS

In linear elasticity, the solutions of stresses and displacements of axisymmetric problems can be derived from Lamé's strain potential Ψ or from Love's strain function Φ (equations 1-6, Selvadurai 2001).

Lamé's strain potential satisfies:

$$\nabla^2\Psi = 0 \tag{6}$$

For axisymmetric problems in spherical coordinates (R, Θ, ϑ), the Laplace operator ∇^2 is given by:

$$\nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R} + \frac{\cot\Theta}{R^2}\frac{\partial}{\partial\Theta} + \frac{1}{R^2}\frac{\partial^2}{\partial\Theta^2} \tag{7}$$

The displacements are given by:

$$2Gu_R = \frac{\partial \Psi}{\partial R}$$
$$2Gu_\Theta = \frac{\partial \Psi}{R\partial \Theta} \tag{8}$$

and the stresses are given by:

$$\sigma_R = \frac{\partial^2 \Psi}{\partial r^2}$$
$$\sigma_\Theta = \frac{1}{R}\frac{\partial \Psi}{\partial R} + \frac{1}{R^2}\frac{\partial^2 \Psi}{\partial \Theta^2}$$
$$\sigma_\vartheta = \frac{1}{R}\frac{\partial \Psi}{\partial r} + \frac{\cot\Theta}{R^2}\frac{\partial \Psi}{\partial \Theta} \tag{9}$$
$$\sigma_{R\Theta} = \frac{\partial^2}{\partial R \partial \Theta}\left[\frac{\Psi}{R}\right]$$

Love's strain function satisfies:

$$\nabla^2 \nabla^2 \Phi = 0 \tag{10}$$

The displacements are given by:

$$2Gu_R = \cos\Theta\left[\frac{\partial^2}{\partial R^2} - 2(1-\nu)\nabla^2\right]\Phi + \frac{\sin\Theta}{R}\frac{\partial}{\partial \Theta}\left[\frac{1}{R} - \frac{\partial}{\partial R}\right]\Phi$$
$$2Gu_\Theta = \sin\Theta\left[2(1-\nu)\nabla^2 - \frac{1}{R}\frac{\partial}{\partial r} - \frac{1}{R^2}\frac{\partial^2}{\partial \Theta^2}\right]\Phi + \frac{\cos\Theta}{R}\frac{\partial}{\partial \Theta}\left[\frac{\partial}{\partial R} - \frac{1}{R}\right]\Phi \tag{11}$$

The stresses are given by:

$$\sigma_R = \cos\Theta \frac{\partial}{\partial R}\left[(2-\nu)\nabla^2 - \frac{\partial^2}{\partial R^2}\right]\Phi + \frac{\sin\Theta}{R}\frac{\partial}{\partial\Theta}\left[-\nu\nabla^2 + \frac{\partial^2}{\partial R^2} - \frac{2}{R}\frac{\partial}{\partial R} + \frac{2}{R^2}\right]\Phi$$

$$\sigma_\Theta = \cos\Theta \frac{\partial}{\partial R}\left[\nu\nabla^2 - \frac{1}{R}\frac{\partial}{\partial R} - \frac{1}{R^2}\frac{\partial^2}{\partial\Theta^2}\right]\Phi + \frac{\sin\Theta}{R}\frac{\partial}{\partial\Theta}\left[-(2-\nu)\nabla^2 + \frac{3}{R}\frac{\partial}{\partial R} - \frac{2}{R^2} + \frac{1}{R^2}\frac{\partial^2}{\partial\Theta^2}\right]\Phi$$

$$\sigma_\vartheta = \left\{\cos\Theta\frac{\partial}{\partial R} - \frac{\sin\Theta}{R}\frac{\partial}{\partial\Theta}\right\}\left[-(1-\nu)\nabla^2 + \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\Theta^2}\right]\Phi$$

$$\sigma_{R\Theta} = \frac{\cos\Theta}{R}\frac{\partial}{\partial\Theta}\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R} - \frac{2}{R}\right]\Phi + \sin\Theta\frac{\partial}{\partial r}\left[-(1-\nu)\nabla^2 + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\Theta^2}\right]\Phi \tag{12}$$

