Bak–Sneppen type models and rank-driven processes

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The Bak–Sneppen model is a well-known stochastic model of evolution that exhibits self-organized criticality; only a few analytical results have been established for it so far. We report a surprising connection between Bak–Sneppen type models and more tractable Markov processes that evolve without any reference to an underlying topology. Specifically, we show that in the case of a large number of species, the long time behaviour of the fitness profile in the Bak–Sneppen model can be replicated by a model with a purely rank-based update rule whose asymptotics can be studied rigorously.

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I. INTRODUCTION

In [1], Bak and Sneppen introduced a very fruitful and simple model of evolution that exhibits interesting dynamics but has proved surprisingly hard to analyse. The classical Bak–Sneppen (BS) model is a stochastic coarse-grained model of evolution of an ecosystem consisting of a fixed number N of evolutionary niches organised in a ring. Each niche is occupied by a species with a particular *fitness* value in $[0, 1]$. Direct inter-species interactions (predation, competition, etc.) occur only between species in neighbouring niches. The dynamics of the system is driven by the removal (extinction) of the least fit species in the entire system, whose niche is taken over by a new species; the extinction of the least fit species induces changes in the fitnesses of the species in the two neighbouring niches. In this contribution, we show we can algorithmically associate with the BS model a stochastic process whose update rule is defined solely in terms of the *ranks* of the fitness values, without any reference to topology of interactions, which exhibits asymptotic behaviour and self-organized criticality statistics similar to those of the BS model.

We call processes of the type we associate to the BS model *rank-driven processes* (RDPs) and analyse them in detail in [2]. RDPs are of independent mathematical interest and can be used to define new evolution models.

In more detail, the BS model [1] is a discrete-time process which advances every time there is a species extinction event. Each species occupying the N niches is initially assigned a fitness $x_k \in [0, 1]$, $k \in \{1, \ldots, N\}$, chosen independently from the uniform distribution on the unit interval, $U[0, 1]$. At each step of the algorithm, we choose the smallest of all the x_k, x_{kmin} say, and replace x_{kmin} and its two nearest neighbours $x_{kmin \pm 1}$ (indices calculated modulo N) by new independent $U[0, 1]$ random numbers. In simulations with large N, the marginal distribution of the fitness at any particular niche is seen to evolve to a $U[s^*, 1]$ distribution, with $s^* \approx 0.667$.

The Bak–Sneppen model has had a considerable impact on the physics community and beyond, as witnessed by more than a thousand citations to it to date. While it is impossible to encompass the whole range of work inspired by this simple model, we will only indicate some directions: applications to evolution modelling [3], economics [4–6], and numerical analysis of aspects of BS dynamics such as avalanche statistics and damage spreading [7], computational complexity of the

model [8] and so on.

A number of variants of Bak and Sneppen's original model have been introduced which evolve according to different criteria. One simple variant is the discrete Bak-Sneppen model, in which fitnesses are only allowed to take the values 0 and 1 [9]; another is the *anisotropic* Bak–Sneppen (aBS) model [10–12], in which, in addition to the least fit species, only its *right-hand* nearest neighbour is replaced. The aBS model also gives rise (according to large-N simulations) to a threshold value $s^* \approx 0.724$ [12]. Another variant on the BS model which eliminates topology is the mean-field version analysed in [13–15], in which one replaces the smallest fitness and $K - 1$ randomly chosen other ones; below we show that such models fall within the RDP framework.

Rigorous results on the BS model include proofs the conjugacy of the discrete BS model to a contact process [16], of non-triviality of the steady-state distribution [17], and a description of duration of avalanches [18]. See also the thesis [19]. In this contribution, by exploiting the tools for analysis of RDP models developed in [2], we provide an approach for establishing new results in this active area.

