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Relative length of longest paths and longest cycles in triangle-free graphs

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Abstract

In this paper, we study triangle-free graphs. Let $G = (V_G, E_G)$ be an arbitrary triangle-free graph with minimum degree at least two and $\sigma_4(G) \geq |V_G| + 2$. We first show that either for any path P in G there exists a cycle C such that $|V_P \setminus V_C| \leq 1$, or G is isomorphic to exactly one exception. Using this result, we show that for any set S of at most δ vertices in G there exists a cycle C such that $S \subseteq V_C$.

1 Introduction

Let $G = (V_G, E_G)$ be a graph, where V_G is a finite set of order $|V_G| = n$ and E_G is a set of unordered pairs of two different vertices, called edges. For graph terminology not defined below we refer to [10]. For simplicity, we sometimes denote $|V_G|$ by $|G|$ and "$u \in V_G$" by "$u \in G$". For a vertex $u \in G$ we denote its *neighborhood*, i.e., the set of adjacent vertices, by $N_G(u) = \{v \mid uv \in E_G\}$. The *degree* $d_G(u)$ of a vertex u is the number of edges incident with it, or equivalently the size of its neighborhood. The minimum degree of G is denoted by δ_G. If no confusion is possible we will omit the subscript $_G$ in the later notations.

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A graph H is a *subgraph* of a graph G, denoted by $H \subseteq G$, if $V_H \subseteq V_G$ and $E_H \subseteq E_G$. For a subset $U \subseteq V_G$ we denote by $G[U]$ the *induced subgraph* of G over U; hence $G[U] = (U, E_G \cap (U \times U))$. For simplicity, we denote $G[V_G \backslash V_H]$ by $G - H$.

We denote the *complement* of a graph $G = (V, E)$ by $\overline{G} = (V, (V \times V)\backslash E)$. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we denote their *union* by $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ and their *join* by $G_1 * G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup (V_1 \times V_2))$. A *complete* graph is a graph with an edge between every pair of vertices. The complete graph on n vertices is denoted by K_n. The *complete bipartite* graph $\overline{K_k} * \overline{K_\ell}$ is denoted by $K_{k,\ell}$.

A graph G is called *connected* if for every pair of distinct vertices u and v, there exists a *path* P connecting u and v, i.e., a sequence $P = v_1 v_2 \dots v_p$ of distinct vertices starting by $u = v_1$ and ending by $v = v_p$, where each pair of consecutive vertices forms an edge of G. The vertices v_1 and v_p are called the *ends* of P. The order of a longest path in G is denoted by p_G. A vertex u is called a *cut vertex* of a connected graph G if $G[V \backslash \{u\}]$ is disconnected. A graph $G = (V, E)$ is called *k-connected* if $G[V \backslash U]$ is connected for any set $U \subseteq V$ of at most $k - 1$ vertices. A *cycle* C is a sequence $v_1 v_2 \dots v_p v_1$ of distinct vertices, where each pair of consecutive vertices forms an edge. The order of a longest cycle in a graph G is called the *circumference* c_G. A cycle C is called *dominating* if $G - C$ is edgeless.

Let $G = (V, E)$ be a graph. A set $U \subseteq V$ is called *independent* if G does not contain edges with both ends in U. The number of vertices in a maximum independent set is called the *independence number* of G. We denote

$$\sigma_k(G) = \min\{\sum_{i=1}^{k} d_G(x_i) \mid x_1, x_2, \dots, x_k \text{ are distinct and independent}\}.$$

If the independence number of G is less than k, then we define $\sigma_k(G) = \infty$.

Previous research

A graph G is called *hamiltonian* if G contains a cycle C with $V_C = V_G$. The problem of finding whether a given graph G is hamiltonian is one of the oldest problems in the history of graph theory and has direct applications to, for example, the travelling salesman problem. See Gould [14] for a survey. For a graph G that is not hamiltonian, a natural question is to ask how close it is to hamiltonicity. To measure this, we can take the difference $p_G - c_G$, called the *relative length*, between the order of a longest path and the circumference of G. We observe that $p_G - c_G = 0$ if and only if G is hamiltonian. Furthermore, $p_G - c_G \leq 1$ implies that all longest cycles are dominating. In order to see this, suppose C is a non-dominating longest cycle of a graph G. So $|C| = c_G$. Since C is non-dominating, $G - C$ contains an edge. We take a shortest path connecting this edge to C and extend it with $c_G - 1$ edges of C, say P. We then find that $p_G - c_G \geq |P| - |C| \geq 2$, a contradiction. In the literature many results on dominating cycles and the relative length $p_G - c_G$ can be found (see, e.g., [17, 18, 22, 23]).

