DYNAMICAL ZETA FUNCTIONS FOR TYPICAL EXTENSIONS OF FULL SHIFTS

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Abstract. We consider a family of isometric extensions of the full shift on p symbols (for p a prime) parametrized by a probability space. Using Heath–Brown's work on the Artin conjecture, it is shown that for all but two primes p the set of limit points of the growth rate of periodic points is infinite almost surely. This shows in particular that the dynamical zeta function is not algebraic almost surely.

1. Introduction

The S-integer dynamical systems were introduced in [2]: they are a natural family of isometric extensions of hyperbolic dynamical systems, parametrized by rings of S-integers in $\mathbb{A}$-fields. Their dynamical properties are governed by arithmetic in algebraic number fields or rational function fields depending on the characteristic. A detailed description is in [2], along with some examples; the "random" approach to their study is outlined in [6] and [7]. Applications to a certain class of cellular automata are described in [5].

Our purpose here is to extend a result from [7] concerning typical behaviour for simple examples associated to $\mathbb{A}$-fields of finite characteristic.

Let $K = \mathbb{F}_p(t)$, and following Weil [8, Chapter III] let $P = \{|\cdot|_v\}$ be the set of places (equivalence classes of inequivalent multiplicative valuations) on K. The "finite" elements of P are in one-to-one correspondence with the irreducible polynomials in $\mathbb{F}_p[t]$, with $|f|_{v(t)} = p^{-\mathrm{ord}_v(f)\cdot\deg(f)}$. The "infinite" place is determined by $|f(t)|_\infty = |f(t^{-1})|_t$. With these normalizations the product formula

$$\prod_{v\leq\infty} |f|_v = 1 \text{ for all } f \in K\backslash\{0\} \tag{1}$$

holds. Enumerate the countable set P in some order

$$P = \{|\cdot|_{v_{-1}=\infty}, |\cdot|_{v_0=t}, |\cdot|_{v_1}, \dots\}. \tag{2}$$

Denote by Ω the probability space $\{0,1\}^{\mathbb{N}}$, equipped with the infinite product measure $\mu_\rho = (\rho, 1-\rho)^{\mathbb{N}}$ for some $\rho \in [0,1]$. Each point $\omega = (\omega(k))_{k\geq 1} \in \Omega$ defines a ring R_ω of S-integers in k defined by

$$R_\omega = \{f \in K : |f|_{v_k} \leq 1 \text{ for all } k \geq 1 \text{ such that } \omega(k) = 0\}. \tag{3}$$

Notice that the infinite place and the place corresponding to the irreducible polynomial t are excluded from the condition imposed in (3).

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Example 1. If $\rho = 1$ then μ_ρ-almost surely $\omega(k) = 0$ for all $k \geq 1$ so

$$R_\omega = \{f \in K : |f|_{v_k} \leq 1 \text{ for all } k \geq 1\} = \mathbb{F}_p[t^{\pm 1}].$$

At the opposite extreme, if $\rho = 0$ then μ_ρ-almost surely $\omega(k) = 1$ for all $k \geq 1$ so

$$R_\omega = \mathbb{F}_p(t).$$

For non-atomic measures μ_ρ (that is, for $0 < \rho < 1$), the ring R_ω is a "random" ring, in which each irreducible polynomial $v(t)$ is invertible with independent probability $1 - \rho$.

Definition 2. The dynamical system $\alpha_\omega : X_\omega \to X_\omega$ is the automorphism α_ω of the compact abelian group X_ω dual to the automorphism $f \mapsto tf$ of the ring R_ω.

A basic formula from [2] gives the number of periodic points in such a dynamical system: the points of period n under the map α_ω comprise the set

$$F_n(\alpha_\omega) = \{x \in X_\omega : \alpha_\omega^n(x) = x\},$$

and [2, Lemma 5.2] shows that

$$|F_n(\alpha_\omega)| = |t^n - 1|_\infty \times \prod_{\omega(k)=1} |t^n - 1|_{v_k} = p^n \times \prod_{\omega(k)=1} |t^n - 1|_{v_k}. \tag{4}$$

Equivalently, this may be written

$$\log_p |F_n(\alpha_\omega)| = n - \sum_{\omega(k)=1} \mathrm{ord}_{v_k}(t^n - 1) \cdot \deg(v_k), \tag{5}$$

which indicates how the valuations *excluded* from the condition in (3) reduce the number of points of given period depending on how the corresponding irreducible polynomials divide $t^n - 1$.

