ON THE CONNECTION BETWEEN PROBABILITY BOXES AND POSSIBILITY MEASURES

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ABSTRACT. We explore the relationship between possibility measures (supremum preserving normed measures) and p-boxes (pairs of cumulative distribution functions) on totally preordered spaces, extending earlier work in this direction by De Cooman and Aeyels, among others. We start by demonstrating that only those p-boxes who have 0–1-valued lower or upper cumulative distribution function can be possibility measures, and we derive expressions for their natural extension in this case. Next, we establish necessary and sufficient conditions for a p-box to be a possibility measure. Finally, we show that almost every possibility measure can be modelled by a p-box, simply by ordering elements by increasing possibility. Whence, any techniques for p-boxes can be readily applied to possibility measures. We demonstrate this by deriving joint possibility measures from marginals, under varying assumptions of independence, using a technique known for p-boxes. Doing so, we arrive at a new rule of combination for possibility measures, for the independent case.

1. INTRODUCTION

Firstly, possibility measures are supremum preserving set functions, and were introduced in fuzzy set theory [39], although earlier appearances exist [28, 20]. Because of their computational simplicity, possibility measures are widely applied in many fields, including data analysis [32], diagnosis [4], cased-based reasoning [18], and psychology [27]. This paper concerns quantitative possibility theory [13], where degrees of possibility range in the unit interval. Interpretations abound [11]: we can see them as likelihood functions [12], as particular cases of plausibility measures [29, 30], as extreme probability distributions [31], or as upper probabilities [37, 6]. The upper probability interpretation fits our purpose best, whence is assumed herein.

Secondly, probability boxes [14, 15], or p-boxes for short, are pairs of lower and upper cumulative distribution functions, and are often used in risk and safety studies, in which they play an essential role. P-boxes have been connected to info-gap theory [16], random sets [19, 26], and also, partly, to possibility measures [1, 6]. P-boxes can be defined on arbitrary finite spaces [9], and, more generally, even on arbitrarily totally pre-ordered spaces [34]—we will use this extensively.

This paper aims to consolidate the connection between possibility measures and p-boxes, making as few assumptions as possible. We prove that almost every possibility measure can be interpreted as a p-box, simply by ordering elements by increasing possibility, whence, p-boxes effectively generalize possibility measures. Conversely, we provide necessary and sufficient conditions for a p-box to be a possibility measure, whence, providing conditions under which the more efficient mathematical machinery of possibility measures is applicable to p-boxes.

To study this connection, we use imprecise probabilities [36], because both possibility measures and p-boxes are particular cases of imprecise probabilities. Possibility measures are explored as imprecise probabilities in [37, 6, 23], and p-boxes were studied as imprecise probabilities briefly in [36, Section 4.6.6] and [33], and in much more detail in [34].

The paper is organised as follows: in Section 2, we give the basics of the behavioural theory of imprecise probabilities, and recall some facts about p-boxes and possibility measures; in Section 3, we first determine necessary and sufficient conditions for a p-box to be maximum preserving, before determining in Section 4 necessary and sufficient conditions for a p-box to be a possibility measure; in Section 5, we show that almost any possibility measure can be seen as particular p-box, and that many p-boxes can be seen as a couple of possibility measures; some special cases are detailed in Section 6. Finally, in Section 7 we apply the work on multivariate p-boxes from [34] to derive multivariate possibility measures from given marginals, and in Section 8 we give a number of additional comments and remarks.

2. PRELIMINARIES

2.1. **Imprecise Probabilities.** We start with a brief introduction to imprecise probabilities (see [2, 38, 36, 22] for more details). Because possibility measures are interpretable as upper probabilities, we start out with those, instead of lower probabilities—the resulting theory is equivalent.

Let Ω be the possibility space. A subset of Ω is called an *event*. Denote the set of all events by $\wp(\Omega)$, and the set of all finitely additive probabilities on $\wp(\Omega)$ by $\mathscr{P}$.

In this paper, an *upper probability* is any real-valued function $\overline{P}$ defined on an arbitrary subset $\mathscr{K}$ of $\wp(\Omega)$. With $\overline{P}$, we associate a *lower probability* $\underline{P}$ on $\{A : A^c \in \mathscr{K}\}$ via the conjugacy relationship

$$\underline{P}(A) = 1 - \overline{P}(A^c).$$

Denote the set of all finitely additive probabilities on $\wp(\Omega)$ that are dominated by $\overline{P}$ by:

$$\mathscr{M}(\overline{P}) = \{P \in \mathscr{P} : (\forall A \in \mathscr{K})(P(A) \leqslant \overline{P}(A))\}$$

Clearly, $\mathscr{M}(\overline{P})$ is also the set of all finitely additive probabilities on $\wp(\Omega)$ that dominate $\underline{P}$ on its domain $\{A : A^c \in \mathscr{K}\}$.

The upper envelope $\overline{E}$ of $\mathscr{M}(\overline{P})$ is called the *natural extension* [36, Thm. 3.4.1] of $\overline{P}$:

$$\overline{E}(A) = \sup\{P(A) : P \in \mathscr{M}(\overline{P})\}$$

for all $A \subseteq \Omega$. The corresponding lower probability is denoted by $\underline{E}$, so $\underline{E}(A) = 1 - \overline{E}(A^c)$. Clearly, $\underline{E}$ is the lower envelope of $\mathscr{M}(\overline{P})$.

We say that $\overline{P}$ is *coherent* (see [36, p. 134, Sec. 3.3.3]) when it coincides with $\overline{E}$ on its domain, that is, when, for all $A \in \mathscr{K}$,

$$\overline{P}(A) = \overline{E}(A).$$

The lower probability $\underline{P}$ is called *coherent* whenever $\overline{P}$ is.

The upper envelope of any set of finitely additive probabilities on $\wp(\Omega)$ is coherent. A coherent upper probability $\overline{P}$ and its conjugate lower probability $\underline{P}$ satisfy the following properties [36, Sec. 2.7.4], whenever the relevant events belong to their domain:

(1) $0 \leqslant \underline{P}(A) \leqslant \overline{P}(A) \leqslant 1$.
(2) $A \subseteq B$ implies $\overline{P}(A) \leqslant \overline{P}(B)$ and $\underline{P}(A) \leqslant \underline{P}(B)$. [Monotonicity]
(3) $\overline{P}(A \cup B) \leqslant \overline{P}(A) + \overline{P}(B)$. [Subadditivity]
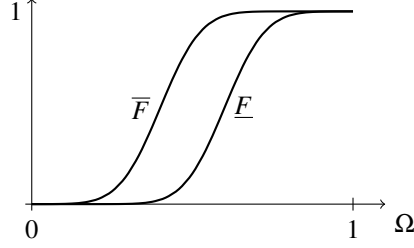
FIGURE 1. Example of a p-box on $[0,1]$.

2.2. **P-Boxes.** In this section, we revise the theory and some of the main results for p-boxes defined on totally preordered (not necessarily finite) spaces. For further details, we refer to [34].

We start with a totally preordered space $(\Omega, \preceq)$. So, $\preceq$ is transitive and reflexive and any two elements are comparable. As usual, we write $x < y$ for $x \preceq y$ and $x \not\succeq y$, $x > y$ for $y < x$, and $x \simeq y$ for $x \preceq y$ and $y \preceq x$. For any two $x, y \in \Omega$ exactly one of $x < y$, $x \simeq y$, or $x > y$ holds. We also use the following common notation for intervals in Ω:

$$[x,y] = \{z \in \Omega \colon x \preceq z \preceq y\}$$
$$(x,y) = \{z \in \Omega \colon x < z < y\}$$

and similarly for $[x,y)$ and $(x,y]$.

We assume that Ω has a smallest element 0_Ω and a largest element 1_Ω. This is not an essential assumption, since we can always add these two elements to the space Ω.

A *cumulative distribution function* is a non-decreasing map $F : \Omega \to [0,1]$ for which $F(1_\Omega) = 1$. For each $x \in \Omega$, $F(x)$ is interpreted as the probability of $[0_\Omega, x]$. No further restrictions are imposed on F.

The quotient set of Ω with respect to $\simeq$ is denoted by $\Omega/\simeq$:

$$[x]_\simeq = \{y \in \Omega \colon y \simeq x\} \text{ for any } x \in \Omega$$
$$\Omega/\simeq = \{[x]_\simeq \colon x \in \Omega\}.$$

Because F is non-decreasing, F is constant on elements $[x]_\simeq$ of $\Omega/\simeq$—we will use this repeatedly.

Definition 1. A *probability box*, or *p-box*, is a pair $(\underline{F}, \overline{F})$ of cumulative distribution functions from Ω to $[0,1]$ satisfying $\underline{F} \leqslant \overline{F}$.

A p-box is interpreted as a lower and an upper cumulative distribution function (see Fig. 1), or more specifically, as an upper probability $\overline{P}_{\underline{F},\overline{F}}$ on the set of events

$$\{[0_\Omega, x] \colon x \in \Omega\} \cup \{(y, 1_\Omega] \colon y \in \Omega\}$$

defined by

$$\overline{P}_{\underline{F},\overline{F}}([0_\Omega,x]) = \overline{F}(x) \text{ and } \overline{P}_{\underline{F},\overline{F}}((y,1_\Omega]) = 1 - \underline{F}(y). \tag{1}$$

We denote by $\overline{E}_{\underline{F},\overline{F}}$ the natural extension of $\overline{P}_{\underline{F},\overline{F}}$ to all events.

We now recall the main results that we shall need regarding the natural extension $\overline{E}_{\underline{F},\overline{F}}$ of $\overline{P}_{\underline{F},\overline{F}}$ (see [34] for further details). First, because $\overline{P}_{\underline{F},\overline{F}}$ is coherent, $\overline{E}_{\underline{F},\overline{F}}$ coincides with $\overline{P}_{\underline{F},\overline{F}}$ on its domain.

Next, to simplify the expression for natural extension, we introduce an element $0_\Omega-$ such that:

$$0_\Omega- \prec x \text{ for all } x \in \Omega$$

$$F(0_\Omega-) = \underline{F}(0_\Omega-) = \overline{F}(0_\Omega-) = 0.$$

Note that $(0_\Omega-, x] = [0_\Omega, x]$. Now, let $\Omega^* = \Omega \cup \{0_\Omega-\}$, and define

$$\mathcal{H} = \{(x_0, x_1] \cup (x_2, x_3] \cup \cdots \cup (x_{2n}, x_{2n+1}] : x_0 \prec x_1 \prec \cdots \prec x_{2n+1} \text{ in } \Omega^*\}.$$

Proposition 2 ([34]). *For any $A \in \mathcal{H}$, that is $A = (x_0, x_1] \cup (x_2, x_3] \cup \cdots \cup (x_{2n}, x_{2n+1}]$ with $x_0 \prec x_1 \prec \cdots \prec x_{2n+1}$ in Ω^*, it holds that $\overline{E}_{\underline{F}, \overline{F}}(A) = \overline{P}_{\underline{F}, \overline{F}}^{\mathcal{H}}(A)$, where*

$$\overline{P}_{\underline{F}, \overline{F}}^{\mathcal{H}}(A) = 1 - \sum_{k=0}^{n+1} \max\{0, \underline{F}(x_{2k}) - \overline{F}(x_{2k-1})\}, \tag{2}$$

with $x_{-1} = 0_\Omega-$ and $x_{2n+2} = 1_\Omega$.

