Extended Deligne-Lusztig varieties for general and special linear groups

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Abstract

We give a generalisation of Deligne-Lusztig varieties for general and special linear groups over finite quotients of the ring of integers in a non-archimedean local field. Previously, a generalisation was given by Lusztig by attaching certain varieties to unramified maximal tori inside Borel subgroups. In this paper we associate a family of so-called extended Deligne-Lusztig varieties to all tamely ramified maximal tori of the group.

Moreover, we analyse the structure of various generalised Deligne-Lusztig varieties, and show that the "unramified" varieties, including a certain natural generalisation, do not produce all the irreducible representations in general. On the other hand, we prove results which together with some computations of Lusztig show that for $\mathrm{SL}_2(\mathbb{F}_q[[\varpi]]/(\varpi^2))$, with odd q, the extended Deligne-Lusztig varieties do indeed afford all the irreducible representations.

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1 Introduction

Let F be a non-archimedean local field with finite residue field $\mathbb{F}_q$. Let $\mathcal{O}_F$ be the ring of integers in F, and let $\mathfrak{p}$ be its maximal ideal. If $r \geq 1$ is a natural number, we write $\mathcal{O}_{F,r}$ for the finite quotient ring $\mathcal{O}_F/\mathfrak{p}^r$. Let $\mathbf{G}$ be a reductive group scheme over $\mathcal{O}_F$. The representation theory of groups of the form $\mathbf{G}(\mathcal{O}_{F,r})$, in particular for $\mathbf{G} = \mathrm{GL}_n$, has recently attracted attention from several different directions. On the one hand, there are the "algebraic" approaches to the construction of representations. These include the method of Clifford theory and conjugacy orbits, which can deal explicitly with the class of regular representations (cf. [13] and [33]). Another approach, due to Onn [25], is based on a generalisation of parabolic induction for general automorphism groups of finite $\mathcal{O}_F$-modules. This approach and the associated notion of cuspidality for $\mathrm{GL}_n(\mathcal{O}_{F,r})$ are developed in [1]. Moreover, by the work of Henniart [3] and Paskunas [26], it is known that every supercuspidal representation of $\mathrm{GL}_n(F)$ has a unique type on $\mathrm{GL}_n(\mathcal{O}_F)$. Hence the representation theory of the finite groups $\mathrm{GL}_n(\mathcal{O}_{F,r})$ encodes important information about the infinite-dimensional representation theory of the p-adic group $\mathrm{GL}_n(F)$.

On the other hand, there is the cohomological approach to constructing representations. The case $r = 1$ corresponds to connected reductive groups over finite fields and was treated in the celebrated work of Deligne and Lusztig [6]. In [30], Springer asks whether the geometric methods employed for $r = 1$ can be used to deal also with groups of the form $\mathbf{G}(\mathcal{O}_{F,r})$, for $r \geq 2$. The first step in this direction was taken by Lusztig [19], where a cohomological construction of certain representations of groups of the form $\mathbf{G}(\mathcal{O}_{F,r})$ was suggested (without proof). More recently, the proof was given in [20] for the case where F is of positive characteristic, and this was generalised to groups over arbitrary finite local rings in [34]. This construction attaches varieties and corresponding virtual representations $R_{T,U}(\theta)$ of $\mathbf{G}(\mathcal{O}_{F,r})$ to certain maximal tori in $\mathbf{G}$. However, this construction has two limitations. Firstly, in contrast to the case $r = 1$, it is not true for $r \geq 2$ that every irreducible representation of $\mathbf{G}(\mathcal{O}_{F,r})$ is a component of some $R_{T,U}(\theta)$. Secondly, the maximal tori in $\mathbf{G}$ correspond to unramified tori in the group $\mathbf{G} \times F$, that is, maximal tori which are split after an unramified extension. However, there also exist ramified maximal tori in $\mathbf{G} \times F$, and these are known to play a role in the representation theory of $\mathrm{GL}_n(\mathcal{O}_{F,r})$ and $\mathrm{SL}_n(\mathcal{O}_{F,r})$ analogous to that of the unramified maximal tori. In particular, since the work of Howe [14] it has been known that tamely ramified supercuspidal representations of $\mathrm{GL}_n(F)$ come in families attached to maximal tori. Given the correspondence between supercuspidal representations of $\mathrm{GL}_n(F)$ and their types on $\mathrm{GL}_n(\mathcal{O}_F)$, it is not surprising that ramified maximal tori should play a role in the representation theory of $\mathrm{GL}_n(\mathcal{O}_{F,r})$.

It is thus natural to ask whether it is possible to generalise the "unramified" construction of [20] and [34] to account also for the ramified maximal tori. The main purpose of this paper is to introduce a family of so-called *extended Deligne-Lusztig varieties*, corresponding to all the tamely ramified maximal tori. Another part of the paper motivates our approach by showing the inadequacy of varieties defined only with respect to unramified extensions of F. Finally, we show in a non-trivial special case that our construction leads to the expected result, namely, that varieties attached to a ramified maximal torus realise in their cohomology a family of representations which is known (by the algebraic construction) to be associated to this maximal torus.

The following is a more detailed outline of the paper. For a scheme $\mathbf{X}$ over $\overline{\mathbb{F}}_q$, and a prime l different from p, we will consider the l-adic étale cohomology groups with compact support $H_c^i(\mathbf{X}, \overline{\mathbb{Q}}_l)$. In what follows, l will be fixed and we will denote $H_c^i(\mathbf{X}, \overline{\mathbb{Q}}_l)$ simply by $H_c^i(\mathbf{X})$. We denote the alternating sum of cohomologies $\sum_{i\geq 0}(-1)^i H_c^i(\mathbf{X})$ by $H_c^*(\mathbf{X})$. Let F^{ur} be the maximal unramified extension of F (inside a fixed algebraic closure of F), and let $\mathcal{O}_{F^{\mathrm{ur}}}$ be its ring of integers. The construction of [20] and [34] considers the finite group $\mathbf{G}(\mathcal{O}_{F,r})$ as the fixed-point subgroup of $\mathbf{G}(\mathcal{O}_{F^{\mathrm{ur}},r})$ under a Frobenius endomorphism $\varphi : G_r \to G_r$, typically induced by the (arithmetic) Frobenius element in $\mathrm{Gal}(F^{\mathrm{ur}}/F)$. The Greenberg functor allows one to view $\mathbf{G}(\mathcal{O}_{F^{\mathrm{ur}},r})$ as a connected affine algebraic group G_r over the algebraic closure $\overline{\mathbb{F}}_q$, and $\mathbf{G}(\mathcal{O}_{F,r})$ is naturally isomorphic to a subgroup $G_{F,r}$ of G_r. For instance, if φ comes from the Frobenius in $\mathrm{Gal}(F^{\mathrm{ur}}/F)$, then $G_r^{\varphi} \cong G_{F,r}$. Similarly, for every subgroup scheme $\mathbf{H}$ of $\mathbf{G}$, we have a connected algebraic subgroup $H_r \cong \mathbf{H}(\mathcal{O}_{F^{\mathrm{ur}},r})$ of G_r. For $r \geq r' \geq 1$ we have a natural map $\rho_{r,r'} : H_r \to H_{r'}$, and we denote its kernel by $H_r^{r'}$.

Suppose that $\mathbf{T}$ is a maximal torus in $\mathbf{G}\times\mathcal{O}_{F^{\mathrm{ur}}}$ contained in a Borel subgroup $\mathbf{B}$ with unipotent radical $\mathbf{U}$ such that T_r and U_r are φ-stable. Let $L : G_r \to G_r$ be the Lang map, given by $g \mapsto g^{-1}\varphi(g)$. For any element w in the Weyl group $N_{G_1}(T_1)/T_1$, and any lift $\hat{w} \in N_{G_r}(T_r)$ of w, we can then define the varieties

$$X_r(w) = L^{-1}(\dot{w}B_r)/B_r \cap \dot{w}B_r\dot{w}^{-1},$$
$$\widetilde{X}_r(\hat{w}) = L^{-1}(\hat{w}U_r)/U_r \cap \hat{w}U_r\hat{w}^{-1},$$

where $\widetilde{X}_r(\hat{w})$ is a finite cover of $X_r(w)$. These varieties were first considered by Lusztig [19], and coincide with classical Deligne-Lusztig varieties for $r = 1$. For $r = 1$ the Bruhat decomposition in G_1 implies that the varieties $X_1(w)$, and hence the corresponding covers $\widetilde{X}_1(\hat{w})$, are attached to double B_1-B_1 cosets.

It was shown by Deligne and Lusztig [6] that every irreducible representation of G_1^{φ} is a component of the cohomology of some variety $\widetilde{X}_1(\hat{w})$. In contrast, using the varieties $\widetilde{X}_r(\hat{w})$ for $r \geq 2$, this is no longer true in general. On the other hand, for $r \geq 2$ there exist double B_r-B_r cosets which are not indexed by elements of the Weyl group. In order to construct the missing representations it therefore seems natural to define the following varieties (first considered by Lusztig)

$$L^{-1}(xB_r)/B_r \cap xB_rx^{-1}, \quad L^{-1}(xU_r)/U_r \cap xU_rx^{-1}, \quad \text{for any } x \in G_r.$$

One may then hope that since these varieties account for all double B_r-B_r cosets in G_r, they may also afford further representations of G_r^{φ}, not obtainable by the varieties $\widetilde{X}_r(\hat{w})$. However, it turns out that this is not the case, and we prove in Section 3 that there are non-trivial cases where these varieties do not afford any new representations beyond those given by the varieties $\widetilde{X}_r(\hat{w})$. In Subsection 3.1 we give an explicit algebraic description of the irreducible representations of $\mathrm{SL}_2(\mathcal{O}_{F,r})$, using Clifford theory and orbits. This construction is well-known for odd q, but the case when q is a power of 2 requires a modification and does not seem to have previously appeared in this form.

Assume for the moment that $\mathbf{G} = \mathrm{SL}_2$, and let $\mathbf{U}$ and $\mathbf{U}^-$ be the upper and lower uni-triangular subgroups, respectively. If G is a finite group acting on two varieties X and Y, we write $X \sim Y$ if $H_c^*(X) \cong H_c^*(Y)$ as virtual G-representations. In Subsection 3.2, we show

Theorem 3.5. Let $y \in (U^-)^1_2$. Then $L^{-1}(yU_2) \sim \widetilde{X}_2(1)$, and hence

$$H^*_c(L^{-1}(yU_2)) \cong \mathrm{Ind}_{U_2^\varphi}^{G_2^\varphi} \mathbf{1}$$

as G_2^φ-representations.

Together with Proposition 3.4 and Proposition 3.6 this result implies that any irreducible representation of $\mathrm{SL}_2(\mathcal{O}_{F,2})$ which appears in the cohomology of a variety of the form $L^{-1}(xB_2)/B_2 \cap xB_2x^{-1}$, $L^{-1}(xU_2)/U_2 \cap xU_2x^{-1}$, or $L^{-1}(xU_2)$ already appears in the cohomology of a variety $\widetilde{X}_2(\hat{w})$, where w is one of the two elements of $N_{G_1}(T_1)/T_1$. Combining this with results of Lusztig on the cohomology of $\widetilde{X}_2(\hat{w})$, for $w \neq 1$ and F of positive characteristic (cf. [20], 3), we deduce as a corollary that there exist certain nilpotent representations of $\mathrm{SL}_2(\mathcal{O}_{F,2})$, for F of positive characteristic, which do not appear in the cohomology of any of the above varieties.

Having shown that the idea of attaching generalised Deligne-Lusztig varieties to double B_r-B_r cosets does not lead to a satisfactory construction, we turn to another point of view. In this paper we will primarily be concerned with the cases $\mathbf{G} = \mathrm{GL}_n$ or $\mathbf{G} = \mathrm{SL}_n$, and where φ is the standard Frobenius. Assume now that we are in one of these cases.

Rather than using the varieties $\widetilde{X}_r(\hat{w})$, the unramified representations $R_{T,U}(\theta)$ of [20] and [34] can also be constructed by using another type of variety. A variety of this kind is attached to a Borel subgroup containing certain maximal torus. Let now $\mathbf{T}$ be any maximal torus of $\mathbf{G} \times \mathcal{O}_{F^{\mathrm{ur}}}$ such that T_r is φ-stable. Let $\mathbf{B}$ be a Borel subgroup containing $\mathbf{T}$, and let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. One can then attach a Deligne-Lusztig variety to the inclusion $T_r \subset B_r$. In the case $r = 1$, the group T_1 is a maximal torus of G_1, but in general T_r is not a maximal torus, but a Cartan subgroup of G_r. A φ-stable Cartan subgroup T_r is the connected centraliser of a regular semisimple element in G_r^φ. This shows the relation between regular semisimple elements in G_r^φ and the unramified Deligne-Lusztig construction. The work of Hill [13] for GL_n, and the results for SL_2 (see Subsection 3.1) clearly show that the regular elements in $\mathbf{G}(\mathcal{O}_{F,r})$ and their centralisers play an important role in the representation theory of $\mathbf{G}(\mathcal{O}_{F,r})$. Among the elements in $\mathbf{G}(\mathcal{O}_{F^{\mathrm{ur}},r})$, there are those with distinct eigenvalues in some extension of the ring $\mathcal{O}_{F^{\mathrm{ur}},r}$. We call such elements, and the corresponding elements in G_r, *separable.* For $r = 1$ they are precisely the regular semisimple elements, but in general there are non-regular unipotent separable elements. The Cartan subgroups T_r are thus the reductions mod $\mathfrak{p}^r$ of the $\mathcal{O}_{F^{\mathrm{ur}}}$-points of unramified maximal tori in $\mathbf{G} \times F^{\mathrm{ur}}$ defined over $\mathcal{O}_{F^{\mathrm{ur}}}$, and correspond to regular semisimple elements. In addition, there exist subgroups of $\mathbf{G}(\mathcal{O}_{F^{\mathrm{ur}},r})$ which come from ramified tori, and these are the centralisers of regular separable elements which are not semisimple.

The idea in Section 4 is that one should attach generalised Deligne-Lusztig varieties not only to unramified maximal tori, but to the centraliser of any regular separable element in G_r^φ. To achieve this, we consider an arbitrary regular separable element $x \in G_r^\varphi$, and its centraliser $C_{G_r}(x)$, called a *quasi-Cartan* subgroup. To generalise the unramified case, we would also need an inclusion of $C_{G_r}(x)$ into a group of the form B_r. However, one feature of general regular separable elements is that they may not be triangulable in G_r, that is, x may not be conjugate in G_r to any element in B_r. This means that unlike the Cartan subgroups T_r, general quasi-Cartans may not lie inside any conjugate of B_r. We are thus lead to extend the base field F to a ramified extension. More precisely, in Section 4 we show that given any element $x \in G_{F,r'}$, for some $r' \geq 1$, there exists a finite extension L/F^{ur}, an integer $r \geq r'$, a connected affine algebraic group $G_{L,r} \cong \mathbf{G}(\mathcal{O}_{L,r})$, and a $\lambda \in G_{L,r}$, such that $G_{F,r'} \subseteq G_{L,r}$ and such that $\lambda^{-1}x\lambda \in B_{L,r}$. This implies that if x is regular separable, then

$$C_{G_r}(x) \subseteq \lambda B_{L,r}\lambda^{-1}.$$

Given a φ-stable quasi-Cartan $C_{G_r}(x)$, and a group $\lambda B_{L,r}\lambda^{-1}$ containing it, and assuming that L/F^{ur} is tamely ramified, we construct a variety $X^{\Sigma}_{L,r}(\lambda)$, where Σ contains two endomorphisms of $G_{L,r}$ (including one Frobenius). The variety $X^{\Sigma}_{L,r}(\lambda)$ is a subvariety of $G_{L,r}/B_{L,r}$, which is a generalisation of the flag variety of Borel subgroups, and is provided with an action of the finite groups of fixed points $G^{\Sigma}_{L,r}$. When L/F^{ur} is tamely ramified, we show that $G^{\Sigma}_{L,r} = G_{F,r'}$.