Ore [19] showed that a graph G with $\sigma_2 \geq n$ is hamiltonian. Bondy [5] studied σ_3 and proved the following result.

Theorem 1 ([5]). *If G is a 2-connected graph with $\sigma_3 \geq n + 2$, then all longest cycles are dominating.*

The lower bound on σ_3 in Theorem 1 is tight. One can see this as follows. Consider the graph $G_k = (K_k \cup K_k \cup K_k) * \overline{K_2}$ of order $n = 3k + 2$ for $k \geq 2$. It is easy to check that G_k is 2-connected and has $\sigma_3(G_k) = 3k + 3 = n + 1$. However, since each cycle in G_k can pass through $\overline{K_2}$ at most twice, any longest cycle does not contain vertices of one K_k, and consequently is not dominating.

Enomoto et al. [12] proved the following.

Theorem 2 ([12]). *If G is a 2-connected graph with $\sigma_3 \geq n + 2$, then $p_G - c_G \leq 1$.*

We already noted that $p_G - c_G \leq 1$ implies that all longest cycles are dominating. Hence, Theorem 2 generalizes Theorem 1. Clearly, the opposite is not true. For example, consider the graph obtained from a cycle $u_1 u_2 \ldots u_p u_1$ by adding two new vertices v and w and two edges vu_1 and wu_2.

Our results

In this paper we are interested in proving a similar result for *triangle-free* graphs (graphs that do not contain K_3) corresponding to Theorem 2 of Enomoto et al. Is it possible to make a jump from σ_3 to σ_4 when we restrict ourselves to this graph class? Triangle-free graphs are the natural generalization of bipartite graphs and therefore have been widely studied in the literature, also in the context of hamiltonian research (cf. [2, 3, 7, 13, 16]). Broersma, Yoshimoto and Zhang [9] showed that a 2-connected triangle-free graph with $\sigma_3 \geq (n+5)/2$ contains a longest cycle that is dominating. The lower bound on σ_3 is tight, even for the existence of dominating cycles. Note that graphs satisfying the conditions of this theorem might contain longest cycles that are not dominating. However, if $\sigma_2 \geq (n+1)/2$, then all longest cycles are dominating [24]. This lower bound is almost best possible by examples due to Ash and Jackson [1].

The main result of this paper is as follows. Its proof is given in Section 2.

Theorem 3. *Let G be a triangle-free graph with $\delta \geq 2$ not isomorphic to the graph in Figure 1(i). If $\sigma_4 \geq n+2$ then for any path P there exists a cycle C such that $|P - C| \leq 1$.*

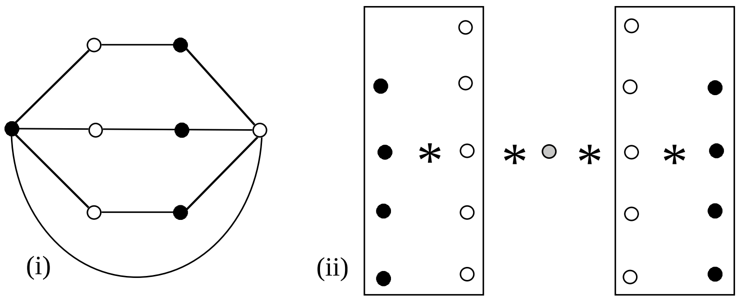

Figure 1: (i) exception for Theorem 3, (ii) the graph H_5.

We note that Theorem 3 immediately implies that $p_G - c_G \leq 1$. Hence, this result for triangle-free graphs is "similar" to Theorem 2 of Enomoto et al. for 2-connected graphs.

The lower bound on σ_4 in Theorem 3 is tight. In order to see this, consider the graph $H_k = \overline{K_{k-1}} * \overline{K_k} * K_1 * \overline{K_k} * \overline{K_{k-1}}$ of order $n = 4k-1$ for $k \geq 2$. For an illustration of the case $k = 5$, see Figure 1(ii). Obviously, H_k is triangle-free. It is easy to check that H_k has minimum degree $2 \leq \delta_{H_k} = k = \frac{n+1}{4}$. Since H_k contains at least four vertices of minimum degree, we find that $\sigma_4(H_k) = n+1$. Furthermore, H_k contains a path P of order $|P| = n$. However, any cycle can pass through K_1 at most once. So a longest cycle C contains all vertices of exactly one $\overline{K_{k-1}}$, one adjacent $\overline{K_k}$ and the vertex of the K_1. Hence, for all $k \geq 2$, the circumference of H_k is $c_{H_k} = 2k = \frac{n+1}{2} \leq n-2$. So, for P there does not exist a cycle C with $|P - C| \leq 1$. This means that the bound on σ_4 is tight indeed.