To calculate $\overline{E}_{\underline{F}, \overline{F}}(A)$ for an arbitrary event $A \subseteq \Omega$, we can use the outer measure [36, Cor. 3.1.9, p. 127] $\overline{P}_{\underline{F}, \overline{F}}^{\mathcal{H}}{}^*$ of the upper probability $\overline{P}_{\underline{F}, \overline{F}}^{\mathcal{H}}$ defined in Eq. (2):

$$\overline{E}_{\underline{F}, \overline{F}}(A) = \overline{P}_{\underline{F}, \overline{F}}^{\mathcal{H}}{}^*(A) = \inf_{C \in \mathcal{H}, A \subseteq C} \overline{P}_{\underline{F}, \overline{F}}^{\mathcal{H}}(C). \tag{3}$$

For intervals, we immediately infer from Proposition 2 and Eq. (3) that

$$\overline{E}_{\underline{F}, \overline{F}}((x, y]) = \overline{F}(y) - \underline{F}(x) \tag{4a}$$

$$\overline{E}_{\underline{F}, \overline{F}}([x, y]) = \overline{F}(y) - \underline{F}(x-) \tag{4b}$$

$$\overline{E}_{\underline{F}, \overline{F}}((x, y)) = \begin{cases} \overline{F}(y) - \underline{F}(x) & \text{if } y \text{ has no immediate predecessor} \\ \overline{F}(y-) - \underline{F}(x) & \text{if } y \text{ has an immediate predecessor} \end{cases} \tag{4c}$$

$$\overline{E}_{\underline{F}, \overline{F}}([x, y)) = \begin{cases} \overline{F}(y) - \underline{F}(x-) & \text{if } y \text{ has no immediate predecessor} \\ \overline{F}(y-) - \underline{F}(x-) & \text{if } y \text{ has an immediate predecessor} \end{cases} \tag{4d}$$

for any $x \prec y$ in Ω,[1] where $\overline{F}(y-)$ denotes $\sup_{z \prec y} \overline{F}(z)$ and similarly for $\underline{F}(x-)$. If $\Omega/\simeq$ is finite, then one can think of $z-$ as the immediate predecessor of z in the quotient space $\Omega/\simeq$ for any $z \in \Omega$. Note that in particular

$$\overline{E}_{\underline{F}, \overline{F}}(\{x\}) = \overline{F}(x) - \underline{F}(x-) \tag{5}$$

for any $x \in \Omega$. We will use this repeatedly.

2.3. **Possibility and Maxitive Measures.** Very briefly, we introduce possibility and maxitive measures. For further information, see [39, 13, 37, 6].

Definition 3. A *maxitive measure* is an upper probability $\overline{P} \colon \wp(\Omega) \to [0, 1]$ satisfying $\overline{P}(A \cup B) = \max\{\overline{P}(A), \overline{P}(B)\}$ for every $A, B \subseteq \Omega$.

It follows from the above definition that a maxitive measure is also maximum-preserving when we consider finite unions of events.

The following result is well-known, but we include a quick proof for the sake of completeness.

Proposition 4. *A maxitive measure $\overline{P}$ is coherent whenever $\overline{P}(\varnothing) = 0$ and $\overline{P}(\Omega) = 1$.*

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[1] In case $x = 0_\Omega$, evidently, $0_\Omega-$ is the immediate predecessor.

Proof. By [25, Theorem 1], a maxitive measure $\overline{P}$ satisfying $\overline{P}(\varnothing) = 0$ is ∞-alternating, and as a consequence also 2-alternating. Whence, $\overline{P}$ is coherent by [35, p. 55, Corollary 6.3]. □

Possibility measures are a particular case of maxitive measures.

Definition 5. A (normed) *possibility distribution* is a mapping $\pi\colon \Omega \to [0,1]$ satisfying $\sup_{x\in\Omega} \pi(x) = 1$. A possibility distribution π induces a *possibility measure* Π on $\wp(\Omega)$, given by:
$$\Pi(A) = \sup_{x\in A} \pi(x) \text{ for all } A \subseteq \Omega.$$

Equivalently, possibility measures can be defined as supremum-preserving upper probabilities, i.e., as functionals Π for which
$$\Pi(\cup_{A\in\mathscr{A}}A) = \sup_{A\in\mathscr{A}} \Pi(A) \ \forall \mathscr{A} \subseteq \mathscr{P}(\Omega).$$

If we write $\underline{E}_\Pi$ for the conjugate lower probability of the upper probability Π, then:
$$\underline{E}_\Pi(A) = 1 - \Pi(A^c) = 1 - \sup_{x\in A^c} \pi(x).$$

A possibility measure is maxitive, but not all maxitive measures are possibility measures.

As an imprecise probability model, possibility measures are not as expressive as for instance p-boxes—for example, the only probability measures that can be represented by possibility measures are the degenerate ones. This poor expressive power is also illustrated by the fact that, for any event A:

$$\Pi(A) < 1 \implies \underline{E}_\Pi(A) = 0, \qquad\qquad \text{and therefore}$$
$$\underline{E}_\Pi(A) > 0 \implies \Pi(A) = 1,$$

meaning that every event has a trivial probability bound on at least one side. Their main attraction is that calculations with them are very easy: to find the upper (or lower) probability of any event, a simple supremum suffices.

In the following sections, we characterize the circumstances under which a possibility measure Π is the natural extension of some p-box $(\underline{F},\overline{F})$. In order to do so, we first characterise the conditions under which a p-box induces a maxitive measure.

3. P-boxes as Maxitive Measures.

We show here that p-boxes $(\underline{F},\overline{F})$ on any totally preordered space where at least one of $\underline{F}$ or $\overline{F}$ is 0–1-valued are maxitive measures, and in this sense are closely related to possibility measures. We then derive a simple closed expression of the (upper) natural extension of such p-boxes.

3.1. **A Necessary Condition for Maxitivity.**

Proposition 6. *If the natural extension $\overline{E}_{\underline{F},\overline{F}}$ of a p-box $(\underline{F},\overline{F})$ is maximum preserving, then at least one of $\underline{F}$ or $\overline{F}$ is 0–1-valued.*

Proof. We begin by showing that there is no $x \in \Omega$ such that $0 < \underline{F}(x) \leqslant \overline{F}(x) < 1$. Assume ex absurdo that there is such an x. For $\overline{E}_{\underline{F},\overline{F}}$ to be maximum preserving, we require that

$$\overline{E}_{\underline{F},\overline{F}}([0_\Omega,x] \cup (x,1_\Omega]) = \max\{\overline{E}_{\underline{F},\overline{F}}([0_\Omega,x]),\overline{E}_{\underline{F},\overline{F}}((x,1_\Omega])\}$$
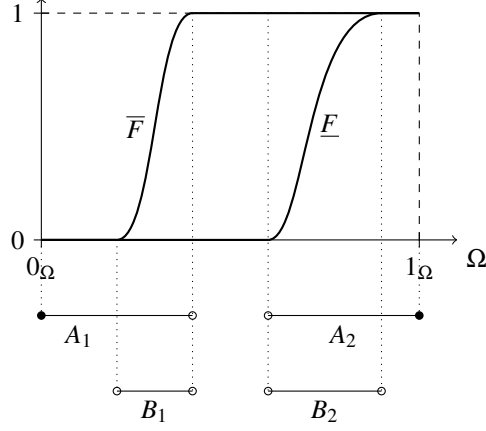
FIGURE 2. A p-box for the proof of Proposition 6.

But this cannot be. The left hand side is 1, whereas the right hand side is strictly less than one, because, by Eq. (1),

$$\overline{E}_{\underline{F},\overline{F}}([0_\Omega, x]) = \overline{F}(x) < 1,$$
$$\overline{E}_{\underline{F},\overline{F}}((x, 1_\Omega]) = 1 - \underline{F}(x) < 1.$$

Whence, for every $x \in \Omega$, at least one of $\underline{F}(x) = 0$ or $\overline{F}(x) = 1$ must hold. In other words, $\underline{F}(x) = 0$ whenever $\overline{F}(x) < 1$, and $\overline{F}(x) = 1$ whenever $\underline{F}(x) > 0$ (see Figure 2). Hence, the sets

$$A_1 := \{x \in \Omega : \overline{F}(x) < 1\}$$
$$A_2 := \{y \in \Omega : \underline{F}(y) > 0\}$$

are disjoint, and $A_1 \prec A_2$ in the sense that $x \prec y$ for all $x \in A_1$ and $y \in A_2$. Indeed, if $x \in A_1$ and $y \in A_2$, then $\overline{F}(x) < 1$, and $\overline{F}(y) = 1$ because $\underline{F}(y) > 0$. These can only hold simultaneously if $x \prec y$.

Note that A_1 is empty when $\overline{F}(x) = 1$ for all $x \in \Omega$, and in this case the desired result is trivially established. A_2 is non-empty because $\underline{F}(1_\Omega) = 1$. Anyway, consider the sets

$$B_1 := \{x \in \Omega : 0 < \overline{F}(x) < 1\} \subseteq A_1$$
$$B_2 := \{y \in \Omega : 0 < \underline{F}(y) < 1\} \subseteq A_2$$

The proposition is established if we can show that at least one of these two sets is empty.

Suppose, ex absurdo, that both are non-empty. Pick any element $c \in B_1$ and $d \in B_2$ and consider the set $C = [0_\Omega, c] \cup (d, 1_\Omega]$—note that $c \prec d$ because $c \in A_1$ and $d \in A_2$, so $(c, d]$ is non-empty. Whence, by Eq. (2),

$$\overline{E}_{\underline{F},\overline{F}}([0_\Omega, c] \cup (d, 1_\Omega]) = 1 - \max\{0, \underline{F}(d) - \overline{F}(c)\}.$$

Also, by Eq. (1),

$$\overline{E}_{\underline{F},\overline{F}}([0_\Omega, c]) = \overline{F}(c),$$
$$\overline{E}_{\underline{F},\overline{F}}((d, 1_\Omega]) = 1 - \underline{F}(d).$$

So, for $\overline{E}_{\underline{F},\overline{F}}$ to be maximum preserving, we require that

$$1 - \max\{0, \underline{F}(d) - \overline{F}(c)\} = \max\{\overline{F}(c), 1 - \underline{F}(d)\}.$$

But this cannot hold. Indeed, because $0 < \overline{F}(c) < 1$ and $0 < 1 - \underline{F}(d) < 1$, the above equality can only hold if $\underline{F}(d) - \overline{F}(c) > 0$—otherwise the left hand side would be 1 whereas the right hand side is strictly less than 1. So, effectively, we require that

$$1 - \underline{F}(d) + \overline{F}(c) = \max\{\overline{F}(c), 1 - \underline{F}(d)\}.$$

This cannot hold, because the sum of two strictly positive numbers (in this case $1 - \underline{F}(d)$ and $\overline{F}(c)$) is always strictly larger than their maximum. We conclude that $\overline{E}_{\underline{F},\overline{F}}$ cannot be maximum preserving if both B_1 and B_2 are non-empty. In other words, at least one of $\underline{F}$ or $\overline{F}$ must be 0–1-valued. $\qquad\square$

3.2. **Sufficient Conditions for Maxitivity.** We derive sufficient conditions for the two different cases described by Proposition 6, starting with 0–1-valued $\underline{F}$.

3.2.1. *Maxitivity for Zero-One Valued Lower Cumulative Distribution Functions.* We first provide a simple expression for the natural extension of such p-boxes over events.

Proposition 7. *Let $(\underline{F}, \overline{F})$ be a p-box with 0–1-valued $\underline{F}$, and let $B = \{x \in \Omega^* : \underline{F}(x) = 0\}$. Then, for any $A \subseteq \Omega$,*

$$\overline{E}_{\underline{F},\overline{F}}(A) = \begin{cases} \inf_{x \in \Omega^* : A \cap B \le x} \overline{F}(x) & \text{if } y < A \cap B^c \text{ for at least one } y \in B^c, \\ 1 & \text{otherwise.} \end{cases} \tag{6}$$

$$= \min_{y \in B^c} \inf_{x \in \Omega^* : A \cap [0_\Omega, y] \le x} \overline{F}(x). \tag{7}$$

In the above, $A \le x$ means $z \le x$ for all $z \in A$, and similarly $y < A$ means $y < z$ for all $z \in A$. For example, it holds that $\varnothing \le x$ and $y < \varnothing$ for all x and y.