As a measure of the regularity of the periodic point behaviour of such a dynamical system, we have the *dynamical zeta function*

$$\zeta_{\alpha_\omega}(z) = \exp \sum_{n=1}^{\infty} |F_n(\alpha_\omega)| \times \frac{z^n}{n}, \tag{6}$$

and the set $\mathcal{L}(\alpha_\omega)$ of limit points of the set $\left\{\frac{1}{n} \log |F_n(\alpha_\omega)|\right\}_{n \in \mathbb{N}}$. It is clear from (5) that the zeta function converges in the disc $|z| < \frac{1}{p}$ and that $\mathcal{L}$ is a subset of $[0, \log p]$.

Example 3. If $\rho = 1$ then (μ_ρ-almost surely) $R_\omega = \mathbb{F}_p[t^{\pm 1}]$, so

$$X_\omega = \{0, 1 \dots, p - 1\}^{\mathbb{Z}},$$

the two-sided shift space on p symbols. The endomorphism α_ω is the left shift

$$(\alpha_\omega(x))_r = x_{r+1}$$

for $x = (x_r) \in X_\omega$. It is clear that there are p^n points of period n under α_ω, which is confirmed by equation (4). The zeta function is rational, $\zeta(z) = \frac{1}{1-pz}$, and $\mathcal{L} = \{\log p\}$ is a singleton.

The opposite example has $\rho = 0$, so (μ_ρ-almost surely) $R_\omega = \mathbb{F}_p(t)$. Here the group X_ω is extremely complicated (it is isomorphic to the quotient of the adele ring $\mathbb{F}_p(t)_{\mathbb{A}}$ by the usual discrete embedded copy of $\mathbb{F}_p(t)$ – see [8, Chapter IV, Section 2]), and $|F_n(\alpha_\omega)| = 1$ for all $n \geq 1$. Once again the zeta function is rational, $\zeta(z) = \frac{1}{1-z}$, and $\mathcal{L} = \{0\}$ is a singleton.

Our result is motivated by two things: an example from [2] and a theorem from [7]. The example corresponds to the (non-random) ring

$$R = \{f \in K : |f|_v \leq 1 \text{ for all } v \neq t^{-1}, t-1\} = \mathbb{F}_p[t][\frac{1}{t-1}].$$

Example 4. (cf. [2, Example 8.5]) The endomorphism $\alpha : X \to X$ dual to $f \mapsto tf$ on the ring $\mathbb{F}_p[t][\frac{1}{t-1}]$ has

$$\mathcal{L}(\alpha) = \left\{ \left(1 - \tfrac{1}{q}\right) \log p : q \in \mathbb{N} \backslash p\mathbb{N} \right\} \cup \{\log p\}. \tag{7}$$

Theorem 5. *Assume that $0 < \rho < 1$. Then, with the possible exception of two primes p, $\mathcal{L}(\alpha_\omega) \supset \{0, \log p\}$ and the zeta function of α_ω is irrational for μ-almost every $\omega \in \Omega$.*

This follows from [7, Theorem 3]. What we prove here is that the infinite collection of limit points seen in Example 4 is typical for the random family of dynamical systems.

Theorem 6. *Assume that $0 < \rho < 1$. Then, with the possible exception of two primes p, $\mathcal{L}(\alpha_\omega)$ is an infinite set containing 0, $\log p$, and a sequence converging to $\log p$, for μ_ρ-almost every $\omega \in \Omega$.*

An element $\ell \in \mathcal{L}(\alpha_\omega)$ corresponds to a singularity at $e^{-\ell}$ for the dynamical zeta function; it follows that Theorem 6 forces the zeta function to be non-algebraic.

Corollary 7. *Assume that $0 < \rho < 1$. Then, with the possible exception of two primes p, the dynamical zeta function of α_ω is μ_ρ-almost surely not an algebraic function.*

2. Proof of Theorem

There are three ingredients to the proof of Theorem 6. The first is a deep result due to Heath–Brown [3] on the Artin conjecture: with the possible exception of two primes p, p is a primitive root mod q for infinitely many primes q.

The second is a trivial consequence of the Borel–Cantelli lemma from probability: if $0 < \rho < 1$ and (n_j) is an increasing sequence of integers, then $\omega(n_j) = 1$ for infinitely many values of j and $\omega(n_j) = 0$ for infinitely many values of j, for μ_ρ-almost every $\omega \in \Omega$. More generally, if (A_j) is an infinite sequence of subsets of $\mathbb{N}$ with bounded cardinality and with $n_j = \min(A_j) \to \infty$, then

$$A_j \subset \{n \in \mathbb{N} : \omega(n) = 1\}$$

for infinitely many j, and

$$A_j \subset \{n \in \mathbb{N} : \omega(n) = 0\}$$

for infinitely many j for μ_ρ-almost every $\omega \in \Omega$.

