A Partition of Unity enriched Dual Boundary Element Method for accurate computations in fracture mechanics

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Abstract

We introduce a novel enriched Boundary Element Method (BEM) and Dual Boundary Element Method (DBEM) approach for accurate evaluation of stress intensity factors (SIFs) in crack problems. The formulation makes use of the Partition of Unity Method (PUM) such that functions obtained from *a priori* knowledge of the solution space can be incorporated in the element formulation. An enrichment strategy is described, in which boundary integral equations formed at additional collocation points are used to provide auxiliary equations in order to accommodate the extra introduced unknowns. In addition, an efficient numerical quadrature method is outlined for the evaluation of strongly singular and hypersingular enriched boundary integrals. Finally, results are shown for mixed mode crack problems; these illustrate that the introduction of PUM enrichment provides for an improvement in accuracy of approximately one order of magnitude in comparison to the conventional unenriched DBEM.

Keywords: BEM, fracture, Partition of unity, enrichment

1 Introduction

Computational fracture mechanics, essentially a subject centred on the problem of modelling the singularity created by a crack tip, is a topic which has been studied extensively over recent years. Many methods are available, but all share the common goal of determining accurate stress intensity

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factors (SIFs) for each of the modes of fracture (K_I, K_{II}, and K_{III}). These parameters, which quantify the strength of the singularity created by the crack tip for a certain geometry and loading, often need to be determined to high levels of accuracy. In particular, fatigue crack growth laws which rely on SIFs raised to certain positive powers magnify any errors obtained in K_N (where $N = I, II, III$ is the mode of fracture) further exacerbating the problem.

The vast majority of computational methods developed for fracture analysis are based on the Finite Element Method (FEM). Watwood [1] demonstrated that if conventional piecewise polynomial shape functions are used to analyse bodes containing cracks, then very high mesh densities are required in the region surrounding the crack tip to obtain relatively low errors in SIFs. Even so, with large numbers of degrees of freedom (DOF), errors in the region of 5% were encountered with difficulties also presented in obtaining reliable K_N values using displacements and stresses extrapolated to the crack tip. However the implementation did provide a method to model general fracture problems. Further improvements to the FEM include special crack-tip shape functions that incorporate the required $\sqrt{\rho}$ (where ρ is the radial distance from the crack tip) variation for displacements around the crack tip and quarter-point elements, independently formulated by Henshell and Shaw [2] and Barsoum [3]. These simply require the repositioning of the mid-node to a quarter-point position. However, Ingraffea and Manu demonstrated the size dependence of quarter-point elements preventing a general strategy for their use being formulated. Furthermore, the extent of the singular region created by the crack tip is restricted to the size of the quarter-point element when in reality it may extend further over a larger region. Further complications arise in the use of quarter-point elements for curved crack geometries.

Other attempts to improve the FEM for fracture include a hybrid-element approach, first introduced by Tong et al. [5] and more recently extended by Karihaloo et al. [6], while a more recent approach, known as the fractal finite element method (FFEM) [7] has been developed. The first uses a complex variable approach in which a special "hybrid" element incorporates the correct crack tip behaviour. The latter technique models the singular region surrounding the crack tip as a self-similar mesh in which several layers, progressively decreasing in size, are used. The large number of unknowns created are transformed into a small number of global unknowns using appropriate interpolation functions. Both methods exhibit accurate results for relatively coarse meshes but do exhibit certain disadvantages. In particular, in the case of multiple cracks with cracks tips in close proximity to one another, problems will occur in the formation of the "hybrid"

element (hybrid-element method) and singular region (FFEM). This would also be the case for any geometrical feature that lay near the crack tip. In these cases Boundary Element Methods, in which only discretisation of the boundary is required, present a distinct advantage in the context of Linear Elastic Fracture Mechanics (LEFM). One further note is made where, in the implementation of [6] in which hybrid elements are used, much emphasis is placed on the evaluation of higher-order times of the Williams expansion. However, current industrial practice in fracture and fatigue assessment is almost entirely based on the first-order values of K_N and it is this approach that is taken in the present work.

The Boundary Element Method (BEM) is another computational method that, in recent years, has found a growing popularity. It offers the advantage that the entire method is based on parameters on the boundary, essentially reducing the dimensionality of the problem by one. It too suffers from the same problems encountered in the FEM in that polynomials are insufficient when trying to model a singular problem such as crack, unless a refined mesh is used. Quarter-point elements have been successfully applied to the BEM [8] although these too suffer from the same limitations that have been encountered in the FEM. However, since the BEM represents tractions independently of displacements, special crack-tip shape functions [9] are required to capture the $\rho^{-\frac{1}{2}}$ variation in tractions seen at the crack-tip. Another technique known as the Subtraction of Singularity Method, originally introduced by Papamichel and Symm [10], removes the singular field of the crack leaving the non-singular field to be modelled numerically. This was extended further by Portela et al. [11] who applied the method in such a way that Stress Intensity Factors (SIFs) were output directly as unknowns. Excellent results were shown but the method encountered difficulties due to the use of the Williams solution which is only valid in a near field region. The problem was overcome by partitioning the domain into near and far-field regions at the cost of ease of implementation. More recently, Watson [12] developed a method in which special singular shape functions are created using eigenfunctions from the Williams expansion [13] that describe a crack tip singularity. In the formation of these shape functions, additional unknowns are introduced requiring the use of auxiliary Boundary Integral Equations (BIEs). Certain restrictions are made on the type of elements used and the implementation of the additional BIEs becomes rather complex, but the method does show an improvement over other boundary element methods. Probably the most popular BEM at present used to model fracture problems is the Dual Boundary Element Method originally developed by Portela et al. [14]. It overcomes the problem encountered when the conventional BEM

is applied to a model containing coincident crack surfaces, where a singular system results from collocation at identical nodal positions. The DBEM instead provides a Traction Boundary Integral Equation (TBIE) that is independent of the conventional Displacement BIE (DBIE) and applies the TBIE for collocation on one of the crack surfaces while the DBIE is applied for collocation at all other points. The DBEM is a robust method that can be applied to various crack geometries while achieving consistently accurate results, though this comes at the cost of a requirement to evaluate some hypersingular boundary integrals.

