ALMOST ALL S–INTEGER DYNAMICAL SYSTEMS HAVE MANY PERIODIC POINTS

T.B. WARD

Abstract. We show that for almost every ergodic S–integer dynamical system the radius of convergence of the dynamical zeta function is no larger than $\exp(-\frac{1}{2}h_{top}) < 1$. In the arithmetic case almost every zeta function is irrational.

We conjecture that for almost every ergodic S–integer dynamical system the radius of convergence of the zeta function is exactly $\exp(-h_{top}) < 1$ and the zeta function is irrational.

In an important geometric case (the S–integer systems corresponding to isometric extensions of the full p–shift or, more generally, linear algebraic cellular automata on the full p–shift) we show that the conjecture holds with the possible exception of at most two primes p.

Finally, we explicitly describe the structure of S–integer dynamical systems as isometric extensions of (quasi–)hyperbolic dynamical systems.

1. Introduction

The S–integer dynamical systems were introduced in [3], and the question of typical behaviour for one family of these systems was considered in [13] (though of course in the arithmetic case such dynamical systems appear in the work of Rokhlin and Halmos). We first define them: a complete description with references and examples is in [3]. They are an arithmetically natural class of isometric extensions of familiar maps like toral endomorphisms or algebraic cellular automata.

Let k be an $\mathbb{A}$–field (that is, an algebraic number field or a rational function field with positive characteristic), with set of places $P(k)$ and infinite places $P_\infty(k)$. Let $S \subset P(k)\backslash P_\infty(k)$ be a set of finite places, define

$$R_S = \{x \in k \mid |x|_\nu \leq 1 \text{ for all } \nu \notin S \cup P_\infty(k)\}$$

to be the associated ring of S–integers, and let ξ be any element of $R_S\backslash\{0\}$. Then the continuous endomorphism $\alpha = \alpha^{(k,S,\xi)}$ of the compact abelian group $X = X^{(k,S)} = \widehat{R_S}$ dual to the monomorphism $x \mapsto \xi x$ of R_S is the S–integer dynamical system associated to the data k, S, ξ. The number of points with period n under α is given by

$$f_n(\alpha) = \prod_{\nu \in S \cup P_\infty(k)} |\xi^n - 1|_\nu \tag{1}$$

Date: April 29 1997.

1991 *Mathematics Subject Classification.* 22D40, 58F20.

The author gratefully acknowledges support from the Royal Society and the London Mathematical Society, and the hospitality of the Erwin Schrödinger Institute in Vienna where this work was completed.

so long as ξ is not a root of unity in k (see Section 5 of [3]; this condition is equivalent to ergodicity for α). Since f_n is finite for all n, the dynamical zeta function of α,

$$\zeta_\alpha(s) = \exp \sum_{n=1}^{\infty} f_n \times \frac{s^n}{n}$$

is a well–defined formal series. In fact simple estimates (see Section 6 of [3]) show that the radius of convergence of the zeta function lies in $(0, 1]$ for any S–integer system.

The topological entropy of α is found in [3],

$$h_{top}(\alpha) = \sum_{\nu \in S \cup P_\infty(k)} \log^+ |\xi|_\nu.$$

From the complete description of the set of places of an $\mathbb{A}$–field in Chapter III, Section 1 of [14], the set $P(k)$ is countably infinite and the set $P_\infty(k)$ is finite. Given $\xi \in k \backslash \{0\}$ not a unit root, let $\omega_1, \dots, \omega_s$ be all the finite places of k for which $|\xi|_{\omega_j} > 1$. Write

$$P(k) \backslash P_\infty(k) = \{\omega_1, \dots, \omega_s, \nu_1, \nu_2, \dots\}, \qquad (2)$$

and define a map ω_k from the subsets of $P(k)\backslash P_\infty(k)$ containing $\{\omega_1, \dots, \omega_s\}$ to $\{0, 1\}^{\mathbb{N}}$ by $\omega_k(S)(n) = 1$ if and only if $\nu_n \in S$. The $(\rho, 1 - \rho)$–independent measure on $\{0, 1\}^{\mathbb{N}}$ with $\rho \in (0, 1)$ defines via the bijection ω_k a probability measure $\mu_k^\rho = \mu_{k,\xi}^\rho$ on the set

$$\Omega_\xi(k) = \{S \mid \{\omega_1, \dots, \omega_s\} \subset S \subset P(k)\backslash P_\infty(k)\}.$$

Let $U : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ be the add–and–carry odometer (or von–Neumann Kakutani adding machine) which preserves the $(\frac{1}{2}, \frac{1}{2})$ independent measure on $\{0, 1\}^{\mathbb{N}}$ and is ergodic (by Theorem 1.9 in [12]: it is enough to know that the subgroup generated by 1 in the compact group $\{0, 1\}^{\mathbb{N}} = \mathbb{Z}_2$ of 2–adic integers is dense). Let $V : \Omega_\xi(k) \to \Omega_\xi(k)$ be defined by $V(S) = \omega_k^{-1}(U\omega_k(S))$. Then V is a $\mu_k^{1/2}$–preserving, invertible, ergodic transformation on $\Omega_\xi(k)$, called the odometer. We shall often be dealing only with the symmetrical measure with $\rho = \frac{1}{2}$, so write $\mu_k = \mu_k^{1/2}$. The phrase "almost every" unadorned will be used for the $\rho = \frac{1}{2}$ measure only.