II.   THE CONSTRUCTION

Consider a process in which at each update the species with the smallest fitness and the R_1-th and R_2-th ranked fitness are replaced, where $R = (R_1, R_2)$ is a random variable on $\{2, 3, \ldots, N\}^2$ sampled independently at each step from a distribution $P[R_1 = k, R_2 = l] = f_N(k, l)$ where $f_N(k, l) \geq 0$, $f_N(k, k) = 0$, $f_N(k, l) = f_N(l, k)$ and $\sum_{k=2}^{N} \sum_{l=2}^{N} f_N(k, l) = 1$. This is an example of a rank-driven process to be defined in the next section: it is a Markov process on the N-simplex

$$\Delta_N = \{(x_{(1)}, \ldots, x_{(N)}) \colon 0 \leq x_{(1)} \leq \cdots \leq x_{(N)} \leq 1\};$$

$x_{(1)}, \ldots, x_{(N)}$ are the (increasing) *order statistics* of $x_1, \ldots, x_N$. The complexity of this RDP is intermediate between that of the BS model and the mean-field model of [13–15] with $K = 2$; the latter is the special case of a RDP with $f_N(k, l) = \frac{1}{(N-1)(N-2)}$ for all distinct $k, l \in \{2, \ldots, N\}$. The RDP has the advantage over BS that it can be analysed rigorously. We have strong numerical

evidence that for a judicious choice of $f_N(k,l)$ this simpler model can replicate the asymptotic behaviour of BS. Similar constructions can be made for other variations of BS, such as aBS, in which case we compare the behaviour of the aBS model with an RDP that replaces the smallest fitness and the R-th ranked fitness chosen from an appropriate distribution $P[R = k] = f_N(k)$.

Specifically, in the BS case, one can choose $f_N(k,l)$ to be $f_N^{\mathrm{BS}}(k,l)$, the *empirical* distribution of the ranks of the pairs of sites chosen in BS: if we let $P^{\mathrm{BS}}(k,l,M)$ be the number of times the pair of k-th and l-th ranked elements, $k, l \geq 2$, is the nearest neighbour pair of the smallest element in M iterations of the BS algorithm,

$$f_N^{\mathrm{BS}}(k,l) = \lim_{M \to \infty} \frac{1}{M} P^{\mathrm{BS}}(k,l,M). \tag{1}$$

Similarly, for aBS we put

$$f_N^{\mathrm{aBS}}(k) = \lim_{M \to \infty} \frac{1}{M} P^{\mathrm{aBS}}(k,M), \tag{2}$$

where $P^{\mathrm{aBS}}(k,M)$ is the number of times in M iterations of the aBS algorithm that the k-th ranked element is the right hand-side neighbour of the smallest element.

Gillett (see Theorem 2.20 of [19]) showed that the BS model, viewed as a Markov process on $[0,1]^N$, has a unique stationary distribution, and that starting from N indpendent $U[0,1]$ variables, the N-dimensional distribution converges to that stationary distribution. Gillett's result does not apply directly to aBS, but similar arguments should be valid. An appropriate ergodic theorem should then imply the existence, with probability one, of the limits (1) and (2); each of these limits will be the appropriate projection of the corresponding stationary distribution on $[0,1]^N$. To put it another way, by transitivity of the underlying graph we may view BS or aBS as Markov processes on $\Delta_N \times S_N$, where Δ_N records the order statistics and S_N, the symmetric group of degree N, records the permutation that maps ranks to sites. Then a version of Gillett's arguments should imply the existence of a stationary distribution $\psi_N \times \theta_N$ on $\Delta_N \times S_N$ for BS and aBS; the limit in (1) or (2) will have component(s) $\theta_N^{-1}(\theta_N(1) \pm 1)$ as appropriate. For our purposes, the $N \to \infty$ behaviour of this distribution is important.