The solution of a rough rigid sphere embedded in a homogenous elastic medium can be obtained by superimposing stresses and displacements derived from Lamé’s strain potential and Love’s strain function. The stresses and displacements should be finite and reduce to zero as $R \to \infty$. If the sphere is taken to be fully-bonded to the surrounding elastic medium, then the relation between the radial and the circumferential components of the displacements at the interface between the sphere and the elastic medium (i.e. at $R=R_0$), is governed by:

$$u_R \sin\Theta = -u_\Theta \cos\Theta \tag{13}$$

The axial component (in the direction of the applied load P) of tractions acting on any spherical surface which encloses the rigid sphere and centred about its origin is governed by:

$$2\pi\int_0^\pi \left[\sigma_R\cos\Theta - \sigma_{R\Theta}\sin\Theta\right]R^2\sin\Theta\text{in} \; + P = 0 \tag{14}$$

A solution for Lamé’s strain potential takes the form of:

$$\Psi = \frac{A}{R^2}\cos\Theta \tag{15}$$

and for Love's strain function:

$$\Phi = BR \tag{16}$$

The coefficients A and B are obtained by satisfying equations (13) and (14) simultaneously. Therefore:

$$\begin{aligned} A &= \frac{PR_0^2}{24\pi(1-\nu)} \\ B &= \frac{P}{8\pi(1-\nu)} \end{aligned} \tag{17}$$

Thus, the displacements are given by:

$$\begin{aligned} 2Gu_R &= P\left[\frac{6R^2(1-\nu)-R_0^2}{12\pi(1-\nu)R^3}\right]\cos\Theta \\ 2Gu_\Theta &= -P\left[\frac{R_0^2+3R^2(3-4\nu)}{24\pi(1-\nu)R^3}\right]\sin\Theta \end{aligned} \tag{18}$$

and the stresses are given by:

$$\begin{aligned} \sigma_R &= -P\left[\frac{R^2(2-\nu)-R_0^2}{4\pi(1-\nu)R^4}\right]\cos\Theta \\ \sigma_\Theta &= P\left[\frac{R^2(1-2\nu)-R_0^2}{8\pi(1-\nu)R^4}\right]\cos\Theta \\ \sigma_\vartheta &= P\left[\frac{R^2(1-2\nu)-R_0^2}{8\pi(1-\nu)R^4}\right]\cos\Theta \\ \sigma_{R\Theta} &= P\left[\frac{R^2(1-2\nu)+R_0^2}{8\pi(1-\nu)R^4}\right]\sin\Theta \end{aligned} \tag{19}$$

UNCOUPLED CONSOLIDATION ANALYSIS

The relationship between the time-dependent volumetric strain ε_v and the pore fluid pressure can be expressed by:

$$\frac{\partial \varepsilon_v}{\partial t} = \frac{k}{\gamma_f} \nabla^2 p \qquad (20)$$

The volumetric strain is related to the mean effective stress σ'_m by:

$$\varepsilon_v = \frac{3(1-2\nu)}{2G(1+\nu)} \sigma'_m \qquad (21)$$

where $\sigma'_m = \dfrac{\sigma'_R + \sigma'_\Theta + \sigma'_\vartheta}{3}$

Immediately after the application of the load P and before the pore fluid pressure starts to dissipate, the volumetric strain is zero and so the initial change in mean effective stress, σ'_m, is also zero. Hence, the initial excess pore pressure, p_0, is equal to the mean total stress change. From equation (19), taking $\nu = 0.5$:

$$p_0 = \frac{P}{4\pi R^2} \cos\Theta \qquad (22)$$

During the consolidation process, the change in volumetric strain is linked to the change in mean effective stress by equation (21), and the latter can be related to the change of the pore fluid pressure. At the end of consolidation, the pore fluid pressure will be zero and the change mean effective stress, from equation (19), is:

$$\sigma'_m = \frac{P}{12\pi R^2} \frac{(1+\nu)}{(1-\nu)} \cos\Theta \qquad (23)$$

Comparing equations (22) and (23), gives:

$$\Delta\sigma_m' = \frac{(1+\nu)}{3(1-\nu)}\Delta p \tag{24}$$

Combining this result with equations (21) leads to:

$$\frac{\partial \varepsilon_v}{\partial t} = \frac{(1-2\nu)}{2(1-\nu)G}\frac{\partial p}{\partial t} \tag{25}$$

