Optimal taxation, critical-level utilitarianism and economic growth*

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Abstract

We analyze tax policies in an intertemporal economy with endogenous fertility under Critical-Level Utilitarianism, both from a positive and a normative standpoint. On the positive side, we analyze the effects of a change in the tax on capital income and on fertility, both separately and combined so as to keep the per-capita public debt constant. On the normative side, we characterize the first- and second-best optimal tax structures, for both exogenous and endogenous labour supply.

Keywords: Taxation, endogenous fertility, critical level utilitarianism, population.

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1. Introduction

The issue of optimal dynamic taxation has a long tradition in economics. However, only recently the consequences of endogenous fertility on tax policies have been explored. In fact, traditionally the two topics have been analysed separately: on the one hand, the problem of optimal taxation in dynamic general equilibrium models has been investigated extensively: see Atkinson and Stiglitz (1972) for the earliest results on finite-time economies; Judd (1985), Chamley (1986) and Judd (1999) for the results in infinite horizon economies based on Ramsey (1928); Erosa and Gervais (2002) and De Bonis and Spataro (2010) for overlapping-generations economies; see also Basu, Marsiliani, and Renström (2004) and Basu and Renström (2007) for indivisible labour economies.

On the other hand, another strand of literature has been focusing on the optimal population growth rate (Samuelson 1975, Deardorff 1976 and, more recently, Jaeger and Kuhle 2009 and Renström and Spataro 2011**)** and on the role of endogenous fertility on optimal welfare state design (in particular social security; see, for example, Cigno and Rosati 1992, Zhang and Nishimura 1992 and 1993, Cremer, Gahvari and Pestieau 2006, Yew and Zhang 2009, Meier and Wrede 2010).

In this paper we aim at addressing the issue of optimal taxation in presence of endogenous fertility in a unified framework. In particular, we tackle such an issue by assuming that agents are entitled with "critical-level utilitarian preferences" (see Blackorby et al. 1995)[1]. Critical-level utilitarianism (CLU henceforth) is an axiomatically founded population principle that can avoid the repugnant conclusion (see Parfit 1976, 1984,

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[1] Among other non utilitarian principles, see, for example, Golosov, Jones and Tertilt (2007).

Blackorby et al. 1995 and 2002).[2] The latter implies that any state in which each member of the population enjoys a life above "neutrality" is declared inferior to a state in which each member of a larger population lives a life with lower utility. Indeed, such a result is likely to emerge in economic models under classical utilitarianism and endogenous fertility, that is in presence of social orderings based on the (sum of) well-being (i.e. utilities) of the individuals who are alive in different states of the world.

There are several ways for avoiding the repugnant conclusion. Some earlier literature assumed objective functions of a particular form.[3] However, such objective functions may not have an axiomatic foundation. We believe an axiomatic foundation is important, especially since tax policy affects births and the government then indirectly is determining whether some individuals will live or not. In fact, in a twin paper (Renström and Spataro 2011) we have shown that CLU[4] can deliver a steady state equilibrium entailing an interior solution for the rate of growth of population, provided that the critical level belongs to a positive, open interval. We recall here that the critical level α can be defined as the utility level of an extra-individual i who, if added to an otherwise unaffected population N with utility distribution u, would make the two alternatives socially indifferent, i.e. (N,u) as good as $(N,u;i,\alpha)$.

In the present work, we contribute to the field of second best taxation and endogenous fertility, in general equilibrium, under CLU, both from a positive and a normative standpoint. To the best of our knowledge, this has not been done before.

The paper is organized as follows: after presenting the model, in section 3 we characterise the steady state equilibrium and, in section 4 we perform a comparative statics analysis in order to assess the effect of taxes on the equilibrium levels of consumption, capital and population growth rate. Moreover, in section 5 we characterize the optimal structure of taxes both in absence and in presence of endogenous labour supply. Finally, in section 6 we will extend the analysis to the case of linear costs for childbearing.

2. The economy

We assume, for the sake of simplicity, that each generation lives for an instant of time, and life-time utility is $u(c_t)$, where c_t is life-time consumption for that individual. We also follow the convention that $u = 0$ represents neutrality at individual level (i.e. if $u < 0$ the individual prefers not to have been born), and denote the critical level as α. We start our analysis by assuming that labour supply, l_t, is exogenously fixed and normalized to 1; we will relax this assumption in section 5.2. An individual family chooses consumption, savings and the number of children (i.e. the change in the cohort size N_t).

We also assume that raising children is costly. There are two approaches in the literature, that assume the cost per family member in the number of children, θ, either linear (as in Becker and Barro, 1988**,** Cremer, Gahvari and Pestieau 2006**)** or strictly convex (as in Tertilt 2005 and Growiec 2006), respectively. Convex cost implies decreasing returns to scale in childrearing. In the present work we follow the convex-cost approach, although we will also discuss the implications of assuming linear costs for childbearing.

As for firms, we assume perfectly competitive markets and constant returns-to-scale technology. The consequence of the assumptions on the production side is that we retain

[2] Therefore, although several authors have criticised CLU, such as Parfit (1976) and (1984), Hurka (1983) and (2000), Arrenius (2000), Hg (1986), Shiell (2008) -see Blackorby (2005) and Renström and Spataro (2011) for a discussion of such critiques- we decided to maintain such an approach.

[3] E.g. Barro and Becker (1988) and Becker and Barro (1989).

[4] Allowing for discounting of the utilities of future generations, as in Blackorby et al. (1997).

the "standard" second-best framework, in the sense that there are no profits and the competitive equilibrium is Pareto efficient in absence of taxation. Otherwise there would be corrective elements of taxation. Finally, we assume the government finances an exogenous stream of expenditure by issuing debt and levying taxes.

To retain the second-best, we levy taxes on the choices made by the families, i.e. savings and population (fertility). Consequently we introduce the capital income tax and a population tax proportional to the number of children. Regarding the population tax, it does not matter if we tax the present generation or the future, because of altruism. For simplicity we assume that the children pay the population tax, making it proportional to N and when parents make choice of number of children they take into account this tax liability and resulting reduction in their children's consumption. Consequently the population tax distorts fertility choice.

2.1. Individuals

The problem of each household is to maximize the following birth-date dependent critical level utilitarian objective function:

$$\int_{t=0}^{\infty} N_t e^{-\rho t}\left[u(c_t)-\alpha\right]dt \tag{1}$$

s.t.

$$\dot{A}_t = \bar{r}_t A_t + w_t N_t - c_t N_t - \tau_t^N N_t - \theta(n_t) N_t \tag{2}$$

where $u(c_t)$ is the instantaneous utility function, increasing and concave in c_t, $\rho > 0$ is the intergenerational discount rate and $\alpha > 0$ is the critical level.

The childrearing cost, $\theta(n)$, is specified over the number of children each parent has, and is then a function of the population growth rate. In equilibrium each parent has the same number of children, so the per family member population growth rate becomes the economy wide one. We assume that such a cost is increasing in the number of children and, thus, in the population growth rate *(n)*, i.e. $\theta'(n) > 0$; moreover, we assume $\theta(0)$, the cost of raising one child, to be positive (when n=0 population is constant, which implies that each adult generates one child). Moreover, in the benchmark model we assume strict convexity of $\theta(n)$, i.e. $\theta''(n) > 0$, while the linear cost case, with $\theta(n) = \theta' \cdot (1+n)$ (θ' constant), will be presented in the extension provided in section 6.[5]

Since we fix neutrality consumption to zero (i.e. $u(0)=0$), this implies that c^{α}, satisfying $u(c^{\alpha})=\alpha$, is strictly positive. Moreover, A_t is household wealth, $\bar{r}_t = r_t\left(1-\tau_t^k\right)$ is net of tax interest rate, and τ_t^k and τ_t^N are the tax rate on capital income and on the population (household) size, respectively.

The population size, N_t, grows at rate n_t, i.e.

$$\frac{\dot{N}_t}{N_t} = n_t . \tag{3}$$

[5] A concave cost function, in our analysis becomes equivalent to a linear one, in the sense the equilibrium population growth rate is at a corner value during the transition. Tertilt (2005) and Growiec (2006) in fact assume convex cost in order to have an interior solution.

We assume that there are lower and upper bounds on the population growth rate: $n_t \in [\underline{n}, \overline{n}]$. Realistically, there is a physical constraint at each period of time on how many children a parent can have. There is also a constraint on how low the population growth can be. The reason for the latter assumption is twofold: first, we do not allow individuals to be eliminated from the population (in that there is no axiomatic foundation for that); moreover, even if nobody wants to reproduce there will always be accidental births and/or accidental deaths. For this reason, while we assume $\overline{n}$ to be positive, we allow $\underline{n}$ to take negative values. Clearly, from eq. (1) the problem has a finite solution only if $\rho > \overline{n}$ which we assume throughout our analysis.

2.2. Firms

Assuming constant returns-to-scale production technology, $F(K_t, L_t)$, zero capital depreciation rate and perfect competition, firms hire capital, *K*, and labour services, *L*, on the spot market and remunerate them according to their marginal productivity, such that

$$F_{K_t} = r_t \tag{4}$$

$$F_{L_t} = w_t . \tag{5}$$

Normalizing individual labour supply to unity implies $L_t = N_t$ (this will be relaxed in section 5.2).

Moreover, the economy resource constraint is:

$$\dot{K}_t = F(K_t, L_t) - c_t N_t - g_t N_t - \theta(n_t) N_t . \tag{6}$$

2.3. The government

We allow the government to finance an exogenous stream of public expenditure by levying taxes, both on capital income and population size, and issuing debt, *B*, following the law of motion:

$$\dot{B}_t = r_t B_t - \tau_t^k r_t A_t - \tau_t^N N_t + g_t N_t . \tag{7}$$

We take g as exogenous (rather than $G=gN$), preserving second-best analysis as N grows. This is a natural assumption when population size is endogenous.

Two comments are worth making: first, the tax τ_t^N, while being lump-sum at the individual level, is a "population tax" (or family size tax) from the point of view of the dynasty head and hence, will distort fertility choices at the family level. Second, we should note a potential externality problem. If the government is fixing a stream of per-capita public spending, the total expenditure will be proportional to the population size. When individuals decide on family sizes, they will not take into account the externality on the government's spending side. Consequently, a system of lump-sum taxation (lump-sum per family) will not implement the first-best (as mentioned before, however, in absence of government spending and taxation, the competitive equilibrium is Pareto-efficient).