Although BEM approaches have advanced to provide better than 1% accuracy with comparatively few elements, it is important to strive for greater accuracy still. In particular, SIFs derived from numerical approximations may be used to determine fatigue lives according to various crack growth laws. These laws typically give the crack growth rate from expressions containing a term K_I^m, in which the exponent m is typically of value 2 to 4 for metals, but can be considerably higher for polymers and other materials, e.g. as high as 16 for a polyethylene [15]. More accurate determination of SIFs will therefore be of great value in improving the quality of fatigue life estimates.

A relatively recent research area that has shown considerable success is the idea of applying enrichment through the Partition of Unity Method (PUM) [16]. In particular, the Extended Finite Element Method (X-FEM) [17] has developed into a prominent computational method, with a wide range of applications and a rapidly expanding research community. Fracture problems, in which a singularity is found at a crack tip and a discontinuity experienced across the crack face, have been used to study enrichment with X-FEM. Heaviside functions and a basis that encompasses the solution space of crack tip displacements are introduced through enrichment providing the *a priori* knowledge that leads to higher accuracy. Also, by representing the crack independently of the mesh, crack propagation simulation times are dramatically reduced without the need to remesh on each crack increment. The use of level sets to represent the crack has been shown to provide a useful methodology for crack propagation studies [18].

Fracture mechanics computations have been considered using meshfree methods from the early papers on methods [19]; Nguyen et al. [20] provide a useful recent review and include a discussion of the practical use of the methods for cracked bodies. Meshfree algorithms for fracture mechanics remain a subject of considerable research activity, with recent work focussing on locally enriched approximations for problems containing material and geometric non-linearity [21, 22]. Liew *et al.*

[23] extended enriched meshless schemes to a meshless boundary integral formulation, using similar enrichment to the current paper.

In this paper enrichment through the PUM is applied to the Boundary Element Method to allow accurate evaluation of SIFs. The formulation closely follows that of the DBEM to allow models to contain coincident crack surfaces but does present entirely new terms within the BIEs to apply enrichment. The technique used to calculate the singular and hypersingular enriched boundary integrals is shown and the method of introducing additional collocation points to solve for auxiliary unknowns is described. Finally, Mode I and Mode II SIFs for various crack configurations are evaluated and comparisons made with the DBEM and other numerical solutions.

2 The Dual Boundary Element Method

The most widely accepted Boundary Element Method to model general fracture problems is the Dual Boundary Element Method in which the conventional Displacement Boundary Integral Equation (DBIE) is used for collocation on one of the crack surfaces and the independent Traction Boundary Integral Equation (TBIE) on the other. We consider a domain $\Omega \in \mathbb{R}^2$, having boundary $\Gamma \equiv \partial\Omega$. The DBIE is given by

$$\begin{aligned} C_{ij}(\mathbf{x}')u_j(\mathbf{x}') + \int\!\!\!\!\!\!-_{\Gamma} T_{ij}(\mathbf{x}', \mathbf{x})u_j(\mathbf{x})d\Gamma(\mathbf{x}) \\ = \int_{\Gamma} U_{ij}(\mathbf{x}', \mathbf{x})t_j(\mathbf{x})d\Gamma(\mathbf{x}), \quad i, j = x, y \end{aligned} \tag{1}$$

where T_{ij} and U_{ij} are the traction and displacement fundamental solutions, $\mathbf{x}'$ and $\mathbf{x}$ are the source (i.e. collocation) and field points that lie on the surface Γ, u_j and t_j are displacements and tractions and $C_{ij}(\mathbf{x}')u_j(\mathbf{x}')$ represents a jump term due to the singular integral at $\mathbf{x}'$. All integrals are taken over the general boundary Γ and the integral $\int\!\!\!\!\!\!-$ represents a Cauchy Principal Value integral. The TBIE is given by

$$\begin{aligned} \frac{1}{2}t_j(\mathbf{x}') + n_i(\mathbf{x}')\!\!\!\!\!\!=\!\!\!\!\!\!\int_{\Gamma} S_{kij}(\mathbf{x}', \mathbf{x})u_k(\mathbf{x})d\Gamma(\mathbf{x}) \\ = n_i(\mathbf{x}')\!\!\!\!\!\!=\!\!\!\!\!\!\int_{\Gamma} D_{kij}(\mathbf{x}', \mathbf{x})t_k(\mathbf{x})d\Gamma(\mathbf{x}), \quad i, j, k = x, y \end{aligned} \tag{2}$$

where S_{kij} and D_{kij} represent fundamental solutions that are derived by differentiating T_{ij} and U_{ij} and the integral $\int\!\!\!\!\!\!=$ represents a Hadamard finite-part integral. In the crack modelling procedure

proposed by Portela et al. an assumption is made that all elements on which the TBIE is applied are discontinuous; these have the property that nodes are not shared between adjacent elements but are instead placed at interior points. This ensures continuity of displacement derivatives at the collocation points (a requirement for the evaluation of Hadamard finite-part integrals) and forces all source points $\mathbf{x}'$ to lie on smooth boundaries. As a consequence of this, the jump term in (2) has been written as $\frac{1}{2}t_j(\mathbf{x}')$. The form $C_{ij}(\mathbf{x}')$ is retained for the jump term in (1) since the DBIE is used when collocating over the non-crack portions of Γ, and this may be discretised using continuous elements.