In Theorem 3 no condition is imposed on the connectivity of a graph. A natural question (cf. Theorem 2) is to ask whether adding such a condition would be helpful for decreasing the lower bound on σ_4. However, this is not the case: we can add all possible edges between the left $\overline{K_{k-1}}$ and the right $\overline{K_{k-1}}$ in H_k. This way we obtain a new graph H'_k that is still triangle-free, has minimum degree $\frac{n+1}{4} \geq 2$ and $\sigma_4(H'_k) = n+1$, and furthermore contains a path of length n. However, a longest cycle C will pass through all vertices except one vertex of each $\overline{K_{k-1}}$, so $|C| = c_{H'_k} = n-2$. We reach the same conclusion as before.

In the literature the following related problem has been studied for general graphs and graph classes (see, e.g., [4, 6, 8, 11, 15, 20, 21]): for a given graph G, does any subset S of vertices of restricted size have some cycle passing through it? As an application of Theorem 3, we obtain the following result for triangle-free graphs. Its full proof is given in Section 3.

Theorem 4. *Let G be a triangle-free graph with $\delta \geq 2$. If $\sigma_4 \geq n+2$, then for any set S of at most δ vertices, there exists a cycle C such that $S \subseteq V_C$.*

This result implies that a triangle-free graph with $\delta \geq 2$ and $\sigma_4 \geq n+2$ is 2-connected. On the other hand, the previously defined graph H_k contains a cut vertex, namely the vertex of the K_1. Hence, the lower bound on σ_4 in Theorem 4 is tight. In Section 3 we show that a triangle-free graph with $\delta \geq 2$ and $\sigma_4 \geq n+1$ is connected. The lower bound on σ_4 is tight due to the graph $K_{k,k} \cup K_{k,k}$ for $k \geq 2$.

Additional notations

Let $G = (V, E)$ be a graph. For a subset $U \subseteq V$ and vertex $u \in V$ we sometimes write "$U \backslash u$" instead of "$U \backslash \{u\}$".

Let H be a subgraph of G. We denote $N_G(x) \cap V_H$ by $N_H(x)$ and its cardinality $|N_H(x)|$ by $d_H(x)$. The set of neighbours $\bigcup_{v \in H} N_G(v) \setminus V_H$ is denoted by $N_G(H)$ or $N(H)$. For an edge $e = uv$ in G, we write $N(e) = N(\{u, v\})$. For a subgraph $F \subseteq G$, we write $N_G(H) \cap V_F$ as $N_F(H)$.

Let $C = v_1 v_2 \ldots v_p v_1$ be a cycle with a fixed orientation. The successor v_{i+1} of v_i is denoted by v_i^+ and its predecessor v_{i-1} by v_i^-. For a vertex subset A in C, we denote $\{v_i^+ \mid v_i \in A\}$ and $\{v_i^- \mid v_i \in A\}$ by A^+ and A^-, respectively. The segment $v_i v_{i+1} \ldots v_j$ is written as $v_i \overrightarrow{C} v_j$, where the subscripts are to be taken modulo $|C|$.

The converse segment $v_j v_{j-1} \dots v_i$ is written as $v_j \overleftarrow{C} v_i$. Similarly, for a path $P = u_1 u_2 \dots u_p$, we use the notations $u_i \overrightarrow{P} u_j = u_i u_{i+1} \dots u_j$ and $u_j \overleftarrow{P} u_i = u_j u_{j-1} \dots u_i$.

2 The Proof of Theorem 3

Let S be a vertex subset of G. If a path P is a longest path over all paths containing S, then we call P a *maximal path for* S. The set of all maximal paths for S is denoted by $\mathcal{P}(S)$. Before proving Theorem 3 we first show the following lemma.

Lemma 5. *Let G be a triangle-free graph with $\delta_G \geq 2$ not isomorphic to the graph in Figure 1i. Then for any path R, there either exists a path in $\mathcal{P}(V_R)$ such that the degree sum of the ends is at least $\sigma_4(G)/2$, or else a cycle C such that $|R - C| \leq 1$.*

Proof. Let G be a triangle-free graph with $\delta_G \geq 2$. Assume that G is not isomorphic to the graph in Figure 1i. Let R be any path in G and $P = u_1 u_2 \dots u_p \in \mathcal{P}(V_R)$ such that the degree sum of the ends is maximal in $\mathcal{P}(V_R)$. Notice that $N(u_1) = N_P(u_1)$ and $N(u_p) = N_P(u_p)$. So all neighbors of u_1 and u_p in G belong to P.

