

THE DEFECTIVE AND MATERIAL CONDITIONALS IN MATHEMATICS: DOES IT MATTER?

Matthew Inglis

Loughborough University

Adrian Simpson

Durham University

In this paper we discuss the relative merits of two different ways of understanding conditional statements of the form ‘if p then q ’. We demonstrate that there is no relationship between an ability to draw modus tollens deductions and having a material conditional understanding, as proposed by Durand-Guerrier (2003). Instead, we suggest that the so-called defective conditional understanding is widespread among high achieving mathematics students. We argue that, despite its name, adopting this understanding does not prevent students from drawing valid logical deductions, or from being successful in university-level mathematics examinations.

DIFFERENT CONCEPTIONS OF “IF P THEN Q ”

Logical implication is fundamental to mathematical proof, and thus of major concern to mathematics educators at all levels. However, it is well known that students find dealing with conditional statements – statements of the form ‘if p then q ’ – to be counterintuitive and difficult (e.g. Hoyles & Küchemann, 2002). One possible reason for these difficulties arises from the different meanings that can be given to such statements. In this paper we focus on two such meanings, the so-called material conditional and defective conditional conceptions.

In formal logic courses university students are taught the formal concept definition of implication which is captured by the truth table shown in Figure 1(a): ‘if p then q ’ is true in all cases except where the antecedent (p) is true and the consequent (q) false.

p	q	if p then q	p	q	if p then q
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	T	I
F	F	T	F	F	I
(a)			(b)		

Figure 1: The material (a) and defective (b) conditional truth tables (T denotes ‘true’, F denotes ‘false’ and ‘I’ denotes ‘irrelevant’).

This understanding – known by logicians as the *material* conditional – leads to some oddities such as statements like “if 3 is even, then π is irrational” being logically true.

Quine (1966) noted that this understanding of the conditional is not generally used in day-to-day life. He wrote that “‘if p then q ’ is commonly felt less as an affirmation of a conditional than as a conditional affirmation of the consequent” (p.12). Wason (1966) described this alternative understanding of the conditional – one focussed on its meaning as a conditional affirmation of the consequent – as the *defective* conditional. Here the conditional is deemed to be irrelevant in the cases where p is false; the corresponding truth table is shown in Figure 1(b). Other researchers have used various other terms for this understanding, such as the ‘hypothetical conditional’ (Mitchell, 1962) or the ‘common understanding’ (Durand-Guerrier, 2003).

Given these two different understandings of a conditional statement, an obvious and important question arises: which is most appropriate for the mathematics classroom? Typically one would expect that the most appropriate concept image for a student to have is one which matches the relevant concept definition (Tall & Vinner, 1981). But Hoyles & Küchemann (2002) disagreed, and directly answered the question in favour of the defective conditional:

“We claim that when studying reasoning in school mathematics, the [defective conditional] is a more appropriate interpretation of logical implication than the [material conditional], since in school mathematics, students have to appreciate the consequence of an implication when the antecedent is taken to be true” (p. 196).

Durand-Guerrier (2003) disagreed with Hoyles & Küchemann, and gave two arguments in support of her position. First, that a material conditional understanding would be required to understand definitions such as that of a diagonal matrix:

An $n \times n$ matrix $[a_{ij}]$ is diagonal if and only if for every i from 1 to n , and every j from 1 to n , if $i \neq j$ then $a_{ij} = 0$.

The crucial case here is that of $i = j$; the material conditional “if $i \neq j$ then $a_{ij} = 0$ ” is true in this case regardless of whether $a_{ij} = 0$ or $a_{ij} \neq 0$, and so, as Durand-Guerrier pointed out, the definition accurately describes our concept image of a diagonal matrix (i.e. where entries are zero everywhere except the diagonal, where they may be either non-zero or zero). However, the same is true of the defective conditional: it is not false in the case $i = j$, but rather irrelevant. That is to say that when $i = j$, nothing can be concluded about a_{ij} from the defective conditional. Thus, in our view at least, a defective conditional interpretation of “if $i \neq j$ then $a_{ij} = 0$ ” does seem to give a concept definition which matches the appropriate concept image of diagonal matrices: when $i \neq j$ we know that $a_{ij} = 0$, but when $i = j$ we don’t know anything as the conditional is irrelevant.