It is also important to define finite covers of $X^{\Sigma}_{L,r}(\lambda)$, generalising $\widetilde{X}_r(\hat{w})$. However, in general there does not seem to be any straightforward way to define such a cover of the whole of $X^{\Sigma}_{L,r}(\lambda)$, but only of a certain subvariety of $X^{\Sigma}_{L,r}(\lambda)$. The covers we construct are denoted $\widetilde{X}^{\Sigma}_{L,r}(\lambda)$, and do indeed reduce to the covers $\widetilde{X}_r(\hat{w})$ in the unramified case. In particular, $\widetilde{X}^{\Sigma}_{L,r}(\lambda)$ also carries an action of $G^{\Sigma}_{L,r}$, and a commuting action of a finite group $S(\lambda)/S(\lambda)^0$. This generalises the action of $G^{\varphi}_r \times T^{\hat{w}\varphi}_r$ on $\widetilde{X}_r(\hat{w})$. We call the varieties $X^{\Sigma}_{L,r}(\lambda)$ and $\widetilde{X}^{\Sigma}_{L,r}(\lambda)$ *extended Deligne-Lusztig varieties.*

In Section 5 we study the extended Deligne-Lusztig varieties for $\mathbf{G} = \mathrm{GL}_2$ and $\mathbf{G} = \mathrm{SL}_2$, with F of odd characteristic and $r = 3$. In this case, only one (tamely) ramified quadratic extension L/F^{ur} occurs, and we have $G^{\Sigma}_{L,3} = G_{F,2} \cong \mathbf{G}(\mathcal{O}_{F,2})$. There are four conjugacy classes of rational quasi-Cartan subgroups of G_2. The two classes of Cartan subgroups give rise to the "unramified" varieties $\widetilde{X}_2(1)$ and $\widetilde{X}_2(\dot{w})$, respectively. The third class gives rise to an extended Deligne-Lusztig variety $\widetilde{X}^{\Sigma}_{L,3}(\lambda)$, and we show the following

Theorem 5.1. Let $\mathbf{Z}$ be the centre of $\mathbf{G}$. Then there exists a $G^{\Sigma}_{L,3}$-equivariant isomorphism

$$\widetilde{X}^{\Sigma}_{L,3}(\lambda)/(Z^1_{L,3})^{\varphi} \cong G^{\Sigma}_{L,3}/(Z^1_{L,3})^{\Sigma}(U^1_{L,3})^{\Sigma}.$$

Here $Z^1_{L,3}$ is the kernel of the natural reduction map $Z_{L,3} \to Z_{L,1}$, and similarly for $U^1_{L,3}$. Combining this result with results of Lusztig [20], we can show in particular that every irreducible representation of $\mathrm{SL}_2(\mathbb{F}_q[[\varpi]]/(\varpi^2))$, with odd q appears in the cohomology of some extended Deligne-Lusztig variety.

In the final section, we state some open problems and indicate several directions in which our results could be taken further.

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2 Notation and general facts

For any discrete valuation field F we denote by $\mathcal{O}_F$ its ring of integers, by $\mathfrak{p}_F$ the maximal ideal of $\mathcal{O}_F$, and by $k = k_F$ the residue field (which we always assume to be perfect). If $r \geq 1$ is a natural number, we let $\mathcal{O}_{F,r}$ denote the quotient ring $\mathcal{O}_F/\mathfrak{p}_F^r$. Throughout the paper $\varpi = \varpi_F$ will denote a fixed prime element of $\mathcal{O}_F$.

Let $\mathbf{X}$ be a scheme of finite type over $\mathcal{O}_{F,r}$. Greenberg [10, 11] has defined a functor $\mathcal{F}_{\mathcal{O}_{F,r}}$ from the category of schemes of finite type over $\mathcal{O}_{F,r}$ to the category of schemes over k, such that there exists a canonical isomorphism

$$\mathbf{X}(\mathcal{O}_{F,r}) \cong (\mathcal{F}_{\mathcal{O}_{F,r}}\mathbf{X})(k),$$

and such that $\mathcal{F}_{\mathcal{O}_{F,1}} = \mathcal{F}_k$ is the identity functor. Moreover, Greenberg has shown that the functor $\mathcal{F}_{\mathcal{O}_{F,r}}$ preserves schemes of finite type, separated schemes, affine schemes, smooth schemes, open and closed subschemes, and group schemes, over the corresponding bases, respectively. If $\mathbf{X}$ is smooth over $\mathcal{O}_{F,r}$ and $\mathbf{X} \times k$ is reduced and irreducible, then $\mathcal{F}_{\mathcal{O}_{F,r}}\mathbf{X}$ is reduced and irreducible ([11], 2, Corollary 2).

Let $\mathbf{G}$ be an affine smooth group scheme over $\mathcal{O}_F$. By definition it is then also of finite type over $\mathcal{O}_F$. For any natural number $r \geq 1$ we define

$$G_{F,r} := \mathcal{F}_{\mathcal{O}_{F,r}}(\mathbf{G} \times_{\mathcal{O}_F} \mathcal{O}_{F,r})(k).$$

By the results of Greenberg, $G_{F,r}$ is then the k-points of a smooth affine group scheme over k. It can thus be identified with the k-points of an affine algebraic group defined over k. Since $\mathbf{G}$ is smooth over $\mathcal{O}_F$, it follows that for any natural numbers $r \geq r' \geq 1$, the reduction map $\mathcal{O}_{F,r} \to \mathcal{O}_{F,r'}$ induces a surjective homomorphism $\rho_{r,r'} : G_{F,r} \to G_{F,r'}$. The kernel of $\rho_{r,r'}$ is denoted by $G_{F,r}^{r'}$. The multiplicative representatives map $k^\times \to \mathcal{O}_{F,r}^\times$ induces a section $i_r : G_{F,1} \to G_{F,r}$. In the case where F is of positive characteristic, there is an inclusion of k-algebras $k \to \mathcal{O}_{F,r}$, and i_r is an injective homomorphism. When F is of characteristic zero i_r is not in general a homomorphism. However, if $\mathbf{G}$ is a split torus, then i_r is always a homomorphism, irrespective of the characteristic of F.

Following [28], XIX 2.7, we call a group scheme $\mathbf{G}$ over a base scheme $\mathbf{S}$ *reductive* if $\mathbf{G}$ is affine and smooth over $\mathbf{S}$, and if its geometric fibres are connected and reductive as algebraic groups. If $\mathbf{G}$ is a reductive group scheme over $\mathbf{S}$, we will speak of maximal tori and Borel subgroups of $\mathbf{G}$, which are also group schemes over $\mathbf{S}$. For any Borel subgroup of $\mathbf{G}$ there is also a well-defined unipotent radical. For these notions, see [28], XXII 1.3, XIV 4.5, and XXVI 1.6, respectively. For more on reductive group schemes, see [34] and its references.

From now on and throughout the paper, let F denote a local field with finite residue field $\mathbb{F}_q$ of characteristic p. We will use the same symbol $\mathfrak{p}_F$ to denote the maximal ideal in $\mathcal{O}_F$, as well as the maximal ideal in any of the quotients $\mathcal{O}_{F,r}$. Let $\mathbf{G}$ be a reductive group scheme over $\mathcal{O}_F$. By definition, $\mathbf{G}$ is affine and smooth over $\mathcal{O}_F$. We fix an algebraic closure of F in which all algebraic extensions are taken. Denote by F^{ur} the maximal unramified extension of F with residue field $\overline{\mathbb{F}}_q$, an algebraic closure of $\mathbb{F}_q$. Suppose that L is a finite extension of F^{ur}. Then L also has residue field $\overline{\mathbb{F}}_q$. We define

$$G_{L,r} := (\mathbf{G} \times_{\mathcal{O}_F} \mathcal{O}_L)_{L,r} = \mathcal{F}_{\mathcal{O}_{L,r}}(\mathbf{G} \times_{\mathcal{O}_F} \mathcal{O}_{L,r})(\overline{\mathbb{F}}_q).$$

Thus $G_{L,r}$ is an affine algebraic group over $\overline{\mathbb{F}}_q$. Since $\mathbf{G}$ has connected fibres (by definition), $G_{L,r}$ is connected. For F^{ur} we will drop the subscript and write G_r for $G_{F^{\mathrm{ur}},r}$, and $G_r^{r'}$ for the kernel $G_{F^{\mathrm{ur}},r}^{r'}$.

If G is a finite group, we denote by $\mathrm{Irr}(G)$ the set of irreducible $\overline{\mathbb{Q}}_l$-representations of G. Since the values of the characters in $\mathrm{Irr}(G)$ all lie in some finite extension of $\mathbb{Q}$, there is a character preserving bijection between $\mathrm{Irr}(G)$ and the set of irreducible complex representations of G. For any finite group G we denote its trivial representation by $\mathbf{1}$.

If x is a real number, we will write $[x]$ for the largest integer $\leq x$.

Many results about l-adic cohomology used in classical Deligne-Lusztig theory are applicable also in the generalised situations we will consider, and throughout we will assume familiarity with the results stated in [7], 10. In what follows, all varieties will be separated reduced schemes of finite type over $\overline{\mathbb{F}}_q$, and we identify every variety with its set of $\overline{\mathbb{F}}_q$-points. Suppose that G is a finite group acting on a variety X. Then each $g \in G$ induces an element of $\mathrm{Aut}_{\overline{\mathbb{Q}}_l}(H_c^i(X))$, for each $i \geq 0$, and this is a representation of G. The quantity

$$\mathscr{L}(g, X) := \sum_{i \geq 0} (-1)^i \operatorname{Tr}(g \mid H_c^i(X)) = \operatorname{Tr}(g \mid H_c^*(X))$$

is called the *Lefschetz number* of X at g. A *virtual representation* of G is an element in the Grothendieck group of the semigroup generated by $\mathrm{Irr}(G)$ under the direct sum operation. The function $\mathscr{L}(-, X) : G \to \overline{\mathbb{Q}}_l$ is the character of the virtual representation $H_c^*(X)$ given by the action of G on X. Let G be a finite group that acts on the varieties X and Y, respectively. Recall that we write $X \sim Y$ if $H_c^*(X) = H_c^*(Y)$ as virtual G-representations. We then have $X \sim Y$ if and only if $\mathscr{L}(-, X) = \mathscr{L}(-, Y)$, and the relation $\sim$ is an equivalence relation.

Lemma 2.1. *Suppose that $f : X \to Y$ is a (set-theoretic) bijection between two varieties such that $f\varphi = \varphi f$, for some Frobenius endomorphisms $\varphi : X \to X$ and $\varphi : Y \to Y$. Let g, g' be automorphisms of finite order of X, Y such that $fg = g'f$. Then $\mathscr{L}(g, X) = \mathscr{L}(g', Y)$.*

Proof. As in the proof of [7], 10.12 (ii), we have that for sufficiently large m,

$$|X^{g\varphi^m}| = \sum_{y\in Y^{g'\varphi^m}} |f^{-1}(y)^{g'\varphi^m}| = |Y^{g'\varphi^m}|,$$

which implies that $\mathscr{L}(g, X) = \mathscr{L}(g', Y)$. □

Let G be an affine algebraic group, and let $X \subseteq G$ be a locally closed subset. Suppose that H is a closed subgroup of G, acting by multiplication on G, such that X is stable under the action of H. Then the quotient X/H is a locally closed subset of G/H. For a proof of this fact, see for example [31], Lemma 1.5. This shows that the quotient X/H has a natural structure of algebraic variety, which ensures that certain sets we will define in the following are indeed varieties.

The following observations will be very useful in our analysis of the cohomology of varieties. Let G be a finite group that acts on the variety X, and let $H \subset G$ be a subgroup such that there exists a G-equivariant morphism

$$\rho : X \longrightarrow G/H,$$

that is, ρ satisfies $\rho(gx) = g\rho(x)$, for all $g \in G$, $x \in X$. It then follows that ρ is a surjection, and for any $a \in G$, the stabiliser in G of the fibre $\rho^{-1}(aH)$ is $H \cap {}^aH$. Let f be the fibre over the trivial coset $H \in G/H$. Then every fibre of ρ is isomorphic to f via translation by an element of G. Hence every $x \in X$ has the form $x = gy$, for $g \in G$ and $y \in f$ which are uniquely determined up to the action of H given by $h(g, y) = (gh^{-1}, hy)$. We thus have a G-equivariant isomorphism

$$X \xrightarrow{\sim} (G \times f)/H, \qquad x \longmapsto (g, y)H.$$

Here G acts on $(G \times f)/H$ via $g'(g, y)H = (g'g, y)H$. It follows that

$$H_c^*((G \times f)/H) \cong \overline{\mathbb{Q}}_l[G] \otimes_{\overline{\mathbb{Q}}_l[H]} H_c^*(f) = \mathrm{Ind}_H^G H_c^*(f),$$

as virtual G-representations.

3 The unramified approach

Let $\mathbf{G}$ be a reductive group scheme over $\mathcal{O}_F$, and let $r \geq 1$ be an integer. A certain generalisation of the construction of Deligne and Lusztig to the case $r \geq 1$ was obtained by Lusztig [20] for F of characteristic p, and in [34] for general F and also for groups over general finite local rings. The generalised Deligne-Lusztig varieties in these constructions are attached to certain maximal tori in $\mathbf{G} \times \mathcal{O}_{F^{\mathrm{ur}}}$, and are close analogues of the classical Deligne-Lusztig varieties. Any maximal torus in $\mathbf{G} \times \mathcal{O}_{F^{\mathrm{ur}}}$ is an unramified torus in $\mathbf{G} \times_{\mathcal{O}_{F^{\mathrm{ur}}}} F^{\mathrm{ur}}$ in the sense that it splits over an unramified extension of F. The construction given by these varieties can thus be seen as an "unramified" generalisation of the construction of Deligne and Lusztig. We give an outline of this construction.

Let $\varphi : G_r \to G_r$ be a surjective endomorphism of algebraic groups such that G_r^{φ} is finite. We call such a map φ a *Frobenius endomorphism*. Let $L : G_r \to G_r$, denote the map $g \mapsto g^{-1}\varphi(g)$. Assume for simplicity that $\mathbf{G} \times \mathcal{O}_{F^{\mathrm{ur}}}$ contains a maximal torus $\mathbf{T}$ and a Borel subgroup $\mathbf{B}$ containing $\mathbf{T}$, such that T_r and B_r are φ-stable. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. By the results in [32], we know that B_r is a self-normalising subgroup of G_r. Note that the assumption that B_r be φ-stable is not necessary for the construction of the representations in [20] and [34], but it simplifies the models of the varieties we consider here.