Suppose there are vertices $u_i \in N(u_1) \setminus u_2$ and $u_j \in N(u_p) \setminus u_{p-1}$ such that $i \leq j$. Then $\{u_1, u_{i-1}, u_{j+1}, u_p\}$ is independent; otherwise there is a triangle (forbidden) or a cycle containing V_R (we are done). Because $d(u_1) + d(u_{i-1}) + d(u_{j+1}) + d(u_p) \geq \sigma_4$, one of the degree sums $d(u_1) + d(u_p)$ and $d(u_{i-1}) + d(u_{j+1})$ is at least $\sigma_4/2$. Hence, at least one of the paths P or $u_{i-1} \overleftarrow{P} u_1 u_i \overrightarrow{P} u_j u_p \overleftarrow{P} u_{j+1}$ is a desired path.

In the remaining case we have

$$i > j \text{ for any two vertices } u_i \in N(u_1) \setminus u_2 \text{ and } u_j \in N(u_p) \setminus u_{p-1}. \tag{1}$$

Suppose there is a vertex $u_s \in N_P(u_1) \setminus \{u_2, u_{p-2}\}$. Since $\delta_G \geq 2$ and $N(u_p) = N_P(u_p)$, vertex u_p has a neighbor $u_t \neq u_{p-1}$ on P. Then we find that the path $P' = u_{t+1} \overrightarrow{P} u_s u_1 \overrightarrow{P} u_t u_p \overleftarrow{P} u_{s+1}$ is a path in $\mathcal{P}(V_R)$. The vertex u_1 is not adjacent to u_{t+1} nor u_{s+1}; otherwise there is a triangle or a cycle containing V_R. Also, the vertex u_p is not adjacent to u_{t+1} nor to u_{s+1} by statement (1) and $u_s \neq u_{p-2}$. Thus $\{u_1, u_{t+1}, u_{s+1}, u_p\}$ is an independent set. Hence, at least one of the paths P and P' is a desired path as in the previous case. Therefore $N(u_1) = \{u_2, u_{p-2}\}$ and, by symmetry, $N(u_p) = \{u_3, u_{p-1}\}$. Furthermore, by the maximality of the degree sum of the ends of P we deduce that

$$\text{the degree of an end of any path in } \mathcal{P}(V_R) \text{ is two.}$$

Because the path $u_1u_2u_3u_p\overleftarrow{P}u_4$ is in $\mathcal{P}(V_R)$, the vertex u_1 has to be adjacent to $u_4^{++} = u_6$; otherwise, as in the above case, we can obtain a desired cycle or path. Therefore $u_6 = u_{p-2}$, i.e., $p = 8$, and so any vertex in $\{u_1, u_2, u_4, u_5, u_7, u_8\}$ is the end of some path in $\mathcal{P}(V_R)$, and consequently has degree two. As G is triangle-free, the vertices u_1, u_5 and u_7 are mutually disjoint. If $G - P$ is not empty, then for any $x \in G - P$, the set $\{x, u_1, u_5, u_7\}$ is independent. Hence we find that

$$d(x) \geq \sigma_4 - (d(u_1) + d(u_5) + d(u_7)) \geq n + 2 - 6 = n - 4.$$

However, x is adjacent to none of the vertices in $\{u_1, u_2, u_4, u_5, u_7, u_8\}$ because their degrees are all equal to two. Thus $d(x) \leq n-7$, a contradiction. Therefore $G-P = \emptyset$ and $n = 8$. As u_3 is adjacent to none of the vertices u_1, u_5, u_7, vertex u_3 has to be adjacent to u_6; otherwise $d(u_1) + d(u_3) + d(u_5) + d(u_7) = 9 < n + 2$. Hence G is isomorphic to the graph in Figure 1i, a contradiction. $\square$

We are ready to prove Theorem 3. Let G be a triangle-free graph with $\delta \geq 2$ and $\sigma_4 \geq n + 2$ that is not isomorphic to the graph in Figure 1i. Let R be any path in G. We prove that G contains a desired cycle, i.e., a cycle C such that $|R - C| \leq 1$.

Suppose the independence number of G is at most three. Then $\sigma_4(G) = \infty$. By Lemma 5, there exists a cycle C such that $|R - C| \leq 1$.

From now on we assume that the independence number of G is at least four. Let $P = u_1u_2 \dots u_p \in \mathcal{P}(V_R)$ such that

$$\text{the degree sum of the ends is maximal in } \mathcal{P}(V_R). \tag{2}$$

Then from Lemma 5, $d(u_1)+d(u_p) \geq \sigma_4/2$. Notice that we may assume that there is no path in $\mathcal{P}(V_R)$ whose ends are adjacent; otherwise obviously there exists a cycle containing V_R.