Durand-Guerrier’s (2003) second argument against Hoyles & Küchemann’s (2002) position was more complex, and related to the kinds of deductions that can be made using each type of conditional. From a material conditional two valid deductions are possible: modus ponens (deducing q from “if p then q ” and p) and modus tollens (deducing $\neg p$ from “if p then q ” and $\neg q$). However, Durand-Guerrier suggested that the second of these can not be made from a defective conditional:

With [the defective conditional] understanding of implication, it is no more possible to interpret the Modus Tollens without using the contrapositive. Indeed, the Modus Tollens conclusion is that antecedent is false; if one accepts only implications with true antecedent, one must use the contrapositive and apply Modus Ponens to it. However, the equivalence between a conditional statement and the corresponding contrapositive requires material implication (p. 29).

Essentially Durand-Guerrier suggested that the modus tollens deduction cannot be made using a defective conditional understanding. With the material conditional understanding two routes are open to reasoners: they may either simply know the modus tollens deduction and apply it directly, or they may convert the conditional into its contrapositive (i.e. convert ‘if p then q ’ into ‘if $\neg q$ then $\neg p$ ’) and then apply modus ponens. Neither of these routes appear to be viable if you have a defective conditional understanding.

However, it could be possible to make a modus tollens deduction via a third route, by using modus ponens and an informal contradiction argument. Suppose a reasoner is given the defective conditional “if p then q ” and the statement $\neg q$. They might suppose p , conclude q by modus ponens, notice that q contradicts the given statement $\neg q$, and so conclude that their supposition p was incorrect, concluding $\neg p$. This admittedly rather long chain of reasoning seems to be entirely accessible to someone who has a defective understanding of the original conditional. Nevertheless, it might well be the case that the length of this chain of deductions hinders students from accurately making the modus tollens deduction, in which case we might agree with Durand-Guerrier’s (2003) arguments against Hoyles & Küchemann (2002).

In this paper we report data from an experiment which directly investigated of whether having a defective conditional conception hinders making logical deductions and, in particular, making modus tollens deductions.

METHOD

The data reported in this paper come from a wider study which investigated the development of logical reasoning skills across the first year of undergraduate mathematics study. Participants were 33 first-year undergraduate students studying mathematics (either mathematics, or a joint degree with a significant mathematics component) at a highly-ranked UK university. All the students had been highly successful during their school mathematics studies: other than the two overseas students in the sample, all participants had been awarded A grades in both A Level Mathematics and A Level Further Mathematics (this represents the highest possible achievement in mathematics for 18 year-old school leavers in England and Wales).

Students participated in two sessions of data collection during the course of their first year studies, once at the very beginning and once at the end. In both sessions participants worked individually through a booklet of tasks, all designed to interrogate logical reasoning behaviour. In this paper we report responses to the two tasks relevant to the debate between Hoyles & Küchemann (2002) and Durand-

Guerrier (2003): the conditional inference task, and the truth table task. In each case we used abstract versions of the tasks to avoid the well-documented confounding effects of realistic/mathematical content (e.g. Stylianides, Stylianides, & Philippou, 2004).

Conditional Inference Task

The conditional inference task used was identical to that used by Inglis & Simpson (2008). In both sessions participants were given 32 problems of the form:

This problem concerns an imaginary letter-number pair. Your task is to decide whether or not the conclusion *necessarily* follows from the rule and the premise.

Rule: If the letter is not G then the number is 6.

Premise: The number is not 6.

Conclusion: The letter is G.

☐ YES (it follows) ☐ NO (it does not follow)

The inferences tested were balanced: half were valid – modus ponens and modus tollens – and half were invalid – denial of the antecedent (concluding $\neg q$ from $\neg p$ and ‘if p then q ’) and affirmation of the consequent (concluding p from q and ‘if p then q ’). Following Inglis & Simpson (2008), the presence of negated statements in the rules were rotated (e.g. the rules of the form ‘if p then $\neg q$ ’, ‘if $\neg p$ then q ’ and ‘if $\neg p$ then $\neg q$ ’ were used in addition to ‘if p then q ’), and half of negated statements were represented explicitly (e.g. “not 3”) and half implicitly (e.g. “8” rather than “not 3”). Participants took the task twice, once at the beginning of the year and once at the end, thus ended up with a score out of 64 together with subscores for each of the four tested inferences. These profiles gave an identification of how fluent participants were at drawing each of the four inferences tested, and of overall inferential fluency.