Substituting in equation (20) gives:

$$\frac{\partial p}{\partial t} = c\nabla^2 p \tag{26}$$

where c is the consolidation constant given by:

$$c = \frac{2kG(1-\nu)}{(1-2\nu)\gamma_w} \tag{27}$$

This expression does indeed turn out to be identical to Terzaghi's one-dimensional consolidation coefficient. It is also similar to that derived from the one-dimensional Biot's coupled analysis provided K(t) is taken to be constant (equation 4).

The boundary conditions for consolidation around a rigid sphere are:

$$p = 0 \quad \text{at} \;\; R \to \infty \tag{28}$$

$$\frac{\partial p}{\partial R} = 0 \quad \text{at} \quad R = R_0 \;\; \text{(for an impervious sphere)} \tag{29a}$$

$$p = 0 \quad \text{at} \quad R = R_0 \;\; \text{(for a pervious sphere)} \tag{29b}$$

$$p = p_0 \quad \text{at} \;\; t = 0 \tag{30}$$

Equation (26) can be solved if the pore fluid pressure p is expressed as a multiplication of two functions:

$$p = \xi(t).\psi(R,\Theta) \tag{31}$$

Equation (26) can then be re-written as two separate equations:

$$\frac{1}{c\xi}\frac{\partial \xi}{\partial t} = -\eta^2 \tag{32}$$

and

$$\frac{\partial^2 \psi}{\partial R^2} + \frac{2}{R}\frac{\partial \psi}{\partial R} + \frac{1}{R^2}\frac{\partial}{\partial \mu}\left\{(1-\mu^2)\frac{\partial \psi}{\partial \mu}\right\} + \eta^2\psi = 0 \tag{33}$$

where η is constant and $\mu = \cos\Theta$.

The solution to equation (32) is:

$$\xi(t) = e^{-c\eta^2 t} \tag{34}$$

and to equation (33) is:

$$\psi(R,\Theta) = \frac{C}{\sqrt{\eta R}}\left[J_{n+1/2}(\eta R) + \alpha Y_{n+1/2}(\eta R)\right]P_n(\mu) \tag{35}$$

where J_v and Y_v are Bessel functions of the first and of the second kinds, respectively, of order v.

Satisfying the boundary condition given by equation (30) implies that the pore pressure distribution in the Θ-direction can be expressed by a cosine function. From the properties of the Legendre polynomial:

$$P_1(\mu) = \mu \tag{36}$$

Therefore, the full expression for the pore fluid pressure is:

$$p = \sum_{k=1}^{\infty} \frac{C_k}{\sqrt{\eta_k R}} \left[J_{3/2}(\eta_k R) + \alpha_k Y_{3/2}(\eta_k R) \right] e^{-c\eta_k^2 t} \cos\Theta \tag{37}$$

Now let us assume that at some radial distance $R^* >> R_0$, the pore fluid pressure is never more than negligibly small (i.e. p=0 at R= R^*). Satisfying the boundary condition given by equation (28) implies that:

$$J_{3/2}(\eta_k R^*) + \alpha_k Y_{3/2}(\eta_k R^*) = 0 \tag{38}$$

so that the coefficient α_k is given by:

$$\alpha_k = -\frac{J_{3/2}(\eta_k R^*)}{Y_{3/2}(\eta_k R^*)} \tag{39}$$

The boundary condition given by equation 29 can be re-written as:

$$\begin{aligned} &2\sin(\eta_k (R^* - R_0)) + 2\eta_k^2 R^* R_0 \sin(\eta_k (R^* - R_0)) + \eta_k^3 R_0^2 R^* \cos(\eta_k (R^* - R_0)) \\ &- \eta_k^2 R_0^2 \sin(\eta_k (R^* - R_0)) - 2\eta_k R^* \cos(\eta_k (R^* - R_0)) + 2\eta_k R_0 \cos(\eta_k (R^* - R_0)) = 0 \end{aligned}$$

(for an impervious sphere) (40a)

$$(\eta_k^2 R^* R_0 + 1)\sin(\eta_k (R^* - R_0)) - \eta_k (R^* - R_0)\cos(\eta_k (R^* - R_0)) = 0$$

(for a pervious sphere) (40b)

where η_k represents the non-zero roots.