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III. RANK-DRIVEN PROCESSES (RDPS)

Following [2], we define an RDP to be a discrete-time Markov process on the N-simplex Δ_N. The RDP evolves according to the following Markovian rule. At each step, K of the x_k-values are selected, according to rank, by sampling according to some specified probability distribution $\kappa_N(i_1, \ldots, i_K)$ on $\{1, 2, \ldots, N\}^K$ which is invariant under permutations of its arguments and such that $\kappa_N(i_1, \ldots, i_K) = 0$ necessarily if $i_m = i_l$ for some $1 \leq m, l \leq K$, $m \neq l$. The sample taken from $\{1, 2, \ldots, N\}^K$ according to κ_N specifies the ranks of the elements that are chosen. The chosen K elements are replaced by new independent $U[0, 1]$ values. Let

$$g_N(i) = \sum_{i_2=1}^{N} \cdots \sum_{i_K=1}^{N} \kappa_N(i, i_2, \ldots, i_K) \text{ and } G_N(n) = \sum_{i=1}^{n} g_N(i).$$

Let us consider a subclass of RDPs relevant to BS-type models, in which at each step we choose the smallest and $(K - 1)$ other elements as described above. Set $\kappa_N(1, i_2, \ldots, i_K) = K^{-1} f_N(i_2, \ldots, i_K)$, where f_N is a symmetric probability distribution on $\{2, \ldots, N\}^{K-1}$, and $f_N(i_2, \ldots, i_K) = 0$ if $i_m = i_l$ for some $2 \leq m, l \leq K$, $m \neq l$. Note that $g_N(1) = 1/K$, and define for all $i \in \{2, \ldots, N\}$

$$\phi_N(i) = \sum_{i_3=2}^{N} \cdots \sum_{i_K=2}^{N} f_N(i, i_3, \ldots, i_K).$$

Assume that $F(n) = \lim_{N \to \infty} \sum_{i=2}^{N} \phi_N(n)$ exists for all n and finally set

$$\alpha = \lim_{n \to \infty} F(n) \in [0, 1]. \tag{3}$$

The quantity α measures the "atomicity" of f_N as $N \to \infty$. It is not hard to see that if at each step we choose the smallest and the second-smallest of all elements, $\alpha = 1$ and if we choose at each step the smallest element and another uniformly random one, $\alpha = 0$.

For RDPs of such form, the following two results proved in [2] are crucial for our purposes:

(a) In the limit $N \to \infty$ the limiting marginal probability distribution function of any arbitrary x_k

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converges to $\pi(x)$, $\pi(x) > 0$ if $x > s^*$ and $\pi(x) = 0$ for $x \in [0, s^*]$, where the threshold s^* satisfies

$$s^* = \frac{1 + (K-1)\alpha}{K}. \tag{4}$$

(b) If $g_N(n)$ is "eventually uniform", i.e. if for large n,

$$g_N(n) \approx \frac{1 - s^*}{N} \longleftrightarrow f_N(n) \approx \frac{1 - \alpha}{N}, \tag{5}$$

then $\pi(x) = \frac{x - s^*}{1 - s^*}$ for $x \in [s^*, 1]$, i.e., π is the uniform distribution $U[s^*, 1]$.

For example, for the mean-field aBS model, the threshold is at $s^* = 1/2$ and since the eventual uniformity condition holds by definition, the limiting distribution is indeed $U[1/2, 1]$ as indicated by [14].

In the Appendix we give a brief derivation of the origin of the threshold formula (4); see [2] for details.

IV.   COMPARISON OF DYNAMICS

In this section we numerically compute the empirical distributions for RDPs associated with BS and aBS, f_N^{BS} and f_N^{aBS}, respectively, and compare various aspects of the behaviour of RDPs defined by these distributions with the Bak–Sneppen type models that gave rise to them.

A.   Computation of distributions

Both f_N^{aBS} and f_N^{aBS} can be accurately numerically computed. Figure 1 shows simulation estimates of $f_N^{\mathrm{aBS}}(k)$ for small values of k and different values of N.

Figure 2 shows a representative example of $f_N^{\mathrm{BS}}(k)$ with a simulation estimate for $N = 250$.

