PERIODIC POINTS FOR EXPANSIVE ACTIONS
OF $\mathbb{Z}^d$ ON COMPACT ABELIAN GROUPS

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ABSTRACT. In this note we show that the periodic points of an expansive $\mathbb{Z}^d$ action on a compact abelian group are uniformly distributed with respect to Haar measure if the action has completely positive entropy. In the general expansive case, we show that any measure obtained as the distribution of periodic points along some sequence of periods necessarily has maximal entropy but need not be Haar measure.

§1. Introduction

Let X be a compact abelian group with normalised Haar measure μ defined on the σ–algebra of Borel sets $\mathcal{B}$. A $\mathbb{Z}^d$ action on X by automorphisms is a homomorphism $T : \mathbb{Z}^d \to \text{Aut}(X)$, where $\text{Aut}(X)$ is the group of continuous automorphisms of X. Such an action may be defined by specifying d commuting automorphisms of X, $U_1, \ldots, U_d$ say, and then setting $T_{\mathbf{n}} = T_{(n_1,\ldots,n_d)} = U_1^{n_1} \ldots U_d^{n_d}$ for $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$.

A period for the action T is a finite index subgroup $\Lambda \subset \mathbb{Z}^d$, and the set of points of period Λ, denoted $\text{Fix}_\Lambda(T)$, is defined by

$$\text{Fix}_\Lambda(T) = \{x \in X \mid T_{\mathbf{n}}x = x \text{ for any } \mathbf{n} \in \Lambda\}. \tag{1.1}$$

Since Λ is a lattice in $\mathbb{Z}^d$, it has d independent generators $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_d$ so we may write $\Lambda = \boldsymbol{\lambda}_1\mathbb{Z} + \cdots + \boldsymbol{\lambda}_d\mathbb{Z}$. Following [5], §7, we measure the size of the period Λ by setting $\|\Lambda\| = d(\mathbf{0}, \Lambda\backslash\{\mathbf{0}\})$ with d the usual metric on $\mathbb{Z}^d$. This has the following consequence: if $\|\Lambda_n\| \to \infty$, then the canonical fundamental domains for $\mathbb{Z}^d/\Lambda_n$ form a Følner sequence in $\mathbb{Z}^d$.

Let $\Gamma = \widehat{X}$ denote the dual group (group of characters) of X; the action T induces a dual action $\widehat{T}$ of $\mathbb{Z}^d$ on Γ defined by $(\widehat{T}_{\mathbf{n}}\gamma)(x) = \gamma(T_{\mathbf{n}}(x))$ for $\gamma \in \Gamma$, $\mathbf{n} \in \mathbb{Z}^d$ and $x \in X$. Since the action T is by continuous automorphisms, $\text{Fix}_\Lambda(T)$ is a closed subgroup of X for any Λ, and it has dual given by

$$\widehat{\text{Fix}_\Lambda(T)} \cong \Gamma/\big((\widehat{T}_{\boldsymbol{\lambda}_1} - I)\Gamma + \cdots + (\widehat{T}_{\boldsymbol{\lambda}_d} - I)\Gamma\big). \tag{1.2}$$

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For any period Λ, the (normalized) Haar measure on the closed subgroup $\mathrm{Fix}_\Lambda(T)$ is a T–invariant probability on X, denoted μ_Λ. By (1.2), the Fourier transform of μ_Λ is given by

$$\widehat{\mu}_\Lambda(\gamma) = \begin{cases} 1, & \text{if } \gamma \in \left((\widehat{T}_{\boldsymbol{\lambda}_1} - I)\Gamma + \cdots + (\widehat{T}_{\boldsymbol{\lambda}_d} - I)\Gamma\right) \\ 0, & \text{if not.} \end{cases} \tag{1.3}$$

Here I is the identity on Γ. Let $\{\Lambda_n\}_{n\in\mathbb{N}}$ be a sequence of periods with $\|\Lambda_n\| \to \infty$ as $n \to \infty$. The corresponding periodic point measures μ_{Λ_n} converge weakly to ν if and only if $\widehat{\mu}_{\Lambda_n}$ converges pointwise on Γ to $\widehat{\nu}$.