2.4. Per-capita formulation

In some instances, it will be convenient to use per-capita notation. We then define the capital intensity $k \equiv \frac{K}{N}$, such that, by exploiting constant-returns-to-scale in the production function we can write: $F(K,L) = Nf(k)$, $F_K(K,L) = f'(k)$, $F_L(K,L) = f(k) - f'(k)k$. Hence, the capital and debt accumulation constraints in per-capita terms can be written as:

$$\dot{k}_t = f(k_t) - c_t - g_t - n_t k_t - \theta(n_t) \tag{8}$$

$$\dot{b}_t = (r_t - n_t)b_t - \tau_t^k r_t a_t - \tau_t^N + g_t . \tag{9}$$

where a_t and b_t are per-capita household assets (A_t/N_t) and per-capita public debt (B_t/N_t) respectively.

3. Decentralized solution

The problem of the individual (household) is to maximize (1) subject to (2), (3) and $n_t \in [\underline{n}, \overline{n}]$, taking A_0 and N_0 as given. The current value Hamiltonian is:

$$H_t = N_t[u(c_t) - \alpha] + q_t[\bar{r}_t A_t + w_t N_t - c_t N_t - \tau^N N_t - \theta(n_t)N] + \lambda_t n_t N_t \tag{10}$$

Note that the last term in the Hamiltonian captures the fact that at each instant of time the population size is given (and thus is a state variable) and can only be controlled by the choice of n (which is a control variable). The law of motion for the population size is provided by (3). Moreover, λ_t is the shadow value of population.

The first-order conditions are the following[6]:

$$\frac{\partial H}{\partial A} = \rho q - \dot{q} \Rightarrow \dot{q} = (\rho - \bar{r})q \tag{11}$$

$$\frac{\partial H}{\partial c} = 0 \Rightarrow u_c = q \Rightarrow \dot{c} = (\rho - \bar{r})\frac{u_c}{u_{cc}} \tag{12}$$

$$\frac{\partial H}{\partial N} \cdot \frac{1}{N} = \rho\lambda - \dot{\lambda} \Rightarrow \dot{\lambda} = (\rho - n)\lambda - (u - \alpha) - q[w - c - \tau^N - \theta] \tag{13}$$

$$\frac{\partial H}{\partial n} \equiv z = -\theta' q + \lambda > (<) 0 \text{ (with equality if interior solution for } n\text{)} \tag{14}$$

and the transversality conditions are

$$\lim_{t\to\infty} e^{-\rho t} q_t A_t = 0, \; \lim_{t\to\infty} e^{-\rho t} \lambda_t N_t = 0 . \tag{15}$$

[6] We omit the subscript referring to time when it causes no ambiguity to the reader.

We now characterize the competitive equilibrium. Supposing that the economy starts at time *t*=0, we recall that a competitive equilibrium is time paths of: a) policies $\pi_t = \left(\tau_t^k, \tau_t^N, B_t\right)$, b) allocations $\chi_t = \left(c_t, N_t, K_t\right)$, c) prices $\sigma_t = \left(w_t, r_t\right)$, such that, at each point in time $t \in [0, \infty)$: b) satisfies max eq. (1) subject to eqs. (2) and (3), given a) and c); c) satisfies eqs. (4) and (5) and, finally, eqs. (8) and (9) are satisfied. Moreover, in a competitive equilibrium, Walras' law holds, such that the following condition applies:

$$a_t = k_t + b_t. \tag{16}$$

As for the population growth rate, by taking the time-derivative of equation (14) and using (11),(12), (13) we get:

$$\dot{n}_t = \frac{1}{\theta''}\left\{\theta'(n_t)\left[\left(1-\tau_t^k\right)f'(k_t) - n_t\right] - \frac{u(c_t)-\alpha}{u_c(c_t)} + c_t + \theta(n_t) + f'(k_t)k_t - f(k_t) + \tau_t^N\right\} \tag{17}$$

if $\theta'' > 0$

and

$$n_t = \begin{cases} \bar{n}, z_t > 0 \\ n_t^*, z_t = 0 \\ \underline{n}, z_t < 0 \end{cases} \tag{18}$$

if $\theta'' = 0$

which, together with eqs. (8), (12) and (15) fully characterize the dynamics of the economy.

Some comments on eqs. (17)-(18) are worth making. The system entails an interior solution for *n*, along the transition path, only if the childbearing cost is strictly convex.

The reason for this can be explained as follows. If θ is linear, then since λ (the co-state for *N*) is the shadow value of population size, from equation (14) we can see that if λ is different from $\theta' q$, either population should be increased as much as possible (if $\lambda > \theta' q$), or as little as possible (if $\lambda < \theta' q$). In fact, by defining $z = \lambda - \theta' q$, such that $\dot{z} = \dot{\lambda} - \theta' \dot{q}$ (given that θ' is a constant), exploiting eqs. (11), (12) and (13) and integrating we get:

$$z_t = \int_t^\infty e^{-\int_t^\tau (\rho - n_s) ds} \left\{u(c_\tau) - \alpha - u_{c_\tau}\left[c_\tau + \left(\bar{r}_\tau - n_\tau\right)\theta' + \theta_\tau - w_\tau + \tau_\tau^N\right]\right\} d\tau \tag{19}$$

The integrand is the difference between two terms. One term, $u(c) - \alpha + u_c w$, is the value (in utility units) a new individual brings to the family (his/her utility in excess of the critical level α plus the utility value of his/her labour endowment), and the other is the value (in utility units) of what the new individual is taking out of the family (consumption plus the population tax and the childrearing cost). If these terms are the same for the entire future, then population size is optimal, and *z* is zero. However, with linear costs (19) can be simplified further, since $\theta(n) = (1+n) \cdot \theta'$, to obtain:

$$z_t = \int_t^{\infty} e^{-\int_t^{\tau}(\rho - n_s)ds} \left\{ u_\tau - \alpha - u_{c_\tau}\left[c_\tau + (1+\bar{r}_\tau)\theta' - w_\tau + \tau_\tau^N\right]\right\} d\tau \gtrless 0 \qquad (20)$$

and clearly the sign of z is independent of n such that there is no possibility to induce z to be zero along the transition path.

In fact, we have shown in another work that, without taxes and childbearing costs, along the transition path population will grow either at the maximum or at the minimum speed, the interior solution arising only at the steady state (see Renström and Spataro 2011).

With convex cost however the integrand in (19) determines the sign of the time derivative of the population growth rate (compare eq. (17) with eq. (19). We will use such a property in the normative analysis of section 5).

In the remainder of the paper we will assume that childbearing costs are strictly convex and we will extend the analysis to the case of linear costs for raising children in section 6.

We will briefly discuss the dynamic properties of the model in the next section.

3.1. Steady state

If the steady state solution for n is interior, then $z=0$ and by exploiting eqs. (12), (8) and (17), we can provide the following three equations which fully characterize the steady state:

$$f'(k^{ss})\left(1-\tau^k\right) = \rho \qquad (21)$$

$$f(k^{ss}) - n^{ss}k^{ss} - g - \theta\left(n^{ss}\right) = c^{ss} \qquad (22)$$

$$\Psi = 0 : \frac{u(c^{ss}) - \alpha}{u_c(c^{ss})} = c^{ss} + \tau^N + \theta\left(n^{ss}\right) - [f(k^{ss}) - f'(k^{ss})k^{ss}] + \left[\left(1-\tau^k\right)f'\left(k^{ss}\right) - n^{ss}\right]\theta'\left(n^{ss}\right). \qquad (23)$$

We first note that the critical utility level α will have to be restricted to allow for any meaningful analysis. For example, if α was so large that the economy could not deliver that level of utility in the long run, it is not meaningful to analyse population choice (in this case, in the long run, it would be optimal to have the smallest physically possible population growth rate i.e. the largest possible population decline). We can derive the lowest and largest values of α that permit an interior solution for n at a steady state (the derivation follows the one in Renström and Spataro 2011). We assume that the critical level utility, α, lies between these two values, i.e.:

$$\alpha \in \left(\bar{\alpha}, \underline{\alpha}\right), \qquad (24)$$

where

$$\begin{aligned} \bar{\alpha} &\equiv u(\bar{c}) - u_c(\bar{c})(\rho - \bar{n})\left[a^{ss} + \theta'(\bar{n})\right] \\ \underline{\alpha} &\equiv u(\underline{c}) - u_c(\underline{c})(\rho - \underline{n})\left[a^{ss} + \theta'(\underline{n})\right] \end{aligned} \qquad (25)$$

and $\bar{c} \equiv F_L^{ss} + \left(F_K^{ss} - \bar{n}\right)k^{ss} - g - \theta(\bar{n})$ and $\underline{c} \equiv F_L^{ss} + \left(F_K^{ss} - \underline{n}\right)k^{ss} - g - \theta(\underline{n})$.

In fact, when $\alpha \leq \bar{\alpha}$, the solution for the population growth rate will be $n^{ss} = \bar{n}$, that is the repugnant conclusion would arise, in that the population should grow at the maximum speed; on the other hand, if $\alpha \geq \underline{\alpha}$ then the solution would be the opposite, $n^{ss} = \underline{n}$, that is the population growth rate should be at its minimum.

We will assume that α is in the interval given by (24), such that the solution entails an interior value for *n* at the steady state. If this were not the case, our model would resemble a Cass-Koopmans-Ramsey model with exogenous fertility.