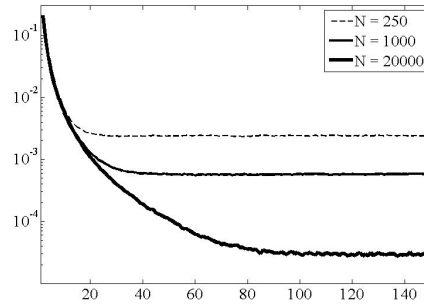From the joint distribution we can compute the (indistinguishable) marginal distribution of ranks

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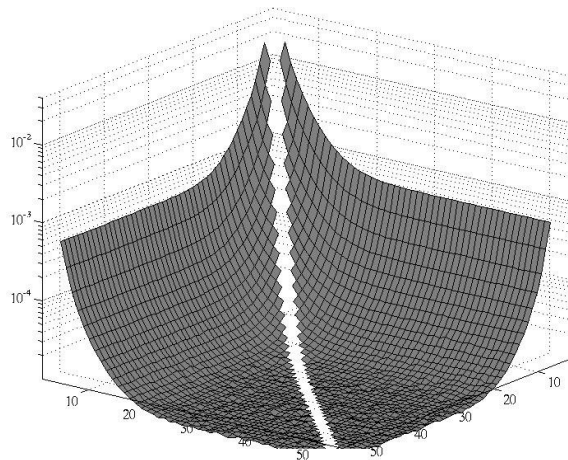FIG. 1. Plot of $f_N^{\mathrm{aBS}}(k)$, $k \in \{2, \ldots, 150\}$ for $N = 250, 1000, 20000$.

FIG. 2. Plot of $f_{250}^{\mathrm{BS}}(k)$, $k \in \{2, \ldots, 50\}$.

of the left and right neighbours, $g_N^{\mathrm{BS}}(k)$. Representative examples are given in Figure 3.

B. Thresholds

From the results of Figures 1–3 we see that for a given N, the empirical distributions decay rapidly for small k before settling down to a uniform value. In fact, it appears that there are constants C_1 and C_2 such that $f_N^{\mathrm{aBS}}(k) \approx C_1/N$ and $g_N^{\mathrm{BS}}(k) \approx C_2/N$ for large enough k. Thus the numerical evidence supports the eventual uniformity condition (5) and we can compute $\alpha = 1 - C_i$. Numerical results give lower bounds of $\alpha \approx 0.496$ for BS and $\alpha \approx 0.445$ for aBS and hence $s_{\mathrm{BS}}^* \approx 0.664$ and $s_{\mathrm{aBS}}^* \approx 0.723$ in close agreement with the simulations of [12].
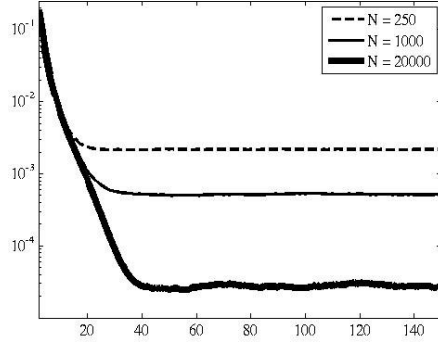
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FIG. 3. Plot of $g_N^{\mathrm{BS}}(k)$, $k \in \{2, \dots, 150\}$ for $N = 250, 1000, 20000$.

C. Avalanches

Following [1] we define the length of an s-avalanche to be t if the number of consecutive steps for which the smallest fitness value stays below s is t. As s approaches s^* we expect $n(\ell)$, the distribution of s-avalanche lengths, to show the power law behaviour characteristic of self-organized criticality. Using $f_N^{\mathrm{aBS}}(k)$ and $f_N^{\mathrm{BS}}(k)$, we can compare the avalanches of RDPs with those of aBS and BS.