Our purpose here is to show that if T is expansive and has completely positive entropy, then for any such sequence of periods $\{\Lambda_n\}_{n\in\mathbb{N}}$, the periodic point measures μ_{Λ_n} converge weakly to Haar measure, in which case we will say that the periodic points are uniformly distributed with respect to Haar measure, or that Haar measure describes the distribution of the periodic points of T. In the absence of completely positive entropy, it may no longer be the case that the periodic point measures converge, but if they do along some sequence then the limiting measure must be one of maximal entropy.

The corresponding considerations for $\mathbb{Z}$ actions come in two flavours: hyperbolic and number–theoretic. In the hyperbolic direction, a result of Bowen, [1], shows that the periodic point measures converge to the maximal measure for Axiom A diffeomorphisms. The algebraic result, due to Lind [4], states that the periodic point measures for ergodic toral automorphisms converge to Haar measure. This recovers a special case Bowen's result when the automorphism is hyperbolic; if the toral automorphism is ergodic but not hyperbolic the argument in [4] is of a quite different kind, and uses some deep Diophantine estimates.

For the case of a $\mathbb{Z}^d$ action, the space is no longer a manifold in the usual sense: the examples we will be interested in will typically have either zero or infinite topological dimension.

We recall some of the general theory of $\mathbb{Z}^d$ actions on compact abelian groups from [2], [5] and [7].

An action $T : \mathbb{Z}^d \to \mathrm{Aut}(X)$ determines (and is determined by) a module $M = M_{(X,T)}$ over the ring $R_d = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ by the correspondence of [2], §11: first define the additive group M to be the dual group of X, then define automorphisms $\times x_1, \ldots, \times x_d$ to be the dual automorphisms of $T_{(1,0,\ldots,0)}, \ldots, T_{(0,\ldots,0,1)}$ respectively. Extending "linearly" (by the structure of M as an additive group or $\mathbb{Z}$–module) makes M into an R_d–module. If the action T is expansive, then (X, T) satisfies the Descending Chain Condition on closed invariant subgroups (Definition 3.1 and Theorem 5.2 of [2]). It follows that the corresponding module $M_{(X,T)}$ is Noetherian (Theorem 11.4 of [2]), and that the set of periodic points (points whose orbit under the action T is finite) is dense in X (Theorem 7.2, [2]).

A prime ideal $\mathfrak{p} \subset R_d$ is associated to the module M if $\mathfrak{p}$ is the annihilator of some element of M. By [7], Theorem 3.9, the action of T on X is expansive if and only if for every prime $\mathfrak{p}$ associated to $M_{(X,T)}$, the set of zeros of $\mathfrak{p}$, $V_{\mathbb{C}}(\mathfrak{p})$, contains no point $(z_1, \ldots, z_d)$ with $|z_1| = \cdots = |z_d| = 1$. By [5], Theorem 6.5, the action of T on X has completely positive entropy if and only if each prime $\mathfrak{p}$ associated to $M_{(X,T)}$ is principal, and not generated by

2

a polynomial of the form $c(x_1^{a_1} \ldots x_d^{a_d})$, where c is some cyclotomic polynomial. Following [5], we call such ideals *positive* and call all other ideal *null*.

We restrict attention to the case of expansive actions for two reasons. Firstly, an expansive action has finitely many points of each period, whereas an ergodic action may have infinitely many periodic points for some periods. This cannot arise in $\mathbb{Z}$ actions, but when $d > 1$ the variety associated to the system may have unit roots without violating ergodicity (see [7], §3 for an explanation of this and for examples). Secondly, the distribution of periodic points for a $\mathbb{Z}^d$ action turns out to follow from the existence of a limiting growth rate for the number of periodic points, and this is only known in the expansive case (see §7 of [5]).