Let $\mathcal{B}_r$ be the set of subgroups conjugate to B_r. Since B_r is self-normalising we have a bijection $\mathcal{B}_r \cong G_r/B_r$, giving $\mathcal{B}_r$ a variety structure. As in the $r = 1$ case, we have a bijection

$$G_r\backslash(\mathcal{B}_r \times \mathcal{B}_r) \xrightarrow{\sim} B_r\backslash G_r/B_r.$$

However, for $r > 1$, the double B_r-B_r cosets are no longer in one-to-one correspondence with elements of the group $N_{G_r}(T_r)/T_r$, and the structure of $B_r\backslash G_r/B_r$ is too complex to admit any straightforward description. Let $x \in G_r$ be an arbitrary element. In analogy with the $r = 1$ case we can define a variety

$$\begin{aligned} X_r(x) &:= \{B \in \mathcal{B}_r \mid (B, \varphi(B)) \in O(x)\} \\ &\cong \{g \in G_r \mid g^{-1}\varphi(g) \in B_r x B_r\}/B_r, \\ &\cong \{g \in G_r \mid g^{-1}\varphi(g) \in xB_r\}/(B_r \cap xB_r x^{-1}), \end{aligned}$$

where $O(x)$ denotes the orbit in $G_r\backslash(\mathcal{B}_r \times \mathcal{B}_r)$ corresponding to the double coset $B_r x B_r$. In the same way as for $r = 1$, the finite group G_r^{φ} acts on $X_r(x)$ by left multiplication. For each $\hat{w} \in N_{G_r}(T_r)$ we also have a variety

$$\begin{aligned} \widetilde{X}_r(\hat{w}) &:= \{g \in G_r \mid g^{-1}\varphi(g) \in \hat{w}U_r\}/U_r \cap \hat{w}U_r\hat{w}^{-1} \\ &= L^{-1}(\hat{w}U_r)/U_r \cap \hat{w}U_r\hat{w}^{-1}. \end{aligned}$$

The variety $\widetilde{X}_r(\hat{w})$ has a left action of G_r^{φ}, and a commuting right action of the group

$$T_r^{\hat{w}\varphi} := \{t \in T_r \mid \hat{w}\varphi(t)\hat{w}^{-1} = t\}.$$

It is then not hard to verify, by the same method as for $r = 1$, that the varieties $\widetilde{X}_r(\hat{w})$ are finite G_r^{φ}-covers of $X_r(\hat{w})$. This depends on the fact that $\hat{w}$ normalises the group T_r. The varieties $\widetilde{X}_r(\hat{w})$ (or rather, certain models isomorphic to them) were used in [20] and [34] to construct certain generalised Deligne-Lusztig representations. However, we will show in Subsection 3.2 that the representations thus constructed leave out a non-trivial subset of $\mathrm{Irr}(G_r^{\varphi})$, for $r \geq 2$. To remedy this situation one would like to define further varieties that would produce the missing representations. Given the above construction and the fact that the elements $\hat{w} \in N_{G_r}(T_r)$ do not account for all of the double cosets in $B_r\backslash G_r/B_r$, it is a priori natural to define the following varieties (first considered by Lusztig)

$$L^{-1}(xU_r) = \{g \in G_r \mid g^{-1}\varphi(g) \in xU_r\}, \quad \text{for any } x \in G_r.$$

Note that $L^{-1}(xU_r)$ has an action of $U_r \cap xU_rx^{-1}$ by right multiplication, and the quotient $L^{-1}(xU_r)/U_r \cap xU_rx^{-1}$ is a variety (see Section 2). For $x = \hat{w} \in N_{G_r}(T_r)$ we have $L^{-1}(\hat{w}U_r)/U_r \cap \hat{w}U_r\hat{w}^{-1} = \widetilde{X}_r(\hat{w})$, and as we observed above, the variety $\widetilde{X}_r(\hat{w})$ is a finite cover of $X_r(\hat{w})$. However, we point out that when $x \notin N_{G_r}(T_r)$, it is not in general the case that $L^{-1}(xU_r)$, or even its quotient $L^{-1}(xU_r)/U_r \cap xU_rx^{-1}$, is a finite cover of $X_r(x)$. One might then hope that in general any irreducible representation of G_r^{φ} is realised by some variety $X_r(x)$ or $L^{-1}(xU_r)$, for some $x \in G_r$. This however, turns out to be not the case in general. In the present section we will show that there exist irreducible representations of $\mathrm{SL}_2(\mathcal{O}_{F,2})$, with F of positive characteristic, which are not realised in the cohomology of any variety of the form $X_2(x)$ or $L^{-1}(xU_2)$. Our proof proceeds as follows. First we give an algebraic description of the irreducible representations of $\mathrm{SL}_2(\mathcal{O}_{F,r})$, with particular emphasis on the so-called nilpotent representations. We then analyse varieties of the form $L^{-1}(xU_2)$ and $X_2(x)$ and compare this to the algebraic description of representations given earlier. Using computations of Lusztig, giving the irreducible components of the cohomology of $\widetilde{X}_2(\hat{w})$, where $B_2\hat{w}B_2 \neq B_2$, we can show that there exist representations in $\mathrm{Irr}(\mathrm{SL}_2(\mathcal{O}_{F,2}))$ which are not afforded by the varieties $L^{-1}(xU_2)$ or $X_2(x)$.

The following results will be applied in Subsection 3.2 to the case where $\mathbf{G} = \mathrm{SL}_2$, $r = 2$.

Lemma 3.1. *The inclusion $L^{-1}(xU_r) \hookrightarrow L^{-1}(U_rxU_r)$ induces an isomorphism*

$$L^{-1}(xU_r)/U_r \cap xU_rx^{-1} \xrightarrow{\sim} L^{-1}(U_rxU_r)/U_r,$$

commuting with the action of G_r^{φ} on both varieties.

Proof. Let f be the composition of the maps

$$L^{-1}(xU_r) \hookrightarrow L^{-1}(U_rxU_r) \to L^{-1}(U_rxU_r)/U_r,$$

where the latter is the natural projection. Clearly f is surjective, because if $gU_r \in L^{-1}(U_rxU_r)/U_r$, with $L(g) \in uxu'$ for $u, u' \in U_r$, then $L(gu) = u^{-1}uxu'\varphi(u) \in xU_r$, so $gu \in L^{-1}(xU_r)$, and $f(gu) = gU_r$.

On the other hand, the fibre of f at gU_r is equal to

$$\begin{aligned}\{gv \in L^{-1}(xU_r) \mid v \in U_r\} &= \{gv \mid v^{-1}L(g)\varphi(v) \in xU_r,\ v \in U_r\} \\ = \{gv \mid v^{-1}ux \in xU_r, v \in U_r\} &= \{gv \mid v^{-1}u \in U_r \cap xU_rx^{-1}\} \\ &= \{gv \mid v = u \bmod U_r \cap xU_rx^{-1}\}.\end{aligned}$$

Factoring $L^{-1}(xU_r)$ by $U_r \cap xU_rx^{-1}$ therefore gives an isomorphism which commutes with the action of G_r^{φ}. □

Lemma 3.2. *Let $x \in G_r$ be an arbitrary element, and let λ be an element such that $L(\lambda) = x$. Then there is an isomorphism*

$$L^{-1}(xU_r) \xrightarrow{\sim} L^{-1}(\varphi(\lambda)U_r\varphi(\lambda)^{-1}), \quad g \longmapsto g\lambda^{-1},$$

commuting with the action of G_r^{φ}.

Proof. Let $g \in L^{-1}(xU_r)$. Then

$$L(g\lambda^{-1}) = \lambda L(g)\varphi(\lambda)^{-1} \in \lambda x U_r \varphi(\lambda)^{-1} = \varphi(\lambda) U_r \varphi(\lambda)^{-1}.$$

It is clear that this map is a morphism of varieties, and it has an obvious inverse. □

Lemma 3.3. *Suppose that $n \in N_{G_r}(T_r)$, and let $x \in B_r n B_r$. Then*

$$L^{-1}(xU_r)/U_r \cap xU_r x^{-1} \sim L^{-1}(nU_r)/U_r \cap nU_r n^{-1}.$$

Proof. We can write x as $utnt'u'$, for some $u, u' \in U_r$ and $t, t' \in T_r$. Since U_r is isomorphic to an affine space, [7], 10.12 (ii) together with Lemma 3.1 imply that

$$\begin{aligned} L^{-1}(xU_r)/U_r \cap xU_r x^{-1} &\sim L^{-1}(U_r utnt'u'U_r) \\ &= L^{-1}(U_r tnt'U_r) \sim L^{-1}(tnt'U_r)/U_r \cap tnt'U_r(tnt')^{-1} \\ &\qquad = L^{-1}(t''nU_r)/U_r \cap nU_r n^{-1}, \end{aligned}$$

for some $t'' \in T_r$. Since $t \mapsto n\varphi(t)n^{-1}$ is a Frobenius map on T_r, The Lang-Steinberg theorem says that there exists a $\lambda \in T_r$ such that $\lambda^{-1}n\varphi(\lambda)n^{-1} = t''$. The map

$$\begin{aligned} L^{-1}(t''nU_r)/U_r \cap nU_r n^{-1} &\longrightarrow L^{-1}(nU_r)/U_r \cap nU_r n^{-1} \\ g(U_r \cap nU_r n^{-1}) &\longmapsto g\lambda^{-1}(U_r \cap nU_r n^{-1}), \end{aligned}$$

is then an isomorphism of varieties which preserves the action of G_r^{φ}. The lemma is proved. □

3.1 The representations of $\mathrm{SL}_2(\mathcal{O}_{F,r})$

Using results from Clifford theory and classification of conjugacy orbits in certain algebras over the rings $\mathcal{O}_{F,r}$, it is possible to completely describe the representations of the groups $\mathrm{SL}_2(\mathcal{O}_{F,r})$, and $\mathrm{GL}_2(\mathcal{O}_{F,r})$. In most cases, these algebras are the Lie algebras of the corresponding group, with SL_2, $p = 2$ being a notable exception, as we will see below. For SL_2 with $p \neq 2$ this method was employed by Kutzko in his thesis (unpublished, see the announcement [17]) and by Shalika (whose results remained unpublished until recently, cf. [29]). Around the same time the representations of $\mathrm{SL}_2(\mathbb{Z}/p^r\mathbb{Z})$, including the case where $p = 2$, were also constructed by Nobs and Wolfart [23, 24], by decomposing Weil representations. For GL_2 with $\mathcal{O}_F = \mathbb{Z}_p$ and p odd, the analogous result was given by Nagornyj [22], and a general construction for all $\mathrm{GL}_2(\mathcal{O}_{F,r})$ can be found in [33]. Recently, the SL_2 case with $p \neq 2$ was also reproduced in [16]. We will focus here on SL_2, using the method of orbits and Clifford theory, and without any restriction on p. The case where $p = 2$ requires special treatment, and does not seem to have previously appeared in the literature in this form. Proofs of the results we use can be found in [29] and [33], and we will therefore omit details that can be found in these references.

Assume until the end of Subsection 3.2 that $\mathbf{G} = \mathrm{SL}_2$, viewed as group scheme over $\mathcal{O}_F$. Let $\mathbf{T}$ be the diagonal split maximal torus in $\mathbf{G}$, $\mathbf{B}$ be the upper-triangular Borel subgroup of $\mathbf{G}$, and $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. Let $\mathbf{U}^-$ be the unipotent radical of the Borel subgroup opposite to $\mathbf{B}$. As usual, we identify $G_{F,r}$ with the matrix group $\mathrm{SL}_2(\mathcal{O}_{F,r})$. Let $\mathfrak{g} = \mathfrak{sl}_2$ be the Lie algebra of SL_2, viewed as a scheme over $\mathcal{O}_F$. Thus $\mathfrak{g}_{F,r} \cong \mathfrak{g}(\mathcal{O}_{F,r})$ is identified with the algebra of 2×2 matrices over $\mathcal{O}_{F,r}$ whose trace is zero. Assume first that $p \neq 2$, and fix a natural number $r > 1$. For any natural number i such that $r \geq i \geq 1$ let $\rho_{r,i} : G_{F,r} \to G_{F,i}$ be the canonical surjective homomorphism. For clarity, we will use the notation K_i for the kernel $G^i_{F,r} = \operatorname{Ker}\rho_{r,i}$. Assume from now on that $i \geq r/2$. Then $K_i = 1 + \mathfrak{p}^i_F\mathfrak{g}_{F,r-i}$ and the map $x \mapsto 1 + \varpi^i x$ induces an isomorphism $\mathfrak{g}_{F,r-i} \xrightarrow{\sim} K_i$. The group $G_{F,r}$ acts on $\mathfrak{g}_{F,r-i}$ by conjugation, via its quotient $G_{F,r-i}$. This action is transformed by the above isomorphism into the action of $G_{F,r}$ on the normal subgroup K_i.

Fix an additive character $\psi : \mathcal{O}_F \to \overline{\mathbb{Q}}_l^\times$ with conductor $\mathfrak{p}^r_F$, and define for any $\beta \in \mathfrak{g}_{F,r-i}$ a character $\psi_\beta : K_i \to \overline{\mathbb{Q}}_l^\times$ by

$$\psi_\beta(x) = \psi(\mathrm{Tr}(\beta(x-1))).$$

Then $\beta \mapsto \psi_\beta$ gives an isomorphism

$$\mathfrak{g}_{F,r-i} \cong \mathrm{Hom}(K_i, \overline{\mathbb{Q}}_l^\times),$$

and for $g \in G_{F,r}$, we have $\rho_{r-i}(g)\beta\rho_{r-i}(g)^{-1} \mapsto (\psi_\beta)^g$.

Set $l = [\frac{r+1}{2}]$, $l' = [\frac{r}{2}]$; thus $l + l' = r$. Let π be an irreducible representation of $G_{F,r}$. By Clifford's theorem, restricting π to K_l determines an orbit of characters on K_l, and hence (by the above isomorphism) an orbit in $\mathfrak{g}_{F,l'}$. If the orbit is in $\mathfrak{p}_F\mathfrak{g}_{F,l'}$, then π is trivial on K_{r-1}, and so factors though $G_{F,r-1}$. We are only concerned with *primitive* representations, that is, those which do not factor through $G_{F,r-1}$. It is therefore enough to consider orbits in $\mathfrak{g}_{F,l'} \setminus \mathfrak{p}_F\mathfrak{g}_{F,l'}$. For any natural number r' such that $r \geq r' \geq 1$ we call an element $\beta \in \mathfrak{g}_{F,r'}$ *regular* if the centraliser $C_{G_1}(\rho_{r',1}(\beta))$ in $G_1 \cong \mathbf{G}(\overline{\mathbb{F}}_q)$ is abelian. We then have $C_{G_{r'}}(\beta) = \mathcal{O}_{r'}[\beta] \cap G_{r'}$, in the connected algebraic group $G_{r'}$. The orbits in $\mathfrak{g}_{F,l'} \setminus \mathfrak{p}_F\mathfrak{g}_{F,l'}$ can be easily classified thanks to the fact that they are all regular. More precisely, the orbits in $\mathfrak{g}_{F,l'} \setminus \mathfrak{p}_F\mathfrak{g}_{F,l'}$ are of three basic types, according to their reductions mod $\mathfrak{p}_F$: There are the orbits with split characteristic polynomial and distinct eigenvalues mod $\mathfrak{p}_F$, the ones which have irreducible characteristic polynomial mod $\mathfrak{p}_F$, and those which are nilpotent mod $\mathfrak{p}_F$. The primitive representations of these three types are called *split, cuspidal,* and *nilpotent,* respectively.