In the implementation proposed by Portela et al. a large portion of the effort required to employ the above equations is focused on the evaluation of the strongly singular and hypersingular integrals. In fact, if flat elements are used along the crack surface these singular terms can be evaluated analytically using rather simple expressions. Other boundaries can be modelled using continuous and semi-discontinuous elements thus creating a robust and accurate method for fracture analysis.

3 Formulation

The Partition of Unity Method, which can be attributed to Melenk and Babuška, provides the basis of enrichment in the present paper. It states, that if a set of functions forms a partition of unity (that is, the sum of those functions is equal to unity at any point within a domain) then an arbitrary set of functions can be incorporated within the approximation. Of course, the functions are chosen to correspond to singularities or discontinuities in the domain from *a priori* knowledge of the solution space, thus allowing fewer degrees of freedom to capture the required field. Using this, displacements for a particular element n can be expressed in the following form

$$u_j^n(\xi) = \sum_{a=1}^{M} N_a(\xi) u_j^{na} + \sum_{a=1}^{M} \sum_{l=1}^{L} N_a(\xi) \psi_l^U(\xi) A_{jl}^{na} \tag{3}$$

where $\xi \in (-1, 1)$ is the local coordinate, N_a is the conventional Lagrangian shape function for local node a, ψ_l^U is the set of L basis functions used for enrichment and M is the number of nodes per element. u_j^{na}, formerly a nodal displacement, is now a nodal coefficient along with A_{jl}^{na}. In the present work where crack tip singularities are encountered, the basis functions are chosen to correspond to Williams' solution for displacements around a crack tip. If the terms of $O(\rho^{1/2})$ are

considered in the expansion, then $L = 4$ and the following enrichment basis vector can be written

$$\psi^U(\rho,\theta) = \left\{ \sqrt{\rho}\cos\left(\frac{\theta}{2}\right), \sqrt{\rho}\sin\left(\frac{\theta}{2}\right), \right.$$
$$\left. \sqrt{\rho}\sin\left(\frac{\theta}{2}\right)\sin(\theta), \sqrt{\rho}\cos\left(\frac{\theta}{2}\right)\sin(\theta) \right\}^{\mathbf{T}} \tag{4}$$

This basis, which incorporates both Mode I and II components, is the same basis vector used by Moës et al. [17] in the implementation of the XFEM. Thus, displacements are enriched in the BEM using the same formulation. However, Boundary Element Methods differ slightly in that tractions are represented independently of displacements creating an opportunity for enrichment of tractions.

It should be made clear that the enrichment of the displacements in this fashion, using the term $\sqrt{\rho}$ whose derivative becomes infinite at $\rho = 0$, therefore implicitly contains the stress singularity at the crack tip. It is not necessary to enrich tractions as well as displacements in order to capture the singularity. Indeed, in the present work only traction free cracks are considered precluding the need for traction enrichment, but it would be entirely possible, using the appropriate expressions for stresses around a crack tip given by Williams, to formulate a Partition of Unity using a basis vector similar in nature to (4).

3.1 Enrichment of the Displacement Boundary Integral Equation

The first step that is required before enrichment can be applied to the DBIE is to express the integral equation (1) in its discretised form. This is given by

$$C_{ij}(\mathbf{x}')u_j(\mathbf{x}') + \sum_{n=1}^{N_e}\sum_{a=1}^{M} P_{ij}^{na}u_j^{na} = \sum_{n=1}^{N_e}\sum_{a=1}^{M} Q_{ij}^{na}t_j^{na} \tag{5}$$

where

$$P_{ij}^{na} = \int_{-1}^{1} N_a(\xi)T_{ij}[\mathbf{x}', \mathbf{x}(\xi)]J^n(\xi)d\xi \tag{6a}$$

$$Q_{ij}^{na} = \int_{-1}^{1} N_a(\xi)U_{ij}[\mathbf{x}', \mathbf{x}(\xi)]J^n(\xi)d\xi \tag{6b}$$

N_e is the number of elements and $J^n(\xi)$ is the Jacobian of the transformation $(x, y) \rightarrow \xi$ for element n. The enriched BIE is then formed by substituting expression (3) for displacements

while also expressing the source point displacement $u_j(\mathbf{x}')$ in the same manner. The jump term is then distributed across nodes within the element containing the source point using the technique described by Perrey-Debain et al. [24] allowing the enriched BIE to be written as

$$\begin{aligned} C_{ij}(\mathbf{x}') &\left(\sum_{a=1}^{M} N_a(\xi_p) u_j^{\overline{n}a} + \sum_{a=1}^{M} \sum_{l=1}^{4} N_a(\xi_p) \psi_l^U(\xi_p) A_{jl}^{\overline{n}a} \right) \\ &+ \sum_{n=1}^{N_e} \sum_{a=1}^{M} P_{ij}^{na} u_j^{na} + \sum_{n=1}^{N_e} \sum_{a=1}^{M} \sum_{l=1}^{4} \tilde{P}_{ijl}^{na} A_{jl}^{na} \\ &= \sum_{n=1}^{N_e} \sum_{a=1}^{M} Q_{ij}^{na} t_j^{na} \end{aligned} \tag{7}$$

where $\overline{n}$ is the number of the element containing $\mathbf{x}'$ and ξ_p refers to the local coordinate of the source point. It will be shown in the next section that the formulation requires collocation at some non-nodal locations within the element. This is accommodated in (7) by expressing the displacement component in the jump term in the enriched form (3), and interpolating over element $\overline{n}$ using Lagrangian shape functions. In this equation, also, the terms P_{ij}^{na} and Q_{ij}^{na} are unchanged from Eqns. (6a) and (6b) while the new enriched term $\tilde{P}_{ijl}^{na}$ is given by

$$\tilde{P}_{ijl}^{na} = \int_{-1}^{1} N_a(\xi) T_{ij}[\mathbf{x}', \mathbf{x}(\xi)] \psi_l^U(\xi) J^n(\xi) d\xi \tag{8}$$