If there is $u_l \in N_P(u_1) \cap N_P(u_p)^+$, then the cycle $u_1\overrightarrow{P}u_l^-u_p\overleftarrow{P}u_lu_1$ is a desired cycle. Thus we can suppose $N_P(u_1) \cap N_P(u_p)^+ = \emptyset$. Similarly, we get $N_P(u_1) \cap N_P(u_p)^{++} = \emptyset$ and $N_P(u_1)^- \cap N_P(u_p)^+ = \emptyset$. If $N_P(u_1)^- \cap N_P(u_p)^{++}$ is also empty, then $N_P(u_1), N_P(u_1)^-, N_P(u_p)^+$ and $(N_P(u_p) \setminus u_p)^{++}$ are mutually disjoint. Hence we find that

$$\begin{aligned} n \geq |P| &\geq |N_P(u_1)| + |N_P(u_1)^-| + |N_P(u_p)^+| + |(N_P(u_p) \setminus u_p)^{++}| \\ &\geq 2d(u_1) + 2d(u_p) - 1 \geq \sigma_4 - 1 > n. \end{aligned}$$

This is a contradiction. Therefore $N_P(u_1)^- \cap N_P(u_p)^{++} \neq \emptyset$.

Let $u_i \in N_P(u_1)^- \cap N_P(u_p)^{++}$.

Claim 1. *If $d(u_i) + d(u_{i-1}) > n/2$, then there is a desired cycle.*

Proof. Let $e_0 = x_1x_2 = u_{i-1}u_i$ and

$$C = u_1 \overrightarrow{P} u_{i-2} u_p \overleftarrow{P} u_{i+1} u_1 = v_1 v_2 \dots v_{p-2} v_1$$

which occur on C in the order of their indices. Notice that $N(e_0) = N(x_1) \cup N(x_2) \setminus \{x_1, x_2\} \subset V_C$ because P is a maximal path for V_R.

If $N(e_0)$ and $N(e_0)^+$ are not disjoint, then there exists a triangle or a desired cycle. Hence $N(e_0) \cap N(e_0)^+ = \emptyset$. In the set of segments $C - N(e_0)$, there are two segments $v_s^+ \overrightarrow{C} v_{s'}^-$ and $v_t^+ \overrightarrow{C} v_{t'}^-$ such that $\{v_s, v_{t'}\} \subset N(x_1)$ and $\{v_{s'}, v_t\} \subset N(x_2)$. Then $v_{s+2}, v_{t+2} \notin N_C(e_0) \cup N_C(e_0)^+$; otherwise there is a desired cycle. Therefore, we find

$$\begin{aligned} n - 2 \geq |C| &\geq |N(e_0)| + |N(e_0)^+| + |\{v_{s+2}, v_{t+2}\}| \\ &= |N_C(x_1)| + |N_C(x_1)^+| + |N_C(x_2)| + |N_C(x_2)^+| + |\{v_{s+2}, v_{t+2}\}| \\ &= 2(d(x_1) - 1) + 2(d(x_2) - 1) + 2 = 2(d(x_1) + d(x_2)) - 2 > n - 2. \end{aligned}$$

This is a contradiction. □

If $\delta \geq (n + 2)/4$, then our proof is completed now by this claim. We divide our argument into two cases.

Case 1. $|N_P(u_1)^- \cap N_P(u_p)^{++}| = 1$

Let $\{u_i\} = N_P(u_1)^- \cap N_P(u_p)^{++}$. We show that $d(u_i) + d(u_{i-1}) > n/2$. Because

$$\begin{aligned} n \geq |P| &\geq |N_P(u_1)| + |N_P(u_1)^-| + |N_P(u_p)^+| + |(N_P(u_p) \setminus u_{p-1})^{++}| \\ &\quad -|N_P(u_1)^- \cap N_P(u_p)^{++}| \\ &= 2d(u_1) + 2d(u_p) - 1 - 1 \geq \sigma_4 - 2 \geq n, \end{aligned}$$

it holds that

$$V_G = V_P = N_P(u_1) \cup N_P(u_1)^- \cup N_P(u_p)^+ \cup (N_P(u_p) \setminus u_{p-1})^{++} \tag{3}$$

and that

$$d(u_1) + d(u_p) = \frac{n}{2} + 1. \tag{4}$$

Hence the order n is even.

Because

$$u_{i-3}\overleftarrow{P}u_1u_{i+1}u_iu_{i-1}u_{i-2}u_p\overleftarrow{P}u_{i+2} \in \mathcal{P}(V_R),$$

we have $u_{i-3}u_{i+2} \notin E_G$. If $u_{i-3}u_1 \in E_G$ then

$$u_{i-2} \notin N_P(u_1) \cup N_P(u_1)^- \cup N_P(u_p)^+ \cup (N_P(u_p) \setminus u_{p-1})^{++}.$$

See Figure 2i. This contradicts (3). Thus $u_{i-3}u_1 \notin E_G$. Especially, u_{i-3} is not u_2.