Recall that the places of the rational function field $\mathbb{F}_p(t)$ are in one–to–one correspondence with the irreducible polynomials together with one "infinite" place with valuation written $|\cdot|_\infty$: this is non–Archimedean and has $|t|_\infty = p$.

The periodic point behaviour for a given ξ is expected to behave as follows.

Conjecture. Given ξ not a unit root in the $\mathbb{A}$–field k, for μ_k^ρ–almost every S in $\Omega_\xi(k)$,

$$\limsup_{n\to\infty} \frac{1}{n} \log f_n \left(\alpha^{(k,S,\xi)}\right) = h_{top}\left(\alpha^{(k,S,\xi)}\right) > 0,$$

$$\liminf_{n\to\infty} \frac{1}{n} \log f_n \left(\alpha^{(k,S,\xi)}\right) = 0,$$

and the dynamical zeta function is irrational.

The excluded atomic measures given by $\rho \in \{0, 1\}$ give the two extremes with exceptional behaviour. For $\rho = 0$, $S = P(k)\backslash P_\infty(k)$ and $f_n(\alpha^{(S)}) = 1$ for all n.

For $\rho = 1$, $S = \emptyset$ and the upper and lower limits are both equal to the entropy by Lemma 5.

Our purpose here is to prove some weaker versions and special cases of this conjecture, and to indicate a connection between the conjecture and a weak generalised version of the Mersenne prime problem.

Theorem 1. *Let k be an $\mathbb{A}$–field, and assume that $\xi \in k\backslash\{0\}$ is not a unit root. Then for μ_k–almost every $S \in \Omega_\xi(k)$, the radius of convergence of the dynamical zeta function of $\alpha^{(k,S,\xi)}$ is no larger than* $\exp(-\frac{1}{2}h_{top}(\alpha^{(k,S,\xi)})) < 1$.

The detail of the proof of Theorem 1 depends on the characteristic of k: when k is an algebraic number field we call the corresponding systems *arithmetic*, when k is a rational function field we call them *geometric*. Some basic estimates from [3] are needed: for completeness these are reproduced in an appendix. The strategy of the proof of Theorem 1 is as follows. First we assume that $\limsup f_n^{1/n} = 0$ μ_k–a.e. A simple argument using the Artin product formula shows that this leads to a contradiction. It follows that the set E of those S for which $\limsup f_n^{1/n}$ is positive has positive measure. On the other hand, the odometer transformation on $\Omega_\xi(k)$ is μ_k–preserving and ergodic, and preserves E. The conclusion is that E is of full μ_k–measure.

A subset $S \subset P(k)\backslash P_\infty(k)$ has density δ if $\frac{1}{n}|\{j \mid \omega_k(S)(j) = 1, j \le n\}| \longrightarrow \delta$ as $n \to \infty$.

Corollary 1. *If a and b are coprime integers, then almost every subset S of $\Omega_{a/b}(\mathbb{Q})$ has density $\frac{1}{2}$ and has*

$$\limsup_{n\to\infty} |a^n - b^n|^{1/n} \times \prod_{p\in S} |a^n - b^n|_p^{1/n} \ge \sqrt{\max\{|a|, |b|\}} > 1. \tag{3}$$

If f and g are coprime elements of $\mathbb{F}_p[t]$, then almost every subset S of $\Omega_{f/g}(\mathbb{F}_p(t))$ has density $\frac{1}{2}$ and has

$$\limsup_{n\to\infty} |f^n - g^n|_\infty^{1/n} \times \prod_{\nu\in S} |f^n - g^n|_\nu^{1/n} \ge \sqrt{\max\{p^{\deg(f)}, p^{\deg(g)}\}} > 1. \tag{4}$$

Proof. Let $k = \mathbb{Q}$, $\xi = \frac{a}{b}$. For any set U containing the finite set $T = \{\nu \mid |b|_\nu \neq 1\}$,

$$f_n\left(\alpha^{(\mathbb{Q},U,\xi)}\right) = \left|\left(\tfrac{a}{b}\right)^n - 1\right| \times \prod_{\nu\in T} \left|\left(\tfrac{a}{b}\right)^n - 1\right|_\nu \times \prod_{\nu\in U\backslash T} \left|\left(\tfrac{a}{b}\right)^n - 1\right|_\nu$$

$$= |a^n - b^n| \times \prod_{\nu\in U\backslash T} |a^n - b^n|_\nu$$

since for any $\nu \in U\backslash T$ we have $|b|_\nu = 1$. The set S may therefore be chosen in the intersection of the full measure set for which (3) holds (by Theorem 1) and the set of those S for which $\omega_{\mathbb{Q}}(S)$ is a normal sequence.