Satisfying the boundary condition at t = 0 (equation (30)) implies that:

$$\sum_{k=1}^{\infty} \frac{C_k}{\sqrt{\eta_k R}} \left[J_{3/2}(\eta_k R) + \alpha_k Y_{3/2}(\eta_k R) \right] = \frac{P}{4\pi R^2} \tag{41}$$

Multiplying both sides of this equation by $R^{3/2}\left[J_{3/2}(\eta_k R) + \alpha_k Y_{3/2}(\eta_k R) \right]$, integrating between R_0 and R^* and using the orthogonal properties of Bessel functions (McLachlan 1957), gives:

$$C_k \int_{R_0}^{R^*} \frac{R}{\sqrt{\eta_k}} \left[J_{3/2}(\eta_k R) + \alpha_k Y_{3/2}(\eta_k R) \right]^2 dR = \int_{R_0}^{R^*} \frac{P}{4\pi\sqrt{R}} \left[J_{3/2}(\eta_k R) + \alpha_k Y_{3/2}(\eta_k R) \right] dR \tag{42}$$

The values of C_k can then be found by integrating both sides of equation (42):

$$C_k = \frac{\dfrac{\eta_k^2 P}{\sqrt{2\pi}} \left[\left(\dfrac{\alpha_k \cos(\eta_k R^*) - \sin(\eta_k R^*)}{R^*} \right) - \left(\dfrac{\alpha_k \cos(\eta_k R_0) - \sin(\eta_k R_0)}{R_0} \right) \right]}{\left\{ \left[\dfrac{2(1+\alpha_k^2)({R^*}^2 \eta_k^2 - 1) - 2(\alpha_k^2 + \alpha_k \eta_k R^* - 1)\cos(2R\eta_k) + (4\alpha_k + \eta_k R^* - R^* \alpha_k^2 \eta_k)\sin(2R^*\eta_k)}{R^*} \right] - \left[\dfrac{2(1+\alpha_k^2)({R_0}^2 \eta_k^2 - 1) - 2(\alpha_k^2 + \alpha_k \eta_k R_0 - 1)\cos(2R_0\eta_k) + (4\alpha_k + \eta_k R_0 - R_0 \alpha_k^2 \eta_k)\sin(2R_0\eta_k)}{R_0} \right] \right\}} \tag{43}$$

The radial displacement can be divided into transient and long-term components. The transient components are time dependent and vary with the change in pore fluid pressure. The long-term components do not vary with time and can be calculated from the elastic solution. Thus, the radial displacement can be written as:

$$u_R = u_R^{\infty} + u_R^{t} \tag{44}$$

where the superscript ∞ donates the long-term component while t indicates the transient component, which is a negative quantity that reduces in magnitude to zero after a long time.

The radial displacements are calculated from the integral of the radial strain ε_R:

$$u_R = \int_{\infty}^{R} \varepsilon_R \, dR \tag{45}$$

The radial strain can be expressed as:

$$\varepsilon_R = \frac{1}{2(1+\nu)G}\left[\sigma'_R - \nu(\sigma'_\Theta + \sigma'_\vartheta)\right] \tag{46}$$

In order to evaluate this, information is needed regarding the changes in σ'_R, σ'_Θ and σ'_ϑ, in addition to the change in σ'_m (now known from equations (24) and (37)). Let us assume that:

$$\begin{aligned}
\frac{\partial \sigma'_R}{\partial t} &= \beta \frac{3\partial \sigma'_m}{\partial t} = \frac{\beta(1+\nu)}{(1-\nu)}\frac{\partial p}{\partial t} \\
\frac{\partial(\sigma'_\Theta + \sigma'_\vartheta)}{\partial t} &= (1-\beta)\frac{3\partial \sigma'_m}{\partial t} = \frac{(1-\beta)(1+\nu)}{(1-\nu)}\frac{\partial p}{\partial t}
\end{aligned} \tag{47}$$

where β is a proportionality factor ($0 \le \beta \le 1$).