3.3. The dynamic system

The dynamic system of the economy is described by eqs. (8), (12), (17) and (15). As for stability of the steady state, we can provide the following proposition:

Proposition 1: W*hen childbearing costs are convex, sufficient for stability of the steady state equilibrium is*

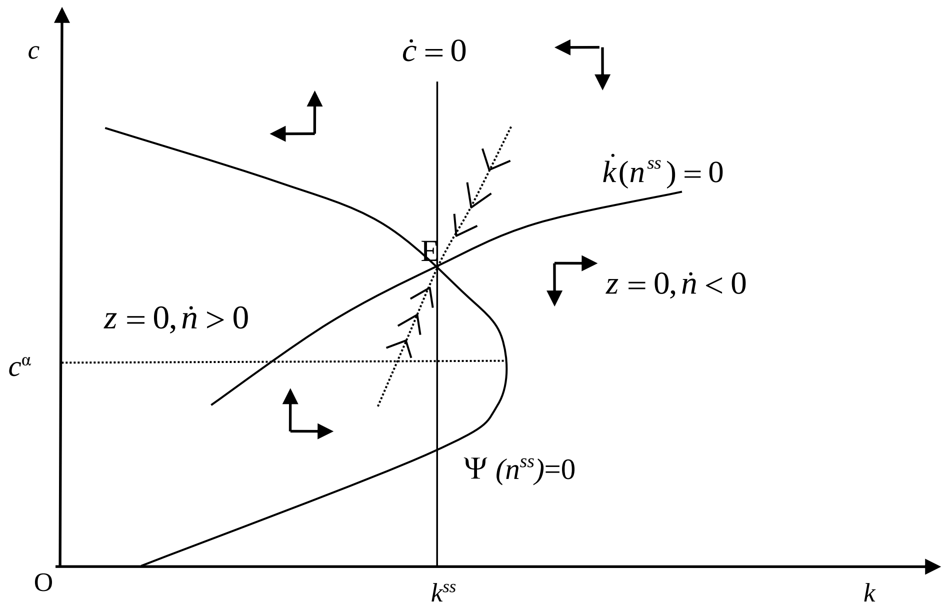
$$\frac{b}{k} \leq \frac{\tau^k}{1-\tau^k}$$

Proof: see Appendix A.1 □

In the remainder of the paper we will assume that such a condition holds.

In Figure 1 we depict the three loci described by eqs. (21)-(23) and the steady state equilibrium is represented by point E. Note that the locus $\Psi = 0$ depends on *n* as well. For this reason in Figure 1 we represent the $\Psi = 0$ locus associated with steady state value of *n*, $\Psi\left(n^{ss}\right) = 0$, whereby $\dot{n} = 0$.

Figure 1: The steady state equilibrium



As for the transition path, we recall that the solution for n is interior at each instant of time, such that $z=0$ for all t. Since by eq. (23), the locus $\Psi(n)=0$ shifts outwards (inwards) if n increases (decreases), for trajectories outside (inside) the $\Psi(n^{ss})=0$ curve, the value of n associated with any such $\Psi(n)=0$ locus is higher (lower) than n^{ss}. Moreover, inside (outside) any $\Psi(n)=0$ locus, the function Ψ is negative (positive) and, in turn, by comparing eq. (17) and (23), $\dot{n} > (<) 0$. Hence, we can conclude that for capital stocks lower than k^{ss}, along the transition path n is increasing, while for capital stocks higher than k^{ss}, n is decreasing. Finally, the dynamics of per-capita consumption and capital intensity are given by Eqs. (12) and (8). From Proposition 1 the equilibrium is saddle-path stable.

We will show in section 5 that under the optimal tax programme (in the first or the second-best) the sum $(a^{ss} + \theta')$ *(i.e. sum of the s*teady state per-capita assets, a^{ss}, and the marginal cost for raising children) are positive. Consequently, the steady state consumption level is greater than the one giving critical-level utility, i.e. $c^{ss} > c^{\alpha}$. To show the latter it is sufficient to see that, by substituting eqs. (4) and (21) into the steady state equation for the household budget constraint (eq. 2), expressed in per-capita terms, and plugging the latter in eq. (23), it follows that

$$u(c^{ss}) - \alpha = u_c(c^{ss})(\rho - n^{ss})(a^{ss} + \theta') > 0 \qquad (26)$$

where the inequality follows from $\rho > \bar{n} \geq n^{ss}$. Hence, by eq. (26) it follows that at the steady state $u(c^{ss}) - \alpha > 0$, that is, $c^{ss} > c^{\alpha}$.

4. Positive analysis of taxation

In this section we aim at analysing the effects of taxation on the equilibrium of our economy. We will perform some comparative statics exercises in which we either let one tax change, keeping the other constant while adjusting public debt, or let both taxes change simultaneously so as to keep the steady state per-capita public debt level the same. We assume non negative capital income tax.

4.1. The effects of a change in the population tax

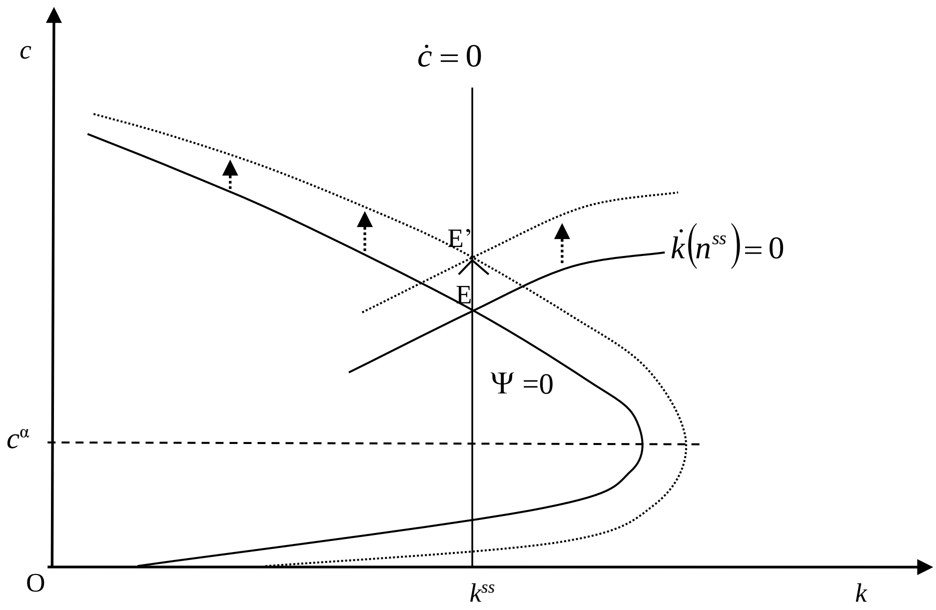
As for the effects of a change in the tax on the population size, the results are summarized by the following proposition:

Proposition 2: *At the steady state, an increase in the tax on the family size increases consumption, decreases the rate of growth of population and leaves capital intensity unchanged.*

Proof: see Appendix A.2 □

The content of Proposition 2 is illustrated in Figure 2. When the population tax is increased (keeping the capital income tax constant), the $\Psi = 0$ locus shifts outwards. As a consequence the new steady state is where the $\dot{c} = 0$ line cuts the new $\Psi = 0$ locus, at E', and the new steady state growth rate for population is lower than previously ($\dot{k} = 0$ shifts upwards). The new steady state level of consumption is higher, while the capital intensity is unaffected. If this policy comes as a surprise tax change for the individual family, per-capita consumption jumps from E to E', and the population growth rate falls to the new level immediately. Consequently there is no transition dynamics in this case.

Figure 2: The effects of an increase of the tax on the population size



4.2. The effects of a change in the capital income tax

As for the effects of a change in the capital income tax we can provide the following proposition:

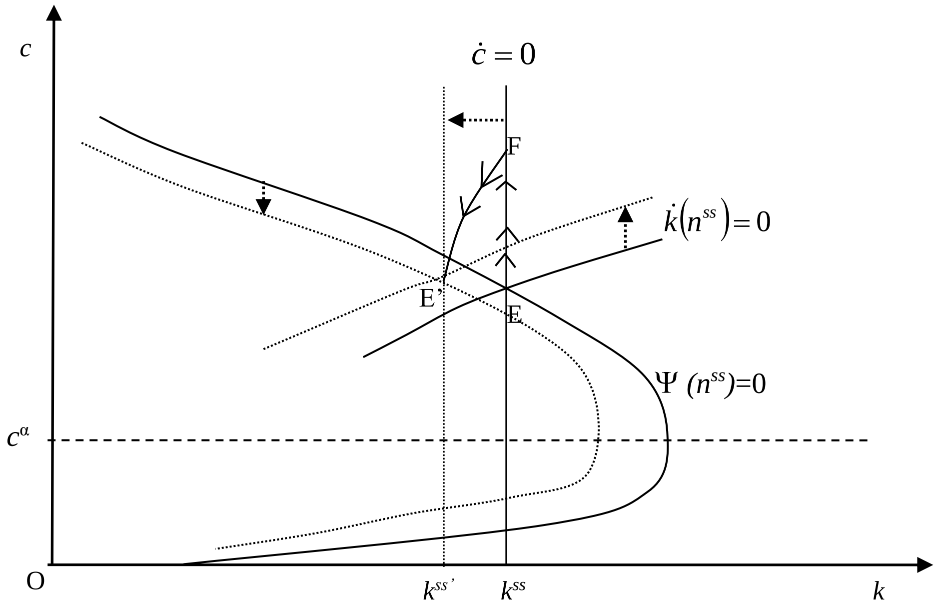
Proposition 3: *At the steady state, an increase of the capital income tax decreases both the rate of growth of population and capital intensity. The sign of the change of steady state consumption is ambiguous.*

Proof: see Appendix A.3. □

Figure 3 illustrates the result summarized in Proposition 3. When the capital income tax increases, the $\dot{c}=0$ line shifts to the left and the $\dot{k}=0$ locus moves up, while the $\Psi(n^{ss})=0$ shifts inwards, and the new equilibrium E' entails both lower population growth rate and lower capital intensity. However, as shown by eq. (32), the change in the level of consumption can be either sign.

If this tax change comes as a surprise, per-capita consumption jumps instantaneously up to point F, that is, on the new trajectory leading to E'. Per-capita consumption first jumps to a high level, and then gradually falls to its new level, creating a consumption boom. Moreover, also the population growth rate first jumps and then starts to decrease towards the new, lower steady state value. Note that the new steady state is reached in infinite time, through a saddle-path-stable trajectory.

Figure 3: The effects of an increase of the capital income



As a general comment on the analysis carried out so far, we can say that the long-run effects on the economy of an increase of either taxes are, to some extent, similar, in that both reduce steady state population growth rate. However, the increase of the capital income tax creates temporary population and consumption bursts and reduces the steady state capital stock, while an increase in the population tax does not.

4.3. The effect of a constant per-capita debt redistribution of taxes

Finally, we analyze the case in which the government changes both taxes in such a way that per-capita debt remains constant. Since the changes in the capital and the population taxes have, to some extent, similar qualitative effects, if we were to increase one of them and decrease the other so as to keep the debt level constant, we may ask which tax dominates.