The geometric case is proved in the same way. □

For integers the order of quantifiers may be reversed: if ξ is an integer in the $\mathbb{A}$–field k, then $\Omega_\xi(k) = \Omega(k) = P(k)\backslash P_\infty(k)$, so we may intersect over the sets in Corollary 1 for all integers.

Corollary 2. *For almost every subset S of the set of rational primes, and for every integer $a \neq \pm 1$,*

$$\limsup_{n\to\infty} |a^n - 1|^{1/n} \times \prod_{p\in S} |a^n - 1|_p^{1/n} \geq \sqrt{|a|}.$$

For almost every subset S of the set of finite places of $\mathbb{F}_p(t)$, and for every non–constant polynomial $f \in \mathbb{F}_p[t]$,

$$\limsup_{n\to\infty} |f^n - 1|_\infty^{1/n} \times \prod_{\nu\in S} |f^n - 1|_\nu^{1/n} \geq \sqrt{p^{\deg(f)}}.$$

Theorem 2. *Let k be an algebraic number field, and assume that $\xi \in k\backslash\{0\}$ is not a unit root. Then for μ_k^ρ–almost every $S \in \Omega_\xi(k)$, the dynamical zeta function of $\alpha^{(k,S,\xi)}$ is irrational.*

Remark 1. (i) In [13] the case $k = \mathbb{Q}$, $\xi = 2$, $\rho = \frac{1}{2}$ is considered: for $S = \emptyset$ this is the circle–doubling map. It is clear that the arithmetic of the case $\xi = 2$ is unique, since expressions of the form $a^n - 1$ can only be prime if $a = 2$. It is shown there that with positive $\mu_{\mathbb{Q}}$–probability the radius of convergence is smaller than one, and that if there is a K for which there are infinitely many values of n for which $2^n - 1$ has no more than K prime factors then with $\mu_{\mathbb{Q}}$–probability one the radius of convergence is exactly $\frac{1}{2}$. This result is generalised in Theorem 4 below. In particular, if there are infinitely many Mersenne primes ($K = 1$) then the radius of convergence is $\frac{1}{2}$. It is also shown in [13] that the zeta function is almost surely irrational.

(ii) There are many sets S for which the radius of convergence is one: according to Example 9.5 of [3], if k is an algebraic number field and S comprises all but finitely many places, then the radius of convergence is one. The simplest instance of this is the case $S = P(k)\backslash P_\infty(k)$: by the Artin product formula (1) shows that $f_n(\alpha) = 1$ for all n.

(iii) Is there a syndetic set S (that is, a set for which $1's$ appear in $\omega_k(S)$ with bounded gaps) with (3)?

The natural geometric analogue of the simplest arithmetic case $k = \mathbb{Q}, \xi = 2$ is the family of isometric extensions of the full p–shift given by $k = \mathbb{F}_p(t), \xi = t$. In this setting the Mersenne prime problem becomes the following: is the polynomial $1 + t + t^2 + \cdots + t^n$ irreducible over $\mathbb{F}_p$ infinitely often? A consequence of Heath–Brown's work on the Artin conjecture is that this is almost solved, and using his work we show that the natural conjectures can all be proved for this one geometric example. The argument immediately extends to the family of isometric extensions of the linear cellular automata given by $k = \mathbb{F}_p(t)$, $\xi = at + b$ ($a \in \mathbb{F}_p\backslash\{0\}$).

Theorem 3. *Let $k = \mathbb{F}_p(t)$, $\xi = at + b$ ($a \in \mathbb{F}_p\backslash\{0\}$), and $\alpha^{(S)} = \alpha^{(k,S,\xi)}$. Then, excepting at most two primes p, for μ_k^ρ–almost every $S \in \Omega_\xi(k)$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log f_n\left(\alpha^{(S)}\right) = \log p = h_{top}\left(\alpha^{(S)}\right),$$

$$\liminf_{n\to\infty} \frac{1}{n} \log f_n\left(\alpha^{(S)}\right) = 0,$$

and the dynamical zeta function of $\alpha^{(S)}$ is irrational.

Corollary 3. *One of the S–integer systems given by $\xi = t$, $k = \mathbb{F}_2(t)$, $\mathbb{F}_3(t)$, or $\mathbb{F}_5(t)$ satisfies the conjecture.*

That is, the property of isometric extensions described by the conjecture holds for one of the full 2–, 3– or 5– shift.

The arithmetic case seems less accessible: Theorem 9.3 in [3] shows that for at least one of the systems given by $k = \mathbb{Q}$, $\xi = 2, 3$ or 5, there is an infinite set S for which $\limsup_{n\to\infty} \frac{1}{n}\log(f_n(\alpha^{(S)})) = h_{top}(\alpha^{(S)})$. Thus the lim sup part of the conjecture holds for an uncountable (but μ_k–null) set of S.

The first two parts of the basic conjecture would follow from the solution to a generalization of the Mersenne prime problem. There does not however seem to be any particular reason to expect such a statement to be true: see [11] for a survey of related questions for the case $k = \mathbb{Q}$, $\xi = 2$.