Consider the RDP induced by $f_N^{\mathrm{aBS}}(k)$ (a similar approach can be adopted for BS, too). An s-avalanche of length l represents the end point of an excursion during the RDP whose starting point was the last time all states had fitnesses greater than s. Each excursion is a Markov chain whose state space is the number of states with fitness values below s and $n(l)$ is the distribution of arrival times at the absorbing state (zero fitnesses below s). The transition probabilities of the Markov chain can easily be calculated from $f_N^{\mathrm{aBS}}(k)$. Let $p_k = \sum_{r=2}^{k} f_N^{\mathrm{aBS}}(r)$ and $q_k = 1 - p_k$. If π_t^k is the probability that there are k states less than s after t steps of the excursion then for $k > 1$,

$$\pi_{t+1}^k = s^2 q_{k-1} \pi_t^{k-1} + (s^2 p_k + 2s(1-s)q_k)\pi_t^k +$$
$$(2s(1-s)p_{k+1} + (1-s)^2 q_{k+1})\pi_t^{k+1} + (1-s)^2 p_{k+2} \pi_t^{k+2}. \tag{6}$$

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For $k = 1$ we omit the first term on the right-hand side of (6) and

$$\pi_{t+1}^0 = \pi_t^0 + (1-s)^2 \pi_t^1 + (1-s)^2 p_2 \pi_t^2.$$

We compute the distribution $n(l)$ of s-avalanche lengths for various values of s for our RDPs and compare them with empirical results for aBS and BS. Representative distributions are given in Figure 4. There is a small but clear difference in the exponents of the two processes, but the RDP shows the characteristic behaviour expected as s approaches s^*.
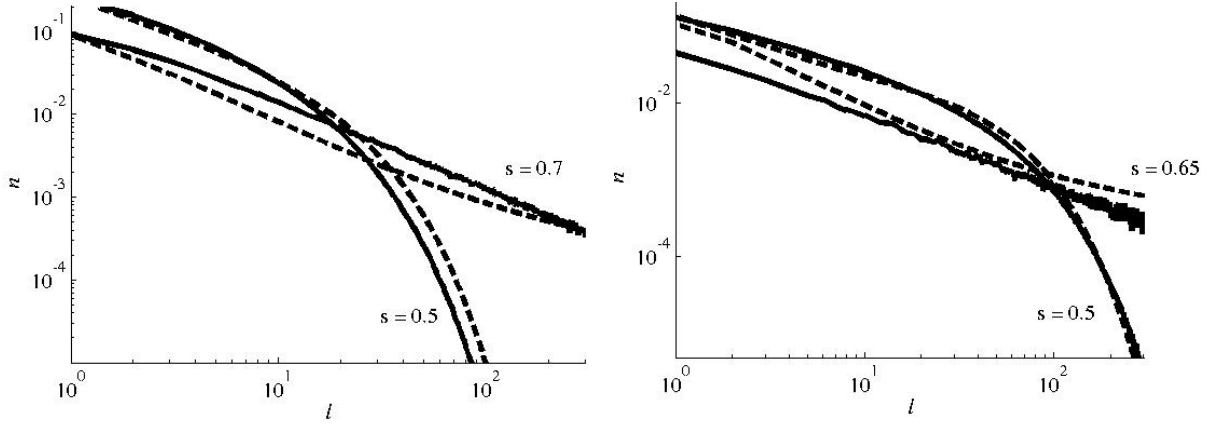

FIG. 4. Size distribution $n(l)$ of s avalanches in RDP (dashed) and aBS (left); and BS (right) for $N = 2000$.

V. REMARKS

The numerical evidence reported above leads to several interesting problems for further investigation, not least the strong suggestion that the RDP with $f_N = f_N^{\text{BS}}$ given by (1) is closely related to BS itself. The exact relationship of the two processes remains to be characterized rigorously. If one wished to define a Markov process on Δ_N whose stationary distribution coincided with the projection onto Δ_N of the stationary distribution of BS, a natural candidate would be a RDP with state-dependent selection distribution: instead of a single $f_N(\cdot)$ one would have a family $f_N(\cdot\,;x)$ of selection distributions conditioned on the state $x \in \Delta_N$. Thus, assuming it exists, one would take $f_N(\cdot\,;x)$ to be $f_N^{\text{BS}}(\cdot\,;x)$, the stationary distribution for BS of the nearest neighbours of the smallest element *conditional* on the projection of the current state onto Δ_N being x. The fact that