Truth Table Task

In the second session of data collection we also collected participants’ responses to 32 tasks of the following form (adapted from earlier studies, Evans & Over, 2004):

This problem relates to a card which has a capital letter on the left and a single-digit number on the right. You will be given a rule together with a picture of a card to which the rule applies. Your task is to determine whether the card conforms to the rule, contradicts the rule, or is irrelevant to the rule.

Rule: If the letter is not E then the number is not 1.

Card:

D	1
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☐ card conforms to the rule ☐ card contradicts the rule ☐ card is irrelevant to the rule

Two cards representing each of the four lines of a truth table (Figure 1) were included, and the positions of negated statements in the conditionals were rotated. Each participant’s responses were scored twice: first, with respect to how consistent they were with the material conditional, and second, with respect to how consistent they were with the defective conditional. Thus each individual ended up with a

material conditional profile score (out of 32) and a defective conditional profile score (out of 32). The higher the profile score the more consistently the participant was using that understanding of the conditional.

To provide a means of controlling for general cognitive skills, during the first session of data collection participants took Part 1 of the AH5 intelligence test (Heim, 1968). This test, designed for high achieving adults, contained 36 items in the categories ‘directions’, ‘verbal analogies’, ‘numerical series’, and ‘similar relationships’. It has been widely used by other researchers interested in individual differences in logical reasoning abilities (e.g. Newstead et al., 2004).

RESULTS

Our primary goal was to determine whether having a defective conditional understanding hinders inference-making, and in particular, making the modus tollens inference. To address this issue we calculated participants’ defective and material conditional profiles on the truth table task (i.e. the number of cards classified according to each conception). Twelve participants consistently adopted the defective conditional understanding (i.e. all their responses were in line with Figure 1(b)). In contrast no participant consistently adopted the material conditional understanding.

The two difference conditional profile scores were correlated with the number of each inference correctly classified on the conditional inference task. If Durand-Guerrier’s (2003) suggestion that a material conditional is required to make the modus tollens deduction we would expect a negative correlation between the defective conditional profiles and modus tollens scores. If, on the other hand, Hoyles & Küchemann’s (2002) suggestion that the defective conditional is “more appropriate” for the mathematics classroom, we might expect positive correlations between defective conditional profiles and each of the inference scores. The various correlations between each of the key indicators are shown in Table 1.

	Def	Mat	MP	DA	AC	MT	All
Def	1	-.54**	.48**	.68**	.60**	.03	.63**
Mat	-.54**	1	-.22	-.57**	-.34	.12	-.39*

Table 1: Pearson correlations between the Defective (Def) and Material (Mat) profile scores, and the number of each inference correctly categorised (respectively modus ponens, denial of the antecedent, affirmation of the consequent and modus tollens).

Significant correlations are denoted by * $p < .05$ and ** $p < .01$.

Participants’ defective conditional profiles were strongly positively correlated with performance on the modus ponens, denial of the antecedent and affirmation of the consequent inferences (all at $p < .01$), i.e. those with a higher defective conditional profile score tended to classify MP, DA and AC inferences more accurately. However, there was no significant relationship with the modus tollens deduction. The material conditional profiles were negatively correlated with performance on the

denial of the antecedent inference, and overall inferential performance. That is, participants with a higher material conditional profile tended, overall, to be more inaccurate on the conditional inference task. There was, however, no relationship between material conditional profiles and accuracy at classifying the modus tollens deduction.

Further analyses revealed a significant positive correlation between defective conditional profiles and AH5 intelligence scores ($r = .40$, $p = .021$), but no relationship between material conditional profiles and AH5 scores ($r = -.03$, $p = .86$). As well as being related to defective conditional profile scores, AH5 scores were found to be borderline significantly related to overall deductive competence on the conditional inference task ($r = .33$, $p = .063$). Consequently, it may have been the case that participants' AH5 scores represented a potential confound. That is to say that any relationship between the different interpretations of "if p then q " and deductive competence merely reflected mutual relationships with intelligence. To test for this possibility we repeated the correlation analyses, this time controlling for AH5 scores. The resulting partial correlations are shown in Table 2.