This paper owes a great deal to Prof. Klaus Schmidt, whose advice greatly simplified the proofs in §2. In particular, the inference of the distribution statements presented here from the growth rate result in §7 of [5] is due to him, and I am grateful for his permission to use it here. I am also grateful to the referee for pointing out an error in the original version of the proof of Theorem 2.2 relating to the algebraic structure of a K action of $\mathbb{Z}^d$, to Prof. David Jabon for a discussion related to Example 3.3, and to Prof. Peter Walters for helpful comments on an earlier version.

§2. Distribution of Periodic Points

We begin by stating two theorems from [5] that determine the entropy and growth rate of periodic points for a $\mathbb{Z}^d$ action. Recall that the expansive $\mathbb{Z}^d$ action T on the compact abelian group X corresponds to a Noetherian R_d–module M, and for any R_d–module M there is a $\mathbb{Z}^d$ action T_M on the compact abelian group $X_M = \widehat{M}$. Choose a prime filtration $0 = M_0 \subset M_1 \subset \ldots \subset M_{r-1} \subset M_r = M$ with $\frac{M_k}{M_{k-1}} \cong \frac{R_d}{\mathfrak{q}_k}$ where each $\mathfrak{q}_k \subset R_d$ is a prime ideal containing an associated prime of M.

Theorem 2.1. ([5], §§1, 3, 4) *The entropy of the $\mathbb{Z}^d$ action T_M is given by*

(1) $h(T_M) = \displaystyle\sum_{k=1}^{r} h(T_{R_d/\mathfrak{q}_k})$, *and*

(2) $h(T_{R_d/\mathfrak{q}})$ *is 0 if $\mathfrak{q}$ is non–principal, infinite if $\mathfrak{q} = \{0\}$, and is equal to*

$$\log M(f) = \int_0^1 \cdots \int_0^1 \log |f(e^{2\pi i s_1}, \ldots, e^{2\pi i s_d})| ds_1 \ldots ds_d \ \text{if } \mathfrak{q} = \langle f \rangle, f \neq 0.$$

Theorem 2.2. ([5], §7) *The growth rate of the number of periodic points of the expansive $\mathbb{Z}^d$ action T_M exists and is given by*

$$\lim_{\|\Lambda\| \to \infty} \frac{1}{|\mathbb{Z}^d/\Lambda|} \log |\mathrm{Fix}_\Lambda(T_M)| = h(T_M).$$

3

Corollary 2.3. *Let $\mathfrak{p} \subset R_d$ be a prime ideal with the property that the $\mathbb{Z}^d$ action corresponding to the module $R_d/\mathfrak{p}$ is expansive. Then for any $h \in R_d \backslash \mathfrak{p}$,*

$$\lim_{\|\Lambda\| \to \infty} \frac{1}{|\mathbb{Z}^d/\Lambda|} \log |\mathrm{Fix}_\Lambda(T_{R_d/(\mathfrak{p}+\langle h \rangle)})| = 0.$$

Proof. Since $h \in R_d \backslash \mathfrak{p}$, any prime ideal associated to the module $R_d/(\mathfrak{p} + \langle h \rangle)$ must be non–principal (since expansiveness implies finite entropy, $\mathfrak{p}$ is not trivial). The corollary follows from Theorems 2.1 and 2.2. □

The distribution theorem follows from Corollary 2.3.

Theorem 2.4. *If (X, T) is an expansive action of $\mathbb{Z}^d$ on a compact abelian group with completely positive entropy, then the periodic points of T are uniformly distributed with respect to Haar measure on X: the measures μ_{Λ_n} converge weakly to Haar measure for any sequence of periods $\{\Lambda_n\}_{n \in \mathbb{N}}$ with $\|\Lambda_n\| \to \infty$ as $n \to \infty$.*

Proof. Let M be the Noetherian R_d–module corresponding to (X, T), and let the associated primes of M be $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$.

In order to prove the theorem, it is sufficient by §1 to show that for any non–trivial character on X, $m \in M \backslash \{0\}$, there is a $K(m) < \infty$ for which $\|\Lambda\| \geq K(m)$ implies that

$$m \notin (\widehat{T}_{\boldsymbol{\lambda}_1} - I)M + \cdots + (\widehat{T}_{\boldsymbol{\lambda}_d} - I)M \tag{2.1}$$

where $\Lambda = \boldsymbol{\lambda}_1 \mathbb{Z} + \cdots + \boldsymbol{\lambda}_d \mathbb{Z}$.