The construction of the representations of $G_{F,r}$ with a given orbit $\Omega \in \mathfrak{g}_{F,l'} \setminus \mathfrak{p}_F\mathfrak{g}_{F,l'}$ proceeds as follows. Pick a representative $\beta \in \Omega$, and consider the corresponding character ψ_β on K_l. The stabiliser in $G_{F,r}$ of ψ_β is given by

$$\mathrm{Stab}_{G_{F,r}}(\psi_\beta) = C_{G_{F,r}}(\hat{\beta})K_{l'},$$

where $\hat{\beta} \in \mathfrak{g}_{F,r}$ is an element such that $\rho_{r,l}(\hat{\beta}) = \beta$. Assume first that r is even so that $l = l'$. Since $C_{G_{F,r}}(\hat{\beta})$ is abelian, the character ψ_β can be extended to a character on $\mathrm{Stab}_{G_{F,r}}(\psi_\beta)$, and all the irreducible representations of $\mathrm{Stab}_{G_{F,r}}(\psi_\beta)$ containing ψ_β are obtained in this way. Inducing a representation of $\mathrm{Stab}_{G_{F,r}}(\psi_\beta)$ containing ψ_β to $G_{F,r}$ gives an irreducible representation, and it is clear that we get all the irreducible representations of $G_{F,r}$ with orbit Ω in this way.

Now assume that r is odd. In this case there are several equivalent variations of the construction, but they all involve (at least for some orbits) a step where a representation of a group is shown to have a unique representation lying above it in a larger group. The other steps consist of various lifts and induction from $\mathrm{Stab}_{G_{F,r}}(\psi_\beta)$, as in the case for r even. For full details, see [29] for SL_2, and [33] for the closely related case of GL_2, respectively.

Now consider the case where $p = 2$. In this case the association $\beta \mapsto \psi_\beta$ does no longer give an isomorphism between $\mathfrak{g}_{F,r-i}$ and the character group of K_i. To remedy this, we first consider the analogous situation for GL_2 where the role of $\mathfrak{g}_{F,r-i}$ is played by the matrix algebra $\mathrm{M}_2(\mathcal{O}_{F,r-i})$, and the analogous map $\beta \mapsto \psi_\beta$ is indeed an isomorphism (for any p). The i^{th} congruence kernel in $\mathrm{GL}_2(\mathcal{O}_{F,r})$ has the form $1 + \mathfrak{p}_F^i \mathrm{M}_2(\mathcal{O}_{F,r-i})$, and so it contains K_i as a subgroup of index $|\mathcal{O}_{F,r-i}|$. For every $\beta \in \mathrm{M}_2(\mathcal{O}_{F,r-i})$ we have a character $\psi_\beta|_{K_i}$ obtained by restricting the character ψ_β on $1 + \mathfrak{p}_F^i \mathrm{M}_2(\mathcal{O}_{F,r-i})$ to K_i. Then $\beta \mapsto \psi_\beta|_{K_i}$ is obviously a surjective homomorphism $\mathrm{M}_2(\mathcal{O}_{F,r-i}) \to \mathrm{Hom}(K_i, \overline{\mathbb{Q}}_l^\times)$. It is easily seen that the kernel of this homomorphism is the subgroup Z_{r-1} of scalar matrices in $\mathrm{M}_2(\mathcal{O}_{F,r-i})$. We therefore have an isomorphism

$$\mathrm{M}_2(\mathcal{O}_{F,r-i})/Z_{r-i} \xrightarrow{\sim} \mathrm{Hom}(K_i, \overline{\mathbb{Q}}_l^\times), \quad \beta + Z_{r-i} \mapsto \psi_\beta|_{K_i}.$$

Since Z_{r-i} is centralised by $G_{F,r}$, we see that for any $g \in G_{F,r}$, we have

$$\rho_{r-i}(g)\beta\rho_{r-i}(g)^{-1} \mapsto (\psi_\beta|_{K_i})^g.$$

As before, let $l = [\frac{r+1}{2}]$, $l' = [\frac{r}{2}]$. If $\beta \in \mathfrak{p}_F \mathrm{M}_2(\mathcal{O}_{F,l'})/Z_{l'}$, then $\psi_\beta|_{K_l}$ is trivial on K_{r-1}, and so an irreducible representation of $G_{F,r}$ whose restriction to K_l contains this $\psi_\beta|_{K_l}$ must factor through $G_{F,r-1}$, and hence is not primitive. To construct the primitive representations, the first task is now to classify the orbits under the action of $G_{F,r}$ on $\mathrm{M}_2(\mathcal{O}_{F,l'})/Z_{l'} \setminus \mathfrak{p}_F \mathrm{M}_2(\mathcal{O}_{F,l'})/Z_{l'}$. The following is a list a representatives of these orbits:

1. $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $a \in \mathcal{O}_{F,l'}^\times$,
2. $\begin{pmatrix} 0 & 1 \\ \Delta & s \end{pmatrix}$, where $\Delta, s \in \mathcal{O}_{F,l'}$, and $x^2 - sx - \Delta$ is irreducible mod $\mathfrak{p}_F$,
3. $\begin{pmatrix} 0 & 1 \\ \Delta & s \end{pmatrix}$, where $\Delta, s \in \mathfrak{p}_F$.

The construction of representations then proceeds as in the case $p \neq 2$.

Remark. Clearly the method used in the case $p = 2$ could also be applied when $p \neq 2$. We have however chosen to give the two separate cases in order to illustrate their contrasts. Note that when $p \neq 2$ the embedding $\mathfrak{g}_{F,l'} \hookrightarrow \mathrm{M}_2(\mathcal{O}_{F,l'})$ induces a $G_{F,r}$-equivariant isomorphism

$$\mathfrak{g}_{F,l'} \xrightarrow{\sim} \mathrm{M}_2(\mathcal{O}_{F,l'})/Z_{l'},$$

so in general the algebra $\mathrm{M}_2(\mathcal{O}_{F,l'})/Z_{l'}$ is the right object, rather than the Lie algebra $\mathfrak{g}_{F,l'}$, in which to consider orbits.

In the following we will be especially interested in the *nilpotent* representations of $G_{F,2} \cong \mathrm{SL}_2(\mathcal{O}_{F,2})$, that is, the irreducible primitive representations whose orbits mod $\mathfrak{p}_F$ are nilpotent, or contain a nilpotent element mod Z_1 when $p = 2$, respectively. We call the corresponding orbits nilpotent (although in the $p = 2$ case, they are strictly speaking only nilpotent mod centre). The construction of representations given above shows that the nilpotent representations are induced from 1-dimensional representations on $\mathrm{Stab}_{G_{F,2}}(\psi_\beta|_{K_1})$, where β is a representative of a nilpotent orbit. When $p \neq 2$ there are exactly two nilpotent orbits in $\mathfrak{g}_{F,1} \setminus \mathfrak{p}_F\mathfrak{g}_{F,1}$, given by the representatives

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \zeta \\ 0 & 0 \end{pmatrix},$$

respectively (here $\zeta \in \mathbb{F}_q^\times$ is a non-square element). When $p = 2$ there is just one nilpotent-mod-Z_1 orbit in $\mathrm{M}_2(\mathcal{O}_{F,1})/Z_1 \setminus \mathfrak{p}_F \mathrm{M}_2(\mathcal{O}_{F,1})/Z_1$, given by the representative $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. If we let β be any of these representatives, then the stabiliser $\mathrm{Stab}_{G_{F,2}}(\psi_\beta|_{K_1})$ is given by

$$S := \mathrm{Stab}_{G_{F,2}}(\psi_\beta|_{K_1}) = \{\pm 1\}U_{F,2}K_1,$$

where $\{\pm 1\}$ denotes a subgroup of scalar matrices (which is equal to the centre of $G_{F,2}$ for $p \neq 2$, and is trivial for $p = 2$), and $U_{F,2}$ is isomorphic to the subgroup of $\mathbf{G}(\mathcal{O}_{F,2})$ of upper unitriangular matrices. The index of S in $G_{F,2}$ is equal to $(q^2-1)/2$ when $p \neq 2$, and equal to q^2-1 when $p = 2$. It is not hard to show that the commutator subgroup of S is $[S, S] = B^1_{F,2} = B_{F,2} \cap K_1$. Thus all nilpotent representations of $G_{F,2}$ are components of the induced representation $\mathrm{Ind}_{B^1_{F,2}}^{G_{F,2}} \mathbf{1}$. Each ψ_β has $|S/K_1|$ extensions to S, and each such extension induces to a distinct nilpotent representation. When $p \neq 2$ we thus have $4q$ nilpotent representations, all of which have dimension $(q^2 - 1)/2$. When $p = 2$ we have q nilpotent representations, all of which have dimension $q^2 - 1$.

We will have occasion to consider the question of which nilpotent representations occur as components of $\mathrm{Ind}_{U_{F,2}}^{G_{F,2}} \mathbf{1}$. By the above we know that any nilpotent representation of $G_{F,2}$ is of the form $\mathrm{Ind}_S^{G_{F,2}} \rho$, for some ρ such that $\rho|K_1$ contains ψ_β, with β one of the above nilpotent representatives. By Mackey's intertwining number formula, we have

$$\langle \mathrm{Ind}_S^{G_{F,2}} \rho, \mathrm{Ind}_{U_{F,2}}^{G_{F,2}} \mathbf{1} \rangle = \sum_{x \in S \backslash G_{F,2}/U_{F,2}} \langle \rho|_{S \cap {}^x U_{F,2}}, \mathbf{1} \rangle,$$

and since S contains K_1 we can identify $S\backslash G_{F,2}/U_{F,2}$ with $U_{F,1}\backslash G_{F,1}/U_{F,1}$. To calculate the value of the right-hand side it is thus enough to let x run through elements in $T_{F,2}$ and elements in $\hat{w}T_{F,2}$, respectively ($\hat{w} \in N_{G_{F,2}}(T_{F,2})$ denotes a lift of the non-trivial element of the Weyl group of $\mathrm{SL}_2(k)$). Since $T_{F,2}$ normalises $U_{F,2}$, it is moreover enough to consider only $x = 1$ and $x = \hat{w}$. For $x = 1$ we get a term $\langle \rho|_{U_{F,2}}, \mathbf{1}\rangle$, and for $x = \hat{w}$ we get a term $\langle \rho|_{(U^-)^1_{F,2}}, \mathbf{1}\rangle$. The latter is always zero, since $\rho|_{(U^-)^1_{F,2}} = \psi_\beta|_{(U^-)^1_{F,2}} \neq \mathbf{1}$ for our choice of β. Hence we conclude that $\mathrm{Ind}_S^{G_{F,2}} \rho$ is contained in $\mathrm{Ind}_{U_{F,2}}^{G_{F,2}} \mathbf{1}$ if and only $\langle \rho|_{U_{F,2}}, \mathbf{1}\rangle = 1$. In particular, since there exist representations of S which are lifts of ψ_β and which are non-trivial on $U_{F,2}$, we see that there exist nilpotent representations which are not components of $\mathrm{Ind}_{U_{F,2}}^{G_{F,2}} \mathbf{1}$.

3.2 Inadequacy of the unramified varieties

We keep the assumption $\mathbf{G} = \mathrm{SL}_2$ until the end of this subsection. We will show that there exist nilpotent representations of $G_{F,2}$ which cannot be realised as components of the cohomology of varieties of the form $L^{-1}(xU_2)$, $L^{-1}(xU_2)/U_2 \cap xU_2x^{-1}$, or $X_2(x)$, for $x \in G_2$. More precisely, we show that the only nilpotent representations which can be realised in this way are the irreducible components of $\mathrm{Ind}_{U_{F,2}}^{G_{F,2}} \mathbf{1}$. As we saw above, these do not account for all the nilpotent representations of $G_{F,2}$.

Let $\varphi : G_2 \to G_2$ be the standard Frobenius endomorphism induced by the map which sends every matrix entry to its q^{th} power. Then $G_{F,2} = G_2^\varphi$, and we will use either of these ways of writing the group, depending on the context. Moreover, each of the subgroups T_2, B_2, U_2, and $(U^-)_2$ is φ-stable. We need a description of the double cosets $B_2\backslash G_2/B_2$. One checks directly that a set of representatives is given by

$$\left\{1,\ w := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ e := \begin{pmatrix} 1 & 0 \\ \varpi & 1 \end{pmatrix}\right\}.$$

Note that $e \in (U^-)^1_2$ and that for any $a \in (U^-)^1_2 - \{1\}$, we have $U_2 \cap aU_2a^{-1} = U_2^1$, which is an affine space. In this case, [7], 10.12 (ii) implies that $L^{-1}(aU_2) \sim L^{-1}(aU_2)/U_2^1$. Note also that $U_2 \cap wU_2w^{-1} = \{1\}$.

Proposition 3.4. *Let $x \in G_2$ be an arbitrary element. Then there exists an element $y \in \{1, w\} \cup (U^-)^1_2$ such that $L^{-1}(xU_2) \sim L^{-1}(yU_2)$.*

Proof. The elements 1 and w normalise T_2 so, by Lemma 3.3, for any element $x \in B_2$ we have $L^{-1}(xU_2) \sim L^{-1}(U_2)$, and for any $x \in B_2wB_2$ we have $L^{-1}(xU_2) \sim L^{-1}(wU_2)$.

In contrast, no element in B_2eB_2 normalises T_2. Assume that $x = utet'u'$, where $u, u' \in U_2$ and $t, t' \in T_2$. Then $L^{-1}(utet'u'U_2) \sim L^{-1}(U_2tet'U_2) \sim L^{-1}(tet'U_2)$, and by Lemma 3.2 we have $L^{-1}(tet'U_2) \sim L^{-1}(\varphi(\lambda)U_2\varphi(\lambda)^{-1})$, where $\lambda \in G_2$ is such that $L(\lambda) = tet'$. Since $tet' \in (U^-)^1_2T_2$ and the group $(U^-)^1_2$ is φ-stable, we can take $\lambda \in (U^-)^1_2T_2$, by the Lang-Steinberg theorem.

Writing $\lambda = vs$, with some $v \in (U^-)^1_2$ and $s \in T_2$, we get

$$L^{-1}(\varphi(\lambda)U_2\varphi(\lambda)^{-1}) = \\ L^{-1}(\varphi(vs)U_2\varphi(vs)^{-1}) = L^{-1}(\varphi(v)U_2\varphi(v)^{-1}) \sim L^{-1}(L(v)U_2).$$

Since the group $(U^-)^1_2$ is φ-stable, we have $L(v) = v^{-1}\varphi(v) \in (U^-)^1_2$. Hence, for every $x \in B_2eB_2$, we have $L^{-1}(xU_2) \sim L^{-1}(yU_2)$, for some $y \in (U^-)^1_2$. □

Theorem 3.5. *Let $y \in (U^-)^1_2$. Then $L^{-1}(yU_2) \sim \widetilde{X}_2(1)$, and hence*

$$H^*_c(L^{-1}(yU_2)) \cong \mathrm{Ind}_{U_2^\varphi}^{G_2^\varphi} \mathbf{1}$$

as G_2^φ-representations.