	Def	Mat	MP	DA	AC	MT	All
Def	1	-.57**	.42*	.62***	.53**	.10	.58**
Mat	-.57**	1	-.21	-.60***	-.35*	.12	-.40*

Table 2: Partial correlations between Defective (Def) and Material (Mat) profile scores and the number of each inference correctly categorised, controlling for AH5 scores. Significant correlations are denoted by * $p < .05$, ** $p < .01$, *** $p < .001$.

After controlling for AH5 scores, essentially the same pattern of correlations emerged. Participants with high defective conditional profile scores tended to be more proficient at accurately categorising (as valid or invalid) modus ponens, denial of the antecedent and affirmation of the consequent deductions. Those with high material conditional profile scores tended to categorise fewer deductions correctly. There was, however, no relationship between either of the two conditional profile scores and the number of MT deductions correctly assigned. This was not because of a floor effect; participants were able to tackle the MT component of the task. Performance on this section was respectable, with a mean of 10.3 inferences correctly categorised (out of 16; $SD = 3.0$).

DISCUSSION

Recall that Durand-Guerrier (2003) argued that the material conditional was necessary to make the modus tollens deduction. Our results call this assertion into question. We found no relationship between accurately categorising modus tollens deductions and the interpretation of "if p then q " participants adopted. However, having a defective interpretation – where "if p then q " is deemed irrelevant when p is false – was associated with higher scores on every other measure of deductive

competence we took (the valid/invalid categorisation of modus ponens, denial of the antecedent and affirmation of the consequent deductions), and with higher AH5 scores. Given this, if our data supports one interpretation of “if p then q ” over the other, it would seem to be the defective conditional ahead of the material conditional.

It is somewhat surprising that defective conditional profile scores were correlated with every deductive competence measure apart from that relating to modus tollens. One possible explanation for this relates to the “informal contradiction argument” route to making modus tollens deductions discussed earlier. This involves rather a long chain of reasoning, so it seems reasonable to propose that the success or failure of such a chain might be more related to factors such as the concentration or motivation levels of the participant rather than would be the case for the modus ponens, denial of the antecedent or affirmation of the consequent deductions. Further investigations would be required to determine exactly which factors influence success or failure at making modus tollens deductions.

Some caution is needed in the interpretation of these results, due to the relatively small number of participants who adopted the material conditional. As noted above, no participant did so consistently, and indeed all but two of the participants had higher defective conditional profiles than they did material conditional profiles. This observation suggests that the defective conditional understanding of the conditional is adopted by the majority of high-attaining mathematics undergraduates, and that it does not prevent them from being relatively successful at drawing logical deductions.

This consideration also suggests a reason for why we found positive correlations between having a defective conditional understanding and performance on the modus ponens, denial of the antecedent and affirmation of the consequent inferences. If the large majority of participants adopted a defective understanding, then those with lower defective conditional profile scores (and hence probably higher material conditional profile scores) would be those participants who found it difficult to apply their understanding consistently across the task. If this interpretation were correct we might expect a correlation between defective profile and AH5 scores (which, indeed, was the case, $r = .40$, $p = .021$). Under these assumptions, the defective conditional profile score could be interpreted as a measure of consistency throughout the task.

It is possible that the participants in this study were atypical of university level mathematics students, and that their near uniform adoption of the defective conditional understanding of the conditional did harm their mathematics achievement. To test for this possibility we obtained each participant’s first year examination marks for the four core mathematics modules taken by each student. Although there was no correlation between students’ average marks and their material or defective conditional profile scores ($r_s = 0.04$, 0.09 respectively), the group as a whole had very high levels of achievement. The mean first-year examination mark obtained by the sample was 76% (SD=12%), well above the 70% threshold for the top grade.

CONCLUSION

Hoyles & Küchemann (2002) suggested that the defective conditional was the most appropriate to develop in the mathematics classroom, as it allows students to appreciate “the consequence of an implication when the antecedent is taken to be true”. Durand-Guerrier (2003) disagreed with Hoyles & Küchemann, by suggesting that the material conditional was necessary in order to be able to make modus tollens deductions. In this paper we have empirically demonstrated that there is no connection between having a material conditional understanding and making modus tollens deductions. Furthermore, we have shown that high achieving mathematics undergraduates almost uniformly adopt the defective conditional understanding, and that this does not seem to adversely affect either their ability to draw inferences from abstract conditional statements, or their performance on university mathematics examinations. In sum, mathematically, there appears to be no disadvantage to holding a defective understanding of the conditional.

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