We will demonstrate this by a contradiction argument. Let $m \in M \backslash \{0\}$ have

$$m \in (\widehat{T}_{\boldsymbol{\lambda}_1^{(n)}} - I)M + \cdots + (\widehat{T}_{\boldsymbol{\lambda}_d^{(n)}} - I)M \tag{2.2}$$

for a sequence of periods $\{\Lambda_n\}_{n \in \mathbb{N}}$, where

$$\Lambda_n = \boldsymbol{\lambda}_1^{(n)} \mathbb{Z} + \cdots + \boldsymbol{\lambda}_d^{(n)} \mathbb{Z} \tag{2.3}$$

and $\|\Lambda_n\| \to \infty$.

From (2.2) we will make two algebraic reductions (effectively replacing the module M firstly with a module whose set of associated primes is a singleton, and then passing from that module to a cyclic module) which will show that (2.2) contradicts Corollary 2.3 above. Let

$$J(\Lambda) = (\mathbf{x}^{\boldsymbol{\lambda}_1} - 1)R_d + \cdots + (\mathbf{x}^{\boldsymbol{\lambda}_d} - 1)R_d$$

where $\mathbf{x}^{\mathbf{n}} = x_1^{n_1} \cdot \ldots \cdot x_d^{n_d}$.

REDUCTION TO $\mathfrak{p}$–PRIMARY CASE. Choose a reduced primary decomposition $\{M_1, \ldots, M_k\}$ of zero in M (see [7], §2). The map

$$\phi : M \longrightarrow \frac{M}{M_1} \oplus \cdots \oplus \frac{M}{M_k} = N; \quad \phi(a) = (a + M_1, \ldots, a + M_k)$$

4

is an injective homomorphism of R_d–modules, and each $\frac{M}{M_j}$ has $\{\mathfrak{p}_j\}$ as its set of associated primes. Then (2.2) implies that

$$\phi(m) \in (\widehat{T}_{\boldsymbol{\lambda}_1^{(n)}} - I)N + \cdots + (\widehat{T}_{\boldsymbol{\lambda}_d^{(n)}} - I)N = J(\Lambda_n)N \tag{2.4}$$

for every n. Thus, in (2.2) we may assume that the module M is $\mathfrak{p}_j$–primary for some j by passing to a subsequence.

REDUCTION FROM $\mathfrak{p}$–PRIMARY TO CYCLIC CASE. Choose and fix a j, with the property that

$$a = (\phi(m))_j \in J(\Lambda_n)\frac{M}{M_j} \tag{2.5}$$

$(a \neq 0)$ for a sequence of periods $\{\Lambda_n\}_{n\in\mathbb{N}}$ with $\|\Lambda_n\| \to \infty$ as $n \to \infty$ (this sequence of periods is a subsequence of the original one in (2.2)). Let $L = \frac{M}{M_j}$ and $\mathfrak{p} = \mathfrak{p}_j$.

Choose an adapted prime filtration of L,

$$L = L_s \supset \cdots \supset L_0 = \{0\} \tag{2.6}$$

with succesive quotients $\frac{L_i}{L_{i-1}} \cong \frac{R_d}{\mathfrak{q}_i}$ where $\mathfrak{q}_i = \mathfrak{p}$ for $i = 1, \ldots, t$ and $\mathfrak{q}_i \supsetneq \mathfrak{p}$ for $i = t+1, \ldots, s$. Choose polynomials $g_i \in \mathfrak{q}_i \backslash p$ for $i = t+1, \ldots, s$ and let $g = g_s \cdot \ldots \cdot g_{t+1}$. Then $g \cdot a \in L_t \backslash \{0\}$.