Theorem 4. *If, for any $\mathbb{A}$–field k and $\xi \in k\backslash\{0\}$ not a unit root, the set*

$$P_n = \{\nu \in P(k) \mid |\xi^n - 1|_\nu \neq 1\}$$

is bounded in cardinality for infinitely many n, then for μ_k–almost every $S \in \Omega_\xi(k)$,

$$\limsup_{n\to\infty} \frac{1}{n} \log f_n\left(\alpha^{(k,S,\xi)}\right) = h_{top}\left(\alpha^{(k,S,\xi)}\right) > 0$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log f_n\left(\alpha^{(k,S,\xi)}\right) = 0.$$

Finally, we describe explicitly the structure of any S–integer dynamical system as an isometric extension of a (quasi–)hyperbolic base system. Non–hyperbolicity in the base can only occur in the infinite places.

Recall from [7] that an ergodic toral endomorphism is called quasihyperbolic if the corresponding integer matrix has an eigenvalue with unit modulus, and from [14] that for each non–Archimedean place ν of an $\mathbb{A}$–field the corresponding completion k_ν has a maximal compact subring $r_\nu = \{x \in k_\nu \mid |x|_\nu \leq 1\}$. For consistency, we call an ergodic S–integer system hyperbolic if it is expansive (this accords with hyperbolicity meaning that the "eigenvalues" are not of unit modulus) and quasihyperbolic if the only unit modulus eigenvalues appear in the infinite places.

Theorem 5. *For any k, S, ξ (ξ not a unit root), let*

$$H = \{\nu \in P(k) \mid |\xi|_\nu \neq 1\} \cap S. \tag{5}$$

Then $\alpha^{(k,S,\xi)}$ is an isometric extension of $\alpha^{(k,H,\xi)}$. The action on the fibre above the identity is isometric to multiplication by ξ on $\prod_{\nu\in S\backslash H} r_\nu$, and this map is an isometry. For each $\nu \in H \cup P_\infty(k)$, the map $x \mapsto \xi \cdot x$ on the field k_ν is hyperbolic unless ν is infinite, in which case the map may be quasihyperbolic.

I thank Sanju Velani for asking if the set E is invariant under an ergodic transformation, Graham Everest for various lessons in arithmetic, and Klaus Schmidt for pointing out that the lower bound $\exp(-\frac{1}{2}h_{top})$ follows from these methods, and for Remark 2(i).

2. Proof of Theorem 1

Let $\mathcal{P} = \{2, 3, 5, 7, \dots\}$ denote the rational primes.

Lemma 1. *For any $\mathbb{A}$–field k, and for ξ not a unit root in k, the set*

$$E = \{S \mid \limsup_{n\to\infty; n\in\mathcal{P}} \frac{1}{n} \log f_n\left(\alpha^{(k,S,\xi)}\right) > 0\}$$

has positive μ_k–measure.

Notice that the set E is measurable: since a given natural number (of periodic points) is divisible by only finitely many primes, for fixed n the function sending S to $\frac{1}{n}\log f_n\left(\alpha^{(k,S,\xi)}\right)$ is continuous (with the product topology on $\{S\}$ identified with $\{0,1\}^{\mathbb{N}}$). It follows that $\limsup_{n\to\infty; n\in\mathcal{P}} \frac{1}{n}\log f_n\left(\alpha^{(k,S,\xi)}\right)$ is a measurable function of S, so the set of points on which it is positive is a measurable set.

Proof. Let $\bar{S} = S \cup P_\infty(k)$, and assume that E has zero measure. Then by (1) we have for a.e. S

$$\lim_{n\to\infty; n\in\mathcal{P}} \frac{1}{n} \log \prod_{\nu\in\bar{S}} |\xi^n - 1|_\nu = 0. \tag{6}$$

By Lemma 5, we know that

$$\lim_{n\to\infty; n\in\mathcal{P}} \frac{1}{n} \log \prod_{\nu\in\bar{S}; |\xi|_\nu\neq 1} |\xi^n - 1|_\nu = h = h_{top}\left(\alpha^{(k,S,\xi)}\right) > 0. \tag{7}$$

Now define a new set of places $\bar{S}'$ by

$$\bar{S}' = \{\nu \in \bar{S} \mid |\xi|_\nu \neq 1\} \cup \{\nu \in P(k) \cup P_\infty(k) \mid \nu \notin \bar{S}, |\xi|_\nu = 1\}.$$

By the product formula, for any $\eta \in k\backslash\{0\}$

$$\prod_{\nu\in\bar{S}'} |\eta|_\nu \times \prod_{\nu\in\bar{S}} |\eta|_\nu = \prod_{\nu\in\bar{S}; |\xi|_\nu\neq 1} |\eta|_\nu. \tag{8}$$

Now (6), (7), (8) together imply that for a.e. S,

$$\lim_{n\to\infty; n\in\mathcal{P}} \frac{1}{n} \log \prod_{\nu\in\bar{S}'} |\xi^n - 1|_\nu = h > 0. \tag{9}$$