Proof. We use the observations from the end of Section 2. Consider the composition of the maps

$$\rho : L^{-1}(yU_2)/U^1_2 \xrightarrow{\rho_{2,1}} X_1(U_1) \longrightarrow G_1^\varphi/U_1^\varphi \cong G_2^\varphi/U_2^\varphi(G^1_2)^\varphi,$$

where the first map is the restriction of $\rho_{2,1} : G_2 \to G_1$, and the second map is given by $g \mapsto gU_1^\varphi$. Then ρ is clearly G_2^φ-equivariant. The fibre $f := \rho^{-1}(U_2^\varphi(G^1_2)^\varphi)$ over the trivial coset in $G_2^\varphi/U_2^\varphi(G^1_2)^\varphi$ is given by

$$f = \{um \in U_2G^1_2 \mid (um)^{-1}\varphi(um) \in yU_2\}/U^1_2.$$

Pick a $\lambda \in (U^-)^1_2$ such that $\lambda^{-1}\varphi(\lambda) = y$. Then the translation $x \mapsto x\lambda^{-1}$ induces a $U_2^\varphi(G^1_2)^\varphi$-equivariant isomorphism

$$f \xrightarrow{\sim} f\lambda^{-1} = \{um \in U_2G^1_2 \mid (um)^{-1}\varphi(um) \in \varphi(\lambda)U_2\varphi(\lambda)^{-1}\}/U^1_2.$$

We now observe that the group $\varphi(\lambda)U_2\varphi(\lambda)^{-1}$ is contained in $U_2T^1_2$. Thus, every element in $f\lambda^{-1}$ is φ-fixed up to right multiplication by some element in $U_2T^1_2$. Hence there is a map

$$\rho' : f\lambda^{-1} \longrightarrow (U_2G^1_2/U_2T^1_2)^\varphi \cong U_2^\varphi(G^1_2)^\varphi/U_2^\varphi(T^1_2)^\varphi, \qquad x \longmapsto xU_2^\varphi(G^1_2)^\varphi,$$

which is clearly $U_2^\varphi(G^1_2)^\varphi$-equivariant. Define f' to be the fibre of ρ' over the trivial coset. Then

$$f' = \{um \in U_2T^1_2 \mid (um)^{-1}\varphi(um) \in \varphi(\lambda)U_2\varphi(\lambda)^{-1}\}/U^1_2,$$

which has a left action of $U_2^\varphi(T^1_2)^\varphi$, and a right action of $(T^1_2)^\varphi$.

We now show that the $U_2^\varphi(T^1_2)^\varphi$-representation afforded by f' is isomorphic to $\mathrm{Ind}_{U_2^\varphi}^{U_2^\varphi(T^1_2)^\varphi} \mathbf{1}$. Define the variety

$$V = \{g \in U_2T^1_2 \mid g^{-1}\varphi(g) \in U_2\} = U_2(T^1_2)^\varphi.$$

This has a left action of $U_2^{\varphi}(T_2^1)^{\varphi}$ and a right action of U_2^{φ}. We have $V/U_2 \cong U_2^{\varphi}(T_2^1)^{\varphi}/U_2^{\varphi}$, so V affords the representation $\mathrm{Ind}_{U_2^{\varphi}}^{U_2^{\varphi}(T_2^1)^{\varphi}} \mathbf{1}$, that is

$$H_c^*(V) \cong \mathrm{Ind}_{U_2^{\varphi}}^{U_2^{\varphi}(T_2^1)^{\varphi}} \mathbf{1},$$

as $U_2^{\varphi}(T_2^1)^{\varphi}$-representations. Now, for every $u \in U_2$ there exists a $t_u \in T_2^1$ such that $ut_u \in f'$, and this t_u is unique up to multiplication by $(T_2^1)^{\varphi}$. Hence, by choosing such a t_{um} for each $um \in f'$, we can write each element in f' uniquely in the form $ut_u a$, where $u \in U_2^1$, $t_u \in T_2^1$, and $a \in (T_2^1)^{\varphi}$. Moreover, we may always choose the same t_u for all elements $vsus^{-1}$, where $v \in U_2^{\varphi}$ and $s \in (T_2^1)^{\varphi}$. Similarly, we may always choose t_u so that $\varphi^m(t_u) = t_{\varphi^m(u)}$, for all natural numbers $m \geq 1$. We can then define a bijective function

$$\eta : f' \longrightarrow V, \qquad ut_u a \longmapsto ua.$$

For $vs \in U_2^{\varphi}(T_2^1)^{\varphi}$ we have

$$\eta(vsut_u a) = \eta(v(sus^{-1}t_u sa)) = v(sus^{-1})sa = vsua,$$

so η is $U_2^{\varphi}(T_2^1)^{\varphi}$-equivariant. Let m be a natural number such that $\varphi^m(\lambda) = \lambda$. Then φ^m is a Frobenius endomorphism on f'. Furthermore, φ^m is clearly a Frobenius endomorphism which stabilises V. The bijection η satisfies

$$\eta(\varphi^m(ut_u a)) = \eta(\varphi^m(u)\varphi^m(t_u)a) = \eta(\varphi^m(u)t_{\varphi^m(u)}a) = \varphi^m(u)a = \varphi^m(ua),$$

so η commutes with the Frobenius endomorphisms φ^m on f' and V, respectively. By Lemma 2.1 f' and V afford the same $U_2^{\varphi}(T_2^1)^{\varphi}$-representation, and so

$$\begin{aligned} &H_c^*(L^{-1}(yU_2)) \\ &\quad\cong \mathrm{Ind}_{U_2^{\varphi}(G_2^1)^{\varphi}}^{G_2^{\varphi}} \mathrm{Ind}_{U_2^{\varphi}(T_2^1)^{\varphi}}^{U_2^{\varphi}(G_2^1)^{\varphi}} \mathrm{Ind}_{U_2^{\varphi}}^{U_2^{\varphi}(T_2^1)^{\varphi}} \mathbf{1} = \mathrm{Ind}_{U_2^{\varphi}}^{G_2^{\varphi}} \mathbf{1} \\ &\qquad \cong H_c^*(\widetilde{X}_2(1)). \end{aligned}$$

□

The representations realised by the variety $\widetilde{X}_2(1)$, that is, the irreducible components of $\mathrm{Ind}_{U_2^{\varphi}}^{G_2^{\varphi}} \mathbf{1}$, are just the irreducible components of the representations obtained by lifting characters of T_2^{φ} to B_2^{φ}, and inducing to G_2^{φ}. As we saw in the end of Section 3.1, not all of the nilpotent representations are of this form.

When F is a local field of characteristic p, Lusztig [20] has identified the representations realised by the variety $\widetilde{X}_2(w)$. In particular, none of them is of dimension $(q^2-1)/2$ when $p \neq 2$, or of dimension $q^2 - 1$ when $p = 2$, so in this case the variety $\widetilde{X}_2(w)$ does not realise any of the nilpotent representations of $G_2^{\varphi} = G_{F,2}$. Thus the results of this section imply that there are nilpotent representations of $\mathrm{SL}_2(\mathbb{F}_q[[\varpi]]/(\varpi^2))$ which are not realised in the cohomology of any of the varieties $L^{-1}(xU_2)$, or equivalently, the varieties $L^{-1}(xU_2)/U_2 \cap xU_2x^{-1}$, for $x \in G_2$.

Remark. It seems likely that Lusztig's result on the representations afforded by $\widetilde{X}_2(w)$ hold in any characteristic, in particular, that $\widetilde{X}_2(w)$ does not afford any nilpotent representation of $G_{F,2}$, for any non-archimedean local field F. More precisely, every irreducible representation of $G_{F,2}$ afforded by $\widetilde{X}_2(w)$ should be either non-primitive or cuspidal. Since the results in this section hold uniformly in any characteristic, this would imply the inadequacy of the varieties $L^{-1}(xU_2)$ also for the group $\mathrm{SL}_2(\mathbb{Z}/p^r\mathbb{Z})$.

As we remarked in the beginning of the section, the variety $L^{-1}(eU_2)/U_2^1$ is not a finite cover of $X_2(e)$, so the representations afforded by the latter are not necessarily all afforded by the former (as is the case for the covers $\widetilde{X}_r(\hat{w})$ of $X_r(\hat{w})$, for $\hat{w} \in N_{G_r}(T_r)$). It is thus a priori conceivable that $X_2(e)$ may yield further representations not obtainable by $L^{-1}(eU_2)$. The following result shows that this is not the case.

Proposition 3.6. *We have*

$$H_c^*(X_2(e)) = \left(\mathrm{Ind}_{B_2^\varphi (G_2^1)^\varphi}^{G_2^\varphi} \mathbf{1}\right) - \mathrm{Ind}_{B_2^\varphi}^{G_2^\varphi} \mathbf{1},$$

as virtual G_2^φ-representations.

Proof. Consider the composition of the maps

$$X_2(e) \xrightarrow{\rho_{2,1}} L^{-1}(B_1)/B_1 \xrightarrow{\sim} G_1^\varphi/B_1^\varphi \xrightarrow{\sim} G_2^\varphi/B_2^\varphi(G_2^1)^\varphi.$$

This gives a G_2^φ-equivariant map $X_2(e) \to G_2^\varphi/B_2^\varphi(G_2^1)^\varphi$. The fibre of the trivial coset under this map is

$$f := \{g \in B_2 G_2^1 \mid g^{-1}\varphi(g) \in B_2 e B_2\}/B_2.$$

Thus we have

$$H_c^*(X_2(e)) = \mathrm{Ind}_{B_2^\varphi (G_2^1)^\varphi}^{G_2^\varphi} H_c^*(f).$$

Now an element in $B_2^\varphi (G_2^1)^\varphi$ must lie in exactly one of the double cosets B_2 and $B_2 e B_2$. Hence

$$f \sqcup \{g \in B_2 G_2^1 \mid g^{-1}\varphi(g) \in B_2\}/B_2 = B_2 G_2^1/B_2.$$

Since $B_2 G_2^1/B_2 \cong G_2^1/B_2^1$ is an affine space, the G_2^φ-representation afforded by it is the trivial representation. Moreover, the variety

$$\{g \in B_2 G_2^1 \mid g^{-1}\varphi(g) \in B_2\}/B_2$$

is isomorphic to $B_2^\varphi (G_2^1)^\varphi / B_2^\varphi$, and so affords the representation $\mathrm{Ind}_{B_2^\varphi}^{B_2^\varphi (G_2^1)^\varphi} \mathbf{1}$. Putting these results together, we get

$$H_c^*(f \sqcup \{g \in B_2 G_2^1 \mid g^{-1}\varphi(g) \in B_2\}/B_2) = H_c^*(f) + \mathrm{Ind}_{B_2^\varphi}^{B_2^\varphi (G_2^1)^\varphi} \mathbf{1} = \mathbf{1},$$

whence the result. ∎

The irreducible components of the representation $\mathrm{Ind}_{B_2^{\varphi}(G_2^1)^{\varphi}}^{G_2^{\varphi}} \mathbf{1}$ are all non-primitive, since they have $(G_2^1)^{\varphi}$ in their respective kernels. Moreover, the irreducible components of $\mathrm{Ind}_{B_2^{\varphi}}^{G_2^{\varphi}} \mathbf{1}$ form a subset of the irreducible components of $\mathrm{Ind}_{U_2^{\varphi}}^{G_2^{\varphi}} \mathbf{1}$. Thus, the variety $X_2(e)$ does not afford any nilpotent representations of $G_2^{\varphi} = G_{F,2}$ which are not already afforded by $L^{-1}(eU_2)$.

4 Extended Deligne-Lusztig varieties

As before, Let F be an arbitrary local field with finite residue field $\mathbb{F}_q$. Let L_0 be a finite totally ramified Galois extension of F, and set $L = L_0^{\mathrm{ur}}$. Then L is a finite extension of F^{ur} (cf. [9], II 4), and thus L is a Henselian discrete valuation field with the same residue field as F^{ur}, namely $\overline{\mathbb{F}}_q$. We have the relation $\mathfrak{p}_F \mathcal{O}_L = \mathfrak{p}_L^e$, where $e = [L_0 : F]$ is the ramification index of L_0/F.

Restriction of automorphisms gives a map

$$\alpha : \mathrm{Gal}(L/F) \longrightarrow \mathrm{Gal}(F^{\mathrm{ur}}/F) \xrightarrow{\sim} \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \supset \mathbb{Z},$$

where the subgroup $\mathbb{Z}$ is generated by the Frobenius map $x \mapsto x^q$. The corresponding Frobenius element in $\mathrm{Gal}(F^{\mathrm{ur}}/F)$ is denoted by φ_F. Let $\Gamma = \Gamma(L/F)$ be the group $\alpha^{-1}(\mathbb{Z}) \subset \mathrm{Gal}(L/F)$. This is a relative variant of the Weil group and sits in the following commutative diagram.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Gal}(L/F^{\mathrm{ur}}) & \longrightarrow & \Gamma(L/F) & \longrightarrow & \langle \varphi_F \rangle & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Gal}(L/F^{\mathrm{ur}}) & \longrightarrow & \mathrm{Gal}(L/F) & \longrightarrow & \mathrm{Gal}(F^{\mathrm{ur}}/F) & \longrightarrow & 1 \\
 & & & & & \nwarrow & \downarrow \cong & & \\
 & & & & & & \mathrm{Gal}(L/L_0) & &
\end{array}$$

We see that $\varphi_{L_0} \in \mathrm{Gal}(L/L_0)$ defines an element in Γ which is not in $\mathrm{Gal}(L/F^{\mathrm{ur}})$. Hence Γ is generated by $\mathrm{Gal}(L/F^{\mathrm{ur}})$ together with the element φ_{L_0}. The group $\mathrm{Gal}(L_0/F)$ is naturally isomorphic to $\mathrm{Gal}(L/F^{\mathrm{ur}})$, and we shall identify elements in the former with their corresponding images in the latter.

From now on, let $\mathbf{G}$ be either GL_n or SL_n, viewed as group schemes over $\mathcal{O}_F$. Let $\mathbf{T}$ be the standard split maximal torus in $\mathbf{G}$. Let $\mathbf{B}$ be the upper-triangular Borel subgroup scheme of $\mathbf{G}$, and let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$.

Let $r \geq 1$ be a natural number. Every automorphism $\sigma \in \mathrm{Gal}(L/F)$ stabilises $\mathcal{O}_L$ and $\mathfrak{p}_L^r$, respectively (cf. [9], II Lemma 4.1). Therefore, each $\sigma \in \mathrm{Gal}(L/F)$ defines a morphism of $\mathcal{O}_F$-algebras $\sigma : \mathcal{O}_{L,r} \to \mathcal{O}_{L,r}$, and hence a homomorphism of groups $\sigma : \mathbf{G}(\mathcal{O}_{L,r}) \to \mathbf{G}(\mathcal{O}_{L,r})$. Moreover, $\mathcal{O}_{L,r}$ has the structure of algebraic ring (isomorphic to affine r-space over $\overline{\mathbb{F}}_q$), and each $\sigma \in \Gamma$ such that $\sigma \in \alpha^{-1}(\mathbb{Z}_{\geq 0})$ gives rise to an algebraic endomorphism of $\mathcal{O}_{L,r}$. Hence each $\sigma \in \mathrm{Gal}(L/F^{\mathrm{ur}})$ and each non-negative power of φ_{L_0} induces (via the canonical isomorphism $\mathbf{G}(\mathcal{O}_{L,r}) \cong G_{L,r}$) an endomorphism of the algebraic group $G_{L,r}$. For $\sigma \in \mathrm{Gal}(L/F^{\mathrm{ur}})$, the resulting endomorphism of $G_{L,r}$ is also denoted by σ. Furthermore, the Frobenius map $\varphi_{L_0} \in \mathrm{Gal}(L/L_0)$ induces a Frobenius endomorphism of the algebraic group $G_{L,r}$, which we denote by φ. It is clear that $T_{L,r}$, $B_{L,r}$, and $U_{L,r}$ are stable under φ and under each of the endomorphisms induced by $\sigma \in \mathrm{Gal}(L/F^{\mathrm{ur}})$.