Proof. We deduce from Eq. (3) and from the conjugacy between $\underline{E}_{\underline{F},\overline{F}}$ and $\overline{E}_{\underline{F},\overline{F}}$ that for any $A \subseteq \Omega$,

$$\underline{E}_{\underline{F},\overline{F}}(A) = \sup_{(x_0,x_1] \cup \cdots \cup (x_{2n},x_{2n+1}] \subseteq A} \sum_{k=0}^{n} \max\{0, \underline{F}(x_{2k+1}) - \overline{F}(x_{2k})\}.$$

All the terms in this sum are zero except possibly for one (if it exists) where $x_{2k} \in B, x_{2k+1} \in B^c$, where we get $1 - \overline{F}(x_{2k})$. Aside, as subsets of Ω^*, note that both B and B^c are non-empty: $0_\Omega - \in B$ and $1_\Omega \in B^c$. Consequently,

$$\underline{E}_{\underline{F},\overline{F}}(A) = 1 - \inf_{x,y:\, x \in B, y \in B^c,\, (x,y] \subseteq A} \overline{F}(x);$$

and therefore

$$\overline{E}_{\underline{F},\overline{F}}(A) = \inf_{x,y:\, x \in B, y \in B^c,\, (x,y] \subseteq A^c} \overline{F}(x)$$

$$= \inf_{x,y:\, x \in B, y \in B^c, A \subseteq [0_\Omega,x] \cup (y,1_\Omega]} \overline{F}(x)$$

where it is understood that the infimum evaluates to 1 whenever there are no $x \in B$ and $y \in B^c$ such that $A \subseteq [0_\Omega, x] \cup (y, 1_\Omega]$.

Now, for any $x \in B$ and $y \in B^c$, it holds that $A \subseteq [0_\Omega, x] \cup (y, 1_\Omega]$ if and only if

$$A \cap B \subseteq ([0_\Omega, x] \cup (y, 1_\Omega]) \cap B = [0_\Omega, x] \text{ and}$$

$$A \cap B^c \subseteq ([0_\Omega, x] \cup (y, 1_\Omega]) \cap B^c = (y, 1_\Omega],$$

that is, if and only if

$$A \cap B \le x \text{ and } y \prec A \cap B^c.$$

Hence, if there is an $y \in B^c$ such that $y \prec A \cap B^c$, then:

  (i) either there is no $x \in B$ such that $A \cap B \le x$, whence

$$\overline{E}_{\underline{F}, \overline{F}}(A) = 1 = \inf_{x \in \Omega^* : A \cap B \le x} \overline{F}(x),$$

  taking into account that for any $x \in \Omega^*$ such that $A \cap B \le x$ it must be that $x \in B^c$, whence $\underline{F}(x) = \overline{F}(x) = 1$;

  (ii) or there is some $x \in B$ such that $A \cap B \le x$, in which case

$$\overline{E}_{\underline{F}, \overline{F}}(A) = \inf_{x \in B : A \cap B \le x} \overline{F}(x) = \inf_{x \in \Omega^* : A \cap B \le x} \overline{F}(x),$$

  where the second equality follows from the monotonicity of $\overline{F}$.

This establishes Eq. (6).

We now turn to proving Eq. (7). In case $y \prec A \cap B^c$ for at least one $y \in B^c$, it follows that

$$\overline{E}_{\underline{F}, \overline{F}}(A) = \inf_{x \in \Omega^* : A \cap B \le x} \overline{F}(x)$$

But in this case, $A \cap B = A \cap [0_\Omega, y']$ for any $y' \in B^c$ such that $y' \le y$, because

$$A \cap [0, y'] = A \cap [0_\Omega, y'] \cap (B \cup B^c) = (A \cap B \cap [0_\Omega, y']) \cup (A \cap B^c \cap [0_\Omega, y']) = A \cap B$$

as $B \cap [0_\Omega, y'] = B$ and $A \cap B^c \cap [0_\Omega, y'] = \emptyset$ because $y' \le y$ and $y \prec A \cap B^c$. So, by the monotonicity of $\overline{F}$, Eq. (7) follows.

In case $y \nprec A \cap B^c$ for all $y \in B^c$, it follows that

$$\overline{E}_{\underline{F}, \overline{F}}(A) = 1 = \overline{F}(x)$$

for all x in B^c—indeed, because $A \cap [0, y] \cap B^c \ne \emptyset$ for every $y \in B^c$, it holds that $A \cap [0, y] \le x$ implies $x \in B^c$, and hence $\overline{F}(x) = 1$. Again, Eq. (7) follows. $\square$

A few common important special cases are summarized in the following corollary:

Corollary 8. *Let $(\underline{F}, \overline{F})$ be a p-box with 0–1-valued $\underline{F}$, and let $B = \{x \in \Omega^* : \underline{F}(x) = 0\}$. If $\Omega/ \simeq$ is order complete, then, for any $A \subseteq \Omega$,*

$$\overline{E}_{\underline{F}, \overline{F}}(A) = \min_{y \in B^c} \overline{F}(\sup A \cap [0_\Omega, y]).$$

If, in addition, B^c has a minimum, then

$$\overline{E}_{\underline{F}, \overline{F}}(A) = \overline{F}(\sup A \cap [0_\Omega, \min B^c]). \tag{8}$$

If, in addition, $B^c = [1_\Omega]_\simeq$ (this occurs exactly when $\underline{F}$ is vacuous, i.e. $\underline{F} = I_{[1_\Omega]_\simeq}$), then

$$\overline{E}_{\underline{F}, \overline{F}}(A) = \overline{F}(\sup A). \tag{9}$$

Note that Eq. (9) is essentially due to [6, paragraph preceeding Theorem 11]—they work with chains and multivalued mappings, whereas we work with total preorders. We are now ready to show that the considered p-boxes are maxitive measures.

Proposition 9. *Let $(\underline{F}, \overline{F})$ be a p-box where $\underline{F}$ is 0–1-valued. Then $\overline{E}_{\underline{F}, \overline{F}}$ is maximum-preserving.*

Proof. Consider a finite collection $\mathscr{A}$ of subsets of Ω. If there are $A \in \mathscr{A}$ such that, for all $y \in B^c$, $y \not\prec A \cap B^c$, then $\overline{E}_{\underline{F},\overline{F}}(A) = 1 = \overline{E}_{\underline{F},\overline{F}}(\cup_{A \in \mathscr{A}} A)$ by Eq. (6), establishing the desired result for this case.

So, from now on, we may assume that, for every $A \in \mathscr{A}$, there is a $y_A \in B^c$ such that $y_A \prec A \cap B^c$. With $y = \min_{A \in \mathscr{A}} y_A \in B^c$, it holds that $y \prec \cup_{A \in \mathscr{A}} A \cap B^c$, and so, by Eq. (6),

$$\overline{E}_{\underline{F},\overline{F}}(A) = \inf_{x \in \Omega^* : A \cap B \leq x} \overline{F}(x) \text{ for every } A \in \mathscr{A}, \text{ and}$$

$$\overline{E}_{\underline{F},\overline{F}}(\cup_{A \in \mathscr{A}} A) = \inf_{x \in \Omega^* : \cup_{A \in \mathscr{A}} A \cap B \leq x} \overline{F}(x).$$

Now, because $\mathscr{A}$ is finite, there is an $A' \in \mathscr{A}$ such that

$$\{x \in \Omega^* : A' \cap B \leq x\} = \cap_{A \in \mathscr{A}} \{x \in \Omega^* : A \cap B \leq x\}$$

and because $\cup_{A \in \mathscr{A}} A \cap B \leq x$ if and only if $A \cap B \leq x$ for all $A \in \mathscr{A}$,

$$= \{x \in \Omega^* : \cup_{A \in \mathscr{A}} A \cap B \leq x\}.$$

Consequently,

$$\max_{A \in \mathscr{A}} \overline{E}_{\underline{F},\overline{F}}(A) = \max_{A \in \mathscr{A}} \inf_{x \in \Omega^* : A \cap B \leq x} \overline{F}(x)$$

$$\geq \inf_{x \in \Omega^* : A' \cap B \leq x} \overline{F}(x) = \inf_{x \in \Omega^* : \cup_{A \in \mathscr{A}} A \cap B \leq x} \overline{F}(x) = \overline{E}_{\underline{F},\overline{F}}(\cup_{A \in \mathscr{A}} A).$$

The converse inequality follows from the coherence of $\overline{E}_{\underline{F},\overline{F}}$. Concluding,

$$\max_{A \in \mathscr{A}} \overline{E}_{\underline{F},\overline{F}}(A) = \overline{E}_{\underline{F},\overline{F}}(\cup_{A \in \mathscr{A}} A)$$

for any finite collection $\mathscr{A}$ of subsets of Ω. $\qquad\square$

3.2.2. *Maxitivity for Zero-One Valued Upper Cumulative Distribution Functions.* Let us now consider the case of 0–1-valued $\overline{F}$.

Proposition 10. *Let $(\underline{F}, \overline{F})$ be a p-box with 0–1-valued $\overline{F}$, and let $C = \{x \in \Omega^* : \overline{F}(x) = 0\}$. Then, for any $A \subseteq \Omega$,*

$$\overline{E}_{\underline{F},\overline{F}}(A) = \begin{cases} 1 - \sup_{y \in \Omega^* : y \prec A \cap C^c} \underline{F}(y) & \text{if } A \cap C \leq x \text{ for at least one } x \in C, \\ 1 & \text{otherwise.} \end{cases} \tag{10}$$

$$= 1 - \max_{x \in C} \sup_{y \in \Omega^* : y \prec A \cap (x, 1_\Omega]} \underline{F}(y). \tag{11}$$

Proof. We deduce from Eq. (3) and from the conjugacy between $\underline{E}_{\underline{F},\overline{F}}$ and $\overline{E}_{\underline{F},\overline{F}}$ that for any $A \subseteq \Omega$,

$$\underline{E}_{\underline{F},\overline{F}}(A) = \sup_{(x_0,x_1] \cup \cdots \cup (x_{2n},x_{2n+1}] \subseteq A} \sum_{k=0}^{n} \max\{0, \underline{F}(x_{2k+1}) - \overline{F}(x_{2k})\}.$$

All the terms in this sum are zero except possibly for one (if it exists) where $x_{2k} \in C, x_{2k+1} \in C^c$, where we get $\underline{F}(x_{2k+1})$. Aside, as subsets of Ω^*, note that both C and C^c are non-empty: $0_{\Omega}- \in C$ and $1_{\Omega} \in C^c$. Consequently,

$$\underline{E}_{\underline{F},\overline{F}}(A) = \sup_{x,y : x \in C, y \in C^c, (x,y] \subseteq A} \underline{F}(y);$$

and therefore

$$\overline{E}_{\underline{F},\overline{F}}(A) = 1 - \sup_{x,y:\, x\in C,\, y\in C^c,\, (x,y]\subseteq A^c} \underline{F}(y)$$

$$= 1 - \sup_{x,y:\, x\in C,\, y\in C^c,\, A\subseteq[0_\Omega,x]\cup(y,1_\Omega]} \underline{F}(y)$$

where it is understood that the supremum evaluates to 0 whenever there are no $x \in C$ and $y \in C^c$ such that $A \subseteq [0_\Omega,x]\cup(y,1_\Omega]$.