It should be noted that if the total radial stress were to remain constant, the rate of change of the radial effective stress would equal the rate of change of the excess pore water pressure, implying an upper limit for β of $(1 - \nu)/(1 + \nu)$. However, here, we do not have conditions of constant total radial stress everywhere (although the radial stress at r_0 is essentially independent of Poisson's ratio (equation 19)). For the purpose of obtaining a closed-form solution for displacements, a constant value for β can be used as a first approximation of the relation between the change in stresses and the change in pore water pressure. As it will be demonstrated later in the note, the error resulting from this simplification is insignificant.

By substituting equation (47) into equation (46), the transient component of the radial strain can be written as:

$$\varepsilon_R^t = \frac{\left(\beta(1+\nu)-\nu\right)}{2(1-\nu)G} p \tag{48}$$

from which the transient component of the radial displacement may be derived as:

$$u_R^t = \int_{R^*}^{R} \frac{\left(\beta(1+\nu)-\nu\right)\cos\Theta}{2(1-\nu)G} \sum_{k=1}^{\infty} \frac{C_k}{\sqrt{\eta_k R}} e^{-c\eta_k^2 t} \left[J_{3/2}(\eta_k R) + \alpha_k Y_{3/2}(\eta_k R)\right] dR \tag{49}$$

Combining equation (49) with equation (18), the general expression for the total radial displacement is therefore:

$$u_R = \frac{P}{2G}\left[\frac{6R^2(1-\nu)-R_0^2}{12\pi(1-\nu)R^3}\right]\cos\Theta + \frac{\left(\beta(1+\nu)-\nu\right)}{2(1-\nu)G}\cos\Theta \sum_{k=1}^{\infty}\sqrt{\frac{2}{\pi}}\frac{C_k e^{-c\eta_k^2 t}}{\eta_k^2}\left(\left[\frac{\sin(\eta_k R^*)-\alpha_k\cos(\eta_k R^*)}{R^*}\right]-\left[\frac{\sin(\eta_n R)-\alpha_k\cos(\eta_k R)}{R}\right]\right) \tag{50}$$

The sphere displacement at t=0 can be estimated from the elastic solution by putting ν=0.5 in equation (18). Therefore, β can be shown to be given by:

$$\beta = \frac{-\dfrac{P}{12\pi R_0}\left[\dfrac{(1-2\nu)}{(1+\nu)}\right]}{\sum_{k=1}^{\infty}\sqrt{\dfrac{2}{\pi}}\dfrac{C_k}{\eta_k^2}\left(\left[\dfrac{\sin(\eta_k R^*)-\alpha_k\cos(\eta_k R^*)}{R^*}\right]-\left[\dfrac{\sin(\eta_k R_0)-\alpha_k\cos(\eta_k R_0)}{R_0}\right]\right)} + \frac{\nu}{(1+\nu)} \tag{51}$$

Therefore, the displacement of the sphere in the direction of the load ($R=R_0, \Theta = 0$) is given by:

$$u_R(R_0, 0, \vartheta, t) = \frac{P}{24\pi G R_0}\left[\frac{(5-6\nu)}{(1-\nu)}\right] - \frac{P}{24\pi G R_0}\left[\frac{(1-2\nu)}{(1-\nu)}\right] \frac{\sum_{k=1}^{\infty}\sqrt{\frac{2}{\pi}}\frac{C_k e^{-c\eta_k^2 t}}{\eta_k^2}\left(\left[\frac{\sin(\eta_k R^*) - \alpha_k \cos(\eta_k R^*)}{R^*}\right] - \left[\frac{\sin(\eta_k R_0) - \alpha_k \cos(\eta_k R_0)}{R_0}\right]\right)}{\sum_{k=1}^{\infty}\sqrt{\frac{2}{\pi}}\frac{C_k}{\eta_k^2}\left(\left[\frac{\sin(\eta_k R^*) - \alpha_k \cos(\eta_k R^*)}{R^*}\right] - \left[\frac{\sin(\eta_k R_0) - \alpha_k \cos(\eta_k R_0)}{R_0}\right]\right)} \quad (52)$$