By inspecting the terms within this integral it can be seen that, with kernel T_{ij} of $O(1/r)$ (where $r := |\mathbf{x} - \mathbf{x}'|$ is the distance between the source and field points) and ψ_l^U of $O(\sqrt{\rho})$, it is clear that an appropriate numerical integration procedure is required that is capable of evaluating integrals which incorporate fundamental solutions which are strongly singular and basis functions which exhibit infinite gradients at the crack tip. Details of such a routine are given in section 4.2.

It might be noted that the coefficients A_{jl}^{na}, that multiply the enrichment functions at the node a on the element n, are each acting as an alias for the stress intensity factors. It is possible to adopt a different enrichment strategy in which the functions are more directly related to the stress intensity factors. Such a basis has been considered in finite element context by Benzley [25] and in a meshfree context by Duflot & Nguyen-Dang [26] and Fleming et al. [27]. The present authors have also considered this approach for enrichment of DBEM approximations, and this is the subject of a different article. It should be noted that the results of the two enrichment approaches are of comparable accuracy.

3.2 Enrichment of the Traction Boundary Integral Equation

The Dual Boundary Element Method of Portela et al. makes use of an independent BIE known as the Traction Boundary Integral Equation (TBIE) formed by differentiating the DBIE. The cost of this is to produce a BIE that contains not only strongly singular integrals of $O(1/r)$ but also hypersingular integrals of $O(1/r^2)$ that require special consideration. Before enrichment is applied to the TBIE, the discretised form of the unenriched boundary integral is

$$\frac{1}{2}t_j(\mathbf{x}') + n_i(\mathbf{x}')\sum_{n=1}^{N_e}\sum_{a=1}^{M} E_{kij}^{na}u_k^{na} = n_i(\mathbf{x}')\sum_{n=1}^{N_e}\sum_{a=1}^{M} F_{kij}^{na}t_k^{na} \tag{9}$$

where the terms E_{kij}^{na} and F_{kij}^{na} are expressed as

$$E_{kij}^{na} = \int_{-1}^{1} N_a(\xi)S_{kij}[\mathbf{x}', \mathbf{x}(\xi)]J^n(\xi)d\xi \tag{10a}$$

$$F_{kij}^{na} = \int_{-1}^{1} N_a(\xi)D_{kij}[\mathbf{x}', \mathbf{x}(\xi)]J^n(\xi)d\xi \tag{10b}$$

The boundary integral equation given in (10a) is hypersingular of $O(1/r^2)$ while that in (10b) is strongly singular of $O(1/r)$. The enriched form of the BIE is then given by substituting Eqn. (3) into (9), yielding

$$\begin{aligned}\frac{1}{2}\left(\sum_{a=1}^{M} N_a(\xi_p)t_j^a\right) + n_i(\mathbf{x}')\sum_{n=1}^{N_e}\sum_{a=1}^{M} E_{kij}^{na}u_k^{na} \\ + n_i(\mathbf{x}')\sum_{n=1}^{N_e}\sum_{a=1}^{M}\sum_{l=1}^{4} \tilde{E}_{kijl}^{na}A_{kl}^{na} \\ = n_i(\mathbf{x}')\sum_{n=1}^{N_e}\sum_{a=1}^{M} F_{kij}^{na}t_k^{na}\end{aligned} \tag{11}$$

where

$$\tilde{E}_{kijl}^{na} = \int_{-1}^{1} N_a(\xi)S_{kij}[\mathbf{x}', \mathbf{x}(\xi)]\psi_l^U(\xi)J^n(\xi)d\xi \tag{12}$$

if element n is enriched, otherwise $\tilde{E}_{kijl}^{na} = 0$. Clearly, with the introduction of the term S_{kij} which is of $O(1/r^2)$, the evaluation of the singular integral becomes more involved than that described in the previous section. However, using a convenient technique to subtract the singularity, it will be shown that in fact the singular term can be evaluated without undue difficulty. Notice that,

although enrichment is not applied to tractions, the jump term in (11) is expressed in terms of nodal tractions and shape functions. This is exactly the same technique which was applied to the DBIE to allow collocation at any general point. Further explanation of this is given in the implementation section.

Equations (7) and (11) are fundamental to the enriched BEM where, in the same manner as the DBEM, one is used for collocation on all boundaries including one of the crack surfaces while the other is used solely for the opposite crack surface. What is new in this method is the ability to incorporate functions that are known to model the local displacement field of a crack tip, thereby increasing accuracy for a given number of degrees of freedom (DOF).

4 Implementation

With each basis function in expression (4) incurring an additional DOF, and each enriched node potentially using four basis functions, it becomes clear that a fully enriched model will lead to a substantial increase in demand for computing resources. Therefore, to optimise both accuracy and efficiency a selective enrichment strategy must be employed. We choose to enrich elements on or near the crack since it is this local region where the basis functions of (4) are valid. Figure 1 illustrates a selective enrichment strategy in which elements that lie within a certain distance from the crack tip are chosen to apply the basis functions. In Figure 1 the crack surfaces Γ^+ and Γ^- are depicted as being separated by a finite crack opening displacement. This is for illustrative purposes only; in practice the crack surfaces are coincident, as are the nodes that lie on these surfaces. The system of equations is then formed by using Eqns. (7) and (11) throughout with (11) used for collocating on Γ^- and (7) for all other points.