Multiplying (2.5) by g shows that

$$g \cdot a \in J(\Lambda_n)L_t \text{ for every } n \geq 1. \tag{2.7}$$

Choose i so that $g \cdot a \in L_i \backslash L_{i-1}$. By Lemma 7.6 of [5], $g \cdot a \in L_i \cap J(\Lambda_n) \cdot L_i = J(\Lambda_n) \cdot L_i$. Let $b \in L_i \backslash L_{i-1}$ be the element that realises the isomorphism $\frac{R_d}{\mathfrak{p}} \cong \frac{L_i}{L_{i-1}}$ via the map $f \mapsto f \cdot b + L_{i-1}$. Choose $h \in R_d$ to have $h \cdot b \in g \cdot a + L_{i-1}$; by our choice of i, h is non–zero in $\frac{R_d}{\mathfrak{p}}$ so $h \in R_d \backslash \mathfrak{p}$.

Now

$$\lim_{n\to\infty} \frac{1}{|\mathbb{Z}^d/\Lambda_n|} \log |\mathrm{Fix}_{\Lambda_n}(T_{R_d/(\mathfrak{p}+\langle h \rangle)})| = \lim_{n\to\infty} \frac{1}{|\mathbb{Z}^d/\Lambda_n|} \log |R_d/(\mathfrak{p} + \langle h \rangle + J(\Lambda_n))|$$

$$= \lim_{n\to\infty} \frac{1}{|\mathbb{Z}^d/\Lambda_n|} \log |L_i/(L_{i-1} + \langle g \cdot a \rangle + J(\Lambda_n)L_i)|$$

$$= \lim_{n\to\infty} \frac{1}{|\mathbb{Z}^d/\Lambda_n|} \log |L_i/(L_{i-1} + J(\Lambda_n)L_i)|$$

$$= \lim_{n\to\infty} \frac{1}{|\mathbb{Z}^d/\Lambda_n|} \log |\mathrm{Fix}_{\Lambda_n}(T_{L_i/L_{i-1}})|$$

$$= \lim_{n\to\infty} \frac{1}{|\mathbb{Z}^d/\Lambda_n|} \log |\mathrm{Fix}_{\Lambda_n}(T_{R_d/\mathfrak{p}})|$$

$$= h(T_{R_d/\mathfrak{p}}) > 0,$$

5

which contradicts Corollary 2.3 since $h \in R_d \backslash \mathfrak{p}$.

The above equalities are seen as follows (in order): (1) follows from the isomorphism between $\operatorname{Fix}_{\Lambda_n}(T_{R_d/(\mathfrak{p}+\langle h \rangle)})$ and the dual group of $\frac{R_d}{\mathfrak{p}+\langle h \rangle+J(\Lambda_n)}$, (2) follows from the isomorphism $\frac{R_d}{\mathfrak{p}} \cong \frac{L_i}{L_{i-1}}$, (3) holds because $g \cdot a \in J(\Lambda_n)L_i$ by assumption, (4) follows from the isomorphism between $\operatorname{Fix}_{\Lambda_n}(T_{L_i/L_{i-1}})$ and the dual group of $\frac{L_i}{(L_{i-1}+J(\Lambda_n)L_i)}$, (5) is the same isomorphism as (2), and finally (6) is an application of Theorem 2.2. $\square$

We now turn to the case of an expansive action that is not assumed to be K. Here the case $d > 1$ differs from the case $d = 1$: if we assume all the actions are ergodic, then for $d = 1$ the K property is automatic by [6], whereas for $d > 1$ we have ergodic actions that are not K (Ledrappier's example for instance: see §3 or [3]).

Theorem 2.5. *If (X, T) is an expansive action of $\mathbb{Z}^d$ on a compact abelian group, then any measure arising as the weak limit of a sequence of periodic point measures $\{\mu_{\Lambda_n}\}_{n\in\mathbb{N}}$ as $\|\Lambda_n\| \to \infty$ has maximal entropy.*

Proof. As before, let M be the corresponding Noetherian module. By [5], Theorem 6.5 there is a unique maximal submodule $N \subset M$ with the property that all the associated prime ideals of N are null, while all the associated prime ideals of M/N are positive. Thus $T_{M/N}$ has completely positive entropy and $h(T_M) = h(T_{M/N})$. If $\mathfrak{q}$ is a prime ideal associated with M/N then it must contain a prime associated with M; it follows that $T_{R_d/\mathfrak{q}}$ is expansive so $T_{M/N}$ is expansive too (see §1 or [7]).