Preliminary, we provide a sufficient condition according to which any such policy implies that taxes move in opposite directions (e.g. an increase of the capital income tax with constant per-capita debt implies a reduction of the tax on population size).

Lemma 1: *At the steady state, an increase (decrease) of capital income tax aiming at maintaining per-capita debt constant, implies a reduction (increase) of the tax on the population size if the capital income tax is lower than a threshold, i.e.:*

$$\frac{d\tau^N}{d\tau^k} < 0 \quad \text{if} \quad \tau^k < \bar{\tau}^k, \quad \text{where} \quad \bar{\tau}^k = -\left[\frac{f''k}{f'}(1+M) + Mu_{cc}\frac{(u-\alpha)}{u_c^{\;2}}\frac{(f'-n)}{f'}\right] \quad \text{and}$$

$$M \equiv \frac{bu_c^{\;2}}{u_{cc}(u-\alpha)(k+\theta') - u_c^{\;2}(\rho - n)\theta''}.$$

Proof: see Appendix A.4. □

Note that $\bar{\tau}^k$ in Lemma 1 is the steady-state Laffer maximum capital income tax rate. For any initial capital income tax rate lower than this level, when the government is increasing the capital tax rate, it can lower the population tax rate and keep the debt level the same. If the initial capital tax rate is higher than the Laffer maximum, then an increase in such a tax rate makes the government to lose revenue, and to maintain the same level of debt, it would have to increase the population tax rate. In fact, in the latter case, there is room for decreasing both taxes while keeping *b* constant. We will assume that the initial capital tax rate is lower than the Laffer maximum, i.e. that $\tau^k < \bar{\tau}^k$[7].

We now focus on the sign of the derivatives of both *c* and *n*, which, a priori and differently from the effect on the capital intensity, are ambiguous. Our findings are summarized by the following Proposition 4:

Proposition 4: *At the steady state, a tax reform consisting in an increase (decrease) of the capital income tax and a reduction (increase) of the tax on the population size in such a way to leave per-capita debt unchanged, implies that both capital intensity and the population growth rate decrease (increase), while the change of per-capita consumption is ambiguous.*

Proof: see Appendix A.5. □

The latter tax reform showed that the effect through the capital income tax dominates. Hence, both the dynamics and the changes in the steady state values are qualitatively similar to the ones stemming from the increase of the capital income tax (see Fig. 3)

5. The Ramsey problem

We now solve the optimal tax problem (Ramsey problem). We shall first find the first-best solution, and then move on to the second-best. Since the first-best is obtained as a solution to the second-best problem, when the second-best constraints do not bind, we formulate the latter problem from the outset. In doing so, we adopt the primal approach, consisting in the maximization of a direct social welfare function through the choice of quantities (i.e. allocations; see Atkinson and Stiglitz 1972)[8]. For this purpose it is necessary to restrict the set of allocations among which the government can choose to those that can be decentralized as a competitive equilibrium. We first find the constraints that must be imposed on the government's problem in order to comply with this requirement.

In our framework there is one implementability constraint associated with the individual family's intertemporal consumption choice. More precisely it is the individual budget constraint with prices substituted for by using the consumption Euler equation (the formal derivation is provided in Appendix A.6):

[7] Incidentally, note that when *b*=0, the condition provided in Lemma 1 boils down to $\tau^k < -\frac{f''k}{f'}$ (and the latter inequality is both necessary and sufficient for $\frac{d\tau^N}{d\tau^k} < 0$).

[8] On the contrary, the dual approach takes prices and tax rates as control variables. For a survey see Renström (1999).

$$A_0 u_{c0} = -\int_0^\infty e^{-\rho t} u_{c_t} \left[w_t - c_t - \tau_t^N - \theta(n_t)\right] N_t dt \tag{27}$$

Finally there are two feasibility constraints, one which requires that private and public consumption plus investment be equal to aggregate output (eq. 6); the other is given by eq. (3).

5.1. Solution

In this section we characterize the solution to the Ramsey problem. As already mentioned, the policymaker has to abide both the implementability and the feasibility constraints in order to insure that the optimal allocation, solution of the Ramsey problem, implements a competitive equilibrium.

Hence, supposing that the policy is introduced in period 0, the problem of the policymaker is to maximize (1) subject to eq. (27), and, $\forall t \geq 0$, eqs. (3), (6) and (17). Under interiority of the solution for *n* at each instant t, the appropriate constraint is (17) and we associate to the latter a multiplier, $\tilde{\omega}$, which is costate variable.

Finally, since a price appears in the latter constraint ($\bar{r}_t$), we add eq. (12) as a further constraint in the government's maximization problem and express consumption as a function of the multiplier *q*, which is a (co)-state variable. Hence, the current value Hamiltonian is:

$$\begin{aligned} H_t &= N_t\left[u(c_t(q_t)) - \alpha\right] + \mu\left[-u_{c_t}\left(F_{L_t} - c_t(q_t) - \tau_t^N - \theta(n_t)\right)N_t\right] + \tilde{\eta}_t\left[(\rho - \bar{r}_t)q_t\right] \\ &- \tilde{\omega}_t\left[\frac{u(c_t(q_t)) - \alpha}{u'_{c_t}(q_t)} + \left(F_{L_t} - c_t(q_t) - \tau_t^N - \theta(n_t) - \theta'(n_t)(\bar{r}_t - n_t)\right)\right] + \\ &+ \gamma_t\left[F_t - c_t(q_t)N_t - g_t N_t - \theta(n_t)N_t\right] + \phi_t n_t N_t \end{aligned} \tag{28}$$

We introduce a state variable *x*, defined as $\theta'(n) \equiv x$, and treat *n* as a function of *x*. Then, $\dot{x} = \theta''\dot{n}$, where $\dot{n}$ is given by eq. (17). Then, $\tilde{\omega}$ is the costate variable associated with *x*.

First order conditions for this problem are the following (we omit the time subscript when it does not cause ambiguity to the reader and the transversality conditions for the sake of brevity, also note that L_t=N_t):

$$\begin{aligned} &\frac{\partial H}{\partial q} = \frac{\partial H}{\partial c}\frac{\partial c}{\partial q} + \frac{\partial H}{\partial q} = \eta(\rho - n) - \dot{\eta} \Rightarrow \\ &\dot{\eta} = -\frac{1}{u_{cc}}\left\{u_c(1+\mu) - \mu\left[F_L - c - \tau^N - \theta\right]u_{cc} + \omega\frac{u-\alpha}{u_c^2}u_{cc} - \gamma\right\} + \eta(\bar{r} - n), \end{aligned} \tag{29}$$

where $\frac{\partial c}{\partial q} = \frac{1}{u_{cc}}$ from $u_c = q, \eta \equiv \frac{\tilde{\eta}}{N}, \omega \equiv \frac{\tilde{\omega}}{N}$

$$\frac{\partial H}{\partial K} = \rho\gamma - \dot{\gamma} \Rightarrow \dot{\gamma} = \gamma(\rho - F_K) + F_{LK}(u_c\mu + \omega)N \tag{30}$$

$$\frac{\partial H}{\partial N} = \rho\phi - \dot{\phi} \Rightarrow \dot{\phi} = (\rho - n)\phi - (u - \alpha) + \mu u_c\left[F_L - c - \tau^N - \theta\right] + \\ - \gamma\left[F_L - c - g - \theta\right] + (u_c\mu + \omega)F_{LL}N \tag{31}$$

$$\frac{\partial H}{\partial \bar{r}} = -\eta q + \omega\theta' = 0 \Rightarrow \eta q = \omega\theta' \tag{32}$$

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial n}\frac{\partial n}{\partial x} = (\rho - n)\omega - \dot{\omega} \Rightarrow \dot{\omega} = (\rho - \bar{r})\omega - \frac{1}{\theta''}\left[\theta'(\mu u_c - \gamma) + \phi\right] \tag{33}$$

where

$$\frac{\partial H}{\partial n} = \theta'(\mu u_c - \gamma) + \omega\theta''(\bar{r} - n) + \phi \tag{34}$$

Recall that γ, being the shadow price of capital, is strictly positive and μ, being a measure of the deadweight loss stemming from distortionary taxation, is zero at the first-best and positive at the second-best.

We can now characterize the first-best policy.

Proposition 5: *The first-best policy implies that at each instant of time capital income tax be zero, the tax on the family size be equal to the per-capita public expenditure and the public debt be equal to zero.*

Proof: At the first-best the government controls *c, n, k,* directly. Consequently (27) and (17) are not binding, which implies that $\mu = \omega = 0$. By eq. (32), $\eta = 0$ as well and, by eq. (29), $\gamma = u_c = q$, such that, by comparing eqs. (12) and (30) it follows that $\tau^k = 0$. Moreover, since in the first-best $\lambda = \phi$ (i.e. the government evaluation of the population is equal to the households' evaluation), by comparing (13) and (31) it descends $\tau^N = g$. Finally, since $\gamma = q$ (the marginal value of capital is equal to marginal value of private assets), it descends that, at each instant *t*, *a=k* and *b=0.* □

A comment on the latter result is worth making. The reason why the population tax implements the first-best rather than a family-level lump-sum tax, is because the externality a family exerts on the government budget when choosing the number of children is perfectly internalised when $\tau^N = g$. If there is any public debt it should be defaulted upon, otherwise the population tax would have to exceed the public expenditure level, and the first best would not be implemented.

Suppose now that the first-best taxation is not implementable; more precisely, we assume that the constraint $\tau^N \le \tau^N{}_{\max} < g$ is binding which happens if

$$\frac{\partial H}{\partial \tau^N} = (\mu u_c + \omega)N > 0 \tag{35}$$

which means that the Hamiltonian is increasing in the population tax as long as the second-best constraint binds. In this situation, only a second-best allocation is implementable, and characterized in the next proposition:

Proposition 6: *The second-best tax structure implies* $\tau^{N} = \tau^{N}{}_{\max} < g$ *and positive capital income tax*

$$\tau^{k} = \frac{(\mu u_{c} + \omega) F_{LK} N}{F_{K} \gamma} > 0 \tag{36}$$

in the steady state. Moreover, the optimal level of debt is negative.