Now, for any $x \in C$ and $y \in C^c$, it holds that $A \subseteq [0_\Omega,x]\cup(y,1_\Omega]$ if and only if

$$A\cap C \subseteq ([0_\Omega,x]\cup(y,1_\Omega])\cap C = [0_\Omega,x] \text{ and}$$
$$A\cap C^c \subseteq ([0_\Omega,x]\cup(y,1_\Omega])\cap C^c = (y,1_\Omega],$$

that is, if and only if

$$A\cap C \leq x \text{ and } y < A\cap C^c.$$

Hence, if there is an $x \in C$ such that $A \cap C \leq x$, then:

(i) either there is no $y \in C^c$ such that $y < A\cap C^c$, whence

$$\overline{E}_{\underline{F},\overline{F}}(A) = 1 = 1 - \sup_{y\in\Omega^*:\, y<A\cap C^c} \underline{F}(y),$$

taking into account that for any $y \in \Omega^*$ such that $y < A\cap C^c$ it must be that $y \in C$, whence $\underline{F}(y) = \overline{F}(y) = 0$;

(ii) or there is some $y \in C^c$ such that $y < A\cap C^c$, in which case

$$\overline{E}_{\underline{F},\overline{F}}(A) = 1 - \sup_{y\in C^c:\, y<A\cap C^c} \underline{F}(y) = 1 - \sup_{y\in\Omega^*:\, y<A\cap C^c} \underline{F}(y),$$

where the second equality follows from the monotonicity of $\underline{F}$.

This establishes Eq. (10).

We now turn to proving Eq. (11). In case $A\cap C \leq x$ for at least one $x \in C$, it follows that

$$\overline{E}_{\underline{F},\overline{F}}(A) = 1 - \sup_{y\in\Omega^*:\, y<A\cap C^c} \underline{F}(y).$$

But in this case, $A\cap C^c = A\cap(x',1_\Omega]$ for any $x' \in C$ such that $x' \geq x$, because

$$A\cap(x',1_\Omega] = A\cap(x',1_\Omega]\cap(C\cup C^c) = (A\cap C\cap(x',1_\Omega])\cup(A\cap C^c\cap(x',1_\Omega]) = A\cap C^c$$

as $C^c\cap(x',1_\Omega] = C^c$ and $A\cap C\cap(x',1_\Omega] = \varnothing$ by assumption. So, by the monotonicity of $\underline{F}$, Eq. (11) follows.

In case $A\cap C \nleq x$ for all $x \in C$, it follows that

$$\overline{E}_{\underline{F},\overline{F}}(A) = 1 = 1 - \underline{F}(y)$$

for all y in C—indeed, because $A\cap(x,1_\Omega]\cap C \neq \varnothing$ for every $x \in C$, it holds that $y < A\cap(x,1_\Omega]$ implies $y \in C$, and hence $\underline{F}(y) = 0$. Again, Eq. (11) follows.   □

A few common important special cases are summarized in the following corollary:

Corollary 11. *Let $(\underline{F},\overline{F})$ be a p-box with 0–1-valued $\overline{F}$, and let $C = \{x \in \Omega^* : \overline{F}(x) = 0\}$. If $\Omega/\simeq$ is order complete, and C has a maximum, then, for any $A \subseteq \Omega$,*

$$\overline{E}_{\underline{F},\overline{F}}(A) = \begin{cases} 1 - \underline{F}(\inf A\cap C^c) & \text{if } A\cap C^c \text{ has no minimum} \\ 1 - \underline{F}((\min A\cap C^c)-) & \text{if } A\cap C^c \text{ has a minimum.} \end{cases} \tag{12}$$

If, in addition, $C = \{0_\Omega-\}$ (this occurs exactly when $\overline{F}$ is vacuous, i.e. $\overline{F} = 1$), then

$$\overline{E}_{\underline{F},\overline{F}}(A) = \begin{cases} 1 - \underline{F}(\inf A) & \text{if A has no minimum} \\ 1 - \underline{F}(\min A-) & \text{if A has a minimum.} \end{cases}$$

Proof. Use Proposition 10, and note that $(\max C, 1_\Omega] = C^c$. $\qquad\qquad\qquad\square$

Using Eq. (10), we can also show that $\overline{E}_{\underline{F},\overline{F}}$ is maximum-preserving when $\overline{F}$ is 0–1-valued:

Proposition 12. *Let $(\underline{F},\overline{F})$ be a p-box where $\overline{F}$ is 0–1-valued. Then $\overline{E}_{\underline{F},\overline{F}}$ is maximum-preserving.*

Proof. Consider a finite collection $\mathscr{A}$ of subsets of Ω. If there are $A \in \mathscr{A}$ such that, for all $x \in C$, $A \cap C \not\preceq x$, then $\overline{E}_{\underline{F},\overline{F}}(A) = 1 = \overline{E}_{\underline{F},\overline{F}}(\cup_{A\in\mathscr{A}}A)$ by Eq. (10), establishing the desired result for this case.

So, from now on, we may assume that, for every $A \in \mathscr{A}$, there is an $x_A \in C$ such that $A \cap C \preceq x_A$. With $x = \max_{A\in\mathscr{A}} x_A \in C$, it holds that $\cup_{A\in\mathscr{A}}A \cap C \preceq x$, and so, by Eq. (10),

$$\overline{E}_{\underline{F},\overline{F}}(A) = 1 - \sup_{y\in\Omega^*\,:\,y\prec A\cap C^c} \underline{F}(y) \text{ for every } A \in \mathscr{A}, \text{ and}$$

$$\overline{E}_{\underline{F},\overline{F}}(\cup_{A\in\mathscr{A}}A) = 1 - \sup_{y\in\Omega^*\,:\,y\prec\cup_{A\in\mathscr{A}}A\cap C^c} \underline{F}(y).$$

Now, because $\mathscr{A}$ is finite, there is an $A' \in \mathscr{A}$ such that

$$\{y \in \Omega^* : y \prec A' \cap C^c\} = \cap_{A\in\mathscr{A}}\{y \in \Omega^* : y \prec A \cap C^c\}$$

and because $y \prec \cup_{A\in\mathscr{A}}A \cap C^c$ if and only if $y \prec A \cap C^c$ for all $A \in \mathscr{A}$,

$$= \{y \in \Omega^* : y \prec \cup_{A\in\mathscr{A}}A \cap C^c\}.$$

Consequently,

$$\max_{A\in\mathscr{A}}\overline{E}_{\underline{F},\overline{F}}(A) = \max_{A\in\mathscr{A}}\left(1 - \sup_{y\in\Omega^*\,:\,y\prec A\cap C^c}\underline{F}(y)\right)$$

$$\geqslant 1 - \sup_{y\in\Omega^*\,:\,y\prec A'\cap C^c}\underline{F}(y) = 1 - \sup_{y\in\Omega^*\,:\,y\prec\cup_{A\in\mathscr{A}}A\cap C^c}\underline{F}(y) = \overline{E}_{\underline{F},\overline{F}}(\cup_{A\in\mathscr{A}}A).$$

The converse inequality follows from the coherence of $\overline{E}_{\underline{F},\overline{F}}$. Concluding,

$$\max_{A\in\mathscr{A}}\overline{E}_{\underline{F},\overline{F}}(A) = \overline{E}_{\underline{F},\overline{F}}(\cup_{A\in\mathscr{A}}A)$$

for any finite collection $\mathscr{A}$ of subsets of Ω. $\qquad\qquad\qquad\square$

3.3. **Summary of Necessary and Sufficient Conditions.** Putting Propositions 6, 9 and 12 together, we get the following conditions.

Corollary 13. *Let $(\underline{F},\overline{F})$ be a p-box. Then, $(\underline{F},\overline{F})$ is maximum-preserving if and only if*

$$\underline{F} \text{ is } 0\text{–}1\text{-valued}$$

or

$$\overline{F} \text{ is } 0\text{–}1\text{-valued.}$$

These simple conditions characterise maximum-preserving p-boxes and bring us closer to p-boxes that are possibility measures, and that we will now study.

4. P-Boxes as Possibility Measures.

In this section, we identify when p-boxes coincide exactly with a possibility measure. By Corollary 13, when $\Omega/\simeq$ is finite, $(\underline{F}, \overline{F})$ is a possibility measure if and only if either $\underline{F}$ or $\overline{F}$ is 0–1-valued. More generally, when $\Omega/\simeq$ is not finite, we will rely on the following trivial, yet important, lemma:

Lemma 14. *For a p-box $(\underline{F}, \overline{F})$ there is a possibility measure Π such that $\overline{E}_{\underline{F}, \overline{F}} = \Pi$ if and only if*

$$\overline{E}_{\underline{F}, \overline{F}}(A) = \sup_{x \in A} \overline{E}_{\underline{F}, \overline{F}}(\{x\}) \text{ for all } A \subseteq \Omega \tag{13}$$

and in such a case, the possibility distribution π associated with Π is $\pi(x) = \overline{E}_{\underline{F}, \overline{F}}(\{x\})$.

Proof. "if". If $\overline{E}_{\underline{F}, \overline{F}}(A) = \sup_{x \in A} \overline{E}_{\underline{F}, \overline{F}}(\{x\})$ for all $A \subseteq \Omega$, then $\underline{E}_{\underline{F}, \overline{F}} = \underline{E}_\Pi$ with the suggested choice of π, because, for all $A \subseteq \Omega$,

$$\underline{E}_{\underline{F}, \overline{F}}(A) = 1 - \overline{E}_{\underline{F}, \overline{F}}(A^c) = 1 - \sup_{x \in A^c} \overline{E}_{\underline{F}, \overline{F}}(\{x\}) = 1 - \sup_{x \in A^c} \pi(x) = 1 - \Pi(A^c) = \underline{E}_\Pi(A).$$

"only if". If $\underline{E}_{\underline{F}, \overline{F}} = \underline{E}_\Pi$, then, for all $A \subseteq \Omega$,

$$\overline{E}_{\underline{F}, \overline{F}}(A) = \Pi(A) = \sup_{x \in A} \pi(x) = \sup_{x \in A} \Pi(\{x\}) = \sup_{x \in A} \overline{E}_{\underline{F}, \overline{F}}(\{x\}).$$

$\square$

We will say that *a p-box $(\underline{F}, \overline{F})$ is a possibility measure* whenever Eq. (13) is satisfied.

Note that, due to Proposition 6, for a p-box to be a possibility measure, at least one of $\underline{F}$ or $\overline{F}$ must be 0–1-valued. Next, we give a characterisation of p-boxes inducing a possibility measure in each of these two cases.

4.1. **P-Boxes with Zero-One-Valued Lower Cumulative Distribution Functions.** As mentioned, by Corollary 13, a p-box with 0–1-valued $\underline{F}$ is maxitive, and its upper natural extension is given by Proposition 7. Whence, we can easily determine when such p-box is a possibility measure:

Proposition 15. *Assume that $\Omega/\simeq$ is order complete. Let $(\underline{F}, \overline{F})$ be a p-box with 0–1-valued $\underline{F}$, and let $B = \{x \in \Omega^* : \underline{F}(x) = 0\}$. Then, $(\underline{F}, \overline{F})$ is a possibility measure if and only if*

(i) $\overline{F}(x) = \overline{F}(x-)$ *for all $x \in \Omega$ that have no immediate predecessor, and*
(ii) B^c *has a minimum,*

and in such a case,

$$\overline{E}_{\underline{F}, \overline{F}}(A) = \sup_{x \in A \cap [0_\Omega, \min B^c]} \overline{F}(x) \tag{14}$$

Note that, in case 1_Ω is a minimum of B^c, condition (i) is essentially due to [6, Observation 9]. Also note that, for $\overline{E}_{\underline{F}, \overline{F}}$ to be a possibility measure, both conditions are still necessary even when $\Omega/\simeq$ is not order complete: the proof in this direction does not require order completeness.