The map $\bar{S} \to \bar{S}'$ induces (by restriction to the finite places) a μ_k–preserving involution on $\Omega_\xi(k)$, so (9) contradicts (6). We conclude that

$$\mu_k\left(\{S \mid \limsup_{n\to\infty; n\in\mathcal{P}} \frac{1}{n} \log f_n\left(\alpha^{(k,S,\xi)}\right) > 0\}\right) > 0.$$

□

Notice that E does not contain any set S with $\omega_k(S)(n) = 1$ for all n. So without loss of generality, any set $S \in E$ may be written

$$S = \{\nu_{n(1)}, \nu_{n(2)}, \nu_{n(3)}, \dots\};$$

with $n(1) < n(2) < n(3) < \dots$ and $n(j) = j$ only finitely often: for $j = 1, \dots, r$ say. Then

$$V(S) = \{\nu_{m(1)}, \nu_{m(2)}, \nu_{m(3)}, \dots\};$$

where $m(1)=n(r)+1, m(\ell)=n(r+\ell-1)$ for $\ell\geq 2$ if $n(1)=1$, and $m(1)=1, m(\ell)=n(\ell-1)$ for $\ell\geq 2$ if $n(1)>1$. By assumption, for any $S\in E$ there is a sequence $n_j\to\infty$ in $\mathcal{P}$ for which

$$\frac{1}{n_j}\log\prod_{\nu\in S\cup P_\infty(k)}|\xi^{n_j}-1|_\nu\longrightarrow h_0>0. \tag{10}$$

Assume first that $n(1)=1$. Then

$$\begin{aligned}\frac{1}{n_j}\log\prod_{\nu\in V(S)\cup P_\infty(k)}|\xi^{n_j}-1|_\nu &= \frac{1}{n_j}\log\prod_{\nu\in S\cup P_\infty(k)}|\xi^{n_j}-1|_\nu\\ &- \frac{1}{n_j}\log\prod_{\ell=1,\dots,r}|\xi^{n_j}-1|_{\nu_{n(\ell)}}\\ &+ \frac{1}{n_j}\log|\xi^{n_j}-1|_{\nu_{m(1)}}.\end{aligned}$$

By the basic estimates in the Appendix (Lemma 6 and Lemma 7), we see that the last two terms above converge along $\mathcal{P}$ to zero, so the left hand side converges along $\mathcal{P}$ to $h_0>0$ by (10), showing that $V(S)\in E$.

If $n(1)>1$ then

$$\frac{1}{n_j}\log\prod_{\nu\in V(S)\cup P_\infty(k)}|\xi^{n_j}-1|_\nu=\frac{1}{n_j}\log\prod_{\nu\in S\cup P_\infty(k)}|\xi^{n_j}-1|_\nu+\frac{1}{n_j}\log|\xi^{n_j}-1|_{\nu_{m(1)}},$$

and the basic estimates in the Appendix show that the last term converges along $\mathcal{P}$ to zero, showing again that $V(S)\in E$.

Indeed, V preserves the value of the upper limit, so it is almost everywhere constant. If

$$\limsup_{n\to\infty;n\in\mathcal{P}}\frac{1}{n}\log f_n\left(\alpha^{(k,S,\xi)}\right)<\frac{1}{2}h,$$

then by (7) and (8)

$$\frac{1}{2}h<\liminf_{n\to\infty;n\in\mathcal{P}}\frac{1}{n}\log f_n\left(\alpha^{(k,S,\xi)}\right)<\limsup_{n\to\infty;n\in\mathcal{P}}\frac{1}{n}\log f_n\left(\alpha^{(k,S,\xi)}\right)$$

almost everywhere.

This proves Theorem 1.

Remark 2. (i) The second part of the proof of Theorem 1 depends only on the following: Lemma 6 and Lemma 7 say that modifying the set S in finitely many places does not affect the upper and lower growth rates. Thus the ergodic μ_k^ρ–preserving action of the finitary symmetric group on $\Omega_\xi(k)$ also preserves the upper limit. Thus, if

$$\limsup_{n\to\infty}\frac{1}{n}\log f_n\left(\alpha^{(k,S,\xi)}\right)=h_{top}\left(\alpha^{(k,S,\xi)}\right)$$

on a positive μ_k^ρ–measure set, then the same is true μ_k^ρ–almost everywhere, and so

$$\liminf_{n\to\infty}\frac{1}{n}\log f_n\left(\alpha^{(k,S,\xi)}\right)=0$$

$\mu_k^{1-\rho}$–almost everywhere. Similarly, if

$$\liminf_{n\to\infty}\frac{1}{n}\log f_n\left(\alpha^{(k,S,\xi)}\right)=0$$

on a positive μ_k^ρ–measure set, then the same is true μ_k^ρ–almost everywhere, and

$$\limsup_{n\to\infty}\frac{1}{n}\log f_n\left(\alpha^{(k,S,\xi)}\right)=h_{top}\left(\alpha^{(k,S,\xi)}\right)$$

$\mu_k^{1-\rho}$–almost everywhere.
(ii) Similarly, if

$$\limsup_{n\to\infty}\frac{1}{n}\log f_n\left(\alpha^{(k,S,\xi)}\right)<h_{top}\left(\alpha^{(k,S,\xi)}\right)=\sum_{\nu\in S\cup P_\infty(k)}\log^+|\xi|_\nu$$

for a positive μ_k^ρ–measure set, then

$$\liminf_{n\to\infty}\frac{1}{n}\log f_n\left(\alpha^{(k,S,\xi)}\right)>0$$

for a positive $\mu_k^{1-\rho}$–measure set.