Proof: see Appendix A.7 □

A final comment on the results is worth making. The nonzero capital income tax is non-standard in the traditional literature on optimal taxation and exogenous population growth, in that, typically, in the long run the second-best result entails zero tax on capital income, stemming from the optimality of uniform commodity taxation (Atkinson and Stiglitz 1972), although some exceptions may arise[9]. The rationale for our result is the following: when labour supply is exogenous there are labour rents present. If those rents are not taxed at 100%, the standard second-best results will not hold, in particular results on uniform commodity taxation. In fact, a capital tax partially taxes those rents (because F_{LK} >0).

As for the negative level of debt in the steady state, under the second-best it is optimal to run primary surpluses at the beginning of the tax programme, arriving at the steady state with public assets. At the steady state, tax receipts fall below the level of public expenditure, though not being zero (i.e. it is still optimal to carry tax burden to the steady state).

5.2. Endogenous labour supply

We now show the solution to the Ramsey problem when individuals can endogenously offer their labour services and (distortionary) taxes on wages are levied. In section 5.2.1 we start from the situation in which labour tax can be chosen freely by the policymaker and later, in section 5.2.2, we examine the consequences of restrisctions on the choice of the labour income tax. The instantaneous utility function is now of the form $u(c_t, l_t)$, assumed to be decreasing in labour supply l_t and strictly concave. Total labour supply is then $L_t = N_t l_t$. The household budget constraint is now

$$\dot{A}_t = \bar{r}_t A_t + \bar{w}_t l_t N_t - c_t N_t - \tau_t^{N} N_t - \theta(n_t) N_t \tag{37}$$

where $\bar{w} = w(1-\tau^{l})$ is the wage rate net of labour income tax τ^{l}**.**

5.2.1. The case of unrestricted taxation of labour income

[9] For example, in OLG economies (as argued by Erosa and Gervais 2002) or in presence of different discounting between government and individuals (see De Bonis and Spataro 2005) or a combination of both (see Spataro and De Bonis 2008).

In this setting the first-order conditions of the individual problem entail the following condition:

$$\frac{\partial H_t}{\partial l_t} = 0 \Rightarrow u_{l_t} = -\overline{w}_t q_t , \tag{38}$$

which, combined with eq. (12) provides the following:

$$\frac{u_{lt}}{u_{c_t}} = -\overline{w}_t . \tag{39}$$

Moreover, recall that the decentralized equilibrium implies that the gross wage rate be equal to the marginal productivity of labour, that is

$$F_{Lt}\left(K_t, l_t \cdot N_t\right) = w_t \tag{40}$$

All this said, the problem of the policymaker becomes:

$$\max \int_{t=0}^{\infty} N_t e^{-\rho t}\left[u\left(c_t, l_t\right) - \alpha\right] dt$$

subject to the implementability constraint, which, in this case, is

$$A_0 u'_0 = \int_0^{\infty} e^{-\rho t}\left[u_{l_t} l_t + u_{c_t}\left(c_t + \tau_t^{\ N} + \theta_t\right)\right] N_t dt , \tag{41}$$

and eqs. (3), (6). The solution for n is always interior and, as in section 5.1 we introduce the state variable x defined as $x \equiv \theta'(n)$, to which we associate the costate variable $\tilde{\omega}$. Hence, by making use of eqs. (38) and (39), and reckoning that, by eq. (12) $u_c\left(c_t, l_t\right) = q_t$, the following relationship holds $c_t = c(q_t, l_t)$. Hence, the current value Hamiltonian is:

$$\begin{aligned} H_t &= N_t\left[u\left(c_t, l_t\right) - \alpha\right] + \mu\left[u_{l_t} l_t + q_t\left(c_t + \tau_\tau^{\ N} + \theta_t\right)\right] N_t + \tilde{\eta}_t\left[\left(\rho - \bar{r}_t\right) q_t\right] \\ &- \tilde{\omega}_t\left[\frac{u\left(c_t, l_t\right) - \alpha}{q_t} - \frac{u_{l_t} l_t}{q_t} - (c_t + \tau_t^N + \theta_t + \theta(\bar{r}_t - n_t))\right] + \gamma_t\left[F_t - c_t N_t - g_t N_t - \theta_t N_t\right] + \phi_t n_t N_t \end{aligned} \tag{42}$$

Then, the first-order condition with respect to K is (dropping time subscripts; see Appendix 8 for the remaining first-order conditions):

$$\frac{\partial H}{\partial K} = \rho\gamma - \dot{\gamma} \Rightarrow \dot{\gamma} = \left(\rho - F_K\right)\gamma \tag{43}$$

As for capital income tax, since at the steady state γ is constant, by equation (43) it descends τ^k =0. Finally, as for τ^N, we can start by observing that the tax structure $\tau^k = 0$

$\tau^l = 0$, $\tau^N = g$ and b=0 would implement the first-best allocation. Hence, in order to get a second-best allocation, we again impose that the constraint $\tau^N \leq \tau^N{}_{\max} < g$ is binding, which implies (35) holds.

Hence, we can provide the following proposition:

Proposition 7: *At the steady state the second-best tax structure implies that capital income tax be zero, the tax on the family size be equal to the maximum possible level* $\tau^N{}_{\max}$. *If leisure is non-inferior, then debt is negative and the labour-income tax is strictly positive.*

Proof: see Appendix 8 □

As a final comment, we conclude that with endogenous labour supply, there are no labour rents, and with (yet distortionary) labour income taxation, the zero capital income tax result is restored (i.e. the optimality of uniform commodity taxation). Also, the labour income tax is positive (at least if leisure is non-inferior) implying that it is optimal to carry tax burden to the steady state.[10]

5.2.1 The case of constraints on the labour income tax

Finally, we investigate the consequences on our results, and in particular on the capital tax, if there are constraints on the labour income taxation[11]. More precisely, we now assume that τ^l cannot exceed a certain positive level $\bar{\tau}^l$. This implies an extra constraint in the Hamiltonian and eq. (42) now becomes

$$\begin{aligned}H_t &= N_t\left[u(c_t,l_t)-\alpha\right]+\mu\left[u_{l_t}l_t+q_t\left(c_t+\tau_t{}^N+\theta_t\right)\right]N_t+\tilde{\eta}_t\left[(\rho-\bar{r}_t)q_t\right]\\&-\tilde{\omega}_t\left[\frac{u(c_t,l_t)-\alpha}{q_t}-\frac{u_{l_t}l_t}{q_t}-(c_t+\tau_t^N+\theta_t+\theta_t(\bar{r}_t-n_t))\right]+\\&+\gamma_t\left[F_t-c_tN_t-g_tN_t-\theta_tN_t\right]+\phi_t n_t N_t+\upsilon_t\left(-u_{l_t}-u_{c_t}\left(1-\tau_t^l\right)F_{L_t}\right)\end{aligned} \tag{44}$$

where υ_t is the Kuhn-Tucker multiplier associated with the constraint on the labour income tax rate and is positive if it is binding. The first order condition with respect to K is:

$$\frac{\partial H}{\partial K}=\rho\gamma-\dot{\gamma}\Rightarrow\dot{\gamma}=\left(\rho-F_K\right)\gamma+\upsilon\left(1-\tau^l\right)F_{LK}\,. \tag{45}$$

Since $\gamma,\upsilon,F_{LK} > 0$, we conclude that at the steady state $\tau^k > 0$.

We recall that the result of nonzero income tax in section 5.1 stems for the presence of rents in the economy that cannot be taxed away (at 100% rate). In this case, with endogenous labour supply, the result on capital income tax stems from restrictions on one of the second-best tax instruments, τ^l, which leads to the violation of the zero tax result (see, for example, Correia 1996).

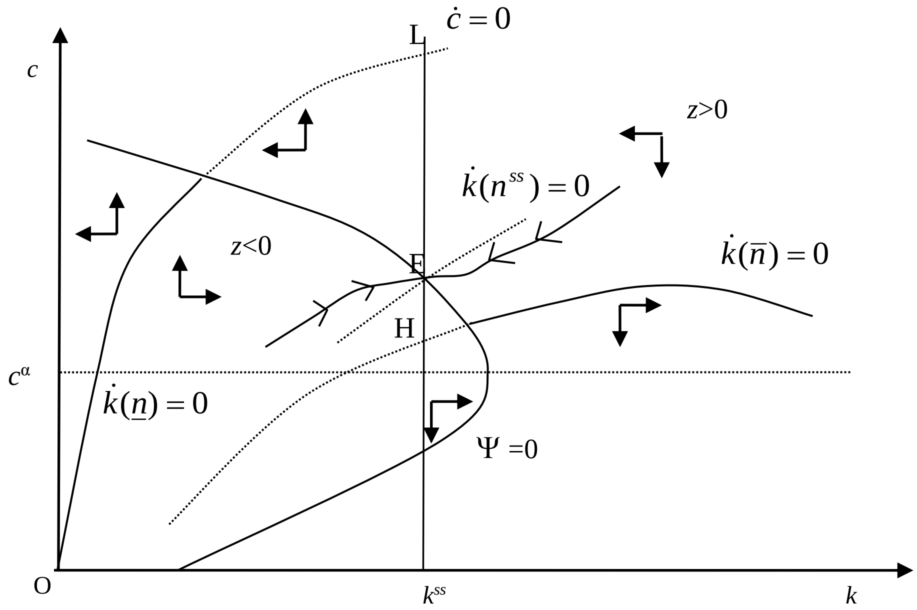
[10] In our endogenous population economy, non-inferiority of leisure is sufficient for the labour tax to be strictly positive at the steady state. On this issue in a Chamley setting, with fixed population, see Renström (1999). For indivisible labour economies, with fixed population, normality is needed for a positive labour tax, see Basu and Renström (2007).

[11] We thank an anonymous referee for pointing us in this direction.

6. An extension: the case of linear costs for raising children

If costs for childbearing are linear the steady state values for c, n and k are provided by eqs. (21)-(23). However, the dynamics is qualitatively different from the convex-cost case.