In Section 3 the finite group $G_{F,r}$ was identified with the fixed points of G_r under a Frobenius map. However, this is not the only way to realise $G_{F,r}$ as a group of fixed points of a connected algebraic group. The following lemma and its corollary make this more precise for tamely ramified extensions. The following is an additive Hilbert 90 for powers of the maximal ideal $\mathfrak{p}_L$.

Lemma 4.1. *Suppose that L_0/F is tamely ramified. Then $\mathrm{Gal}(L_0/F)$ is cyclic. Let σ be a generator of $\mathrm{Gal}(L_0/F)$, $m \geq 1$ be a natural number, and $y \in \mathfrak{p}_{L_0}^m$ be an element such that $\mathrm{Tr}_{L_0/F}(y) = 0$. Then there exists an element $x \in \mathfrak{p}_{L_0}^m$ such that $x - \sigma(x) = y$.*

Proof. Since L_0/F is totally and tamely ramified, the Galois group $\mathrm{Gal}(L_0/F)$ is cyclic of order e (cf. [9], II 4.4). Tamely ramified extensions are characterised by the fact that Tr maps units to units. In particular $e = \mathrm{Tr}_{L_0/F}(1)$ is a unit in $\mathcal{O}_{L_0}$, and $\mathrm{Tr}_{L_0/F}(1/e) = 1$. Let

$$x = \sum_{n=1}^{e-1} \left(\sigma^n(1/e) \cdot \sum_{i=0}^{n-1} \sigma^i(y) \right).$$

Then $x \in \mathfrak{p}_L^m$, and it is easily verified that $x - \sigma(x) = y$. □

Corollary 4.2. *Suppose that L_0/F is tamely ramified, and let $r \geq 1$ be a natural number. Then $\mathcal{O}_{L,r}^{\Gamma} = \mathcal{O}_{F,r'}$, where $r' = [\frac{r-1}{e}] + 1$.*

Proof. Since L_0/F is totally and tamely ramified, it is cyclic, and we choose a generator σ of $\mathrm{Gal}(L_0/F)$. Following our convention, we also use σ to denote the corresponding generator of $\mathrm{Gal}(L/F^{\mathrm{ur}})$. Now Γ is generated by φ_{L_0} and σ and since $\mathcal{O}_{L,r}^{\varphi_{L_0}} = \mathcal{O}_{L_0,r}$, it is enough to show that $\mathcal{O}_{L_0,r}^{\sigma} = \mathcal{O}_{F,r'}$. It is well-known that $(\mathfrak{p}_{L_0}^r)^{\sigma} = \mathfrak{p}_{L_0}^r \cap \mathcal{O}_F = \mathfrak{p}_F^{r'}$, where $r' = [\frac{r-1}{e}] + 1$. The functor of σ-invariants is left exact, so we have an injection $\mathcal{O}_{F,r'} = \mathcal{O}_{L_0}^{\sigma}/(\mathfrak{p}_{L_0}^r)^{\sigma} \hookrightarrow \mathcal{O}_{L_0,r}^{\sigma}$. Lemma 4.1 shows that $H^1(L_0/F, \mathfrak{p}_{L_0}^m) = 0$, and so this injection is surjective, and this yields the result. □

Recall that a Bézout domain is an integral domain in which every finitely generated ideal is principal.

Lemma 4.3. *Let R be a Bézout domain, and let $x \in \mathrm{GL}_n(R)$ be an arbitrary element, where $n \geq 2$. Suppose that the characteristic polynomial of x splits into linear factors over R. Then there exists an element $\lambda \in \mathrm{SL}_n(R)$, such that $\lambda^{-1}x\lambda \in \mathbf{B}(R)$.*

Proof. Let $a_1 \in R$ be an eigenvalue of x with corresponding eigenvector $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in R^n$, so that $xv = a_1 v$. If $g \in \mathrm{GL}_n(R)$, then gv is obviously an eigenvector of $g^{-1}xg$. We claim that we can choose g such that gv has an entry equal to 1. Without loss of generality, we may assume that there exists two integers $1 \leq m, m' \leq n$, such that $\gcd(v_m, v_{m'}) = 1$. Then, since R is a Bézout domain, there exist elements $\alpha, \beta \in R$ such that

$$\alpha v_m + \beta v_{m'} = 1.$$

Let $g = (g_{ij})$ be the matrix such that $g_{mm} = \alpha$, $g_{mm'} = \beta$, $g_{m'm} = -v_{m'}$, $g_{mm'} = v_m$, $g_{ii} = 1$ for all $i \notin \{m, m'\}$, and all other entries equal to 0. We have $g \in \mathrm{SL}_n(R)$, and the m^{th} entry of gv equals 1, which proves the claim. This implies that there exists a matrix $\lambda_1 \in \mathrm{SL}_n(R)$ matrix whose first column is the vector gv. We then have

$$\lambda_1^{-1} g^{-1} x g \lambda_1 = \begin{pmatrix} * & * & \cdots & * \\ 0 & & & \\ \vdots & & x_1 & \\ 0 & & & \end{pmatrix},$$

where $x_1 \in \mathrm{GL}_{n-1}(R)$. We can now repeat the process by choosing an eigenvalue of x_1. Working inductively, we obtain an element $\lambda \in \mathrm{SL}_n(R)$ such that $\lambda^{-1}x\lambda \in \mathbf{B}(R)$. □

The above lemma shows in particular that for any $x \in \mathbf{G}(\mathcal{O}_{F^{\mathrm{ur}}})$, there exists a finite field extension L/F^{ur}, and an element $\lambda \in \mathbf{G}(\mathcal{O}_L)$ such that $\lambda^{-1}x\lambda \in \mathbf{B}(\mathcal{O}_L)$. Reducing modulo $\mathfrak{p}_L^r$ we see that for any $x \in G_{F,r'}$ with r' such that $G_{F,r'} \subseteq G_{L,r}$, there exists a $\lambda \in G_{L,r}$ such that $\lambda^{-1}x\lambda \in B_{L,r}$.

Recall that an element $x \in G_r$ is called *regular* if its centraliser $C_{G_r}(x)$ has minimal dimension (cf. [13] or [7], 14). Note that this is a more general definition than that given in [2], 12.2 (which coincides with the notion of regular semisimple).

Definition 4.4. An element in $\mathbf{G}(\mathcal{O}_{F^{\mathrm{ur}},r})$ is called *separable* if it has distinct eigenvalues. Similarly, an element in G_r is called *separable* if its corresponding element in $\mathbf{G}(\mathcal{O}_{F^{\mathrm{ur}},r})$ (via the canonical isomorphism $G_r \cong \mathbf{G}(\mathcal{O}_{F^{\mathrm{ur}},r})$) is separable. If $x \in G_r$ is a regular separable element, we call its centraliser $C_{G_r}(x)$ a *quasi-Cartan* subgroup (of G_r). Similarly, we call the finite group $C_{G_{F,r}}(x)$ a *quasi-Cartan* subgroup (of $G_{F,r}$).

Note that if $r = 1$, then an element is regular semisimple if and only if it is separable. In general, regular semisimple elements in G_r are separable, but there also exist unipotent regular separable elements.

From now on, let $x \in G_r$ be a regular separable element. Since x is regular we then have

$$C_{G_r}(x) = \mathcal{O}_{F^{\mathrm{ur}},r}[x] \cap G_r.$$

Let L/F^{ur} be a finite field extension and $r' \geq r$ a natural number such that $G_{r'}$ is a subgroup of $G_{L,r}$ and such that there exists an element $\lambda \in G_{L,r}$ such that $\lambda^{-1}x\lambda \in B_{L,r}$ (which is possible thanks to Lemma 4.3). From now on, let $r' = [\frac{r-1}{e}]+1$. Let Σ_0 be a set of generators of the finite group $\mathrm{Gal}(L/F^{\mathrm{ur}})$, and put $\Sigma := \{\varphi\} \cup \Sigma_0$. Notice that if L_0/F is tamely ramified, then Lemma 4.1 and Corollary 4.2 show that we can take Σ_0 to be a one-element set, and that $G_{L,r}^{\Sigma} = G_{L,r}^{\Gamma} = G_{r'}$.

A subgroup of $G_{L,r}$ conjugate to $B_{L,r}$ will be called a *strict Borel subgroup.* Strict Borel subgroups are solvable, but are not in general Borel subgroups of the algebraic group $G_{L,r}$. Since x is regular, we see that the group $C_{G_r}(x)$ lies in the strict Borel $\lambda B_{L,r}\lambda^{-1}$.

Lemma 4.5. *Assume that* $\mathbf{G}$ *is either* GL_n *or* SL_n. *Then strict Borel subgroups in* $G_{L,r}$ *are self-normalising, that is, if* $g \in G_{L,r}$ *and* $gB_{L,r}g^{-1} \subseteq B_{L,r}$, *then* $g \in B_{L,r}$.

Proof. It is sufficient to prove the assertion for the group $B_{L,r}$. In [18], Lemma 1.2, it is shown that $\mathbf{B}(R)$ is self-normalising in $\mathrm{GL}_n(R)$, when R is a finite local PIR. The same proof goes through for rings of the form $\mathcal{O}_{L,r}$, so the assertion holds for $\mathbf{G} = \mathrm{GL}_n$. Since for any ring R we have $\mathrm{GL}_n(R) = Z(R)\,\mathrm{SL}_n(R)$, where $Z(R)$ is the subgroup of scalar matrices, the corresponding assertion for $\mathbf{G} = \mathrm{SL}_n$ follows. It remains to use the isomorphisms $\mathbf{G}(\mathcal{O}_{L,r}) \cong G_{L,r}$ and $\mathbf{B}(\mathcal{O}_{L,r}) \cong B_{L,r}$. □

Lemma 4.6. *Let* G *be a connected algebraic group, and* $\varphi : G \to G$ *a Frobenius endomorphism, that is,* φ *is surjective and* G^{φ} *is finite. Then the corresponding Lang map* $L : G \to G$, $g \mapsto g^{-1}\varphi(g)$ *is an open and closed morphism.*

Proof. By the Lang-Steinberg theorem L is surjective, so it is in particular a dominant map of irreducible varieties. Let $W \subseteq G$ be a closed irreducible subset. Since the fibres of L are all of the form $G^{\varphi}x$, for $x \in G$, the map $L : L^{-1}(W) \to W$ is an orbit map. By [2], II 6.4, G^{φ} then acts transitively on the set of irreducible components of $L^{-1}(W)$, and hence they all have the same dimension, equal to the dimension of $G^{\varphi}\backslash L^{-1}(W) \cong W$. By [15], Theorem 4.5, the map L is thus open.

Now let $X \subseteq G$ be a closed subset. The set $G^{\varphi}X$ is then a closed subset which is a union of fibres. Hence

$$L(G - G^{\varphi}X) = L(G) - L(G^{\varphi}X) = G - L(X),$$

and since $G - X$ is open, and L is open, $L(X)$ is closed in G. □

Let $\mathcal{B}_{L,r}$ denote the set of strict Borel subgroups of $G_{L,r}$. Since $B_{L,r}$ is self-normalising in $G_{L,r}$, strict Borels are in one-to-one correspondence with points of the variety $X_{L,r} := G_{L,r}/B_{L,r}$. Consider the product $\prod_{\sigma\in\{1\}\cup\Sigma} X_{L,r}$, with $G_{L,r}$ acting diagonally. For $(B_\sigma)_{\sigma\in\{1\}\cup\Sigma} \in \prod_{\sigma\in\{1\}\cup\Sigma} X_{L,r}$, we thus have the corresponding $G_{L,r}$-orbit $G_{L,r}(B_\sigma)_{\sigma\in\{1\}\cup\Sigma}$.

Definition 4.7. We define the variety

$$\begin{aligned} &X^{\Sigma}_{L,r}(\lambda) = \{B \in \mathcal{B}_{L,r} \mid G_{L,r}(\sigma(B))_{\sigma\in\{1\}\cup\Sigma} = G_{L,r}(\sigma(\lambda B_{L,r}\lambda^{-1}))_{\sigma\in\{1\}\cup\Sigma}\} \\ &= \{B \in \mathcal{B}_{L,r} \mid h(\sigma(B))_{\sigma\in\{1\}\cup\Sigma} = (\sigma(\lambda B_{L,r}\lambda^{-1}))_{\sigma\in\{1\}\cup\Sigma} \text{ for some } h \in G_{L,r}\}. \end{aligned}$$

Identifying $\mathcal{B}_{L,r}$ with $X_{L,r}$ we can rewrite the variety as

$$\begin{aligned} &X^{\Sigma}_{L,r}(\lambda) \\ &= \{g \in G_{L,r} \mid \sigma(\lambda)^{-1}h\sigma(g) \in B_{L,r} \text{ for all } \sigma \in \{1\}\cup\Sigma \text{ and some } h \in G_{L,r}\}/B_{L,r} \\ &= \{g \in G_{L,r} \mid g^{-1}\sigma(g) \in b\lambda^{-1}\sigma(\lambda)B_{L,r} \text{ for all } \sigma \in \Sigma \text{ and some } b \in B_{L,r}\}/B_{L,r}, \end{aligned}$$

and by making the substitution $g \mapsto gb^{-1}$, we can normalise the defining relations so that

$$X^{\Sigma}_{L,r}(\lambda) = \{g \in G_{L,r} \mid g^{-1}\sigma(g) \in \lambda^{-1}\sigma(\lambda)B_{L,r}\ \forall\, \sigma \in \Sigma\}/B_{L,r}(\lambda),$$

where

$$B_{L,r}(\lambda) := \bigcap_{\sigma\in\{1\}\cup\Sigma} \lambda^{-1}\sigma(\lambda)B_{L,r}\sigma(\lambda)^{-1}\lambda.$$

From now on we will use this last model for $X^{\Sigma}_{L,r}(\lambda)$. The finite group $G^{\Sigma}_{L,r} = G^{\Gamma}_{L,r}$ acts on $X^{\Sigma}_{L,r}(\lambda)$ by left multiplication.

We would now like to define finite covers of the varieties $X^{\Sigma}_{L,r}(\lambda)$ in a way that naturally generalises the finite covers $\widetilde{X}_r(\hat{w})$, defined in the unramified case where $L = F^{\mathrm{ur}}$, and $\hat{w} \in N_{G_r}(T_r)$. In general, however, there does not seem to be any straightforward way to define an analogous cover of the whole of $X^{\Sigma}_{L,r}(\lambda)$, but only of a certain $G^{\Gamma}_{L,r}$-stable subvariety. For ease of notation, write ε for $\lambda^{-1}\varphi(\lambda)$. Let

$$A := \{\varepsilon^{-1}b\varepsilon\varphi(b)^{-1} \mid b \in B_{L,r}(\lambda)\}.$$

Clearly, A is the image of $B_{L,r}(\lambda)$ under the morphism $G_{L,r} \to G_{L,r}$ given by the map $g \mapsto \varepsilon^{-1}g\varepsilon\varphi(g)^{-1}$. Thus A is conjugate to the image of the map $g \mapsto g\varepsilon\varphi(g)^{-1}\varepsilon^{-1}$, which in turn is equal to the image of the map $g \mapsto g^{-1}\varepsilon\varphi(g)\varepsilon^{-1}$. This last map is the Lang map corresponding to the Frobenius endomorphism $g \mapsto \varepsilon\varphi(g)\varepsilon^{-1}$, so by Lemma 4.6, it sends $B_{L,r}(\lambda)$ to a closed set. Hence A is a closed subset of $G_{L,r}$.