As a special case, we mention that $\overline{E}_{\underline{F}, \overline{F}}$ is a possibility measure with possibility distribution

$$\pi(x) = \begin{cases} \overline{F}(x) & \text{if } x \leq \min B^c \\ 0 & \text{otherwise.} \end{cases}$$

whenever $\Omega/\simeq$ is finite.

Proof. "only if". Assume that $(\underline{F}, \overline{F})$ is a possibility measure. For every $x \in \Omega$ that has no immediate predecessor,

$$\overline{F}(x-) = \sup_{x' \prec x} \overline{F}(x')$$

and because $\overline{E}_{\underline{F}, \overline{F}}(\{x'\}) = \overline{F}(x') - \underline{F}(x'-)$ (see Eq. (5)),

$$\geqslant \sup_{x' \prec x} \overline{E}_{\underline{F}, \overline{F}}(\{x'\})$$

and because $(\underline{F}, \overline{F})$ is a possibility measure, by Lemma 14,

$$= \overline{E}_{\underline{F}, \overline{F}}([0_\Omega, x)) = \overline{F}(x)$$

using that x has no immediate predecessor and Eqs. (4). The converse inequality follows from the non-decreasingness of $\overline{F}$.

Next, assume that, ex absurdo, $B^c = \{x \in \Omega^* : \underline{F}(x) = 1\}$ has no minimum. This simply means that for every $x \in B^c$ there is an $x' \in B^c$ such that $x' \prec x$. So, in particular, $\underline{F}(x) = \underline{F}(x-) = 1$ for all x in B^c, and hence,

$$\overline{E}_{\underline{F}, \overline{F}}(B^c) = \sup_{x \in B^c} \overline{E}_{\underline{F}, \overline{F}}(\{x\}) = \sup_{x \in B^c}(\overline{F}(x) - \underline{F}(x-)) = 0.$$

Yet, also,

$$\overline{E}_{\underline{F}, \overline{F}}(B^c) = 1$$

by Eq. (6). We arrived at a contradiction.

Finally, we show that Eq. (14) holds. By Eq. (7),

$$\overline{E}_{\underline{F}, \overline{F}}(A) = \min_{y \in B^c} \inf_{x \in \Omega^* \,:\, A \cap [0_\Omega, y] \leq x} \overline{F}(x) = \inf_{x \in \Omega^* \,:\, A \cap [0_\Omega, \min B^c] \leq x} \overline{F}(x) = \overline{E}_{\underline{F}, \overline{F}}(A'),$$

with $A' := A \cap [0_\Omega, \min B^c]$. Since $\overline{E}_{\underline{F}, \overline{F}}$ is a possibility measure, we conclude that

$$\overline{E}_{\underline{F}, \overline{F}}(A) = \overline{E}_{\underline{F}, \overline{F}}(A') = \sup_{x \in A'} \overline{E}_{\underline{F}, \overline{F}}(\{x\}) = \sup_{x \in A'} \overline{F}(x),$$

because $\overline{E}_{\underline{F}, \overline{F}}(\{x\}) = \overline{F}(x) - \underline{F}(x-) = \overline{F}(x)$, since $\underline{F}(x-) = 0$ for all $x \in [0, \min B^c]$. Hence, Eq. (14) holds.

"if". The claim is established if we can show that Eq. (14) holds, because then

$$\overline{E}_{\underline{F}, \overline{F}}(A) = \sup_{x \in A \cap [0_\Omega, \min B^c]} \overline{F}(x) = \sup_{x \in A \cap [0_\Omega, \min B^c]} \overline{E}_{\underline{F}, \overline{F}}(\{x\}) \leq \sup_{x \in A} \overline{E}_{\underline{F}, \overline{F}}(\{x\}),$$

and the converse inequality follows from the monotonicity of $\overline{E}_{\underline{F}, \overline{F}}$.

Consider any event $A \subseteq \Omega$, and let y be a supremum of $A' = A \cap [0_\Omega, \min B^c]$ (which exists because $\Omega/ \simeq$ is order complete), so $\overline{E}_{\underline{F}, \overline{F}}(A) = \overline{F}(y)$ by Eq. (8). If y has an immediate predecessor, then A' has a maximum (as we will show next), and

$$\overline{E}_{\underline{F}, \overline{F}}(A) = \overline{F}(y) = \overline{F}(\max A') = \max_{x \in A'} \overline{F}(x) = \sup_{x \in A'} \overline{F}(x).$$

If y has no immediate predecessor, then either A' has a maximum, and the above argument can be recycled, or A' has no maximum, in which case

$$\overline{E}_{\underline{F}, \overline{F}}(A) = \overline{F}(y) = \overline{F}(y-) = \sup_{x \in A'} \overline{F}(x).$$

The last equality holds because

$$\overline{F}(y-) = \sup_{x < \sup A'} \overline{F}(x)$$

and, A' has no maximum, so for every $x < \sup A'$, there is an $x' \in A'$ such that $x < x' < \sup A'$, whence

$$= \sup_{x' \in A'} \overline{F}(x').$$

We are left to prove A' has a maximum whenever y has an immediate predecessor. Suppose that A' has no maximum. Then it must hold that

$$x < y \text{ for all } x \in A'$$

since otherwise $x \simeq y$ for some $x \in A'$, whereby x would be a maximum of A'.

But, since y has an immediate predecessor $y-$, the above equation implies that

$$x \le y- \text{ for all } x \in A'.$$

Hence, $y-$ is an upper bound for A', yet $y- < y$: this implies that y is not a minimal upper bound for A', or in other words, that y is not a supremum of A': we arrived at a contradiction. We conclude that A' must have a maximum. $\square$

4.2. **P-Boxes with Zero-One-Valued Upper Cumulative Distribution Functions.** Similarly, we can also determine when a p-box with 0–1-valued $\overline{F}$ is a possibility measure:

Proposition 16. *Assume that $\Omega/\simeq$ is order complete. Let $(\underline{F}, \overline{F})$ be a p-box with 0–1-valued $\overline{F}$, and let $C = \{x \in \Omega^* : \overline{F}(x) = 0\}$. Then, $(\underline{F}, \overline{F})$ is a possibility measure if and only if*

(i) $\underline{F}(x) = \underline{F}(x+)$ for all $x \in \Omega$ that have no immediate successor, and
(ii) C has a maximum,

and in such a case,

$$\overline{E}_{\underline{F},\overline{F}}(A) = 1 - \inf_{y \in A \cap C^c} \underline{F}(y-). \tag{15}$$

Again, for $\overline{E}_{\underline{F},\overline{F}}$ to be a possibility measure, both conditions are still necessary even when $\Omega/\simeq$ is not order complete: the proof in this direction does not require order completeness.

As a special case, we mention that $\overline{E}_{\underline{F},\overline{F}}$ is a possibility measure with possibility distribution

$$\pi(x) = \begin{cases} 1 - \underline{F}(x-) & \text{if } x \in C^c \\ 0 & \text{otherwise,} \end{cases}$$

whenever $\Omega/\simeq$ is finite.

Proof. "only if". Assume that $(\underline{F}, \overline{F})$ is a possibility measure. For every $x \in \Omega$ that has no immediate successor,

$$\underline{F}(x+) = \inf_{x' > x} \underline{F}(x') = \inf_{x' > x} \underline{F}(x'-)$$

where the latter equality holds because for every $x' > x$ there is an x'' such that $x' > x'' > x$; otherwise, x would have an immediate successor. Now, because $\overline{E}_{\underline{F},\overline{F}}(\{x'\}) = \overline{F}(x') - \underline{F}(x'-)$ (see Eq. (5)), $\underline{F}(x'-) \le 1 - \overline{E}_{\underline{F},\overline{F}}(\{x'\})$, whence

$$\le \inf_{x' > x} (1 - \overline{E}_{\underline{F},\overline{F}}(\{x'\})) = 1 - \sup_{x' > x} \overline{E}_{\underline{F},\overline{F}}(\{x'\})$$

and because $(\underline{F}, \overline{F})$ is a possibility measure, by Lemma 14,

$$= 1 - \overline{E}_{\underline{F},\overline{F}}((x, 1_\Omega]) = \underline{F}(x),$$

where last equality follows from Eq. (1). The converse inequality follows from the non-decreasingness of $\underline{F}$.

Next, assume that, ex absurdo, $C = \{x \in \Omega^* : \overline{F}(x) = 0\}$ has no maximum. Since $\overline{F}(x) = \underline{F}(x-) = 0$ for all x in C,

$$\overline{E}_{\underline{F},\overline{F}}(C) = \sup_{x \in C} \overline{E}_{\underline{F},\overline{F}}(\{x\}) = \sup_{x \in C}(\overline{F}(x) - \underline{F}(x-)) = 0.$$

Yet, also,

$$\overline{E}_{\underline{F},\overline{F}}(C) = 1$$

by Eq. (10)—indeed, the second case applies because there is no $x \in C$ such that $C \leq x$, as C has no maximum. We arrived at a contradiction.

Finally, we show that Eq. (15) holds. By Eq. (11),

$$\overline{E}_{\underline{F},\overline{F}}(A) = 1 - \max_{x \in C} \sup_{y \in \Omega^* : y \prec A \cap (x, 1_\Omega]} \underline{F}(y) = 1 - \sup_{y \in \Omega^* : y \prec A \cap (\max C, 1_\Omega]} \underline{F}(y) = \overline{E}_{\underline{F},\overline{F}}(A'),$$

with $A' := A \cap (\max C, 1_\Omega] = A \cap C^c$. Since $\overline{E}_{\underline{F},\overline{F}}$ is a possibility measure, we conclude that

$$\overline{E}_{\underline{F},\overline{F}}(A) = \overline{E}_{\underline{F},\overline{F}}(A') = \sup_{y \in A'} \overline{E}_{\underline{F},\overline{F}}(\{y\}) = \sup_{y \in A'}(1 - \underline{F}(y-)) = 1 - \inf_{y \in A'} \underline{F}(y-),$$

because $\overline{E}_{\underline{F},\overline{F}}(\{y\}) = \overline{F}(y) - \underline{F}(y-) = 1 - \underline{F}(y-)$, since $\overline{F}(y) = 1$ for all $y \in C^c$. Hence, Eq. (15) holds.

"if". The claim is established if we can show that Eq. (15) holds, because then

$$\overline{E}_{\underline{F},\overline{F}}(A) = 1 - \inf_{y \in A \cap C^c} \underline{F}(y-) = \sup_{y \in A \cap C^c}(1 - \underline{F}(y-)) \leq \sup_{y \in A} \overline{E}_{\underline{F},\overline{F}}(\{y\}),$$

and the converse inequality follows from the monotonicity of $\overline{E}_{\underline{F},\overline{F}}$.