3. Zeta functions in the arithmetic case

Let k be a fixed algebraic number field and ξ a non–zero element of k that is not a unit root. For each finite place ν of k, the valuation $|\ |_\nu$ restricted to $\mathbb{Q}\subset k$ is equivalent to a p–adic valuation $|\ |_p$ for a unique rational prime $p\in\mathcal{P}$; in this case write $\nu|p$. By Theorem 1, Chapter III§1 of [14] there are only finitely many places ν with $\nu|p$ for a fixed p; indeed by Chapter III§4 of [14] the number of places above a given p is bounded by $[k:\mathbb{Q}]$.

Lemma 2. *If*

$$\mu_k^\rho\left(\{S\in\Omega_\xi(k)\mid\zeta_{\alpha^{(S)}}\ \textit{is irrational}\ \}\right)<1$$

then there is a function ζ for which

$$\{S\in\Omega_\xi(k)\mid\zeta_{\alpha^{(S)}}=\zeta\}$$

has positive μ_k^ρ–measure.

As in the discussion after Lemma 1, it should be pointed out that the set in question is measurable. By the same argument, after identifying the set of S's with $\{0,1\}^{\mathbb{N}}$ and the set of dynamical zeta functions with $\mathbb{N}^{\mathbb{N}}$ (both with product topology), the function $S\mapsto\zeta_\alpha(\cdot)$ is continuous. On the other hand, there are only countably many rational zeta functions by [1], so the set of irrational ones is measurable.

Proof. According to [1] there are only countably many rational dynamical zeta functions. It follows that the complement of the set $\{S\in\Omega_\xi(k)\mid\zeta_{\alpha^{(S)}}$ is irrational $\}$ has positive μ_k^ρ–measure and is a countable union of sets on which the dynamical zeta function is constant (and rational). One of these sets must therefore have positive measure. □

Lemma 3. *In any positive μ_k^ρ–measure subset of $\Omega_\xi(k)$ there are elements S_0, S_1 for which $\alpha^{(S_0)}$ and $\alpha^{(S_1)}$ have distinct dynamical zeta functions.*

Proof. Let $C\subset\Omega_\xi(k)$ have $\mu_k^\rho(C)>0$. Since the number of ν above each p is globally bounded by $d=[k:\mathbb{Q}]$, the independent sets

$$A_p=\{S\mid\exists\text{ exactly one }\nu\in S,\nu|p\}$$

all have $\mu_k^\rho(A_p) \in (d\rho^d, 1]$. It follows that

$$\mu_k^\rho \left(\{S \mid \exists\ P_0 \text{ infinite such that}, \forall\ p \in P_0\ \exists \text{ exactly one } \nu \in S, \nu | p\ \}\right) = 1$$

by Borel–Cantelli. It follows that in C we may find S_0 with the property that

$$P_0 = \{p \in \mathcal{P} \mid\ \exists \text{ one } \nu \in S_0, \nu | p\}$$

is infinite. Then by Borel–Cantelli, the set $\{S \in \Omega_\xi(k) \mid \forall\ p \in P_0,\ \exists\ \nu \in S, \nu | p\}$ is a null set. So there is a set $S_1 \in C$, and infinitely many primes p for which there is exactly one place $\nu \in S_0$ with $\nu | p$ but there is no place $\nu \in S_1$ with $\nu | p$. Pick any one of these primes and consider the distinguished place $\nu | p$ of k for which $\nu \in S_0$ and $\nu \notin S_1$.

If $|\xi|_\nu > 1$ then since $\xi \in R_{S_0} \cap R_{S_1}$ we have $\nu \in S_0 \cap S_1$, which is impossible by construction.

If $|\xi|_\nu < 1$ then $|\xi^n - 1|_\nu = 1$ for all $n \geq 1$. This means that the p–part of the periodic point data for the two systems is identical. In this case, move to the next prime p in the infinite set constructed above. Since $\{\nu \mid |\xi|_\nu < 1\}$ is finite for any $\xi \in k\backslash\{0\}$, this process must terminate with a ν for which $|\xi|_\nu \geq 1$.

If $|\xi|_\nu = 1$, then choose a prime element $\pi \in k_\nu$ and write

$$\xi = a_0 + a_1\pi + a_2\pi^2 + \dots$$

where each $a_j \in \mathbb{F}_q$, the residue class field of k_ν. Since $\mathbb{F}_q^*$ is cyclic, it follows that $\xi^{(p-1)} = 1 + \epsilon$, with $|\epsilon|_\nu < 1$.

It is clear that $|\xi^n - 1|_\nu$ is some (rational) power of p, so in either case the prime decomposition of f_n shows that the zeta functions are distinct. □

Theorem 2 follows.