Figure 4: The steady state equilibrium in the case of linear costs for raising children



In Figure 4 we depict the three loci described by eqs. (21)-(23) where the steady state equilibrium is represented by point E. Equation (23) gives all combinations of per-capita consumption and per-capita capital that constitute an optimal population size. As anticipated above, this is the case when what an individual brings to the family (utility above α plus the labour endowment) is equal to what he/she takes out (consumption plus the population tax plus childrearing costs). These combinations are depicted by the $\Psi=0$ locus in Figure 4. Note that, with linear childbearing costs, by substituting for $\theta(n)=(1+n)\cdot\theta'$ into (23), it turns out that the $\Psi=0$ locus is independent of *n*. Moreover, for trajectories inside the $\Psi=0$ locus, *z* is negative and consequently *n* is at its lower corner $\underline{n}$. For trajectories outside the $\Psi=0$ locus, *z* is positive and *n* is at its higher corner $\bar{n}$. The steady state value of per-capita capital is given by $\rho=f'(k^{ss})\left(1-\tau^{k}\right)$, giving the vertical $\dot{c}=0$ line. Finally, the steady state population growth rate n^{ss} is such that $\dot{k}=0$ line cuts in point E.

We should notice that the trajectories leading to E are not the usual saddle-paths. The reason is that we are in a corner with respect to *n* along the transition. For capital stocks lower than k^{ss} it is optimal to pick an unstable trajectory in a system where $n=\underline{n}$ and when reaching E, to switch from $\underline{n}$ to n^{ss}. Similarly, for capital stocks greater than k^{ss} it is optimal to take an unstable trajectory in a system where $n=\bar{n}$ and when reaching E, to let *n* jump from $\bar{n}$ to n^{ss}. The reason why the stable trajectory in this system cannot be taken is because the latter trajectory, say, for $k< k^{ss}$, ends in the steady state where $\dot{k}=0$ for $n=\underline{n}$, -point L in Fig. 4-, but there z>0, would prescribe $n=\bar{n}$. Instead there is an (unstable) trajectory with lower initial consumption moving away from the otherwise stable trajectory reaching the point (k^{ss}, c^{ss}) in finite time, say, at time t_1. When this point is reached, the control previously being kept at $n = \underline{n}$ switches to n^{ss}, yielding $\dot{k} = 0$.[12] A similar argument applies for trajectories starting from $k > k^{ss}$.

6.1 Positive analysis of taxation

As for the results of the positive analysis of taxation, when costs for childbearing are linear, both the steady state and the transitional effects of an increase of population tax are quite the same as those arising in the convex-cost case. Thus Proposition 2 holds here as well. However, Proposition 3 changes slightly. An increase in the capital income tax now raises the steady-state levels of both the population growth rate and the capital intensity and consumption. The latter result stems from equation (A.6) of Appendix 3, which now becomes:

$$\frac{dc}{d\tau^k} = -\frac{f'}{\left(1-\tau^k\right)}\frac{ku^{\;2}}{(u-\alpha)u_{cc}} > 0. \tag{46}$$

As for the dynamics of the economy, the capital income tax increases, the $\dot{c} = 0$ line shifts to the left and the $\dot{k} = 0$ locus moves up, while the $\Psi = 0$ locus shifts inwards, such that the new steady state equilibrium entails a higher consumption level and both lower population growth rate and lower capital intensity.

If this tax change comes as a surprise, per-capita consumption first jumps to a high level, and then gradually falls to its new higher level, creating a consumption boom. During the transition, the economy is outside the $\Psi = 0$ locus and consequently population growth is at its maximum, $\bar{n}$. When reaching the new steady state in finite time population growth falls to its lower new steady state value. Thus, the economy experiences a population growth burst (“baby boom”) and then a fall in the population growth rate.

As for the effects of a constant per-capita-debt redistribution of taxes, the steady state results are the same as those described in Proposition 4, although such a policy, implying an increase (decrease) of the capital income tax, unambiguously increases (decreases) per-capita consumption, provided that debt is non negative. In fact, equation (A.11) of Appendix 5 now becomes:

$$\frac{dc}{d\tau^k} = \frac{1}{b}\frac{M}{1+M}\left[\left(f'-\rho\right)\left(k+\theta'\right)+b(f'-n)\right]\frac{dk}{d\tau^k} > 0. \qquad \square \tag{47}$$

6.2 Normative analysis of taxation

As for the normative taxation, the linear costs case implies a different implementability constraint. Instead of equation (17) we have:

$$n_t = \begin{cases} \bar{n}, z_t > 0 \\ n_t^*, z_t = 0 \\ \underline{n}, z_t < 0 \end{cases} \tag{48}$$

[12] For details see Renström and Spataro (2011).

where z_t is according to (19), and n_t^* is any level of *n*. In fact, there is a corner along the transition path for *n*, such that the latter constraint (48), involving *z* (eq. 19), entails both an integral and an inequality, which is difficult to deal with. However, as already mentioned, for trajectories inside (outside) the $\Psi=0$ locus, the expression in square brackets in equation (19) must be negative (positive) at each instant *t* (for details see Renström and Spataro 2011). We associate this latter inequality with the Kuhn-Tucker multiplier $\tilde{\omega}$. The Hamiljtonian is now identical to that of Section 5.1, with the only difference being that $\tilde{\omega}$ is a Kuhn-Tucker multiplier instead of a co-state (in fact the two cases can be nested in one formulation). All derivatives of the Hamiltonian remain the same, apart from equation (33) which no longer applies. Equation (34) is now the first-order variation with respect to *n*, which now can be a corner, i.e.

$$\frac{\partial H}{\partial n}\overset{>}{\underset{<}{-}}0 \Leftrightarrow n=\begin{cases}\overline{n}\\ n^*\\ \underline{n}\end{cases} \tag{49}$$

Weather of not the implementability constraint in equation (48) is binding at the optimum, i.e. whether or not $\tilde{\omega}$ is non-zero, makes no difference to the proofs of Propositions 5 and 6. Consequently Propositions 5 and 6 hold also in this case (i.e. regardless the shape or even the presence of the cost function).

Finally, when labour supply is endogenous, equations (3), (6) and (48) hold, where *z* now is given by the following expression:

$$z_t=\int_t^\infty e^{-\int_t^\tau(\rho-n_s)ds}\left\{u(c_\tau,l_\tau)-\alpha+u_{c_\tau}\left[\overline{w}_\tau l_\tau-c_\tau-\tau_\tau^{\,N}-\theta_\tau-\theta_\tau(\overline{r}_\tau-n_\tau)\right]\right\}d\tau \overset{>}{\underset{<}{-}}0\,. \tag{50}$$

Again, if the cost for raising children is linear, it can be shown that, for trajectories inside (outside) the $\Psi=0$ locus, the expression in brackets in the integral above must be negative (positive) in each instant *t* along the transition path. Now, the steady state relationship given by equation (A.19) holds as well, although stemming from the Kuhn-Tucker complementary slackness condition. Again, Proposition 7 holds also in the linear cost case.

6. Conclusions

In the present work we tackle the issue of taxation in presence of endogenous fertility and under critical level utilitarian preferences and childbearing costs. From a positive standpoint we show that a rise of the tax on the family size decreases the population growth rate and increases steady state per-capita consumption, and does not affect capital; on the other hand, a rise of the capital income tax reduces both steady state capital and population growth rate, while has ambiguous effects on per-capita consumption. However, the increase of the capital income tax creates temporary population and consumption bursts and reduces the steady state capital stock, while an increase in the population tax does not.

We have also analysed the effects of a fiscal policy aiming at redistributing the tax burden in such a way to maintain per-capita debt unchanged. The latter tax reform implies that capital and the population growth rate move in the same direction as the change in the tax on population size, while consumption can move in either direction. Surprisingly enough, on policy grounds the latter result suggests that an economy that aims to increase population growth but is burdened by high public debt (such as Italy) could reduce capital income taxes *and* increase the tax on the family size correspondently, such that, in the long run, both the rate of growth of population and the capital intensity would be increased. Unfortunately, this could happen at the cost of experiencing a reduction in the long run per-capita consumption and a temporary reduction of the same population rate of growth.

As far as the normative analysis is concerned, we show that, at the steady state the first-best policy entails zero capital income tax and zero debt and positive taxation of the family size, no matter whether labour supply is endogenous or not. However, when only a second-best tax structure can be implemented, then positive taxation of capital income and negative debt turn out to be optimal in case labour supply is exogenously fixed.

Finally, the zero capital income tax result arises also in our model when labour supply is endogenous, provided that any constraints on the labour income taxes are not binding. If leisure is a non-inferior good the labour income tax is positive and debt negative. We show that the above normative tax results hold regardless the shape of the childbearing cost function.

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Appendix A.1

Proof of Proposition 1: Let us write the Jacobian matrix to the dynamic system (12), (8) and (17), evaluated at the steady state:

$$J = \begin{bmatrix} 0 & -\frac{u_c}{u_{cc}} f''\left(1-\tau^k\right) & 0 \\ -1 & f'-n & -k \\ \frac{(u-\alpha)u_{cc}}{u_c^{\,2}\theta''} & \frac{f''}{\theta''}\left[k+\left(1-\tau^k\right)\theta''\right] & \rho-n \end{bmatrix}$$

and

$$|J|=\frac{u_c}{u_{cc}}\left(1-\tau^k\right)f''\left[(k+\theta')\frac{(u-\alpha)u_{cc}}{{u_c}^2\theta''}-(\rho-n)\right]=(\rho-n)\left(1-\tau^k\right)f''\left[(k+\theta')\frac{\theta'+a}{\theta''}-\frac{u_c}{u_{cc}}\right].$$

is the determinant of the Jacobian, where the second equality above stems from (26). The characteristic equation associated with the Jacobian matrix is the following:

$$O(\varepsilon)=R(\varepsilon)+|J|=0,$$

where ε are the eigenvalues, $R(\varepsilon)\equiv-\varepsilon\left[(f'-n-\varepsilon)(\rho-n-\varepsilon)-P+\frac{|J|}{\rho-n}\right]$ and $P\equiv-f''\frac{k+\theta'}{\theta''}\left[k-\left(1-\tau^k\right)a\right]$. Since $O(\varepsilon)$ is cubic (since $R(\varepsilon)$ is cubic), we have three eigenvalues.

In order to have a saddle-path stable equilibrium the function $O(\varepsilon)$ must have one negative real root and two positive real roots.