Consider any event $A \subseteq \Omega$, and let x be an infimum of $A' = A \cap C^c$ (which exists because $\Omega/ \simeq$ is order complete). If x has an immediate successor, then A' has a minimum (as we will show next), and by Eq. (12),

$$\overline{E}_{\underline{F},\overline{F}}(A) = 1 - \underline{F}(\min A'-) = 1 - \min_{y \in A'} \underline{F}(y-) = 1 - \inf_{y \in A'} \underline{F}(y-).$$

If x has no immediate successor, then either A' has a minimum, and the above argument can be recycled, or A' has no minimum, in which case Eq. (12) implies that

$$\overline{E}_{\underline{F},\overline{F}}(A) = 1 - \underline{F}(x) = 1 - \underline{F}(x+) = 1 - \inf_{y \in A'} \underline{F}(y-).$$

Here the second equality follows from assumption (i) and the last equality holds because

$$\underline{F}(x+) = \inf_{y > \inf A'} \underline{F}(y)$$

and, A' has no minimum, so for every $y > \inf A'$, there is a $y' \in \Omega$ such that $y > y' > \inf A'$, whence

$$= \inf_{y > \inf A'} \sup_{y > y' > \inf A} \underline{F}(y') = \inf_{y > \inf A'} \underline{F}(y-)$$

and, again, A' has no minimum, so for every $y > \inf A'$, there is a $y'' \in A'$ such that $y > y'' > \inf A'$, whence

$$= \inf_{y'' \in A'} \underline{F}(y''-).$$

We are left to prove A' has a minimum whenever x has an immediate successor. Suppose that A' has no minimum. Then it must hold that

$$y > x \text{ for all } y \in A'$$

since otherwise $y \simeq x$ for some $y \in A'$, whereby y would be a minimum of A'.

But, since x has an immediate successor $x+$, the above equation implies that

$$y \geq x+ \text{ for all } y \in A'.$$

Hence, $x+$ is a lower bound for A', yet $x+ > x$: this implies that x is not a maximal lower bound for A', or in other words, that x is not an infimum of A': we arrived at a contradiction. We conclude that A' must have a minimum. $\qquad\square$

4.3. **Necessary and Sufficient Conditions.** Merging Corollary 13 with Propositions 15 and 16 we obtain the following necessary and sufficient conditions for a p-box to be a possibility measure:

Corollary 17. *Assume that $\Omega/\simeq$ is order complete and let $(\underline{F},\overline{F})$ be a p-box. Then $(\underline{F},\overline{F})$ is a possibility measure if and only if either*

(L1) $\underline{F}$ is 0–1-valued,
(L2) $\overline{F}(x) = \overline{F}(x-)$ for all $x \in \Omega$ that have no immediate predecessor, and
(L3) $\{x \in \Omega^* : \underline{F}(x) = 1\}$ has a minimum,

or

(U1) $\overline{F}$ is 0–1-valued,
(U2) $\underline{F}(x) = \underline{F}(x+)$ for all $x \in \Omega$ that have no immediate successor, and
(U3) $\{x \in \Omega^* : \overline{F}(x) = 0\}$ has a maximum.

This result settles the cases where p-boxes reduce to possibility measures. We can now go the other way around, and characterise those cases where possibility measures are p-boxes. Similarly to what happens in the finite setting, we will see that almost all possibility measures can be represented by a p-box.

5. FROM POSSIBILITY MEASURES TO P-BOXES

In this section, we discuss and extend some previous results linking possibility distribution to p-boxes. We show that possibility measures correspond to specific kinds of p-boxes, and that some p-boxes correspond to the conjunction of two possibility distribution.

5.1. **Possibility Measures as Specific P-boxes.** Baudrit and Dubois [1] already discuss the link between possibility measures and p-boxes defined on the real line with the usual ordering, and they show that any possibility measure can be approximated by a p-box, however at the expense of losing some information. We substantially strengthen their result, and even reverse it: we prove that any possibility measure with compact range can be *exactly* represented by a p-box with vacuous lower cumulative distribution function, that is, $\underline{F} = I_{[1_\Omega]_\simeq}$. In other words, generally speaking, possibility measures are a special case of p-boxes on totally preordered spaces.

Theorem 18. *For every possibility measure Π on Ω with possibility distribution π such that $\pi(\Omega) = \{\pi(x) : x \in \Omega\}$ is compact, there is a preorder $\leq$ on Ω and an upper cumulative distribution function $\overline{F}$ such that the p-box $(\underline{F} = I_{[1_\Omega]_\simeq}, \overline{F})$ is a possibility measure with possibility distribution π, that is, such that for all events A:*

$$\overline{E}_{\underline{F},\overline{F}}(A) = \sup_{x \in A} \pi(x).$$

In fact, one may take the preorder $\leq$ to be the one induced by π (so $x \leq y$ whenever $\pi(x) \leqslant \pi(y)$) and $\overline{F} = \pi$.

Proof. Let $\leq$ be the preorder induced by π. Order completeness of $\Omega/\simeq$ is satisfied because $\pi(\Omega)$ is compact with respect to the usual topology on $\mathbb{R}$. Indeed, for any $A \subseteq \Omega$, the supremum and infimum of π over A belong to $\pi(\Omega)$ by its compactness, whence $\pi^{-1}(\inf_{x \in A} \pi(x))$ consists of the infima of A, and $\pi^{-1}(\sup_{x \in A} \pi(x))$ consists of its suprema.

Consider the p-box $(I_{[1_\Omega]_\simeq}, \pi)$. Then, for any $A \subseteq \Omega$, we deduce from Eq. (9) that

$$\overline{E}_{\underline{F},\overline{F}}(A) = \overline{F}(\sup A) = \pi(\sup A) = \sup_{x \in A} \pi(x)$$

because $x \leq y$ for all $x \in A$ if and only if $\pi(x) \leqslant \pi(y)$ for all $x \in A$, by definition of $\leq$, and hence, a minimal upper bound, or supremum, y for A must be one for which $\pi(y) = \sup_{x \in A} \pi(x)$ (and, again, such y exists because $\pi(\Omega)$ is compact). $\qquad\square$

The representing p-box is not necessarily unique:

Example 19. Let $\Omega = \{x_1, x_2\}$ and let Π be the possibility measure determined by the possibility distribution

$$\pi(x_1) = 0.5 \qquad\qquad \pi(x_2) = 1.$$

As proven in Theorem 18, this possibility measure can be obtained if we consider the order $x_1 < x_2$ and the p-box $(\underline{F}_1, \overline{F}_1)$ given by

$$\underline{F}_1(x_1) = 0 \qquad\qquad \underline{F}_1(x_2) = 1$$
$$\overline{F}_1(x_1) = 0.5 \qquad\qquad \overline{F}_1(x_2) = 1.$$

However, we also obtain it if we consider the order $x_2 < x_1$ and the p-box $(\underline{F}_2, \overline{F}_2)$ given by

$$\underline{F}_2(x_1) = 1 \qquad\qquad \underline{F}_2(x_2) = 0.5$$
$$\overline{F}_2(x_1) = 1 \qquad\qquad \overline{F}_2(x_2) = 1.$$

The p-box $(\underline{F}_2, \overline{F}_2)$ induces a possibility measure from Corollary 13, also taking into account that Ω is finite. Moreover, by Eq. (5),

$$\overline{E}_{\underline{F}_2,\overline{F}_2}(x_2) = \overline{F}(x_2) - \underline{F}(x_2-) = 1$$
$$\overline{E}_{\underline{F}_2,\overline{F}_2}(x_1) = \overline{F}(x_1) - \underline{F}(x_1-) = 0.5$$

as with the given ordering, $x_2- = 0_\Omega-$ and $x_1- = x_2$. As a consequence, $\overline{E}_{\underline{F}_2,\overline{F}_2}$ is a possibility measure associated to the possibility distribution π.

There are possibility measures which cannot be represented as p-boxes when $\pi(\Omega)$ is not compact:

Example 20. Let $\Omega = [0,1]$, and consider the possibility distribution given by $\pi(x) = (1+2x)/8$ if $x < 0.5$, $\pi(0.5) = 0.4$ and $\pi(x) = x$ if $x > 0.5$; note that $\pi(\Omega) = [0.125, 0.25) \cup \{0.4\} \cup (0.5, 1]$ is not compact. The ordering induced by π is the usual ordering on $[0,1]$. Let Π be the possibility measure induced by π. We show that there is no p-box $(\underline{F}, \overline{F})$ on $([0,1], \leq)$, regardless of the ordering $\leq$ on $[0,1]$, such that $\overline{E}_{\underline{F},\overline{F}} = \Pi$.

By Corollary 13, if $\overline{E}_{\underline{F},\overline{F}} = \Pi$, then at least one of $\underline{F}$ or $\overline{F}$ is 0-1–valued. Assume first that $\underline{F}$ is 0-1–valued. By Eq. (5), $\overline{E}_{\underline{F},\overline{F}}(\{x\}) = \overline{F}(x) - \underline{F}(x-) = \pi(x)$. Because $\pi(x) > 0$ for all x, it must be that $\underline{F}(x-) = 0$ for all x, so $\overline{F} = \pi$. Because $\overline{F}$ is non-decreasing, $x \leq y$

if and only if $\overline{F}(x) \leqslant \overline{F}(y)$; in other words, $\preceq$ can only be the usual ordering on $[0,1]$ for $(\underline{F}, \overline{F})$ to be a p-box. Hence, $\underline{F} = I_{\{1\}}$.

Now, with $A = [0, 0.5)$, we deduce from Proposition 7 that

$$\overline{E}_{\underline{F}, \overline{F}}(A) = \inf_{A \preceq x} \overline{F}(x) = 0.4 > 0.25 = \sup_{x \in A} \pi(x) = \Pi(A),$$

where the second equality follows because $B^c = \{1\}$. Hence, $\overline{E}_{\underline{F}, \overline{F}}$ does not coincide with Π.

Similarly, if $\overline{F}$ would be 0–1-valued, then we deduce from Eq. (5) that $\overline{F}(x) = 1$ for every x, again because $\pi(x) > 0$ for all x. Therefore, $\underline{F}(x-) = 1 - \pi(x)$ for all x. But, because $\underline{F}$ is non-decreasing, $\preceq$ can only be the inverse of the usual ordering on $[0,1]$ for $(\underline{F}, \overline{F})$ to be a p-box.

This deserves some explanation. We wish to show that $\underline{F}(x-) < \underline{F}(y-)$ implies $x \prec y$. Assume ex absurdo that $x > y$. But, then,

$$\underline{F}(x-) = \sup_{z < x} \underline{F}(z) \geqslant \sup_{z < y} \underline{F}(z) = \underline{F}(y-),$$

a contradiction. It also cannot hold that $x \simeq y$, because in that case $z \prec x$ if and only if $z \prec y$, and whence it would have to hold that $\underline{F}(x-) = \underline{F}(y-)$. Concluding, it must hold that $x \prec y$ whenever $\underline{F}(x-) < \underline{F}(y-)$, or in other words, whenever $x > y$. So, $\preceq$ can only be the inverse of the usual ordering on $[0,1]$ and, in particular, $[0,1]/\simeq$ is order complete.

Now, for $(\underline{F}, \overline{F})$ to induce the possibility measure Π, we know from Corollary 17 that $\underline{F}(x) = \underline{F}(x+)$ for every x that has no immediate successor in with respect to $\preceq$, that is, for every $x < 0$, or equivalently, for every $x > 0$. Whence,

$$\underline{F}(x) = \underline{F}(x+) = \inf_{y > x} \underline{F}(y) = \inf_{y > x} \underline{F}(y-) = 1 - \sup_{y > x} \pi(y) = 1 - \sup_{y < x} \pi(y)$$

for all $x > 0$. This leads to a contradiction: by the definition of π, we have on the one hand,

$$\underline{F}(0.5-) = \sup_{x < 0.5} \underline{F}(x) = \sup_{x > 0.5} \underline{F}(x) = \sup_{x > 0.5} \left(1 - \sup_{y < x} \pi(y)\right) = 0.5$$

and on the other hand,

$$\underline{F}(0.5-) = 1 - \pi(0.5) = 0.6.$$

Concluding, $\overline{E}_{\underline{F}, \overline{F}}$ coincides with Π in neither case.

Another way of relating possibility measures and p-boxes goes via random sets (see for instance [19] and [9]). Possibility measures on ordered spaces can also be obtained via upper probabilities of random sets (see for instance [6, Sections 7.5–7.7] and [21]).