Let ν be the weak limit of the sequence of periodic point measures μ_{Λ_n} as $\|\Lambda_n\| \to \infty$. Since each of the functions $\widehat{\mu}_{\Lambda_n}$ is 0–1 valued, so too is $\widehat{\nu}$. It follows that ν is an idempotent measure on X with $\widehat{\nu}(0) = 1$ so ν is Haar measure on some closed subgroup $Y \subset X$. Since each of the measures μ_{Λ_n} is T–invariant, the subgroup Y on which ν is supported is also T–invariant, so $\{m \in M \mid \widehat{\nu}(m) = 1\} = Y^{\perp}$ is a submodule of M.

If $\widehat{\nu}(m) = 1$ then (passing to a subsequence if need be) $\widehat{\mu}_{\Lambda_n}(m) = 1$ for all $n \geq 1$, so $m \in J(\Lambda_n) \cdot M$ for all $n \geq 1$. Since M/N corresponds to an expansive action with completely positive entropy, Theorem 2.4 shows that m must be the trivial character in M/N, so $m \in N$. Hence $Y^{\perp} \subset N$ and so $Y \supset N^{\perp}$. The addition formula for $\mathbb{Z}^d$ actions (see Appendix B of [5]), together with the maximality of Haar measure (see §6 of [5]) then shows that $h(T_M) \geq h_\nu(T_M) = h(T|_Y) = h(T|_{N^{\perp}}) + h(T \text{ induced on } Y/N^{\perp}) \geq h(T|_{N^{\perp}}) = h(T_{M/N}) = h(T_M)$. $\square$

§3. Examples

We describe examples to show that in the absence of completely positive entropy the periodic point measures need not converge, though we are not at this stage able to settle the question of whether Theorem 2.4 has a converse. Two examples are given as $\mathbb{N}^2$ actions; the natural extensions to $\mathbb{Z}^2$ actions inherit expansiveness, and if the $\mathbb{N}^2$ action has zero entropy then the corresponding $\mathbb{Z}^2$ action also has zero entropy and is therefore not K. The algebraic nature of these examples makes the extension very transparent, as described below. The

6

first and third examples are very well–known. Example 3.1 is the $\times 2$, $\times 3$ example due to Furstenberg who raised the (still open) question of finding all non–atomic invariant measures for the action. Example 3.3 is due to Ledrappier, who gave it as an example of a zero entropy $\mathbb{Z}^2$ Markov shift which is mixing but not 2–mixing.

We will consider rectangular periods; the period $(n, 0)\mathbb{Z} + (0, m)\mathbb{Z} \subset \mathbb{Z}^2$ will be denoted (n, m).

3.1. Generate an action of $\mathbb{N}^2$ on $\mathbb{T}$ by defining the generators to be $T_{(1,0)}x = 2x \pmod 1$ and $T_{(0,1)}x = 3x \pmod 1$. The periodic points are dense because $(\mathbb{T}, T)$ has the Descending Chain Condition on closed invariant subgroups for instance. Equivalently, note that $\bigcap_{(j,k)} (2^j - 1)\mathbb{Z} + (3^k - 1)\mathbb{Z} = \{0\}$. First notice that there are sequences of periods along which the periodic point measures converge to Lebesgue measure on $\mathbb{T}$. Construct, by the Chinese Remainder Theorem, sequences j_n and k_n going to infinity with the property that 2^{j_n} and 3^{k_n} are congruent to 1 modulo $p_1, p_2, \ldots, p_n$ where p_i is the ith prime exceeding 3: $p_1 = 5$, $p_2 = 7$ and so on. Then the Fourier transform of $\mu_{(j_n, k_n)}$ converges pointwise to the characteristic function of $\{0\} \subset \mathbb{Z}$ so $\mu_{(j_n, k_n)}$ converges weakly to Haar measure.