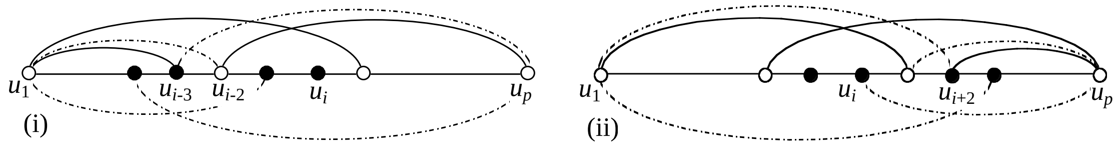

Figure 2:

Similarly, if $u_{i+2}u_p \in E_G$, then

$$u_{i+2} \notin N_P(u_1) \cup N_P(u_1)^- \cup N_P(u_p)^+ \cup (N_P(u_p) \setminus u_{p-1})^{++}.$$

See Figure 2ii. This also contradicts (3). Hence, $u_{i+2}u_p \notin E_G$ and especially $u_{i+2} \neq u_{p-1}$. As $u_1u_p \notin E_G$, $\{u_1, u_{i-3}, u_{i+2}, u_p\}$ is an independent set.

Let $x_1x_2 = u_{i-1}u_i$ and $w_1 = u_{i-3}$ and $w_2 = u_{i+2}$. Because $d(u_1)+d(u_p)+d(w_1)+d(w_2) \geq \sigma_4 \geq n+2$, we have

$$d(w_1) + d(w_2) = \frac{n}{2} + 1$$

by (2) and (4). Notice that none of u_1, u_p, w_1, w_2 are adjacent to x_1 nor x_2; otherwise easily we can find a triangle or a desired cycle. Hence for each i, j,

$$d(u_1) + d(u_p) + d(x_i) + d(w_j) \geq n + 2.$$

Assume that $n/2$ is even, say $2l$. Then $d(u_1) + d(u_p) = d(w_1) + d(w_2) = 2l + 1$. By symmetry, we can suppose that $d(w_1) \leq l$. Because

$$d(u_1) + d(u_p) + d(x_i) + d(w_1) \geq 4l + 2,$$

we have $d(x_i) \geq l+1$ for $i = 1, 2$. Hence $d(x_1) + d(x_2) \geq 2l + 2 > n/2$.

Suppose $n/2$ is odd, say $2l+1$. Then $d(u_1) + d(u_p) = d(w_1) + d(w_2) = 2l + 2$. By symmetry, we may assume that $d(w_1) \leq l + 1$. Because

$$d(u_1) + d(u_2) + d(w_1) + d(x_i) \geq 4l + 4,$$

we have $d(x_i) \geq l+1$ for $i = 1, 2$. Thus $d(x_1) + d(x_2) \geq 2l + 2 > n/2$.

Therefore, in either cases, $d(u_i) + d(u_{i-1}) > n/2$, and hence we are done by Claim 1.

Case 2. $|N_P(u_1)^- \cap N_P(u_p)^{++}| \geq 2$.

Let $u_i, u_j \in N_P(u_1)^- \cap N_P(u_p)^{++}$ $(i > j)$. If u_{i-1} is adjacent to u_{j-1}, then the cycle $u_1 \overrightarrow{P} u_{j-1} u_{i-1} u_i u_i^+ \overrightarrow{P} u_p u_{i-2} \overleftarrow{P} u_j^+ u_1$ is a desired cycle. See Figure 3i. Therefore

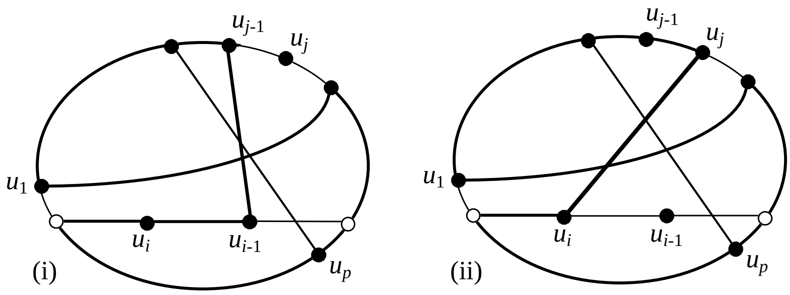

Figure 3:

$u_{i-1}u_{j-1} \notin E_G$. Similarly we can obtain $u_i u_j \notin E_G$, see Figure 3ii. Hence we find that

$$\begin{aligned} & (d(u_1) + d(u_p) + d(u_{i-1}) + d(u_{j-1})) + (d(u_1) + d(u_p) + d(u_i) + d(u_j)) \\ \geq \ & \sigma_4 + \sigma_4 \geq 2n + 4. \end{aligned}$$