5.2. **P-boxes as Conjunction of Possibility Measures.** In [9], where p-boxes are studied on finite spaces, it is shown that a p-box can be interpreted as the conjunction of two possibility measures, in the sense that $\mathscr{M}(\overline{P}_{\underline{F}, \overline{F}})$ is the intersection of two sets of additive probabilities induced by two possibility measures. The next proposition extends this result to arbitrary totally preordered spaces.

Proposition 21. *Let $(\underline{F}, \overline{F})$ be a p-box such that $(\underline{F}_1 = \underline{F}, \overline{F}_1 = I_\Omega)$ and $(\underline{F}_2 = I_{[1_\Omega]_\simeq}, \overline{F}_2 = \overline{F})$ are possibility measures. Then, $(\underline{F}, \overline{F})$ is the intersection of two possibility measures defined by the distributions*

$$\pi_1(x) = 1 - \underline{F}(x-) \qquad\qquad \pi_2(x) = \overline{F}(x)$$

in the sense that $\mathscr{M}(\overline{P}_{\underline{F}, \overline{F}}) = \mathscr{M}(\Pi_1) \cap \mathscr{M}(\Pi_2)$.

Proof. Using Propositions 15 and 16, and the fact that, by construction, $[0_\Omega, \min B^c] = C^c = \Omega$, it follows readily that π_1 and π_2 are the possibility distributions corresponding to the p-boxes $(\underline{F}_1, \overline{F}_1)$ and $(\underline{F}_2, \overline{F}_2)$.

Thus, by assumption, $\overline{E}_{\underline{F}_1, \overline{F}_1} = \Pi_1$ and $\overline{E}_{\underline{F}_2, \overline{F}_2} = \Pi_2$. Because natural extensions of two coherent lower previsions can only coincide when their credal sets are the same [36, §3.6.1], it follows that

$$\mathscr{M}(\overline{P}_{\underline{F}_1, \overline{F}_1}) = \mathscr{M}(\Pi_1) \qquad\qquad \mathscr{M}(\overline{P}_{\underline{F}_2, \overline{F}_2}) = \mathscr{M}(\Pi_2)$$

We are left to prove that

$$\mathscr{M}(\overline{P}_{\underline{F}, \overline{F}}) = \mathscr{M}(\overline{P}_{\underline{F}_1, \overline{F}_1}) \cap \mathscr{M}(\overline{P}_{\underline{F}_2, \overline{F}_2})$$

but this follows almost trivially after writing down the constraints for each p-box.   □

This suggests a simple way (already mentioned in [9]) to conservatively approximate $\underline{E}_{\underline{F}, \overline{F}}$ by using the two possibility distributions:

$$\max\{\underline{E}_{\pi_{\overline{F}}}(A), \underline{E}_{\pi_{\underline{F}}}(A)\} \leqslant \underline{E}_{\underline{F}, \overline{F}}(A) \leqslant \overline{E}_{\underline{F}, \overline{F}}(A) \leqslant \min\{\overline{E}_{\pi_{\overline{F}}}(A), \overline{E}_{\pi_{\underline{F}}}(A)\}.$$

This approximation is computationally attractive, as it allows us to use the supremum preserving properties of possibility measures. However, as next example shows, the approximation will usually be very conservative, and hence not likely to be helpful.

Example 22. Consider $x < y \in \Omega$. The distance between $\overline{E}_{\underline{F}, \overline{F}}$ and its approximation $\min\{\overline{E}_{\pi_{\overline{F}}}, \overline{E}_{\pi_{\underline{F}}}\}$ on the interval $(x, y]$ is given by

$$\begin{aligned}
&\min\{\overline{E}_{\pi_{\overline{F}}}((x,y]), \overline{E}_{\pi_{\underline{F}}}((x,y])\} - \overline{E}((x,y]) \\
&= \min\{\overline{F}(y), 1 - \underline{F}(x)\} - (\overline{F}(y) - \underline{F}(x)) \\
&= \min\{\underline{F}(x), 1 - \overline{F}(y)\}.
\end{aligned}$$

Therefore, the approximation will be close to the exact value on this set only when either $\underline{F}(x)$ is close to zero or $\overline{F}(y)$ is close to one.

6. NATURAL EXTENSION OF 0–1-VALUED P-BOXES

From Proposition 7, we can derive an expression for the natural extension of a 0–1-valued p-box (see Figure 3):

Proposition 23. *Let* $(\underline{F}, \overline{F})$ *be a p-box where* $\overline{F} = I_{C^c}, \underline{F} = I_{B^c}$ *for some* $C \subseteq B \subseteq \Omega$. *Then for any* $A \subseteq \Omega$,

$$\overline{E}_{\underline{F}, \overline{F}}(A) = \begin{cases} 0 & \text{if there are } x \in C \text{ and } y \in B^c \text{ such that} \\ & A \cap C \leq x \text{ and } A \cap B \cap C^c = \varnothing \text{ and } y < A \cap B^c \\ 1 & \text{otherwise.} \end{cases} \qquad (16)$$

Proof. From Proposition 7, the natural extension of $(\underline{F}, \overline{F})$ is given by

$$\overline{E}_{\underline{F}, \overline{F}}(A) = \begin{cases} \inf_{x \in \Omega^* : A \cap B \leq x} \overline{F}(x) & \text{if } y < A \cap B^c \text{ for at least one } y \in B^c, \\ 1 & \text{otherwise.} \end{cases}$$

Now, if $\overline{F} = I_{C^c}$, the infimum in the first equation is equal to 0 if and only if there is some $x \in C$ such that $A \cap B \leq x$, and equal to 1 otherwise. Finally, note that $A \cap B \leq x$ if and only
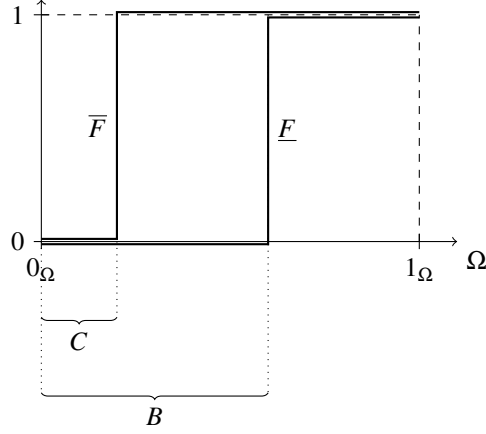
FIGURE 3. A 0–1-valued p-box.

if

$$A \cap B \cap C \subseteq [0_\Omega, x] \cap C \text{ and}$$
$$A \cap B \cap C^c \subseteq [0_\Omega, x] \cap C^c$$

and observe that $B \cap C = C$, $[0_\Omega, x] \cap C = [0_\Omega, x]$, and $[0_\Omega, x] \cap C^c = \emptyset$, to arrive at the conditions in Eq. (16).   □

Moreover, Propositions 15 and 16 allow us to determine when this p-box is a possibility measure:

Proposition 24. *Assume that $\Omega/\simeq$ is order complete. Let $(\underline{F}, \overline{F})$ be a p-box where $\overline{F} = I_{C^c}$, $\underline{F} = I_{B^c}$ for some $C \subseteq B \subseteq \Omega$. Then $(\underline{F}, \overline{F})$ is a possibility measure if and only if C has a maximum and B^c has a minimum. In such a case,*

$$\overline{E}_{\underline{F}, \overline{F}}(A) = \begin{cases} 1 & \text{if } A \cap (\max C, \min B^c] \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

or, in other words, in such a case, $\overline{E}_{\underline{F}, \overline{F}}$ is a possibility measure with possibility distribution

$$\pi(x) = \begin{cases} 1 & \text{if } x \in (\max C, \min B^c] \\ 0 & \text{otherwise.} \end{cases}$$

Proof. "only if". Immediate by Propositions 15 and 16.

"if". By Proposition 15, $(\underline{F}, \overline{F})$ is a possibility measure if and only if B^c has a minimum and $\overline{F}(x) = \overline{F}(x-)$ for every x with no immediate predecessor. The latter condition holds for every $x \in C$, and for every $x \in C^c$ with a predecessor in C^c. Whence, we only need to check whether C^c has a minimum—if not, then every $x \in C^c$ has a predecessor in C^c—and if so, that this minimum has an immediate predecessor (because obviously $1 = \overline{F}(\min C) = \overline{F}(\min C-) = 0$ cannot hold).

Indeed, because C has a maximum, $C^c = (\max C, 1_\Omega]$. So, either C^c has a minimum, in which case $\max C$ must be the immediate predecessor of this minimum, or C^c has no minimum.

The expression for $\overline{E}_{\underline{F},\overline{F}}(A)$ follows from Eq. (16). In that equality, without loss of generality, we can take $x = \max C$ and $y = \min B^c$, and $A \cap C \leq \max C$ is obviously always satisfied, so

$$\overline{E}_{\underline{F},\overline{F}}(A) = \begin{cases} 0 & \text{if } A \cap B \cap C^c = \varnothing \text{ and } \min B^c \prec A \cap B^c \\ 1 & \text{otherwise} \end{cases}$$

So, to establish the desired equality, it suffices to show that $A \cap B \cap C^c = \varnothing$ and $\min B^c \prec A \cap B^c$ if and only if $A \cap (\max C, \min B^c] = \varnothing$.

Indeed, $\min B^c \prec A \cap B^c$ precisely when $\min B^c \notin A$. Moreover,

$$B \cap C^c = [0_\Omega, \min B^c) \cap (\max C, 1_\Omega] = (\max C, \min B^c).$$

So, the desired equivalence is established.   □

In particular, we can characterise under which conditions a precise p-box, i.e., one where $\underline{F} = \overline{F} := F$, induces a possibility measure. The natural extension of precise p-boxes on the unit interval was considered in [24, Section 3.1]. From Proposition 6, the natural extension of F can only be a possibility measure when F is 0–1-valued. If we apply Proposition 23 with $B = C$ we obtain the following:

Corollary 25. *Let $(\underline{F}, \overline{F})$ be a precise p-box where $\overline{F} = \underline{F}$ is 0–1-valued, and let $B = \{x \in \Omega^* : \underline{F}(x) = 0\}$. Then, for every subset A of Ω,*

$$\overline{E}_{\underline{F},\overline{F}}(A) = \begin{cases} 0 & \text{if there are } x \in B, y \in B^c \text{ such that } A \cap B \leq x \text{ and } y \prec A \cap B^c \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Immediate from Proposition 23.   □

Moreover, Proposition 24 allows us to determine when this p-box is a possibility measure:

Corollary 26. *Assume that $\Omega/ \simeq$ is order complete. Let $(\underline{F}, \overline{F})$ be a precise p-box where $\overline{F} = \underline{F}$ is 0–1-valued, and let $B = \{x \in \Omega^* : \underline{F}(x) = 0\}$. Then, $(\underline{F}, \overline{F})$ is a possibility measure if and only if B has a maximum and B^c has a minimum. In that case, for every $A \subseteq \Omega$,*

$$\overline{E}_{\underline{F},\overline{F}}(A) = \begin{cases} 0 & \text{if } \min B^c \notin A \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Immediate by Proposition 24 and Corollary 25.   □

As a consequence, we deduce that a precise 0–1-valued p-box on $(\Omega, \leq) = ([0,1], \leqslant)$ never induces a possibility measure—except when $F = I_{[0,1]}$. Indeed, if $F \neq I_{[0,1]}$, then $B \cap [0,1] \neq \varnothing$, and the maximum of B would need to have an immediate successor (the minimum of B^c), which cannot be for the usual ordering $\leqslant$.