To see that other limits are possible, choose sequences p_n and q_n going to ∞ with the property that the highest common factor of $(2^{p_n} - 1)$ and $(3^{q_n} - 1)$ is 5. This is possible: first choose a sequence $q_n \to \infty$ with $3^{q_n} \equiv 1 \bmod 5$ but with $3^{q_n} - 1$ not divisible by 25. Then choose, by the Chinese Remainder Theorem, a sequence $p_n \to \infty$ with $2^{p_n} \equiv 1 \bmod 5$, and $2^{p_n} \equiv 2 \bmod p$ if p is any other prime factor of $3^{q_n} - 1$.

The periodic point measure $\mu_{(p_n, q_n)}$ has Fourier transform supported on the subgroup $(2^{p_n} - 1)\mathbb{Z} + (3^{q_n} - 1)\mathbb{Z} = 5\mathbb{Z}$ and so converges to (and is) the atomic measure

$$\frac{1}{5}(\delta_0 + \delta_{\frac{1}{5}} + \delta_{\frac{2}{5}} + \delta_{\frac{3}{5}} + \delta_{\frac{4}{5}})$$

where δ_x is the point mass at x.

The extension to a $\mathbb{Z}^2$ action may be constructed as follows. The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[\frac{1}{6}]$ has as dual the surjective homomorphism $Y = \widehat{\mathbb{Z}[\frac{1}{6}]} \to \mathbb{T}$. The compact group Y has commuting automorphisms $\widehat{\times 2}$ and $\widehat{\times 3}$ that generate a $\mathbb{Z}^2$ action extending the given $\mathbb{N}^2$ action. The sequences of periods considered above give identical behaviour in the extension, with $\mu_{(j_n, k_n)}$ converging weakly to Haar measure on the 6–adic solenoid, and $\mu_{(p_n, q_n)}$ constantly atomic.

3.2. We have shown above that expansive & K implies that the periodic points are uniformly distributed with respect to Haar measure. This example shows that expansive & uniform distribution does not imply the K property. Generate an action T of $\mathbb{Z}^2$ on $\widehat{\mathbb{Z}[\frac{1}{2}]} \times \widehat{\mathbb{Z}[\frac{1}{3}]}$ by setting $\widehat{T_{(1,0)}}(x, y) = (2x, y)$ and $\widehat{T_{(0,1)}}(x, y) = (x, 3y)$. This action has zero entropy, is expansive and also has uniformly distributed periodic points. To see this, notice that for instance $\widehat{\times 2}$ on $\widehat{\mathbb{Z}[\frac{1}{2}]}$ is expansive and K as a $\mathbb{Z}$ action. Let p_n and q_n be any sequences going to ∞. Then if (x, y) is in $\mathbb{Z}[\frac{1}{2}] \times \mathbb{Z}[\frac{1}{3}]$ and $(x, y) \neq (0, 0)$, the relation $(x, y) \in (2^{p_n} - 1)\mathbb{Z}[\frac{1}{2}] \times (3^{q_n} - 1)\mathbb{Z}[\frac{1}{3}]$ can only occur for finitely many values of n. Thus the periodic points are uniformly distributed with respect to Haar measure.

Notice that the automorphism $T_{(n,m)}$ is not ergodic if either one of n and m is zero. We conjecture that this is the only way in which an expansive action of $\mathbb{Z}^d$ on a compact abelian group whose periodic points are uniformly distributed can fail to have completely positive entropy.

Conjecture. *If T is an expansive action of $\mathbb{Z}^d$ on a compact abelian group, with the property that the periodic points are uniformly distributed with respect to Haar measure, and each automorphism $T_{(n,m)}$ is ergodic for $(n,m) \neq (0,0)$, then T has completely positive entropy.*