COMPARISON WITH COUPLED CONSOLIDATION ANALYSIS

de Josselin de Jong (1957) derived a solution for the displacement of a rigid sphere in an infinite porous medium using coupled consolidation analysis. The solution is obtained by introducing four types of stress functions E_1, E_2, Ω_1 and Ω_2 such that:

$$\nabla^2 E_1 = c\frac{\partial E_1}{\partial t} \qquad \nabla^2 E_2 = 0$$
$$\nabla^2\nabla^2 \Omega_1 = 0 \qquad \nabla^2 \Omega_2 = 0 \quad (53)$$

Stresses, displacements and pore fluid pressure are derived from these functions. Functions E_2 and Ω_2 are identical so that three stress functions are needed to satisfy the boundary conditions. The Laplace transformation of the displacement of the sphere in the direction of the load was found to be given by:

$$2G\tilde{u}_R(R_0, 0, \vartheta, s) = \frac{P}{3\pi R_0 s}\left[1 + \frac{(1-2\nu)}{4(1-\nu)N}\right] \quad (54)$$

where

$$\tilde{\mathbf{u}} = \int_0^\infty e^{-st}\,\mathbf{u}\,dt$$

$$N = 1 + qR_0 + \frac{(qR_0)^2}{2} \quad \text{(for an impervious sphere)}$$

$$N = 1 + qR_0 \quad \text{(for a pervious sphere)}$$

$$q = \sqrt{\frac{s}{c}}$$

For a pervious sphere, the inverse Laplace transformation of equation (54) was shown to be given by:

$$2Gu_R(R_0, 0, \vartheta, t) = \frac{P}{3\pi R_0}\left[1 + \frac{(1-2\nu)}{4(1-\nu)}\left(1 - \exp\left(\frac{ct}{R_0^2}\right)\mathrm{erfc}\left(\sqrt{\frac{ct}{R_0^2}}\right)\right)\right] \qquad (55)$$

where

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp\left(-y^2\right)dy$$

It should be noted from the properties of the Laplace transformation that in equation (54) when t=0, $N \rightarrow \infty$ and when $t \rightarrow \infty$, $N \rightarrow 1$. Therefore, both the de Josselin de Jong's coupled consolidation analysis and the author's uncoupled analysis (equation (52)) give identical expressions for the initial and final displacement of the sphere.

Figures 2 compares the sphere displacement calculated from the coupled analysis of de Josselin de Jong (equation 52) with that calculated from the uncoupled analysis (equation 54).

The sphere displacement is normalised by its initial value and plotted against the non-dimensional time factor (ct/R_0^2). The results are shown for different Poisson's ratios. The calculations are carried out for the two different drainage conditions at the interface between the sphere and the surrounding porous medium. Figure 2a shows the results for an impervious sphere and the results for a previous sphere are shown in Figure 2b. In the uncoupled analysis, pore fluid pressure and displacements are taken to be vanish at an outer radius $R^*=60R_0$. A summation of 11268 terms of equation (52) was used to plot uncoupled analysis results shown in Figure 2a. This number of terms corresponds to the number of non-zero roots η_k of equations (40a) for values of ηR_0 between 0 and 600. Also, this number of terms gives an insignificant error of 0.1619% for the value of the initial pore fluid pressure (at t = 0 and $R = R_0$) calculated using equation (22). For the results shown in Figure 2b, 11267 non-zero roots of equation (40b) for values of ηR_0 between 0 and 600 were used to ensure that the pore fluid pressure is almost zero at $R=R_0$. For the coupled analysis, the inversion of the Laplace transform is carried out by numerical integration using the efficient scheme devised by de Hoog et al. (1982). The efficiency of the numerical integration algorithm is checked against the existing analytical solution for a pervious sphere (equation 55).

The uncoupled consolidation calculations in Figures 2a and 2b show excellent agreement with the coupled analysis. In the extreme case of the compressibility of a material (v=0), the maximum difference between the two analyses was found to be 0.4558% in the case of an impervious sphere and 0.4555% for a pervious sphere. This also demonstrates the efficiency of the proposed simplified procedure for deriving radial displacements in the uncoupled consolidation analysis (equations 47-51).