Preliminarily note that, without taxes and public debt, $P=0$, $\rho=f'$ and

$$O(\varepsilon)\equiv(\rho-n-\varepsilon)\left[f''\frac{(k+\theta')^2}{\theta''}-f''\frac{u_c}{u_{cc}}+\varepsilon^2-(\rho-n)\varepsilon\right]$$

which has three roots $\varepsilon_1=\rho-n>0$ $\varepsilon_{2,3}=\frac{\rho-n}{2}\pm\sqrt{\left(\frac{\rho-n}{2}\right)^2-f''\left[\frac{(k+\theta')^2}{\theta''}-\frac{u_c}{u_{cc}}\right]}$, $\varepsilon_2>0,\varepsilon_3<0$, such that the steady state equilibrium is saddle-path stable.

In presence of taxes and public debt, it is convenient to start by noting that the roots of $R(\varepsilon)$ are $\Lambda_1=0$ and $\Lambda_{2,3}=\frac{f'-n+\rho-n}{2}\pm\sqrt{\left(\frac{f'-n+\rho-n}{2}\right)^2+P-\frac{|J|}{\rho-n}}$. Sufficient condition for the latter roots to be real is that $P\geq0\Leftrightarrow\frac{b}{k}\leq\frac{\tau^k}{1-\tau^k}$, such that $\Lambda_2>0,\Lambda_3<0$ and $R(\varepsilon)=-\varepsilon(\varepsilon-\Lambda_2)(\varepsilon-\Lambda_3)$. In this situation, by reckoning that $O(0)=|J|<0$, $\left.\frac{\partial R}{\partial\varepsilon}\right|_0=-\Lambda_2\Lambda_3>0$, $\left.\frac{\partial R}{\partial\varepsilon}\right|_{\Lambda_2}=-\Lambda_2(\Lambda_2-\Lambda_3)<0$ and $\left.\frac{\partial R}{\partial\varepsilon}\right|_{\Lambda_3}=\Lambda_3(\Lambda_2-\Lambda_3)<0$, it descends that $O(\varepsilon)$ has one (and only one) negative root. Finally, sufficient for $O(\varepsilon)$ to have two positive roots[13] is that in the interval (Λ_1,Λ_2) there exists a value of ε, say $\hat{\varepsilon}$, such that $O(\hat{\varepsilon})>0$. In fact, by taking $\hat{\varepsilon}=\frac{f'-n+\rho-n}{2}$, it is easy to show that $O(\hat{\varepsilon})=\frac{f'-n+\rho-n}{2}\left[\left(\frac{\tau^kf'}{2}\right)^2+P-\frac{|J|}{\rho-n}\right]+|J|=\frac{f'-n+\rho-n}{2}\left[\left(\frac{\tau^kf'}{2}\right)^2+P\right]-\frac{1}{2}\frac{\tau^kf'|J|}{\rho-n}$.

[13] Consequently, the steady state is always stable. However, in order to avoid cycles, two positive real roots are necessary.

Hence, sufficient for $O(\hat{\varepsilon}) > 0$ is that $P \geq 0 \Leftrightarrow \frac{b}{k} \leq \frac{\tau^k}{1-\tau^k}$. □

Appendix A. 2. Proof of Proposition 2

Using Cramer's rule, we can write:

$$\frac{dn}{d\tau^N} = \frac{|J_n|}{|J|} = \frac{1}{(k+\theta')\frac{(u-\alpha)u_{cc}}{u_c^{\,2}} - (\rho - n)\theta''} < 0 \tag{A.1}$$

where J is the Jacobian matrix to the dynamical system (12) (8) and (17) (as shown in Appendix A.1) and J_n is the Jacobian matrix in which the third column is substituted out by the derivatives of eqs. (21) to (23) (with negative sign) with respect to τ^N.

By the same method we can compute

$$\frac{dc}{d\tau^N} = \frac{|J_c|}{|J|} = -\frac{(k+\theta')}{(k+\theta')\frac{(u-\alpha)u_{cc}}{u_c^{\,2}} - (\rho - n)\theta''} > 0 \tag{A.2}$$

and $\frac{dk}{d\tau^N} = \frac{|J_k|}{|J|} = 0$ □ (A.3)

Appendix A.3. Proof of Proposition 3

Using Cramer's rule, we can write:

$$\frac{dk}{d\tau^k} = \frac{|J_k|}{|J|} = \frac{f'}{f''(1-\tau^k)} < 0. \tag{A.4}$$

where J_k is the Jacobian matrix in which the second column is substituted out by the derivatives of eqs. (21) to (23) (with negative sign) with respect to τ^k.
Analogously, we get:

$$\frac{dn}{d\tau^k} = \frac{|J_n|}{|J|} = \frac{f'}{(1-\tau^k)f''}\frac{\left[f''k + (f'-n)\frac{(u-\alpha)u_{cc}}{u_c^{\,2}}\right]}{\left[(k+\theta')\frac{(u-\alpha)u_{cc}}{u_c^{\,2}} - (\rho-n)\theta''\right]} < 0 \tag{A.5}$$

and

$$\frac{dc}{d\tau^k} = \frac{|J_c|}{|J|} = -\left(\frac{f'}{(1-\tau^k)f''}\right)\frac{\left[(k+\theta')f''k + (\rho-n)(f'-n)\theta''\right]}{\left[(k+\theta')\frac{(u-\alpha)u_{cc}}{u_c^{\,2}} - (\rho-n)\theta''\right]} \gtrless 0 \quad \square \tag{A.6}$$

Appendix A.4. Proof of Lemma 1

By exploiting eqs. (16) and (21), the steady state government budget constraint can be written as:

$$b(\rho-n)=k(f'-\rho)+\tau^N-g\ ;$$

totally differentiating the above expression with respect to τ^k yields:

$$\frac{dn}{d\tau^k}=-\frac{1}{b}\frac{d\tau^N}{d\tau^k}+\frac{1}{b}(\rho-f'-f''k)\frac{dk}{d\tau^k}. \tag{A.7}$$

Recalling that $\frac{dn}{d\tau^k}=\frac{\partial n}{\partial\tau^N}\frac{d\tau^N}{d\tau^k}+\frac{\partial n}{\partial\tau^k}$, and exploiting eqs. (A.1), (A.4) and (A.5) we get also:

$$\frac{dn}{d\tau^k}=\frac{M}{b}\frac{d\tau^N}{d\tau^k}+\frac{M}{b}\left(\frac{(f'-n)(u-\alpha)u_{cc}}{u_c^{\,2}}+f''k\right)\frac{dk}{d\tau^k} \tag{A.8}$$

where $M\equiv\frac{bu_c^{\,2}}{u_{cc}(u-\alpha)(k+\theta')-u_c^{\,2}(\rho-n)\theta''}$.

Hence, by equating eqs. (A.7) and (A.8) and collecting terms it follows that:

$$\frac{d\tau^N}{d\tau^k}=\frac{f'}{1+M}\left\{-\tau^k-\frac{f''k}{f'}-\frac{M}{f'}\left[\frac{(f'-n)(u-\alpha)u_{cc}}{u_c^{\,2}}+f''k\right]\right\}\frac{dk}{d\tau^k} \tag{A.9}$$

Since $\frac{dk}{d\tau^k}<0$, it descends:

$$\frac{d\tau^N}{d\tau^k}<0\Leftarrow\tau^k<\bar{\tau}^k.$$ □

Appendix A.5. Proof of Proposition 4

By plugging eq. (A.9) into eq. (A.8) and collecting terms it descends that[14]

$$\frac{dn}{d\tau^k}=\left(\frac{M}{1+M}\right)\left(\frac{1}{b}\right)\left[f'-\rho-\frac{u_{cc}(u-\alpha)(f'-n)}{u_c^{\,2}}\right]\frac{dk}{d\tau^k}<0. \tag{A.10}$$

Moreover, by differentiating (22) with respect to τ^k and exploiting eq. (A.10), we obtain

[14] Note that, under non negative capital income tax, $\tau^k<\bar{\tau}^k$ implies that *(1+M)>0*. Moreover, *(M/b)<0*.

$$\frac{dc}{d\tau^k} = \frac{1}{b}\frac{M}{1+M}\left[(f'-\rho)(k+\theta') + b(f'-n) - (\rho - n)(f'-n)\theta''\right]\frac{dk}{d\tau^k} \gtrless 0 . \qquad \text{(A.11)}$$

□

Appendix A. 6. The derivation of the implementability and feasibility constraints

As for implementability (eq. 27), by multiplying both sides of eq. (2) by $q_t e^{-\int_0^t \bar{r}_s ds}$, integrating out the household's budget constraint and using the transversality condition we get:

$\left(\dot{A}_t - \bar{r}_t A_t\right) q_t e^{-\int_0^t \bar{r}_s ds} = q_t e^{-\int_0^t \bar{r}_s ds}\left[w_t - c_t - \tau_t^N - \theta(n_t)\right]$, that is

$$A_0 u_{c_0} = -\int_0^\infty e^{-\rho t} u_{c_t}\left(w_t - c_t - \tau_t^N - \theta(n_t)\right) N_t dt .$$

As for feasibility, write eq. (2) as $\dot{A}_t = r_t\left(1-\tau_t^k\right)A_t + w_t N_t - c_t N_t - \tau_t^N N_t - \theta(n_t)N_t$. Using market clearing condition (eq. 16) we get:

$$\dot{K}_t + \dot{B}_t = r_t\left(1-\tau_t^k\right)\left(K_t + B_t\right) + w_t N_t - c_t N_t - \tau_t^N N_t - \theta(n_t)N_t$$

$$\dot{K}_t + \dot{B}_t = r_t\left(1-\tau_t^k\right)\left(K_t + B_t\right) + w_t N_t - c_t N - \tau_t^N N_t ;$$

moreover, by exploiting CRS and using eqs. (4) and (5) we get:

$$\dot{K}_t = F_t - \dot{B}_t + r_t B_t - c_t N_t - \tau_t^N N_t - \tau_t^k r_t\left(K_t + B_t\right) - \theta(n_t)N_t ;$$

and, finally, by exploiting debt equation (eq. 7) it descends that

$$\dot{K}_t = F_t - c_t N_t - g_t N_t - \theta(n_t)N_t . \qquad \square$$

Appendix A.7: Proof of Proposition 6

Eq. (35) gives the level of τ^N. By (11), $(1-\tau^k)F_K = \rho$ in steady state. Using this in (30) and evaluating it at $\dot{\gamma} = 0$ gives (36), which is positive by eq. (35).