When $F = I_{[0,1]}$, we obtain $\max B = 0-$ and $\min B^c = 0$, whence applying Corollary 26 we deduce that (F, F) is a possibility measure, with possibility distribution

$$\pi(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

To see why the possibility distribution

$$\pi(x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

for any $y > 0$ does not correspond to the precise p-box (F, F) with $F = I_{[y,1]}$, first note that

$$\Pi([0, y)) = \sup_{x \prec y} \pi(x) = 0.$$

But, for Π to be the p-box $\overline{E}_{F,F}$, we also require that

$$\overline{E}_{F,F}([0, y)) = F(y) - F(0-) = 1$$

using Eq. (4), because y has no immediate predecessor. Whence, we arrive at a contradiction.

7. CONSTRUCTING MULTIVARIATE POSSIBILITY MEASURES FROM MARGINALS

In [34], multivariate p-boxes were constructed from marginals. We next apply this construction together with the p-box representation of possibility measures, given by Theorem 18, to build a joint possibility measure from some given marginals. As particular cases, we consider the joint,

   (i) either without any assumptions about dependence or independence between variables, that is, using the Fréchet-Hoeffding bounds [17],
  (ii) or assuming epistemic independence between all variables, which allows us to use the factorization property [7].

Let us consider n variables $X_1, \ldots, X_n$ assuming values in $\mathscr{X}_1, \ldots, \mathscr{X}_n$. Assume that for each variable X_i we are given a possibility measure Π_i with corresponding possibility distribution π_i on $\mathscr{X}_i$. We assume that the range of all marginal possibility distributions is $[0, 1]$; in particular, Theorem 18 applies, and each marginal can be represented by a p-box on $(\mathscr{X}_i, \preceq_i)$, with vacuous $\underline{F}_i$, and $\overline{F}_i = \pi_i$. Remember that the preorder $\preceq_i$ is the one induced by π_i.

7.1. **Multivariate Possibility Measures.**
The construction in [34] employs the following mapping Z, which induces a preorder $\preceq$ on $\Omega = \mathscr{X}_1 \times \cdots \times \mathscr{X}_n$:[2]

$$Z(x_1, \ldots, x_n) = \max_{i=1}^{n} \pi_i(x_i). \tag{17}$$

With this choice of Z, we can easily find the possibility measure which represents the joint as accurately as possible, under any rule of combination of coherent lower probabilities:

Lemma 27. *Let $\odot$ be any rule of combination of coherent upper probabilities, mapping the marginals $\overline{P}_1, \ldots, \overline{P}_n$ to a joint coherent upper probability $\bigodot_{i=1}^{n} \overline{P}_i$ on all events. If there is a continuous function u for which*

$$\bigodot_{i=1}^{n} \overline{P}_i \left(\prod_{i=1}^{n} A_i \right) = u(\overline{P}_1(A_1), \ldots, \overline{P}_n(A_n))$$

for all $A_1 \subseteq \mathscr{X}_1, \ldots, A_n \subseteq \mathscr{X}_n$, then the possibility distribution π defined by

$$\pi(x) = u(Z(x), \ldots, Z(x))$$

induces the least conservative upper cumulative distribution function on $(\Omega, \preceq)$ that dominates the combination $\bigodot_{i=1}^{n} \Pi_i$ of $\Pi_1, \ldots, \Pi_n$.

---

[2]With $\max_{i=1}^{n} \pi_i(x_i)$, we mean $\max\{\pi_i(x_i) : i \in \{1, \ldots, n\}\}$.

Proof. To apply [34, Lem. 22], we first must consider the upper cumulative distribution functions, which in our case coincide with the possibility distributions, as functions on the unit interval $z \in [0,1]$. For the marginal possibility distribution π_i, the preorder is the one induced by π_i itself, so, as a function of z, π_i is simply the identity map:

$$\pi_i(z) = \pi_i(\pi_i^{-1}(z)) = z.$$

Using [34, Lem. 22], the least conservative upper cumulative distribution function on the space $(\Omega, \preceq)$ that dominates the combination $\bigodot_{i=1}^n \Pi_i$ is given by

$$\overline{F}(z) = u(\pi_1(z), \ldots, \pi_n(z)) = u(z, \ldots, z) \; \forall z \in [0,1].$$

As a function of $x \in \Omega$, this means that

$$\overline{F}(x) = u(Z(x), \ldots, Z(x))$$

with the Z that induced $\preceq$, that is, the one defined in Eq. (17).

Now, by Proposition 15, such upper cumulative distribution function corresponds to a possibility measure with possibility distribution

$$\pi(x) = u(Z(x), \ldots, Z(x))$$

whenever $\overline{F}(x) = \overline{F}(x-)$ for all $x \in \Omega$ that have no immediate predecessor, that is, whenever

$$\overline{F}(x) = \sup_{y\,:\,Z(y) < Z(x)} \overline{F}(y)$$

for all x such that $Z(x) > 0$. But this must hold, because (i) the range of Z is $[0,1]$, so $Z(y)$ can get arbitrarily close to $Z(x)$ from below, and (ii) u is continuous, so $\overline{F}(y) = u(Z(y), \ldots, Z(y))$ gets arbitrarily close to $\overline{F}(x) = u(Z(x), \ldots, Z(x))$. $\qquad\square$

7.2. **Natural Extension: The Fréchet Case.** The *natural extension* $\boxtimes_{i=1}^n \overline{P}_i$ of $\overline{P}_1, \ldots, \overline{P}_n$ is the upper envelope of all joint (finitely additive) probability measures whose marginal distributions are compatible with the given marginal upper probabilities. So, the model is completely vacuous (that is, it makes no assumptions) about the dependence structure, as it includes all possible forms of dependence. See [8, p. 120, §3.1] for a rigorous definition. In this paper, we only need the following equality, which is one of the Fréchet bounds (see for instance [36, p. 122, §3.1.1]):

$$\boxtimes_{i=1}^n \overline{P}_i \left(\prod_{i=1}^n A_i \right) = \min_{i=1}^n \overline{P}_i(A_i) \tag{18}$$

for all $A_1 \subseteq \mathscr{X}_1, \ldots, A_n \subseteq \mathscr{X}_n$.

Theorem 28. *The possibility distribution*

$$\pi(x) = \max_{i=1}^n \pi_i(x_i) \tag{19}$$

induces the least conservative upper cumulative distribution function on $(\Omega, \preceq)$ that dominates the natural extension $\boxtimes_{i=1}^n \Pi_i$ of $\Pi_1, \ldots, \Pi_n$.

Proof. Immediate, by Lemma 27 and Eq. (18). $\qquad\square$

In other words, if we consider the marginal credal sets $\mathscr{M}(\Pi_1), \ldots, \mathscr{M}(\Pi_n)$ and consider the set $\mathscr{M}$ of all the finitely additive probabilities on Ω whose $\mathscr{X}_i$-marginals belong to $\mathscr{M}(\Pi_i)$ for $i = 1, \ldots, n$, then $\mathscr{M}$ is included in the credal set of the possibility distribution π as defined in Eq. (19).

Since it is based on very mild assumptions, it is not surprising that the possibility distribution given by Eq. (19) is very uninformative (that is, very close to a vacuous model

where $\pi(x) = 1$ for every x): we shall have $\pi(x) = 1$ as soon as $\pi_i(x_i) = 1$ for some i, even if $\pi_j(x_j) = 0$ for every $j \neq i$. In particular, if one of the marginal possibility distributions is vacuous, then so is π. This also shows that the corresponding possibility measure Π may not have $\Pi_1, \ldots, \Pi_n$ as its marginals.

7.3. **Independent Natural Extension.** In contrast, we can consider joint models which satisfy the property of *epistemic independence* between the different $X_1, \ldots, X_n$. These have been studied in [23] in the case of two marginal possibility measures. The most conservative of these models is called the *independent natural extension* $\otimes_{i=1}^{n} \underline{P}_i$ of $\underline{P}_1, \ldots,$ $\underline{P}_n$. See [7] for a rigorous definition and properties, and [23] for a study of joint possibility measures that satisfy epistemic independence in the case of two variables. In this paper, we only need the following equality for the independent natural extension:

$$\bigotimes_{i=1}^{n} \overline{P}_i \left(\prod_{i=1}^{n} A_i \right) = \prod_{i=1}^{n} \overline{P}_i(A_i) \tag{20}$$

for all $A_1 \subseteq \mathscr{X}_1, \ldots, A_n \subseteq \mathscr{X}_n$.

Theorem 29. *The possibility distribution*

$$\pi(x) = \left(\max_{i=1}^{n} \pi_i(x_i) \right)^n \tag{21}$$

induces the least conservative upper cumulative distribution function on $(\Omega, \preceq)$ that dominates the independent natural extension $\otimes_{i=1}^{n} \Pi_i$ of $\Pi_1, \ldots, \Pi_n$.

Proof. Immediate, by Lemma 27 and Eq. (20). $\qquad\square$

Note, however, that there is no least conservative possibility measure that corresponds to the independent natural extension of possibility measures [23, Sec. 6].

We do not consider the minimum rule and the product rule

$$\min_{i=1}^{n} \pi_i(x_i) \text{ and } \prod_{i=1}^{n} \pi_i(x_i),$$

as their relation with the theory of coherent lower previsions is still unclear. However, we can compare the above approximation with the following outer approximation given by [10, Proposition 1]:

$$\pi(x) = \min_{i=1}^{n} (1 - (1 - \pi_i(x_i))^n). \tag{22}$$

The above equation is an outer approximation in case of *random set independence*, which is slightly more conservative than the independent natural extension [5, Sec. 4], so in particular, it is also an outer approximation of the independent natural extension. Essentially, each distribution π_i is transformed into $1 - (1 - \pi_i)^n$ before applying the minimum rule. It can be expressed more simply as

$$1 - \max_{i=1}^{n} (1 - \pi_i(x_i))^n.$$

If for instance $\pi(x_i) = 1$ for at least one i, then this formula provides a more informative (i.e., lower) upper bound than Theorem 29. On the other hand, when all $\pi(x_i)$ are, say, less than $1/2$, then Theorem 29 does better.

Finally, note that neither Eq. (21) nor Eq. (22) are proper joints, in the sense that, in both cases, the marginals of the joint are outer approximations of the original marginals, and will in general not coincide with the original marginals.

8. Conclusions

Both possibility measures and p-boxes can be seen as coherent upper probabilities. We used this framework to study the relationship between possibility measures and p-boxes. Following [34], we allowed p-boxes on arbitrary totally preordered spaces, whence including p-boxes on finite spaces, on real intervals, and even multivariate ones.

We began by considering the more general case of maxitive measures, and proved that a necessary and sufficient condition for a p-box to be maxitive is that at least one of the cumulative distribution functions of the p-box must be 0–1 valued. Moreover, we determined the natural extension of a p-box in those cases and gave a necessary and sufficient condition for the p-box to be supremum-preserving, i.e., a possibility measure. As special cases, we also studied degenerate p-boxes, and precise 0–1 valued p-boxes.

Secondly, we showed that almost every possibility measure can be represented as a p-box simply by ordering elements by increasing possibility. Hence, in general, p-boxes are more expressive than possibility measures, while still keeping a relatively simple representation and calculus [34], unlike many other models, such as for instance lower previsions and credal sets, which typically require far more advanced techniques, such as linear programming.

Finally, we considered the multivariate case in more detail, by deriving a joint possibility measure from given marginals using the p-box representation established in this paper and results from [34].

In conclusion, we established new connections between both models, strengthening known results from literature, and allowing many results from possibility theory to be embedded into the theory of p-boxes, and vice versa.

As future lines of research, we point out the generalisation of a number of properties of possibility measures to p-boxes, such as the connection with fuzzy logic [39] or the representation by means of graphical structures [3], and the study of the connection of p-boxes with other uncertainty models, such as clouds and random sets.

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