4.1 Additional collocation points

Displacements that lie within the enrichment region are expressed in terms of u_j^{na} and A_{jl}^{na} and, since each term A_{jl}^{na} represents an additional degree of freedom, additional boundary integral equations are required to yield a square system. Watson [12] derived three additional BIEs by differentiating the fundamental solutions U_{ij} and T_{ij} with respect to the source points $\mathbf{x}'$ but made the restriction that Hermitian elements are used. The Enriched Boundary Element Method makes no such restrictions and is therefore much simpler to implement in an existing BEM code. Instead,

the method makes use of additional collocation points located on elements where enrichment is applied, an approach which has been successfully applied to the PUM boundary element analysis of wave problems [24]. Figure 2 illustrates the additional points applied to four enriched elements (two on the upper surface and two on the lower) that lie on flat crack elements. In the case of flat elements aligned with the crack tip, only three additional points are required for each enriched element. This can be explained by considering the basis functions seen in (3) and referring to the flat enriched elements illustrated in Figure 2. All nodes either lie at $\theta = \pi$ (on the upper crack surface) or $\theta = -\pi$ (on the lower crack surface). The constant θ has the effect of reducing the basis to $\psi_l^U = \{\sqrt{\rho}\}$. As a result, since only one additional DOF is introduced for each enriched node, three DOF will be introduced for each enriched element, requiring three additional collocation points. In the general case of curved elements, the full set of four basis functions is available requiring twelve additional points to be applied to each enriched element.

It should be pointed out that the reduction of the basis to $\psi_l^U = \{\sqrt{\rho}\}$ for flat elements causes the approximation to resemble the use of quarter-point elements. However, the new formulation is imposing the enrichment in a more general way that can be readily applied to curved cracks, which will be the subject of another paper, and with multiple enriched elements. Moreover, we will proceed in Section 5 to show that the new formulation provides for improved accuracy and more rapid convergence in comparison with quarter-point elements.

Numerous tests were run to investigate the sensitivity of the SIF results to the positions of these additional collocation points where positions that lay both on the boundary ($\mathbf{x}' \in \Gamma$) and external to this boundary ($\mathbf{x}' \notin \Omega$) were tried. These points were also placed on unenriched and enriched elements to determine the optimum location. It was found that the best results were obtained by placing the points on enriched elements (on the boundary) and that the location of the points within these elements had little effect on the results. Increased errors arise if additional collocation points are located very close to nodes, where collocation also takes place. This is to be expected, since eventually, in the limit as the additional point approaches a nodal point, two identical equations are produced leading to a singular system.

4.2 Numerical integration of strongly singular and hypersingular enriched integrals

Much of the effort required to implement the DBEM is focused on evaluating the hypersingular integrals of the TBIE. The same is true in the Enriched Boundary Element Method but with the added complication of developing methods for evaluating singular integrals that not only must cope with collocation points lying at any general position within an element, but must also include the enrichment functions ψ_l^U. This is true also of the strongly singular integrals seen in the DBIE. The evaluation of the non-enriched singular integrals is well documented, with analytical expressions or other singular numerical quadrature techniques available. There are also various methods available for the evaluation of singular and hypersingular integrals. Some use quadrature, for example Ioakimidis [28], but in the present work we use the technique developed by Guiggiani et al. [29] because it is both computationally efficient and easily adapted, as we show herein, for the inclusion of general enrichment functions. This draws upon Aliabadi & Hall [30], who were the first to expand the integrand in a Taylor series, and further works by Guiggiani & Gigante [31] and Guiggiani & Casalini [32]. Assuming that enrichment is applied to discontinuous elements along the crack edge, the following formula is used to evaluate the enriched hypersingular integrals (8) and (12) which are of $O(1/r)$ and $O(1/r^2)$ respectively,

$$\begin{aligned} I = &\int_{-1}^{+1} \left[F(\xi_p, \xi) - \left(\frac{F_{-2}(\xi_p)}{(\xi - \xi_p)^2} + \frac{F_{-1}(\xi_p)}{\xi - \xi_p} \right) \right] d\xi \\ &+ F_{-1}(\xi_p) \ln \left| \frac{1 - \xi_p}{-1 - \xi_p} \right| \\ &+ F_{-2}(\xi_p) \left(-\frac{1}{1 - \xi_p} + \frac{1}{-1 - \xi_p} \right) \quad \xi_p \in (-1, 1) \end{aligned} \tag{13}$$

where the singular integrand $F(\xi_p, \xi)$ can be expressed as

$$F(\xi_p, \xi) = \frac{F_{-2}(\xi_p)}{(\xi - \xi_p)^2} + \frac{F_{-1}(\xi_p)}{(\xi - \xi_p)} \tag{14}$$