By symmetry, we may without loss of generality assume that

$$d(u_1) + d(u_p) + d(u_{i-1}) + d(u_i) \geq n + 2. \tag{5}$$

Let $e_0 = x_1x_2 = u_{i-1}u_i$ and C be the cycle $u_1 \overrightarrow{P} u_{i-2} u_p \overleftarrow{P} u_{i+1} u_1 = v_1 v_2 \dots v_{p-2} v_1$ which occur on C in the order of their indices. Notice that a vertex in $N_C(e_0)^+ \cup \{x_1, x_2\}$ has no neighbours in $G - P$; otherwise P is not maximal. Let $v_s \in N_C(x_2)$

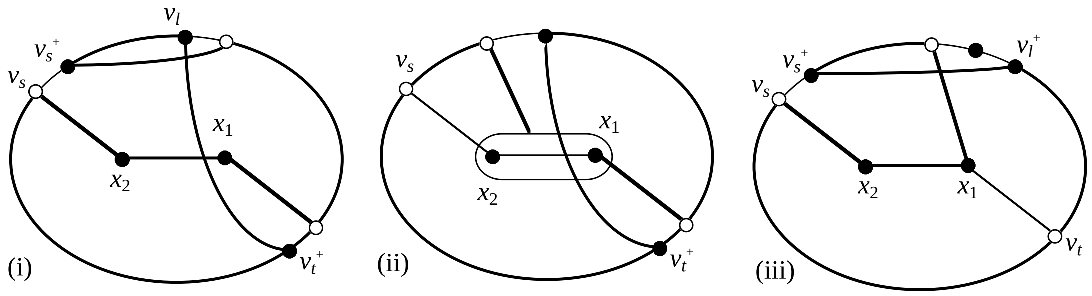

Figure 4:

and $v_t \in N_C(x_1)$ and $I_s = v_s^+\overrightarrow{C}v_t$ and $I_t = v_t^+\overrightarrow{C}v_s$. If there is a vertex $v_l \in N_{I_s}(v_s^+)^- \cap N_{I_s}(v_t^+)$, then the cycle $v_s^+\overrightarrow{C}v_lv_t^+\overrightarrow{C}v_sx_2x_1v_t\overleftarrow{C}v_l^+v_s^+$ is a desired cycle. See Figure 4i. Hence $N_{I_s}(v_s^+)^- \cap N_{I_s}(v_t^+) = \emptyset$. Similarly, we have that

$$N_{I_s}(e_0)^+ \cap N_{I_s}(v_t^+) = \emptyset \text{ and } N_{I_s}(v_s^+)^- \cap N_{I_s}(x_1)^+ = \emptyset.$$

See Figure 4ii-iii. Hence we obtain that

$$|I_s| \geq |N_{I_s}(v_s^+)^-| + |N_{I_s}(v_t^+)| + |(N_{I_s}(e_0) \setminus v_t)^+| - |N_{I_s}(v_s^+)^- \cap N_{I_s}(x_2)^+|.$$

Let $L = N_{I_s}(v_s^+)^- \cap N_{I_s}(x_2)^+$. If L is not empty, then for any vertex $v_l \in L$, $v_l^+ \notin N_{I_s}(v_s^+)^-$ because G is triangle-free. If $v_l^+v_t^+ \in E_G$, then the cycle $v_l^-x_2x_1v_t\overleftarrow{C}v_l^+v_t^+\overrightarrow{C}v_l^-$ is a desired cycle. Since $v_l^+ \notin N_C(e_0)^+$,

$$v_l^+ \notin N_{I_s}(v_s^+)^- \cup N_{I_s}(v_t^+) \cup N_{I_s}(e_0)^+,$$

and so we deduce that

$$L^+ \cap (N_{I_s}(v_s^+)^- \cup N_{I_s}(v_t^+) \cup N_{I_s}(e_0)^+) = \emptyset.$$

Similarly, the vertex v_s^{++} is not contained in $N_{I_s}(v_s^+)^- \cup N_{I_s}(v_t^+) \cup N_{I_s}(e_0)^+$. Therefore we find that

$$\begin{aligned} |I_s| &\geq |N_{I_s}(v_s^+)^-| + |N_{I_s}(v_t^+)| + |(N_{I_s}(e_0) \setminus v_t)^+| - |L| + |L^+| + |\{v_s^{++}\}| \\ &\geq |N_{I_s}(v_s^+)| + |N_{I_s}(v_t^+)| + |N_{I_s}(e_0) \setminus v_t| + 1 \\ &= d_{I_s}(v_s^+) + d_{I_s}(v_t^+) + d_{I_s}(x_1) + d_{I_s}(x_2). \end{aligned}$$