4. Extensions of linear cellular automata on the full p–shift

Notice that the dynamical systems given by $k = \mathbb{F}_p(t)$, $\xi = at + b$ $(a \in \mathbb{F}_p\backslash\{0\})$ comprise a family of isometric extensions of linear algebraic cellular automata. To see this, recall from [14] that $\mathbb{F}_p(t)$ has one distinguished "infinite" place (so–called despite the fact that the corresponding completion is non–Archimedean) labelled t^{-1}; the corresponding valuation has $|t|_{t^{-1}} = p$. For $S = \emptyset$, $\alpha^{(k,S,\xi)}$ is the map given by

$$\left(\alpha^{(k,S,\xi)}x\right)_n = ax_{n+1} + bx_n \quad \text{on} \quad \{0, 1, \dots, p-1\}^{\mathbb{N}}. \tag{11}$$

For $S = \{t^{-1}\}$, $\alpha^{(k,S,\xi)}$ is the map given by

$$\left(\alpha^{(k,S,\xi)}x\right)_n = ax_{n+1} + bx_n \quad \text{on} \quad \{0, 1, \dots, p-1\}^{\mathbb{Z}}. \tag{12}$$

For other sets S, $\alpha^{(k,S,\xi)}$ is an isometric extension of the map (11) (if $t^{-1} \notin S$) or the map (12) (if $t^{-1} \in S$).

Turning to the proof of Theorem 3, first assume that $a = 1$, $b = 0$, so $\xi = t$; by equation (1) we need to understand the irreducible factors of the polynomial

$$t^q - 1 = (t-1)(1 + t + t^2 + \dots + t^{q-1}) = (t-1)c_q(t)$$

for various values of q. Assume that q is prime. By Theorem 2.47 in [5], the polynomial $c_q(t)$ splits over $\mathbb{F}_p$ into $\left(\frac{q-1}{f}\right)$ irreducible factors, where f is the least positive integer for which $p^f \equiv 1 \bmod q$. Using the result of Heath–Brown in [4], eliminate two possible primes p for which the Artin conjecture may fail; we may then assume that if p is one of the remaining primes, then for infinitely many values of q, p is a primitive root mod q, so $c_q(t)$ is irreducible over $\mathbb{F}_p$ infinitely often. The first two parts of Theorem 3 now follow from Theorem 4; to motivate that argument we prove it here for this simple case.

By Borel–Cantelli, for μ_k^ρ–almost every $S \in \Omega_\xi(k)$ there is an infinite sequence of primes q_j with the property that the place corresponding to the irreducible polynomial $c_{q_j}(t)$ lies in S for all j, so

$$f_{q_j}\left(\alpha^{(S)}\right) = \prod_{\nu \in S} |t^{q_j} - 1|_\nu = p^{q_j} \times p^{-(q_j-1)} \times e_j,$$

where $e_j = p^{-1}$ or 1 depending on whether the place corresponding to $(1-t)$ lies in S or not. In either case,

$$\liminf_{n\to\infty} \frac{1}{n} \log f_n\left(\alpha^{(S)}\right) \le \lim_{j\to\infty} \frac{1}{q_j} \log f_{q_j}\left(\alpha^{(S)}\right) = \lim_{j\to\infty} -\frac{1}{q_j} \log e_j = 0,$$

which proves the second statement in Theorem 3.

Equally, we may find an infinite sequence r_j of primes with the property that the place corresponding to the irreducible polynomial $c_{r_j}(t)$ does not lie in S for any j, so

$$f_{r_j}\left(\alpha^{(S)}\right) = \prod_{\nu \in S} |t^{r_j} - 1|_\nu = p^{r_j} \times e_j,$$

and therefore

$$\limsup_{n\to\infty} \frac{1}{n} \log f_n\left(\alpha^{(S)}\right) \ge \lim_{j\to\infty} \frac{1}{r_j} \log f_{r_j}\left(\alpha^{(S)}\right) = \lim_{j\to\infty} \frac{1}{r_j} \log(p^{r_j} \times e_j) = \log p,$$

proving the first statement in Theorem 3.

Now consider the dynamical zeta function of $\alpha^{(S)}$.

Lemma 4. *If*

$$\mu_k^\rho\left(\{S \in \Omega_\xi(k) \mid \zeta_{\alpha^{(S)}} \textit{ is rational }\}\right) > 0$$

then there is a pair c, d of integers with no common factor with the property that the set

$$\{g \in \mathbb{F}_p[t] \mid g \textit{ divides } t^{cn+d} - 1 \textit{ for some } n \in \mathbb{N}\}$$

is finite and, for infinitely many n, the polynomial $c_{cn+d}(t)$ is irreducible.

The conclusion of Lemma 4 is clearly absurd, so the third statement in Theorem 3 follows.