Define the following subvariety of $X^{\Sigma}_{L,r}(\lambda)$, given by

$$X^{\Sigma}_{L,r}(\lambda, A) := \Big(\{g \in G_{L,r} \mid g^{-1}\varphi(g) \in \varepsilon A U_{L,r}\} \cap X^{\Sigma}_{L,r}(\lambda)\Big)/B_{L,r}(\lambda).$$

Note that $B_{L,r}(\lambda)$ acts on $\{g \in G_{L,r} \mid g^{-1}\varphi(g) \in \varepsilon A U_{L,r}\}$ by right multiplication, and that $G^{\Gamma}_{L,r}$ acts on $X^{\Sigma}_{L,r'}(\lambda, A)$ by left multiplication. Since $G^{\Gamma}_{L,r}$ and $B_{L,r}(\lambda)$ act on $X^{\Sigma}_{L,r}(\lambda)$ and $X^{\Sigma}_{L,r}(\lambda, A)$, the complement $X^{\Sigma}_{L,r}(\lambda) \setminus X^{\Sigma}_{L,r}(\lambda, A)$ is also stable under these actions. We can now normalise the defining relations in $X^{\Sigma}_{L,r}(\lambda, A)$ by using the action of $B_{L,r}(\lambda)$, so that

$$X^{\Sigma}_{L,r}(\lambda, A) = \Big(\{g \in G_{L,r} \mid g^{-1}\varphi(g) \in \varepsilon U_{L,r}\} \cap X^{\Sigma}_{L,r}(\lambda)\Big)/S(\lambda),$$

where

$$S(\lambda) := \{b \in B_{L,r}(\lambda) \mid \varepsilon^{-1}b^{-1}\varepsilon\varphi(b) \in U_{L,r}\}.$$

Using the fact that $B_{L,r}(\lambda) \subseteq B_{L,r}$ normalises $U_{L,r}$, it is easy to see that $S(\lambda)$ is a subgroup of $B_{L,r}(\lambda)$. Moreover, $S(\lambda)$ contains $U_{L,r} \cap \varepsilon U_{L,r}\varepsilon^{-1} \cap B_{L,r}(\lambda)$ and acts on $\{g \in G_{L,r} \mid g^{-1}\varphi(g) \in \varepsilon U_{L,r}\}$ by right multiplication. Let $S(\lambda)^0$ denote the connected component of $S(\lambda)$. We define the finite cover

$$\widetilde{X}^{\Sigma}_{L,r}(\lambda) := \Big(\{g \in G_{L,r} \mid g^{-1}\varphi(g) \in \varepsilon U_{L,r}\} \cap X^{\Sigma}_{L,r}(\lambda)\Big)/S(\lambda)^0 \longrightarrow X^{\Sigma}_{L,r}(\lambda, A).$$

We see that the finite group $S(\lambda)/S(\lambda)^0$ acts on $\widetilde{X}^{\Sigma}_{L,r}(\lambda)$. Together with the respective $G^{\Gamma}_{L,r}$-actions this clearly makes $\widetilde{X}^{\Sigma}_{L,r}(\lambda) \to X^{\Sigma}_{L,r}(\lambda, A)$ a $G^{\Gamma}_{L,r} \times S(\lambda)/S(\lambda)^0$-equivariant cover.

Remark. We call the varieties $X^{\Sigma}_{L,r}(\lambda)$ and the covers $\widetilde{X}^{\Sigma}_{L,r}(\lambda)$ *extended* Deligne-Lusztig varieties, for the following reasons. Firstly, the varieties typically correspond to a (non-trivial) extension of the maximal unramified extension. Secondly, the various groups involved are iterated extensions of groups over the corresponding residue fields. Thirdly, there are at least three other constructions which could be referred to as generalisations of (certain) Deligne-Lusztig varieties, neither of which is in the direction given here. One of these is the varieties of Deligne associated to elements in certain braid monoids (cf. [5]); another is the affine Deligne-Lusztig varieties of Kottwitz and Rapoport (cf. [27]), and the third is the varieties of Digne and Michel [8], defined with respect to not necessarily connected, reductive groups.

We close this section by showing that extended Deligne-Lusztig varieties are a natural generalisation of classical Deligne-Lusztig varieties as well as of the varieties which appear in [20] and [34] (in the case of general and special linear groups over finite local PIRs with their standard Frobenius maps φ).

Let $\mathbf{T}'$ be a maximal torus in $\mathbf{G} \times \mathcal{O}_{F^{\mathrm{ur}}}$ such that the group T'_r is φ-stable. Then $T'_r = C_{G_r}(x)$, for some regular semisimple element $x \in G^{\varphi}_r$, and by [34], 2 we have $T'_r = \lambda T_r \lambda^{-1}$ for some $\lambda \in G_r$. Hence λ is an element such that $\lambda^{-1}x\lambda \in T_r \subseteq B_r$, and the condition that T'_r be φ-stable implies that $\lambda^{-1}\varphi(\lambda) \in N_{G_r}(T_r)$. Let $\hat{w} := \lambda^{-1}\varphi(\lambda)$. Take $L_0 = F$ (i.e., $L = F^{\mathrm{ur}}$), $r' = r$, so that $\Gamma = \langle\varphi\rangle$, and $\Sigma = \{\varphi\}$. The resulting extended Deligne-Lusztig variety attached to this data is

$$X^{\{\varphi\}}_{F^{\mathrm{ur}},r}(\lambda) = \{g \in G_r \mid g^{-1}\varphi(g) \in \hat{w}B_r\}/(B_r \cap \hat{w}B_r\hat{w}^{-1}),$$

and since $\hat{w}$ normalises T_r it follows that $B_r(\lambda) = T_r(U_r \cap \hat{w}U_r\hat{w}^{-1})$, and the Lang-Steinberg theorem implies that $A \supseteq T_r$. Hence $X^{\{\varphi\}}_{F^{\mathrm{ur}},r}(\lambda, A) = X^{\{\varphi\}}_{F^{\mathrm{ur}},r}(\lambda)$. Furthermore, we have

$$\begin{aligned} S(\lambda) = \{tu \in T_r(U_r \cap \hat{w}U_r\hat{w}^{-1}) \mid \hat{w}^{-1}u^{-1}t^{-1}\hat{w}\varphi(tu) \in U_r\} \\ = \{tu \in T_r(U_r \cap \hat{w}U_r\hat{w}^{-1}) \mid \hat{w}t^{-1}\hat{w}\varphi(t) \in U_r\} \\ = \{t \in T_r \mid \hat{w}t^{-1}\hat{w}\varphi(t) = 1\}(U_r \cap \hat{w}U_r\hat{w}^{-1}), \end{aligned}$$

and so $S(\lambda)^0 = U_r \cap \hat{w}U_r\hat{w}^{-1}$ and $S(\lambda)/S(\lambda)^0 \cong \{t \in T_r \mid \hat{w}t^{-1}\hat{w}\varphi(t) = 1\}$. The corresponding cover is

$$\widetilde{X}^{\{\varphi\}}_{F^{\mathrm{ur}},r}(\lambda) = \{g \in G_r \mid g^{-1}\varphi(g) \in \hat{w}U_r\}/(U_r \cap \hat{w}U_r\hat{w}^{-1}),$$

and hence $X^{\{\varphi\}}_{F^{\mathrm{ur}},r}(\lambda) = X_r(\hat{w})$ and $\widetilde{X}^{\{\varphi\}}_{F^{\mathrm{ur}},r}(\lambda) = \widetilde{X}_r(\hat{w})$ are the varieties we considered in Section 3. We thus see that the classical Deligne-Lusztig varieties as well as the generalisations in [20] and [34] (in the case of general or special linear groups over finite local PIRs with their standard Frobenius maps φ) appear as special cases of the construction of extended Deligne-Lusztig varieties given in this section.

5 Extended Deligne-Lusztig varieties for GL_2 and SL_2

Throughout this section $\mathbf{G}$ will denote either of the groups GL_2 or SL_2, over $\mathcal{O}_F$. The subgroups $\mathbf{T}$, $\mathbf{B}$, and $\mathbf{U}$ of $\mathbf{G}$ are the same as in Section 4. As in the preceding section we treat the two types of groups simultaneously in a uniform way. Assume that F is a local function field (i.e., $\operatorname{char} F = p$). Assume also that F has residue characteristic different from 2. In this section we will study extended Deligne-Lusztig varieties for groups of the form $G_{F,2}$.

Let ζ denote an arbitrary fixed non-square unit in $\mathcal{O}_{F,2}$. In $G_{F,2}$ the four distinct conjugacy classes of quasi-Cartans are given by the following representatives:

$$\begin{aligned} &T_{F,2}, \\ &C_{G_{F,2}}\begin{pmatrix} 0 & 1 \\ \zeta & 0 \end{pmatrix} = \left\{\begin{pmatrix} a & b \\ \zeta b & a \end{pmatrix}\right\} \cap G_{F,2}, \\ &C_{G_{F,2}}\begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} = \left\{\begin{pmatrix} a & b \\ \varpi b & a \end{pmatrix}\right\} \cap G_{F,2}, \\ &C_{G_{F,2}}\begin{pmatrix} 0 & 1 \\ \zeta\varpi & 0 \end{pmatrix} = \left\{\begin{pmatrix} a & b \\ \zeta\varpi b & a \end{pmatrix}\right\} \cap G_{F,2}. \end{aligned}$$

The first two of these quasi-Cartans are *unramified* in the sense that each of them is the $\mathcal{O}_{F,2}$-points of some maximal torus of the group scheme $\mathbf{G}$. They are also unramified in the sense that they can be brought into triangular form over $\mathcal{O}_{F^{\mathrm{ur}},2}$, that is, there exists a $\lambda \in G_2$ such that $\lambda^{-1} C_{G_{F,2}} \begin{pmatrix} 0 & 1 \\ \zeta & 0 \end{pmatrix} \lambda \subseteq B_2$ (for $T_{F,2}$ this is a trivial fact). For the maximal torus $T_{F,2}$, we can take $\lambda = 1$, and this gives rise to the variety $X_2(1)$. Each λ that triangulises $C_{G_{F,2}} \begin{pmatrix} 0 & 1 \\ \zeta & 0 \end{pmatrix}$ gives rise to the variety $X_2(\lambda) = X_2(\hat{w})$, where w is the non-trivial Weyl group element in G_1. Now the cover $\widetilde{X}_2(\lambda)$ of $X_2(\lambda)$ depends on λ, that is, on the choice of strict Borel subgroup containing the Cartan subgroup in question. However, it is known that the possible finite covers of $X_2(1)$ and $X_2(\hat{w})$ of the type we are considering all give rise to equivalent representations $R_{\mathbf{T},\theta}$ in their cohomology (cf. [34], Corollary 3.4).

We will refer to the last two of the above quasi-Cartans as *ramified.* We now attach extended Deligne-Lusztig varieties and corresponding representations also to the ramified quasi-Cartans. Let $L_0 = F(\sqrt{\varpi})$ be one of the two ramified quadratic extensions of F (recall that $p \neq 2$, so we have only tame ramification). Then $L = L_0^{\mathrm{ur}}$ is independent of the choice of ramified quadratic extension of F. The group Γ is generated by the Frobenius φ_{L_0} together with an involution $\sigma \in \mathrm{Gal}(L/F^{\mathrm{ur}})$, so we take $\Sigma = \{\varphi, \sigma\}$. Let $r = 3$, so that $\mathcal{O}_{L,3}^{\Gamma} = \mathcal{O}_{L,3}^{\Sigma} = \mathcal{O}_{F,2}$. We then have $G_{L,3}^{\Gamma} = G_{F,2}$. Define the following elements of $\mathbf{G}(\mathcal{O}_{L,3})$:

$$\lambda = \begin{pmatrix} 1 & 0 \\ \sqrt{\varpi} & 1 \end{pmatrix}, \qquad \mu = \begin{pmatrix} 1 & 0 \\ \sqrt{\zeta\varpi} & 1 \end{pmatrix}.$$

Then we clearly have

$$\lambda^{-1} C_{G_2} \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} \lambda \subseteq \mathbf{B}(\mathcal{O}_{L,3}), \qquad \mu^{-1} C_{G_2} \begin{pmatrix} 0 & 1 \\ \zeta\varpi & 0 \end{pmatrix} \mu \subseteq \mathbf{B}(\mathcal{O}_{L,3}).$$

This defines the associated extended Deligne-Lusztig varieties

$$X_{L,3}^{\Sigma}(\lambda) = \{g \in G_{L,3} \mid g^{-1}\varphi(g) \in B_{L,3},\ g^{-1}\sigma(g) \in \lambda^{-1}\sigma(\lambda) B_{L,3}\}/B_{L,3}(\lambda),$$
$$X_{L,3}^{\Sigma}(\mu) = \{g \in G_{L,3} \mid g^{-1}\varphi(g) \in \mu^{-1}\varphi(\mu) B_{L,3},\ g^{-1}\sigma(g) \in \mu^{-1}\sigma(\mu) B_{L,3}\}/B_{L,3}(\mu),$$

(note that $\varphi(\lambda) = \lambda$, and that $\varphi(\mu) = \sigma(\mu) = \mu^{-1}$).

The corresponding covers are given by

$$\widetilde{X}_{L,3}^{\Sigma}(\lambda) = \{g \in G_{L,3} \mid g^{-1}\varphi(g) \in U_{L,3},\ g^{-1}\sigma(g) \in \lambda^{-1}\sigma(\lambda) B_{L,3}\}/S(\lambda)^0,$$
$$\widetilde{X}_{L,3}^{\Sigma}(\mu) = \{g \in G_{L,3} \mid g^{-1}\varphi(g) \in \mu^{-1}\varphi(\mu) U_{L,3},\ g^{-1}\sigma(g) \in \mu^{-1}\sigma(\mu) B_{L,3}\}/S(\mu)^0,$$

where

$$S(\lambda) = \{b \in B_{L,r}(\lambda) \mid b^{-1}\varphi(b) \in U_{L,r}\},$$
$$S(\mu) = \{b \in B_{L,r}(\lambda) \mid \varphi(\mu)^{-1}\mu b^{-1}\mu^{-1}\varphi(\mu)\varphi(b) \in U_{L,r}\}.$$

Theorem 5.1. *Let $\mathbf{Z}$ be the centre of $\mathbf{G}$. Then there exists a $G_{L,3}^{\Sigma}$-equivariant isomorphism*

$$\widetilde{X}_{L,3}^{\Sigma}(\lambda)/(Z_{L,3}^1)^{\varphi} \cong G_{L,3}^{\Sigma}/(Z_{L,3}^1)^{\Sigma}(U_{L,3}^1)^{\Sigma}.$$

Proof. We begin by determining $S(\lambda)$ explicitly. For simplicity we shall write e for $\lambda^{-1}\sigma(\lambda)$, in what follows. First consider $B_{L,r}(\lambda) = B_{L,r} \cap eB_{L,r}e^{-1}$. We write elements in $\mathcal{O}_{L,3}$ in the form $a_0 + a_1\sqrt{\varpi} + a_2\varpi$, where $a_i \in \overline{\mathbb{F}}_q$. We then have

$$\varphi(a_0 + a_1\sqrt{\varpi} + a_2\varpi) = a_0^q + a_1^q\sqrt{\varpi} + a_2^q\varpi,$$
$$\sigma(a_0 + a_1\sqrt{\varpi} + a_2\varpi) = a_0 - a_1\sqrt{\varpi} + a_2\varpi.$$

Note in particular that φ and σ commute. As usual, we identify subgroups of $\mathbf{G}(\mathcal{O}_{L,3})$ with their corresponding subgroups in $G_{L,3}$. Then

$$B_{L,r}(\lambda) = \left\{ \begin{pmatrix} a_0 + a_1\sqrt{\varpi} + a_2\varpi & \frac{d_1-a_1}{2} + b_1\sqrt{\varpi} + b_2\varpi \\ 0 & a_0 + d_1\sqrt{\varpi} + d_2\varpi \end{pmatrix} \mid a_i, b_i \in \overline{\mathbb{F}}_q \right\} \cap G_{L,r},$$

and so

$$S(\lambda) = \left\{ \begin{pmatrix} a_0 + a_1\sqrt{\varpi} + a_2\varpi & \frac{d_1-a_1}{2} + b_1\sqrt{\varpi} + b_2\varpi \\ 0 & a_0 + d_1\sqrt{\varpi} + d_2\varpi \end{pmatrix} \mid a_i^q = a_i,\ d_i^q = d_i \right\} \cap G_{L,r}.$$

Hence, the connected component of $S(\lambda)$ is

$$S(\lambda)^0 = U^1_{L,3},$$

and $S(\lambda)/S(\lambda)^0 \cong Z^{\varphi}_{L,1}(T^1_{L,3})^{\varphi} = Z^{\varphi}_1(T^1_{L,3})^{\varphi}$.