The third is an elementary fact from Galois theory (see [4, Theorem 2.47] for instance): if q is prime, then the polynomial $t^{q-1} + t^{q-2} + \cdots + 1$ splits over $\mathbb{F}_p$ into $(q-1)/r$ irreducible factors, where r is the least positive integer for which $p^r \equiv 1$ mod q. Writing

$$\pi_n(t) = \prod_{\gcd(s,n)=1} (t - \xi^s) \tag{8}$$

for the nth cyclotomic polynomial (where ξ is a primitive nth root of unity and $\gcd(n, p) = 1$), π_n factorizes over $\mathbb{F}_p[t]$ into $\phi(n)/d$ distinct irreducibles of the same degree, where d is the least positive integer such that $p^d \equiv 1 \bmod n$.

It will be notationally convenient to pass to subsequences without using additional suffixes: the sequence (n_j) below is progressively thinned out as the proof proceeds.

Turning to the proof, we can find an infinite sequence (n_j) of primes greater than p with the property that

$$t^{n_j} - 1 = (t-1)\pi_{n_j}(t),$$

where

$$\pi_{n_j}(t) = t^{n_j-1} + t^{n_j-2} + \cdots + 1 \tag{9}$$

is an irreducible polynomial in $\mathbb{F}_p[t]$. By (9),

$$t^{n_j} \equiv 1 \text{ in } \mathbb{F}_p[t]/\langle \pi_{n_j} \rangle,$$

a field of order p^{n_j-1}. It follows that

$$t^{qn_j} \equiv 1 \text{ in } \mathbb{F}_p[t]/\langle \pi_{n_j} \rangle$$

for any fixed prime $q \neq p$, so

$$|t^{qn_j} - 1|_{\pi_{n_j}} \leq p^{-(n_j-1)}. \tag{10}$$

The first step is to refine (10) by computing the exact order of π_{n_j} in $t^{qn_j} - 1$; it is greater than or equal to one by (10). Assume that

$$t^{qn_j} - 1 = (t-1)\pi_{n_j}^2(t) \cdot h(t) \tag{11}$$

for some polynomial $h \in \mathbb{F}_p[t]$.

Recall that the *order* of a polynomial $g \in \mathbb{F}_p[t]$ with $g(0) \neq 0$ is defined to be the least positive integer e for which g divides $t^e - 1$ (cf. [4, Lemma 3.1, Definition 3.2]). By Theorem 3.11 *ibid*, we have

$$\text{order}(\pi_{n_j}^2) = c_j \cdot p^d, \tag{12}$$

where d is the least positive integer with $p^d \geq 2$ (so $d = 1$). On the other hand, (11) shows that

$$\text{order}(\pi_{n_j}^2) \text{ divides } qn_j. \tag{13}$$

We deduce that $c_j p | qn_j$, so $p | n_j$ and hence $p = n_j$, which is a contradiction. It follows that (11) is impossible, so we obtain a refinement of (10):

$$|t^{qn_j} - 1|_{\pi_{n_j}} = p^{-(n_j-1)}, \tag{14}$$

or equivalently the order of π_{n_j} in $t^{qn_j} - 1$ is exactly one for all j.

This allows the number of points of period qn_j under α_ω to be approximately calculated μ_ρ-almost surely for infinitely many values of j.

Without loss of generality each n_j exceeds q; factorize $t^{qn_j} - 1$ into cyclotomic factors

$$t^{qn_j} - 1 = \prod_{d|qn_j} \pi_d(t) = \pi_{n_j}(t) \cdot \pi_q(t) \cdot \pi_1(t) \cdot \pi_{qn_j}(t). \tag{15}$$

The last factor splits into $\frac{\phi(qn_j)}{d} = \frac{(q-1)(n_j-1)}{d}$ irreducible factors, where d is the least positive integer with $p^d \equiv 1 \bmod qn_j$. By construction, the order of p mod n_j

is (n_j-1), so $d \geq (n_j-1)$. It follows that π_{qn_j} splits into no more than $(q-1)$ irreducible polynomials, each of degree no smaller than (n_j-1). By Borel–Cantelli, we may assume therefore (by passing to a subsequence in j) that

$$k \in \{r \in \mathbb{N} : |\pi_{qn_j}|_{v_r} \neq 1 \text{ for some } j\} \implies \omega(k)=0 \tag{16}$$

μ_ρ-almost surely.