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the numerical evidence described above suggests that one can proceed not with a state-dependent RDP based on $f_N^{\mathrm{BS}}(\,\cdot\,;x)$ but with the simpler RDP based on $f_N^{\mathrm{BS}}(\,\cdot\,)$ (which is an average of the $f_N^{\mathrm{BS}}(\,\cdot\,;x)$) seems to point to some important underlying property of BS itself. Two possible explanations are:

(a) $f_N^{\mathrm{BS}}(\,\cdot\,) = f_N^{\mathrm{BS}}(\,\cdot\,;x)$ for all x, i.e., at stationarity there is some independence between the order statistics and the permutation that maps sites to ranks; or

(b) $f_N^{\mathrm{BS}}(\,\cdot\,;x)$ satisfies (uniformly in x) the same asymptotic conditions as $f_N^{\mathrm{BS}}(\,\cdot\,)$ that are central to the limit behaviour, namely (3) and (5).

The stronger fact (a) would suggest that the stationary distribution of the RDP coincides with the projection of the stationary distribution of BS onto Δ_N, so that the two processes share the same detailed equilibrium properties. The weaker fact (b) would suffice to explain why the two processes share the same threshold and characteristic $U[s^*, 1]$ limit distribution. We remark that the distributions $f_N^{\mathrm{BS}}(\,\cdot\,;x)$ seem to be very difficult to evaluate numerically.

In conclusion, we have indicated how the distribution $f_N^{\mathrm{BS}}(k, l)$ and the quantity α of (3) capture the build-up of correlations in Bak–Sneppen type algorithms, the threshold behaviour of which can be analysed exactly by considering the appropriate RDP.

The class of RDPs that we have introduced is of interest in its own right. The remaining analytical challenge is to clarify the relationship between BS and the RDP. This involves at least two main parts: (i) proving the existence of the distributions f_N^{BS} given by (1) and of the limit α defined by (3); and (ii) determining the property of BS that allows us to use $f_N^{\mathrm{BS}}(\,\cdot\,)$ instead of the conditional version $f_N^{\mathrm{BS}}(\,\cdot\,;x)$. In respect to challenge (i) above, it is interesting to note that if an explicit description of f_N^{BS} could be obtained, one might be able to obtain an explicit formula for the threshold s^* via (3) and (4).

Appendix A: Derivation of the threshold formula

Consider the s-counting process $C_t^N(s)$ defined to be the number of x_k-values in the interval $[0, s]$ after t iterations of the RDP defined by f_N. Then $C_t^N(s)$ is a Markov chain on the finite state-space $\{0, 1, \ldots, N\}$. The threshold s^* relates to the limiting ($t \to \infty$ then $N \to \infty$) marginal distribution of an arbitrary x_k. To evaluate s^*, we compute the mean drift of $C_t^N(s)$, $E[C_{t+1}^N(s) - C_t^N(s) \mid C_t^N(s) = n]$, where E is the expectation operator.

It can be shown that

$$E[C_{t+1}^N(s) - C_t^N(s) \mid C_t^N(s) = n] = K(s - G_N(n)).$$

Hence the drift is zero (asymptotically, as $n \to \infty$ and $N \to \infty$) at

$$s = s^* = \lim_{n \to \infty} \lim_{N \to \infty} G_N(n).$$

In terms of the functions $f_N(i)$, $F_N(n)$ and the quantity α (3), we have that

$$g_N(i) = \frac{K-1}{K} f_N(i) \text{ and } G_N(n) = \frac{1 + (K-1)F_N(n)}{K},$$

so that

$$s^* = \lim_{n \to \infty} \lim_{N \to \infty} G_N(n) = \frac{1 + (K-1)\alpha}{K}.$$

The drift being zero indicates the threshold behaviour, because a positive (negative) drift would mean $C_t^N(s)$ increases (decreases). In this argument there are several limits involved (n, N, t all going to ∞) that need to be handled with care. In [2] we exploit techniques from Markov process theory, such as Foster–Lyapunov ideas [20], to do this.

ACKNOWLEDGMENTS

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