Let $Y := \{g \in G_{L,3} \mid g^{-1}\varphi(g) \in U_{L,3},\ g^{-1}\sigma(g) \in eB_{L,3}\}$, so that $\widetilde{X}^{\Sigma}_{L,3}(\lambda) = Y/U^1_{L,3}$. For $g \in Y$ we have $g^{-1}\varphi(g) = u$, and $g^{-1}\sigma(g) = eb$, for some $u \in U_{L,3}$, $b \in B_{L,3}$. The commutativity of φ and σ yields $\sigma(gu) = \varphi(geb)$, and since $\varphi(e) = e$ this implies

$$eb\sigma(u) = ue\varphi(b).$$

Hence we obtain $e^{-1}ue \in B_{L,3}$, so that $u \in U_{L,3} \cap eB_{L,3}e^{-1} = U^1_{L,3}$. We thus have $Y = \{g \in G_{L,3} \mid g^{-1}\varphi(g) \in U^1_{L,3},\ g^{-1}\sigma(g) \in eB_{L,3}\}$. If we set

$$Y' := \{g \in G^{\varphi}_{L,3} \mid g^{-1}\sigma(g) \in eB_{L,3}\}/(Z^1_{L,3})^{\varphi}(U^1_{L,3})^{\varphi},$$

we then have a natural $G^{\Sigma}_{L,3}$-equivariant isomorphism

$$\widetilde{X}^{\Sigma}_{L,3}(\lambda)/(Z^1_{L,3})^{\varphi} = Y/(Z^1_{L,3})^{\varphi}U^1_{L,3} \xrightarrow{\sim} Y'.$$

Now the translation map $g \mapsto g\lambda^{-1}$ is an equivariant isomorphism $Y' \xrightarrow{\sim} Y'\lambda^{-1}$, and we have

$$Y'\lambda^{-1} = \{g \in G^{\varphi}_{L,3} \mid g^{-1}\sigma(g) \in \sigma(\lambda)B_{L,3}\sigma(\lambda)^{-1}\}/(Z^1_{L,3})^{\varphi}\lambda(U^1_{L,3})^{\varphi}\lambda^{-1}.$$

If $g \in Y'\lambda^{-1}$, then $g^{-1}\sigma(g) \in \sigma(\lambda)B_{L,3}\sigma(\lambda)^{-1}$, and we then also have $g^{-1}\sigma(g) \in \lambda B_{L,3}\lambda^{-1}$, since σ has order 2. Therefore $g^{-1}\sigma(g) \in \sigma(\lambda)B_{L,3}\sigma(\lambda)^{-1} \cap \lambda B_{L,3}\lambda^{-1}$, which is equivalent to

$$\lambda^{-1}g^{-1}\sigma(g)\lambda \in eB_{L,3}e^{-1} \cap B_{L,3} = B_{L,3}(\lambda).$$

We thus have $g^{-1}\sigma(g) \in \lambda B_{L,3}(\lambda)\lambda^{-1}$. Now, the image of the map $L_\sigma : G_{L,3} \to G_{L,3}$ given by $g \mapsto g^{-1}\sigma(g)$ clearly lies in $G^1_{L,3}$. Thus

$$\begin{aligned} & g^{-1}\sigma(g) \in \lambda B_{L,3}(\lambda)\lambda^{-1} \cap G^1_{L,3} \\ & \quad = \lambda \left\{ \begin{pmatrix} 1 + a_1\sqrt{\varpi} + a_2\varpi & b_1\sqrt{\varpi} + b_2\varpi \\ 0 & 1 + a_1\sqrt{\varpi} + d_2\varpi \end{pmatrix} \mid a_i, b_i \in \overline{\mathbb{F}}_q \right\} \lambda^{-1} \cap G^\varphi_{L,3}, \end{aligned}$$

and since λ normalises the above set of matrices, we get

$$\begin{aligned} g^{-1}\sigma(g) \in & \left\{ \begin{pmatrix} 1 + a_1\sqrt{\varpi} + a_2\varpi & b_1\sqrt{\varpi} + b_2\varpi \\ 0 & 1 + a_1\sqrt{\varpi} + d_2\varpi \end{pmatrix} \mid a_i, b_i \in \overline{\mathbb{F}}_q \right\} \cap G^\varphi_{L,3} \\ & \qquad = (Z^1_{L,3})^\varphi (T^2_{L,3})^\varphi (U^1_{L,3})^\varphi. \end{aligned}$$

Now we can obviously replace the relation $g^{-1}\sigma(g) \in (Z^1_{L,3})^\varphi (T^2_{L,3})^\varphi (U^1_{L,3})^\varphi$ by $g^{-1}\sigma(g) \in (Z^1_{L,3})^\varphi (T^2_{L,3})^\varphi (U^1_{L,3})^\varphi \cap L_\sigma(G^\varphi_{L,3})$, without loss of generality. We thus have

$$\begin{aligned} & Y'\lambda^{-1} \\ & = \{g \in G^\varphi_{L,3} \mid g^{-1}\sigma(g) \in (Z^1_{L,3})^\varphi (T^2_{L,3})^\varphi (U^1_{L,3})^\varphi \cap L_\sigma(G^\varphi_{L,3})\}/(Z^1_{L,3})^\varphi \lambda (U^1_{L,3})^\varphi \lambda^{-1}. \end{aligned}$$

One shows by direct computation that

$$L_\sigma((Z^1_{L,3})^\varphi \lambda (U^1_{L,3})^\varphi \lambda^{-1}) \supseteq (Z^1_{L,3})^\varphi (T^2_{L,3})^\varphi (U^1_{L,3})^\varphi \cap L_\sigma(G^\varphi_{L,3}).$$

This implies that there is a natural equivariant isomorphism

$$Y'\lambda^{-1} \xrightarrow{\sim} G^\Sigma_{L,3}/((Z^1_{L,3})^\varphi \lambda (U^1_{L,3})^\varphi \lambda^{-1})^\Sigma = G^\Sigma_{L,3}/(Z^1_{L,3})^\Sigma (U^1_{L,3})^\Sigma = G_{F,2}/Z^1_{F,2}U^1_{F,2}.$$

Since $\widetilde{X}^\Sigma_{L,3}(\lambda)/(Z^1_{L,3})^\varphi \cong Y'\lambda^{-1}$, the theorem is proved. □

The above theorem, together with [7], 10.10 (i) shows that the variety $\widetilde{X}^\Sigma_{L,3}(\lambda)$ affords the representation

$$\mathrm{Ind}^{G_{F,2}}_{Z^1_{F,2}U^1_{F,2}} \mathbf{1}$$

as a subrepresentation of its cohomology. In particular, for $\mathbf{G} = \mathrm{SL}_2$, we have $Z^1_{F,2} = \{1\}$ (using $p \neq 2$). Moreover, it is easy to show that for $\mathbf{G} = \mathrm{GL}_2$, each nilpotent representation of $\mathrm{GL}_2(\mathcal{O}_{F,2})$ is an irreducible constituent of $\mathrm{Ind}^{G_{F,2}}_{B^1_{F,2}} \mathbf{1}$ (cf. [12], Lemma 2.12; note that we have defined nilpotent representations to be primitive). Thus $\widetilde{X}^\Sigma_{L,3}(\lambda)$ affords in particular all the nilpotent representations of $G_{F,2}$, both for $\mathbf{G} = \mathrm{SL}_2$ and $\mathbf{G} = \mathrm{GL}_2$. Together with the results of Lusztig [20], Section 3, this proves that every irreducible representation of $\mathrm{SL}_2(\mathbb{F}_q[[\varpi]]/(\varpi^2))$, with p odd, appears in the cohomology of some extended Deligne-Lusztig variety attached to a (possibly ramified) quasi-Cartan subgroup.

6 Further directions

In the proof of Theorem 5.1, the hypothesis that F be a function field was only used to calculate the explicit form of the various groups involved, and the image of L_σ. It is therefore likely that the argument can be extended to any non-archimedean local field F with $p \neq 2$, using similar methods. Furthermore, the question of whether the action of the finite group $S(\lambda)/S(\lambda)^0$ on $\widetilde{X}^{\Sigma}_{L,3}(\lambda)$ can be used to decompose $\mathrm{Ind}^{G_{F,2}}_{Z^1_{F,2}U^1_{F,2}} \mathbf{1}$ into irreducible components, remains open. However, the techniques used in the proof of Theorem 5.1 should prove useful for answering this. Provided Lusztig's computations in [20], Section 3 could be carried out for GL_2, it would follow from the results of this paper that every irreducible representation of $\mathrm{GL}_2(\mathbb{F}_q[[\varpi]]/(\varpi^2))$, with p odd, is realised by an extended Deligne-Lusztig variety.

A natural problem is to generalise the construction of extended Deligne-Lusztig varieties to reductive group schemes $\mathbf{G}$ over $\mathcal{O}_F$ other than GL_n or SL_n. The ingredients required for such a generalisation are as follows. First, one needs a generalisation of Lemma 4.5 to any $\mathbf{G}$. This has recently been given in [32]. Moreover, one would need the result that any quasi-Cartan is contained in a strict Borel subgroup of some $G_{L,r}$, which requires a version of Lemma 4.3 for a Borel subgroup of $\mathbf{G}$.

It is also a natural question to ask whether our construction can be extended to the wildly ramified case. When L/F is tamely ramified, we have shown that $G^{\Sigma}_{L,r} = G_{F,r'}$, but in the wildly ramified case this may no longer hold. The difficulties in the wildly ramified case are perhaps a reflection of the fact that the representation theory of the p-adic group $\mathbf{G}(F)$ is radically different in the wildly ramified case. In particular, one cannot expect in this case that all the interesting representations are parametrised in a straightforward way by data attached to maximal tori. Our present construction can thus be seen as dealing efficiently only with the cases where L/F is tamely ramified. It should however be noted that the only obstacle to defining extended Deligne-Lusztig varieties in the wildly ramified case it due to the problem of descending from $G_{L,r}$ to $G_{F,r'}$ by taking fixed-points. This is therefore mainly a problem about Galois theoretic properties of finite ring extensions. To go further in the wildly ramified case, it seems that one has to consider either elements in $\mathrm{Aut}_{\mathcal{O}_{F,r'}}(\mathcal{O}_{L,r})$ other than those coming from elements in $\mathrm{Gal}(L/F)$, or a larger field extension E/L, such that E/F is tamely ramified.

A fundamental result of Deligne and Lusztig (cf. [6], Corollary 7.7) is that every irreducible representation of G_1^{φ} appears in the l-adic cohomology of some variety $\widetilde{X}_1(\hat{w})$. An important question is whether something similar holds for the groups $G^{\varphi}_{r'} = G^{\Sigma}_{L,r}$, with respect to the extended Deligne-Lusztig varieties $\widetilde{X}^{\Sigma}_{L,r}(\lambda)$. Some aspects of the representation theory of the groups $\mathrm{GL}_n(\mathcal{O}_F)$ are analogous to the representation theory of the p-adic group $\mathrm{GL}_n(F)$. In particular, the construction of tamely ramified supercuspidal representations via certain characters of maximal tori, due to Howe [14], provides some of the motivation for attaching extended Deligne-Lusztig varieties to quasi-Cartans. Given this analogy, and the results obtained for nilpotent representations in Section 5, we state the following open problem:

> Suppose that n is prime to p. Is it true that any irreducible representation of $\mathrm{GL}_n(\mathcal{O}_{F,r'})$ which is a type for a supercuspidal representation of $\mathrm{GL}_n(F)$, appears in the l-adic cohomology of some extended Deligne-Lusztig variety $\widetilde{X}^{\Sigma}_{L,r}(\lambda)$?

Here $r' = [\frac{r-1}{e}]+1$, with $e = e(L/F^{\mathrm{ur}})$, as before. For the definition of types, see [3] and [4]. In particular, any depth zero supercuspidal type on $\mathrm{GL}_n(\mathcal{O}_F)$ factors through $\mathrm{GL}_n(k)$, corresponds to an unramified maximal torus, and is realised in the cohomology of some variety $\widetilde{X}_1(\hat{w})$, by the result of Deligne and Lusztig mentioned above. Moreover, the results in Section 5 show that every nilpotent representation of $\mathrm{GL}_2(\mathcal{O}_{F,2})$, for F a function field, is realised by some $\widetilde{X}^{\Sigma}_{L,r}(\lambda)$. Thus, the answer to the question is affirmative at least as far as nilpotent types on $\mathrm{GL}_2(\mathcal{O}_{F,2})$ are concerned.

It is interesting to ask about the possible connections between the constructions in this paper, and the theory of character sheaves. In [21], Lusztig discusses, among other things, the possibility of defining character sheaves on G_r, where F is a function field, and $\mathbf{G}$ is a reductive group scheme over k_F. The conjecture in [21], 8 predicts that there is a theory of character sheaves on G_r for generic principal series representations (i.e., those that correspond to regular characters of a split unramified Cartan). However, Lusztig remarks that one cannot expect to have a complete theory of character sheaves on G_r, citing the irreducible representations of dimension q^2-1 of $G_{F,2}$ (for $\mathbf{G} = \mathrm{GL}_2$, F a function field) as a reason for this. Note that these representations are nilpotent. By the results in Section 3.2 for the closely related case where $\mathbf{G} = \mathrm{SL}_2$, one may indeed expect that the nilpotent representations cannot all be accounted for by character sheaves on G_r. One of the principal aims of this paper has been to demonstrate that the correct algebraic groups for constructing nilpotent representations of $G^{\varphi}_{r'} = G^{\Sigma}_{L,r}$ for $\mathbf{G} = \mathrm{GL}_2$ or $\mathbf{G} = \mathrm{SL}_2$ in the tamely ramified case, are not the "unramified" groups $G_{r'}$, but groups of the form $G_{L,r}$, where L is a finite non-trivial extension of F^{ur}. One may therefore ask whether there exists a theory of character sheaves on the groups $G_{L,r}$, pertaining to (some of) the representations which do not correspond to character sheaves on groups of the form $G_{r'}$.

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