The second and third factors $\pi_1\pi_q$ in (15) are fixed as j varies so we can ignore them: there is a constant $A>0$ for which

$$A \leq \prod_{\omega(k)=1} |\pi_1 \cdot \pi_q|_{v_k} \leq 1 \tag{17}$$

for all $\omega \in \Omega$.

The first factor in (15) is by construction itself irreducible, and by (14) does not appear in any of the other three factors. This fact, together with (16) and (17) gives by (4)

$$p^{qn_j}p^{-(n_j-1)}A \leq |F_{qn_j}(\alpha_\omega)| \leq p^{qn_j}p^{-(n_j-1)} \tag{18}$$

along some infinite sequence of j's, μ_ρ-almost surely. This may be written

$$|F_{qn_j}(\alpha_\omega)| = p^{(q-1)n_j} \cdot B_j, \tag{19}$$

where $pA \leq B_j \leq p$. The growth rate of periodic points along this sequence is then

$$\begin{aligned}\lim_{j\to\infty}\frac{1}{qn_j}\log|F_{qn_j}(\alpha_\omega)| &= \lim_{j\to\infty}\frac{1}{qn_j}\log p^{(q-1)n_j} + \lim_{j\to\infty}\frac{1}{qn_j}\log B_j \\ &= \left(1-\frac{1}{q}\right)\log p.\end{aligned}$$

That is, after excluding two possible values of p, for any prime q distinct from p we have constructed a sequence of times showing that $\left(1-\frac{1}{q}\right)\log p$ lies in $\mathcal{L}(\alpha_\omega)$ for almost every ω. Together with Theorem 5, this proves Theorem 6.

Remark 8. (1) The exact determination of the set $\mathcal{L}(\alpha_\omega)$ for almost every ω in the general case is open. Methods from [7] suggest that there is a single subset $A \subset [0, \log p]$ with the property that $\mathcal{L}(\alpha_\omega) = A$ almost surely, but it is not clear what A is, save that with few exceptions it contains 0 and $\log p$ and the infinite sequence constructed above. In the characteristic zero case [7] again shows that there is a single set A which gives the limit points almost surely, but it is even less accessible: indeed, no single element of A is known (cf. [6], [7]).
(2) That the dynamical zeta function is typically irrational (second part of Theorem 5) would follow at once if it were known that $\omega \neq \omega' \implies \zeta_{\alpha_\omega} \neq \zeta_{\alpha_{\omega'}}$ (since there are only countably many rational zeta functions by [1]). This is clear in simple zero-characteristic examples (see [6]), but it is not clear whether this implication holds in the current setting.
(3) The product measure μ_ρ on Ω is used above for simplicity, but it is clear from the proofs that all that is needed is some kind of Borel–Cantelli property. Thus, for instance, if μ is any probability measure on Ω that is positive on open sets, invariant under the left shift action $T : \Omega \to \Omega$ defined by $T(\omega)_k = \omega_{k+1}$, and ergodic for T, then Theorem 6 holds with respect to μ.

References

[1] R. Bowen and O.E. Lanford III. Zeta functions of restrictions of the shift transformation. In *Global Analysis*, volume 14 of *Proceedings of Symposia in Pure Mathematics*, pages 43–49, Providence, R.I., 1970. American Mathematical Society.

[2] V. Chothi, G. Everest, and T. Ward. S-integer dynamical systems: periodic points. *Journal für die Reine und. angew. Math.*, 489:99–132, 1997.

[3] D.R. Heath-Brown. Artin's conjecture for primitive roots. *Quart. J. Math. Oxford*, 37:27–38, 1986.

[4] R. Lidl and H. Niederreiter. *Introduction to Finite Fields and their Applications*. Cambridge University Press, Cambridge, 1986.

[5] T. Ward. Additive cellular automata and local fields. *Preprint*.

[6] T. Ward. An uncountable family of group automorphisms, and a typical member. *Bulletin of the London Math. Soc.*, 29:577–584, 1997.

[7] T. Ward. Almost all S–integer dynamical systems have many periodic points. *Ergodic Theory and Dynamical Systems*, 18:471–486, 1998.

[8] A. Weil. *Basic Number Theory*. Springer, New York, third edition, 1974.

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