The terms $F_{-2}(\xi_p)$ and $F_{-1}(\xi_p)$ are regular functions determined by a Taylor series expansion of the integrand about the source point, and are given by

$$F_{-2}(\xi_p) = DS_{-2}(\xi_p)N_a(\xi_p)\psi_l^U(\xi_p) \tag{15a}$$

$$\begin{aligned} F_{-1}(\xi_p) =& D\Bigg[S_{-2}(\xi_p)\Bigg(N_a(\xi_p)h(\xi_p)\frac{d\psi_l^U(\xi_p)}{d\xi} \\ &+ \psi_l^U(\xi_p)\left(h(\xi_p)\frac{dN_a(\xi_p)}{d\xi} + N_a(\xi_p)g(\xi_p)\right)\Bigg) \\ &+ S_{-1}(\xi_p)N_a(\xi_p)h(\xi_p)\psi_l^U(\xi_p)\Bigg] \end{aligned} \tag{15b}$$

where D is a constant and the functions $S_{-1}(\xi_p)$, $S_{-2}(\xi_p)$, $h(\xi_p)$ and $g(\xi_p)$ are algebraic expressions involving the components $r_{,i}$, $n_{,i}$ and the Jacobian of transformation $J^n(\xi_p)$. A full definition of these terms is given in the Appendix. Once these expressions are substituted in, the first term in expression (13) then becomes regular and can be evaluated using standard quadrature formulae while the latter two are analytical expressions for evaluating the singular components. Conveniently, expression (13) can be modified easily to cope with integrals of $O(1/r)$ by letting $F_{-2}(\xi_p) = 0$. This allows it to be used for all strongly singular and hypersingular integrals within the enriched boundary element method.

One feature of using Eqn. (13) to evaluate singular integrals is that, due to its general nature, there are no restrictions on the type of element over which the integral is being taken. Thus, when implementing the method, it is entirely possible to use curved elements along the crack faces.

A common feature among PUM implementations is the effect seen on the conditioning of the system due to enrichment. This was also the case in the present work where it was found that as the number of enriched elements was increased, the conditioning deteriorated. For example, in the typical case containing a crack modelled with four elements on each of Γ^+ and Γ^- and enrichment only applied to the crack-tip elements, a condition number of 7.9×10^9 was experienced. When all elements along the crack edge were enriched this increased to 2.6×10^{14}. In the case of a fully enriched BEM model, the problem is a serious issue. However, by implementing the selective enrichment strategy described previously, the situation is much improved.

The above scheme differs from standard integration schemes used for singular integrations over quarter-point boundary elements, which use a vanishing Jacobian to cancel the singularity. It is important to note that the above scheme allows for collocation of the BIE at non-nodal points on the element, which is a feature of our algorithm, and for the distribution of the associated jump

term to the nodes.

5 Results

5.1 Mode I problems

To illustrate the improvements seen by enriching elements around the crack tip an edge crack in a square plate was modelled using a geometry $a/w = 0.5$ and $h/w = 0.5$ as shown in Fig. 3. All elements are 3-noded, quadratic, discontinuous boundary elements. Initially enrichment was applied to elements adjacent to the crack tip, but the enrichment region was increased beyond this to other crack elements in subsequent tests. The J-integral technique, as originally developed by Rice [33], was applied to find stress intensity factors. The boundary of the model was split into lines where, in each step of refinement, additional elements were added to each line with equal element divisions. The crack itself had a minimum of two elements on each surface up to a maximum of eight, with no grading towards the crack tip. This is in contrast to the previous study by Portela et al. [14] in which all models were graded; it is expected that similar grading would improve the results of the current algorithm. After varying the enrichment region it was found that optimum results were obtained by enriching only the elements adjacent to the crack tip. As the number of enriched elements was increased beyond this, the conditioning of the system degraded adversely affecting the accuracy of the results. Furthermore, by adopting the strategy of solely enriching crack tip elements, the number of additional DOF introduced was also kept to a minimum. We note that the conditioning may be related to the relationships between the coefficients A_{jl}^{na} and the SIFs. The errors in normalised SIFs are given by

$$\varepsilon = \left| \frac{K_{N_{norm}} - K_{N_{norm}}^{ref}}{K_{N_{norm}}^{ref}} \right| \times 100\% \tag{16}$$

where

$$K_{N_{norm}} = \frac{K_N}{\sigma\sqrt{\pi a}} \tag{17}$$

and $K_{N_{norm}}^{ref}$ denotes the normalised reference solution. For the edge crack, results evaluated using the DBEM and the enriched BEM with elements adjacent to the crack tip enriched are illustrated in Figure 4. For only twelve additional DOF it can be seen that enrichment provides a substantial

increase in accuracy of approximately one order of magnitude. For example, with forty nine elements on the boundary, an error of 0.03% is achieved while the DBEM exhibits an error of 0.49%. Both methods were found to converge to the reference value of 3.0103 [34] while errors for the enriched BEM were consistently lower than those for the DBEM.

The performance of the PUM enriched DBEM is compared, in Figure 5, against the unenriched DBEM and against the use of quarter-point boundary elements, in converging to the reference solution of Civelek and Erdogan [34]. This comparison shows the use of PUM enrichment to have a striking improvement over both earlier algorithms.

The problem of a central crack was also considered where, due to symmetry, only half the rectangular sheet was meshed for analysis. Figure 6 illustrates the central crack problem along with a mesh used for analysis indicating the line of symmetry. SIF results are compared to the unenriched DBEM in Table 1. In a similar fashion to the edge crack problem, the normalised SIFs are consistently more accurate than those evaluated using the DBEM. Even with a coarse mesh of 17 elements, the enriched BEM is capable of evaluating the SIF to within 1%. The improvement offered by the PUM enrichment is not so striking in this example. However, this is not a general feature of enriched DBEM solutions for centre-cracked plate problems, and we will proceed to show some clearer improvements in section 5.2.

5.2 Mixed mode problems

In the previous two studies all cracks were subject to purely mode I loading while in many cases, mixed mode loading is more realistic. Before mixed mode problems can be analysed however, certain modifications are made to the J-integral to allow the decomposition into each of the modes of fracture. This decomposition technique, illustrated by Aliabadi [36] allows the J-integral to be split into the following components

$$J_I = \frac{K_I^2}{E'} \quad , \quad J_{II} = \frac{K_{II}^2}{E'} \tag{18}$$

where E' represents modified Young's Modulus for plane strain or plane stress. In this way, each of the two SIFs can determined.