To prove the sign of the optimal debt level, use (30) and (31) evaluated at the steady state, the steady state relationships $\left(\rho - \bar{r}\right) = 0$ (from eq. 12), eq. (26), $\left[F_L - c - \tau^N - \theta\right] = -(\bar{r} - n)a$ (from eq. 6 in per capita terms), $\left[F_L - c - g - \theta\right] = -(f'-n)k$ (from eq. 8) and CRS (such that $F_{LL}N = -F_{LK}Nk$) to get:

$$\phi = (1+\mu)u_c a + u_c\theta' - \gamma k . \qquad \text{(A.12)}$$

Moreover, by plugging eq. (32) into (29) and using (12) we get:

$$\gamma = (1+\mu)u_c + u_{cc}(\rho - n)a\left(\mu + \frac{\omega}{u_c}\right) \tag{A.13}$$

Furthermore, by substituting the expression for ϕ stemming from eq. (33) into (A.12) it descends

$$\gamma\theta' = (1+\mu)u_c(a+\theta') - \gamma k \tag{A.14}$$

which implies $(a+\theta')$>0. Finally, by substituting for $(1+\mu)u_c$ from eq. (A.13) into (A.14) and exploiting the market clearing condition *a*=*k*+*b*, we get:

$$b = \frac{(a+\theta')u_{cc}(\rho - n)k\left(\mu + \frac{\omega}{u_c}\right)}{\gamma - (a+\theta')u_{cc}(\rho - n)\left(\mu + \frac{\omega}{u_c}\right)} < 0\,.$$

Appendix A.8: Proof of Proposition 7

The first-order conditions of the problem are:

$$\frac{dH}{dq} = \frac{1}{N}\left[\frac{\partial H}{\partial c}\frac{\partial c}{\partial q} + \frac{\partial H}{\partial q}\right] = \eta(\rho - n) - \dot{\eta};$$

$$\Rightarrow \dot{\eta} = -\frac{1}{N}\frac{\partial H}{\partial c}\frac{1}{u_{cc}} - \mu\left(c + \tau^N + \theta\right) - \frac{\omega}{u_c}\left(\frac{u(c,l) - \alpha}{u_c} - \frac{u_l l}{u_c}\right) + \eta(\bar{r} - n) \tag{A.15}$$

where $\frac{\partial H}{\partial c} = \left[(1+\mu)u_c + \mu u_{cl} l + \frac{\omega}{u_c}u_{cl} l - \gamma\right]N$, $\frac{\partial c}{\partial q} = \frac{1}{u_{cc}}$ from $u_c = q$ and

$$\frac{\partial H}{\partial q} = \left[\mu\left(c + \tau^N + \theta\right) + \eta(\rho - \bar{r}) + \frac{\omega}{q}\left(\frac{u-\alpha}{q} - \frac{u_l l}{q}\right)\right]N$$

$$\frac{dH}{dl} = \frac{\partial H}{\partial c}\frac{\partial c}{\partial l} + \frac{\partial H}{\partial l} = 0\,, \tag{A.16}$$

where $\frac{\partial H}{\partial l} = N\left[(1+\mu)u_l + \left(\mu + \frac{\omega}{u_c}\right)u_{ll} l + \gamma F_L\right]$, $\frac{\partial c}{\partial l} = -\frac{u_{cl}}{u_{cc}}$ from $u_c(c,l) = q$

$$\frac{\partial H}{\partial N} = \rho\phi - \dot{\phi} \Rightarrow \dot{\phi} = (\rho - n)\phi - (u-\alpha) - \mu\left[u_l l + u_c(c + \tau^N + \theta)\right] - \gamma(F_L l - c - g - \theta) \tag{A.17}$$

$$\frac{\partial H}{\partial \bar{r}} = -\eta q N + \omega\theta' N = 0 \Rightarrow \eta q = \omega\theta' \tag{A.18}$$

and conditions (33), (34), and (43). By focusing on the steady state, whereby $\dot{\omega} = \dot{\eta} = \dot{\gamma} = 0$, and plugging eq. (A.18) and the steady state relationship

$$\left[\frac{u-\alpha}{u_c} - \frac{u_l l}{u_c} - (c + \tau^N + \theta + \theta'(\bar{r} - n))\right] = 0 \tag{A.19}$$

(stemming from the condition $\dot{n} = 0$) into, eq. (A.15) yields:

$$u_c\left[1 + \mu + \left(\mu + \frac{\omega}{u_c}\right)\Delta_c\right] = \gamma \tag{A.20}$$

where $\Delta_c \equiv \frac{u_{cc}\left[c + \tau^N + \theta\right] + u_{lc} l}{u_c}$ is usually referred to as the "general equilibrium elasticity" of consumption; moreover, by eqs. (A15), (A.18) and (A.19) it follows that $\frac{\partial H}{\partial c} = -\left(\mu + \frac{\omega}{u_c}\right)(c + \tau^N + \theta)u_{cc}N$. Substituting the latter relationship into (A.16) one gets:

$$u_l\left[1 + \mu + \left(\mu + \frac{\omega}{u_c}\right)\Delta_l\right] = -\gamma F_L \tag{A.21}$$

where $\Delta_l \equiv \frac{u_{cl}\left[c + \tau^N + \theta\right] + u_{ll} l}{u_l}$ is usually referred to as the "general equilibrium elasticity" of leisure.

Since the other results are clear-cut, here we provide the proof for the level of debt and for the labour income tax. As for debt, plugging eqs. (37) and (6) in per-capita terms, eq. (26) and the steady state relationships $\left(\rho - \bar{r}\right) = 0,$ into eq. (A.17), it follows that:

$$\phi = \left(a + \theta'\right)q + \mu q a - \gamma k$$

and, by eq. (33) it turns out:

$$1 + \mu = \frac{\gamma}{q}\frac{\theta' + k}{\theta' + a}.$$

(whereby $\left(\theta' + a\right)$ >0). Moreover, by eq. (A.20) the following relationship holds:

$$1 + \mu = \frac{\gamma}{u_c} - \left(\mu + \frac{\omega}{u_c}\right)\Delta_c.$$

Equating the last two equations yields

$$\frac{b}{a + \theta'} = \frac{q}{\gamma}\left(\mu + \frac{\omega}{u_c}\right)\Delta_c. \tag{A.22}$$

Moreover, by eq. (37), expressed in per-capita terms, at the steady state we get that

$c+\tau^N+\theta=\overline{w}l+(\rho-n)a$; plugging the latter and eq. (39) into the definition of Δ_c yields:

$$\Delta_c u_c=\left(\frac{u_{lc}}{u_l}-\frac{u_{cc}}{u_c}\right)u_l l+u_{cc}(\rho-n)a\,;$$

substituting into (A.22) yields

$$b=\frac{(a+\theta')}{\gamma}\frac{\left(\mu+\frac{\omega}{u_c}\right)\left[\left(\frac{u_{lc}}{u_l}-\frac{u_{cc}}{u_c}\right)u_l l+u_{cc}(\rho-n)k\right]}{\left[1-\frac{(a+\theta')}{\gamma}\left(\mu+\frac{\omega}{u_c}\right)u_{cc}(\rho-n)\right]}.$$

If leisure is non-inferior, then the term $\left(\frac{u_{lc}}{u_l}-\frac{u_{cc}}{u_c}\right)$ is positive, such that *b* is negative.

As for the labour income tax, by dividing equation (A.20) by (A.21) we obtain:

$$\frac{u_c\left[1+\mu+\left(\mu+\frac{\omega}{u_c}\right)\Delta_c\right]}{u_l\left[1+\mu+\left(\mu+\frac{\omega}{u_c}\right)\Delta_l\right]}=-\frac{1}{F_L}$$

and, finally, using eq. (39) and $\overline{w}\equiv(1-\tau^l)w$ yields:

$$\tau^l=\frac{\left(\mu+\frac{\omega}{u_c}\right)(\Delta_l-\Delta_c)}{1+\mu+\left(\mu+\frac{\omega}{u_c}\right)\Delta_l}. \qquad \text{(A.23)}$$

Recall that, exploiting the definitions of Δ_l and Δ_c:

$$\Delta_l-\Delta_c=-\left(\frac{u_{cc}}{u_c}-\frac{u_{cl}}{u_l}\right)\left(c+\tau^N+\theta\right)+\left(\frac{u_{ll}}{u_l}-\frac{u_{lc}}{u_c}\right)l\ . \qquad \text{(A.24)}$$

Hence, eq. (A.24) can be written as:

$$\Delta_l-\Delta_c=-\left(\frac{u_{cc}}{u_c}-\frac{u_{cl}}{u_l}\right)(\rho-n)a+l\left[\frac{u_l}{u_c}\left(\frac{u_{cc}}{u_c}-\frac{u_{cl}}{u_l}\right)+\frac{u_{ll}}{u_l}-\frac{u_{lc}}{u_c}\right] \qquad \text{(A.25)}$$

Next, since under non-inferiority of leisure debt is negative, from eq. (A.22) it descends that $\Delta_c < 0$, and by exploiting the definition of Δ_c, $c + \tau^N + \theta = \overline{w} l + (\rho - n)a$ and $\overline{w} l = -\frac{u_l l}{u_c}$ we can write:

$$-\frac{u_{cc}}{u_c}(\rho - n)a - \frac{u_{cc}}{u_c}\overline{w} l - \frac{u_{lc} l}{u_c} > 0 \Rightarrow -\frac{u_{cc}}{u_c}\left[(\rho - n)a - \frac{u_l l}{u_c} + \frac{u_{lc} l}{u_{cc}}\right] > 0 \qquad \text{(A.26)}$$

Rewriting eq. (A.25) as:

$$\Delta_l - \Delta_c = -\left(\frac{u_{cc}}{u_c} - \frac{u_{cl}}{u_l}\right)\left[(\rho - n)a - \frac{u_l l}{u_c} + \frac{u_{lc} l}{u_{cc}}\right] + \frac{l}{u_l u_{cc}}\left[u_{ll} u_{cc} - u_{lc}{}^2\right]$$

since by concavity of the utility function $\left(u_{ll} u_{cc} - u_{lc}{}^2\right)$>0, and, exploiting (A.26), we can conclude that, if leisure is non-inferior, $\Delta_l - \Delta_c$ >0 and thus the labour income tax is strictly positive. □