CONCLUSIONS

A linear elastic solution for an axially loaded sphere in infinite space was derived by formulating the problem in spherical coordinates and utilizing the properties of Lamé's strain potential and Love's strain potential. An expression for the initial excess pore water pressure generated immediately after the application of the load and before the start of consolidation was then derived from this elastic solution assuming incompressible conditions.

A closed-form solution for time-dependent displacement was derived using uncoupled consolidation analysis. The stresses, strains, and displacements are divided into transient and long-term components. The long-term components are calculated from the elastic solution. A simplified procedure for relating the transient components to the change in pore fluid pressure was outlined.

A comparison was made with the existing coupled consolidation analysis for a variety of drainage conditions and material properties. The uncoupled consolidation calculations were found to be in excellent agreement with the coupled analysis.

REFERENCES

de Josselin de Jong, G. (1957). Application of stress functions to consolidation problems. Proceedings of the 4th International Conference on Soil Mechanics and Foundation Engineering, London, 1957, Vol. 1, pp. 320–323.

de Hoog, F. R., Knight, J. H., and Stokes, A. N. (1982). An improved method for numerical inversion of Laplace transforms. S.I.A.M. J. Sci. and Stat. Comput., 3(3): 357-366.

McLachlan N. W. (1955), Bessel Functions for Engineers. Oxford, Clarendon Press.2nd edition.

Randolph, M.F. Cassidy, M.J., Gourvenec, S. and Erbrich, C.T. (2005) The Challenges of Offshore Geotechnical Engineering (Keynote) Proc. 16th Int. Symp. Soil Mech. and Geotech. Engng (ISSMGE), Osaka, Japan, Balkema. 1, 123-176.

Rendulic, L. (1936). Porenziffer und Porenwasserdruck in Tonen. Der Bauingenieur 17, No. 51/53, 559-564.

Selvadurai, A.P.S. (2001) On Boussinesq's problem, International Journal of Engineering Science, 39(3): 317-322.

Sills, G. C. (1975). Some conditions under which Biot's equation of consolidation reduced to Terzaghi's equation, Geotechnique (25) 1:129-132.

Skempton, A.W, Bjerrum, L (1957). Contribution to the settlement analysis of foundations on clay. *Geotechnique*, 7(4): 168-178.

Terzaghi, K. (1923). Die Berechnung der Durchlässigkeitsziffer des Tones aus dem Verlauf der hydrodynamischen Spannungserscheinungen. Sitzungsber. Akad. Wiss. Wien. Math-Naturwiss. Kl., Part Iia, 32, 125–138.