The first example used to illustrate the accuracy of the enriched BEM for mixed mode fracture is that of an oblique edge crack as studied by Wilson [37] using the boundary collocation technique.

The geometry of the problem is illustrated in Fig. 7 where crack angles, β, of 45° and 62.5° were analysed with crack lengths varying from $a/w = 0.3$ to 0.6. An analytical solution is not available for this problem. Therefore instead of undertaking an error analysis, the results are compared graphically against those presented by Wilson. Both Mode I and II normalised SIFs are plotted in Fig. 8 for the varying crack lengths where excellent agreement with Wilson's results is observed.

Finally, a mixed mode problem of an inclined centre crack (see Fig. 9) in a finite plate was analysed with accurate results published by Murakami [38] and additional results given by Portela et al. [14] using the DBEM. In the analysis, various J-integral paths were used to test the robustness of the method where in each, the path described a circular contour centred at the crack tip starting and finishing on nodal points lying on the crack surface. This is illustrated in Fig. 10 along with the path numbering used in the analysis. Using a crack length of $a/w = 0.5$ inclined at 45°, normalised K_I and K_{II} values were determined for J-integral paths 2 to 5. Two meshes were used with four and six elements on each line where, in contrast to the work carried out in [14], *no grading of the mesh was used*. The results for Mode I and Mode II SIFs are illustrated in Figs. 11 and 12 respectively along with the results obtained using the unenriched DBEM in [14] using six elements on each crack line.

For the Mode I and II results, it can be seen that in all but one case the enriched BEM gives more accurate results for both meshes. Bearing in mind that in both meshes uniform grading is used, it is expected that even more accurate results would be obtained if grading was used. It can also be seen from the plots that the results of the enriched BEM are more consistently accurate as the J-integral path is varied in comparison to the DBEM, so that no particular strategy is required in order to determine the optimum J-integral contour.

For the above results, only one crack tip was enriched, but improved results can be gained by enriching the elements at both crack tips. Figure 13 shows the convergence behaviour of the mode I SIF results for the case $h/w = 2$, $a/w = 2$, $\beta = 45°$, comparing the unenriched dual BEM, and the two cases of the enriched dual BEM, i.e. with one tip enriched and with both tips enriched. It is evident that the convergence rate is markedly improved when both tips are enriched. This leads us to suggest that this is a very promising approach for multi-site damage problems.

6 Conclusions

The Enriched BEM shows an improvement in accuracy over the DBEM for evaluating SIFs at the cost of only a small number of additional DOF. The inclusion of enrichment terms requires the implementation of singular and hypersingular enriched integrals at general collocation points; a general procedure is given for their evaluation. By using this method, a general basis that is known to model a discontinuity or singularity can be included within the BEM which will return a more accurate solution. The enriched BEM, combined with the path independent J-integral, also presents an attractive method for obtaining accurate SIFs for general crack problems using meshes that are considerably coarser that those used in polynomial based element formulations. It has been demonstrated that the method is accurate for both Mode I and II fracture problems. The natural extension of the work will be the adaptation of the algorithms presented for the evaluation of SIFs for 3D problems, which the authors believe will be enabled by using the enrichment approach of Sukumar et al. [39].

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APPENDIX

The procedure outlined in [29] is used to allow the evaluation of the hypersingular integrals arising in the present method where, for illustration purposes, the kernel S_{kij} is used. For 2D elastostatics,

S_{kij} is given by

$$S_{kij} = \frac{\mu}{2\pi(1-\nu)r^2}\Big\{2\frac{\partial r}{\partial n}[(1-2\nu)\delta_{ij}r_{,k} + \nu(r_{,j}\delta_{ik} + r_{,i}\delta_{jk} - 4r_{,i}r_{,j}r_{,k}] + 2\nu(n_i r_{,j}r_{,k} + n_j r_{,i}r_{,k}) + (1-2\nu)(2n_k r_{,i}r_{,j} + n_j\delta_{ik} + n_i\delta_{jk}) - (1-4\nu)n_k\delta_{ij}\Big\} \tag{A.1}$$

All hypersingular integrals involving this term are multiplied by the shape function $N_a(\xi)$ and the Jacobian of transformation $J^n(\xi)$. We are then left with an integral of the form

$$\int_{-1}^{+1} N_a(\xi) S_{kij}\psi_l^U(\xi) J^n(\xi) d\xi \tag{A.2}$$

which is of $O(1/r^2)$ when the source and field point coincide. The method is based on expressing the integrand seen in (A.2) in a Taylor series form where definitions are made to simplify later expressions.

If the components of the field and source point locations are expressed as x_i and y_i respectively (in keeping with the notation of [29]), then the following Taylor series expansion about the point ξ_p can be written

$$\begin{aligned} x_i - y_i &= \left.\frac{dx_i}{d\xi}\right|_{\xi=\xi_p}(\xi-\xi_p) + \left.\frac{d^2x_i}{d\xi^2}\right|_{\xi=\xi_p}\frac{(\xi-\xi_p)^2}{2} + \cdots \\ &= A_i(\xi-\xi_p) + B_i(\xi-\xi_p)^2 + \cdots \\ &= A_i\delta + B_i\delta^2 + O(\delta^3), \end{aligned} \tag{A.3}$$

which defines the constants A_i and B_i along with the term $\delta := \xi - \xi_p$. The constants A and C are also defined as

$$A := \left(\sum_{k=1}^{2} A_k^2\right)^{1/2} \tag{A.4}$$

$$C := \sum_{k=1}^{2} A_k B_k \tag{A.5}$$

However, to determine A_i and B_i (and therefore A and C), the first and second derivatives about the source point must be found. This is achieved by utilising the relevant shape functions and the nodal coordinates in the following way