By symmetry, we get $|I_t| \geq d_{I_t}(v_s^+) + d_{I_t}(v_t^+) + d_{I_t}(x_1) + d_{I_t}(x_2)$. By (5),

$$\begin{aligned} n - 2 \geq |C| = |I_s| + |I_t| &\geq d_{I_s}(v_s^+) + d_{I_s}(v_t^+) + d_{I_s}(x_1) + d_{I_s}(x_2) \\ &\quad + d_{I_t}(v_s^+) + d_{I_t}(v_t^+) + d_{I_t}(x_1) + d_{I_t}(x_2) \\ &= d(v_s^+) + d(v_t^+) + (d(x_1) - 1) + (d(x_2) - 1) \geq n, \end{aligned}$$

which is a contradiction. This completes the proof of Theorem 3.

3 The Proof of Theorem 4

Let $G = (V, E)$ be a triangle-free graph with $\delta \geq 2$ and $\sigma_4 \geq n+2$. If G is isomorphic to the exception of Theorem 3, then obviously for any two vertices, there is a cycle containing the specified vertices. By Theorem 3 and the following lemma, it is enough to show that G is connected. A cycle C is called a *swaying* cycle of a subset $S \subseteq V$ if $|C \cap S|$ is maximum over all cycles of G.

Lemma 6. *Let G be a connected graph such that for any path P, there exists a cycle C such that $|P - C| \leq 1$. Then for any set S with at most δ vertices, there exists a cycle C such that $S \subset V_C$.*

Proof. Let $S \subseteq V_G$ and let C be a longest swaying cycle of S. Suppose $S - C \neq \emptyset$. For any vertex $x \in S - C$, there is a path Q joining x and C. Let P be a longest path containing $V_{C \cup Q}$. Then there exists a cycle D such that $|P - D| \leq 1$. If x has neighbours in $G - C$, then $|P| \geq |C| + 2$ and so $|D| \geq |C| + 1$. Because $|D \cap S| \geq |C \cap S|$, this contradicts the assumption that C is a longest swaying cycle. Hence $N_{G-C}(x) = \emptyset$.

Because $|C \cap S| < \delta$ and $d_C(x) = d(x) \geq \delta$, there exist two vertices $v_i, v_j \in N(x)$ such that $v_{i+1} = v_j$ or $v_i^+ \overrightarrow{C} v_j^- \subset C - S$. Hence the cycle $v_i x v_j \overrightarrow{C} v_i$ contains at least $|C \cap S| + 1$ vertices of S. This contradicts the assumption that C is a swaying cycle. $\square$

Before we can prove that G is connected we first need to show the following lemma.

Lemma 7. *Let H be a connected component of a triangle-free graph G. If $|H| \geq 3$, then H contains non-adjacent vertices x and y such that $|H| \geq \max\{2d(x), 2d(y)\}$.*

Proof. Let $P = u_1 u_2 \ldots u_p$ be a longest path of H. If $u_1 u_p \notin E_G$, then $|P| \geq |N(u_1)| + |N(u_1)^-| + |\{u_p\}| = 2d(u_1) + 1$. Hence by symmetry, we have $|H| \geq \max\{2d(u_1) + 1, 2d(u_p) + 1\}$, and so $\{u_1, u_p\}$ is a desired pair. If $u_1 u_p \in E_G$, then $u_1 u_{p-1} \notin E_G$, and $V_H = V_P$ as P is a longest path. Then, we have

$$|P - u_p| \geq |N(u_{p-1}) \setminus u_p| + |(N(u_{p-1}) \setminus u_p)^+| + |u_1| = 2d(u_{p-1}) - 1.$$

Therefore $|H| \geq 2d(u_{p-1})$. As in the above case, we can have $|H| \geq 2d(u_1)$, and so $\{u_1, u_{p-1}\}$ is a desired pair. $\square$

By using Lemma 7 we can show that G is indeed connected. This finishes the proof of Theorem 4.

Lemma 8. *Let G be a triangle-free graph with $\delta \geq 2$. If $\sigma_4 \geq n+1$, then G is connected.*

Proof. Suppose G contains two connected components H_1 and H_2. Then the assumption that G is triangle-free and $\delta \geq 2$ implies $H_i \geq 3$ for $i = 1, 2$. Therefore there are non-adjacent vertices x_i, y_i in H_i such that $|H_i| \geq \max\{2d(x_i), 2d(y_i)\}$ for $i = 1, 2$ by Lemma 7. Hence $d(x_1) + d(y_1) + d(x_2) + d(y_2) \geq \sigma_4 \geq n + 1$. By symmetry, we may assume $d(x_1) + d(x_2) \geq (n+1)/2$. Thus $n \geq |H_1| + |H_2| \geq 2(d(x_1) + d(x_2)) \geq n + 1$, a contradiction. $\square$

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