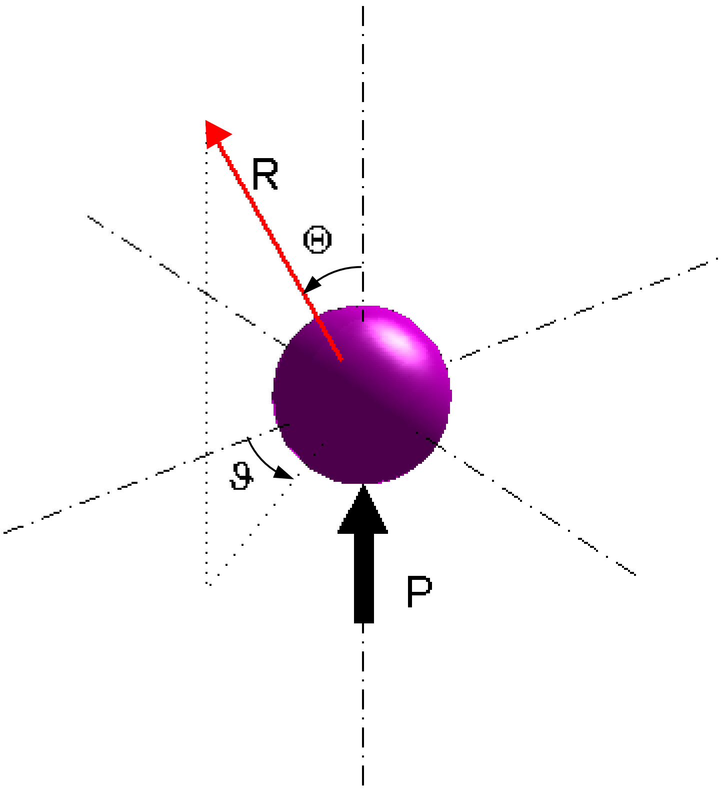

Figure 1 Nomenclature for a sphere under axial load

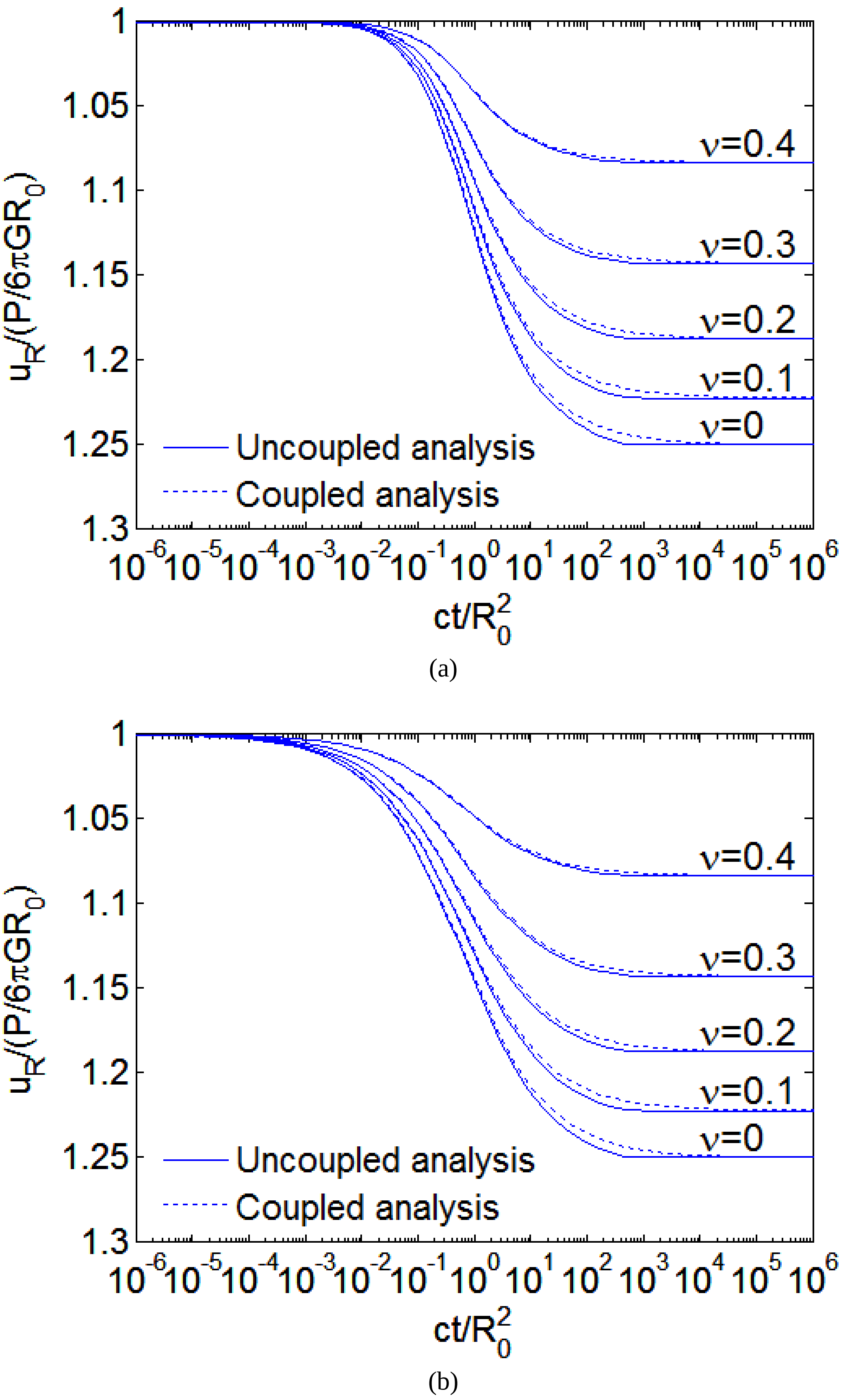

Figure 2 Variation of sphere displacement with time (a) impervious sphere (b) previous sphere