$$\frac{dx_i}{d\xi} = \frac{dN_a}{d\xi} x_i^a \tag{A.6a}$$

$$\frac{d^2x_i}{d\xi^2} = \frac{d^2N_a}{d\xi^2} x_i^a \tag{A.6b}$$

Now the derivative $r_{,i}$ can then be expressed as

$$\begin{aligned} r_{,i} = \frac{x_i - y_i}{r} &= \frac{A_i}{A} + \left(\frac{B_i}{A} - A_i \frac{A_k B_k}{A^3} \right) \delta + O(\delta^2) \\ &=: d_{i0} + d_{i1}\delta + O(\delta^2) \end{aligned} \tag{A.7}$$

while the term $1/r^2$ can also be rewritten as

$$\frac{1}{r^2} = \frac{1}{A^2\delta^2} - \frac{2C}{A^4\delta} + O(1) \tag{A.8}$$

$$=: \frac{S_{-2}}{\delta^2} + \frac{S_{-1}}{\delta} + O(1) \tag{A.9}$$

It is also useful to express the Jacobian of transformation in terms of its components $J_i(\xi)$ where $J^n(\xi) = \sqrt{J_1(\xi)^2 + J_2(\xi)^2}$ and

$$J_1 = A_2 + 2B_2\delta + O(\delta^2) \tag{A.10a}$$

$$J_2 = -A_1 - 2B_1\delta + O(\delta^2) \tag{A.10b}$$

As a generalisation, these are written as

$$J_k = J_{k0} + J_{k1}\delta + O(\delta^2) \tag{A.11}$$

Finally, we express the shape functions N_a and the enrichment functions ψ_l^U as Taylor expansions

$$\begin{aligned} N_a(\xi) &= N_a(\xi_p) + \left. \frac{dN_a}{d\xi} \right|_{\xi=\xi_p} (\xi - \xi_p) + \cdots \\ &= N_{a0} + N_{a1}\delta + O(\delta^2) \end{aligned} \tag{A.12}$$

and

$$\psi_l^U(\xi) = \psi_l^U(\xi_p) + \left.\frac{d\psi_l^U}{d\xi}\right|_{\xi=\xi_p}(\xi-\xi_p) + \cdots$$
$$= \psi_{l0}^U + \psi_{l1}^U\delta + O(\delta^2). \tag{A.13}$$

The integrand in (A.2) can now be expressed as a Taylor series by substituting in expressions (A.7), (A.8), (A.11), (A.12) and (A.13) while also noting that $J_i = n_i J^n$. By collecting all the terms that contain $1/\delta^2$ and $1/\delta$ where, due to the use of quadratic shape functions, any higher order terms are zero, the following expression can be written for the integrand

$$\begin{aligned} N_a(\xi)S_{kij}\psi_l^U(\xi)J^n(\xi) = D\Bigg\{ & \frac{S_{-2}(\xi_p)N_{a0}h(\xi_p)\psi_{l0}^U}{\delta^2} \\ & + \Big[S_{-2}(\xi_p)\big[N_{a0}h(\xi_p)\psi_{l1}^U \\ & \quad + \psi_{l0}^U\left(N_{a1}h(\xi_p) + g(\xi_p)N_{a0}\right)\big] \\ & \quad + S_{-1}N_{a0}h(\xi_p)\psi_{l0}^U\Big]/\delta\Bigg\} \end{aligned} \tag{A.14}$$

where the constant D is defined as $\mu/2\pi(1-\nu)$, and the terms $h(\xi_p)$ and $g(\xi_p)$ are given by

$$\begin{aligned} h(\xi_p) = {} & 2\nu(J_{i0}d_{j0}d_{k0} + J_{j0}d_{i0}d_{k0}) \\ & + (1-2\nu)(2J_{k0}d_{i0}d_{j0} + J_{j0}\delta_{ik} + J_{i0}\delta_{jk}) \\ & - (1-4\nu)J_{k0}\delta_{ij} \end{aligned} \tag{A.15}$$

$$\begin{aligned} g(\xi_p) = {} & 2(d_{l1}J_{l0} + d_{l0}J_{l1})\big[(1-2\nu)d_{k0}\delta_{ij} \\ & \quad + \nu(d_{j0}\delta_{ik} + d_{i0}\delta_{jk}) - 4d_{i0}d_{j0}d_{k0}\big] \\ & + 2\nu\big[J_{i0}(d_{j1}d_{k0} + d_{j0}d_{k1}) + J_{i1}d_{j0}d_{k0} \\ & \quad + J_{j0}(d_{i1}d_{k0} + d_{i0}d_{k1}) + J_{j1}d_{i0}d_{k0}\big] \\ & + (1-2\nu)\big[2(J_{k1}d_{i0}d_{j0} + J_{k0}(d_{i1}d_{j0} \\ & \quad + d_{i0}d_{j1})) + J_{j1}\delta_{ik} + J_{i1}\delta_{jk}\big] \\ & - (1-4\nu)J_{k1}\delta_{ij} \end{aligned} \tag{A.16}$$

where the summation rule is applied to the first two terms in expression (A.16).

BIOGRAPHIES

Robert Simpson has recently completed his postgraduate studies in the School of Engineering & Computing Sciences at Durham University UK. Having graduated with a 1st Class Master of Engineering in 2006, he embarked on his PhD studies in Partition of Unity enriched BEM approximations for fracture mechanics. He has spent one year at the University of California, Berkeley, and also made a research visit to Ecole Centrale de Nantes, France.

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