3.3. [$\mathbb{N}^2$ version of Ledrappier's example] Define an action of $\mathbb{N}^2$ on a compact abelian group by duality as follows. Let $\mathbb{F}_2$ denote the field with two elements, and $\overline{\mathbb{F}}_2$ its algebraic closure. Start with the $\mathbb{F}_2[x,y]$–module $M = \mathbb{F}_2[x,y]/\langle 1 + x + y \rangle$, with commuting endomorphisms $\times x$ and $\times y$. The dual of M is a compact abelian group X carrying an action of $\mathbb{N}^2$ generated by $\widehat{\times x}$ and $\widehat{\times y}$. This action has zero entropy ([3], also see Example 5.2 of [5] and Examples 12.2 (8) of [2]). The number of points of period (n,m) may be shown to be $2^{\#V(n,m)}$ where

$$V(n,m) = V_{\overline{\mathbb{F}}_2}(\langle 1 + x + y \rangle) \cap V_{\overline{\mathbb{F}}_2}(\langle 1 + x^n \rangle) \cap V_{\overline{\mathbb{F}}_2}(\langle 1 + y^m \rangle).$$

We construct sequences of periods along which the periodic point measures converge to Haar measure and to the point mass at the identity. In order to do this, notice that $M \cong \mathbb{F}_2[x]$ (as additive groups) by the map θ sending $g(x,y)$ to $f(x) = g(x, 1 + x)$. Recall that the binomial coefficient $\binom{2^k}{n}$ is even for $n = 1, \ldots, 2^k - 1$, so $\binom{2^k - 1}{n}$ is odd for $n = 0, \ldots, 2^k - 1$. The dual of the group of points with period (n,n) is $M/\langle 1 + x^n, 1 + y^n \rangle$, whose image under θ is $\mathbb{F}_2[x]/\langle 1 + x^n, 1 + (1+x)^n \rangle$.

Consider the sequence of periods $(2^k - 1, 2^k - 1)$. Then

$$\langle 1 + x^n, 1 + (1+x)^n \rangle = \langle 1 + x^n, x + x^2 + \cdots + x^n \rangle = \langle 1 + x^n, 1 + x + x^2 + \cdots + x^{n-1} \rangle$$

where $n = 2^k - 1$. Thus a given character $m \in M$ can have $\theta(m) \in \langle 1 + x^n, 1 + (1+x)^n \rangle$ for only finitely many values of k. It follows that the periodic point measure $\mu_{(2^k-1, 2^k-1)}$ converges weakly to Haar measure as $k \to \infty$. This algebraic argument corresponds to the following dynamical picture: the map $\widehat{\theta}$ sends a point $\mathbf{x} \in X \subset \{0,1\}^{\mathbb{Z}^2}$ to the restriction to the horizontal axis. This determines the point completely since elements of X have $x(i,j) + x(i+1,j) + x(i,j+1) = 0 \bmod 2$ for all i,j. The above shows that points of period $(2^k - 1, 2^k - 1)$ correspond under $\widehat{\theta}$ to strings of 0's and 1's subject only to the restriction that they repeat with period n and the sum of any sequence of digits of length n be $0 \bmod 2$. It is clear that as $n \to \infty$, this collection of points is uniformly distributed in $\{0,1\}^{\mathbb{N}}$, from which it follows that the points of period $(2^k - 1, 2^k - 1)$ are uniformly distributed with respect to Haar measure.

We now claim that there can only be one point with period $(2^k, 2^k)$ for any k, so that $\mu_{(2^k, 2^k)}$ is always the point mass at the identity. Using the above description, if $n = 2^k$, then

$$\langle 1 + x^n, 1 + (1+x)^n \rangle = \langle 1 + x^n, 1 + 1 + x^n \rangle = \langle 1 \rangle$$

8

so that there is only one point with period $(2^k, 2^k)$.

In this example the extension to a $\mathbb{Z}^2$ action is made by replacing $\mathbb{F}_2[x, y]/\langle 1 + x + y \rangle$ with $\mathbb{F}_2[x^{\pm 1}, y^{\pm 1}]/\langle 1 + x + y \rangle$. In the framework of [2] and [5], this is the module $R_2/\langle 2, 1 + x + y \rangle$. As in 3.1, the sequences of periods exhibit the same properties in the $\mathbb{Z}^2$ case: the measures $\mu_{(2^k-1, 2^k-1)}$ converge weakly to Haar measure, while the measures $\mu_{(2^k, 2^k)}$ are all the point mass at the identity.

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