Proof. By the argument used for the lim inf above, we know that, μ_k^ρ–almost surely, there is an infinite sequence s_j of primes with the property that $f_{s_j}(\alpha^{(S)}) = p$ or 1 for all j. It follows that, with positive μ_k–probability, the zeta function is rational and there is an infinite sequence of primes s_j for which $f_{s_j}(\alpha^{(S)}) = A$ for all j, where A is one of p or 1. It follows by the Mahler–Lech theorem [8] or [2] p.88, that $f_k(\alpha^{(S)}) = A$ for all k in some arithmetic progression taking on some prime values; say $k = cn + d$. That is, for every S in some positive μ_k^ρ–measure set, there is a co–prime pair c, d for which all (if $A = 1$) or all but one (if $A = p$) of the factors of $t^{cn+d} - 1$ lie in S for all n. Since there are only countably many such arithmetic progressions, it follows that there is a single pair c, d with the property that every S in a set of positive measure has the property that all (if $A = 1$) or all but one (if $A = p$) of the factors of $t^{cn+d} - 1$ lie in S for all n. By Borel–Cantelli, this can only be possible if the set of factors of $t^{cn+d} - 1$ for all n is itself finite. □

For the general case $\xi = at+b$, the same proof works since $c_q(at+b)$ is irreducible if and only if $c_q(t)$ is irreducible. This completes the proof of Theorem 3.

5. Proof of Theorem 4

Fix k and ξ, and let $n_j \to \infty$ be a sequence with the property that $|P_{n_j}| = L$ for $j = 1, 2, \dots$. Choose, if possible, a subsequence $m_1 = n_{j(1)}, m_2 = n_{j(2)}, \dots$ with the property that

$$P_{m_k} \backslash \bigcup_{\ell < k} P_{m_\ell} \neq \emptyset \tag{13}$$

for all k. If this is not possible, then $\bigcup_{j \in \mathbb{N}} P_{n_j}$ is finite, and therefore with positive μ_k^ρ–probability the set S does not intersect any P_{n_j}, so on a set of positive μ_k^ρ–measure

$$f_{n_j}(\alpha) = \prod_{\nu : |\xi|_\nu \neq 1} |\xi^n - 1|_\nu$$

and hence

$$\limsup_{n \to \infty} \frac{1}{n} \log f_n \left(\alpha^{(k,S,\xi)} \right) = h_{top} \left(\alpha^{(k,S,\xi)} \right) \tag{14}$$

by Lemma 5. It follows by Remark 2(i) that (14) holds for μ_k^ρ–almost every S. Similarly, with positive μ_k^ρ–probability the set S contains all the P_{n_j}, so on a set of positive μ_k^ρ–measure $f_{n_j}(\alpha) = 1$, and hence

$$\liminf_{n \to \infty} \frac{1}{n} \log f_n \left(\alpha^{(k,S,\xi)} \right) = 0$$

and the lower limit is 0 almost everywhere by Remark 2(i) again.

So we may assume (13). Let

$$S_0 = \{ \nu \in P(k) \mid \nu \in P_{m_k} \text{ infinitely often} \};$$

by (13), $|S_0| < L$.

Let $P'_{m_k} = P_{m_k} \backslash S_0$, and choose a further subsequence $s_1 = m_{k(1)}, s_2 = m_{k(2)}, \dots$ with the property that

$$P'_{s_j} \cap \bigcup_{\ell < j} P'_{s_\ell} = \emptyset. \tag{15}$$

By construction,

$$1 \leq L - |S_0| \leq |P'_{s_k}| \leq L \tag{16}$$

for all k. By (15) the sets

$$A_j = \{ \omega_k(S) \mid \omega_k(S)(n) = 1 \text{ if and only if } \nu_n \in P'_{s_j} \}$$

are independent, and by (16) $\mu_k^\rho(A_j) \in [\rho^L, \rho]$, so by the Borel–Cantelli lemma, for μ_k^ρ–almost every S there is a sequence $t(j) = s_{k'_j}$ such that $t(j) \to \infty$ as $j \to \infty$ and $S \cap P'_{t(j)} = \emptyset$ for all j.

Since S_0 is finite, it follows that there is a positive μ_k–measure set on which $S \cap S_0 = \emptyset$ and the above sequence exists. For such an S, let

$$I(n) = \prod_{|\xi|_\nu \neq 1} |\xi^n - 1|_\nu, \text{ and } J(n) = \prod_{\nu \in S : |\xi|_\nu = 1} |\xi^n - 1|_\nu.$$

Notice that $f_n\left(\alpha^{(k,S,\xi)}\right) = I(n) \times J(n)$ since for these S, $S \cap S_0 = \emptyset$.

By Lemma 5,

$$\lim_{n\to\infty} \frac{1}{n} \log I(n) = h_{top}\left(\alpha^{(k,S,\xi)}\right).$$

On the other hand, along the sequence $t(j)$, we have $J(t(j)) = 1$ since the set S_0 has been removed. It follows that

$$\lim_{j\to\infty} \frac{1}{t(j)} \log f_{t(j)}\left(\alpha^{(k,S,\xi)}\right) = h_{top}\left(\alpha^{(k,S,\xi)}\right)$$

for all S in a set of measure at least ρ^L. By Remark 2(i) this implies that the upper limit is $h_{top}\left(\alpha^{(k,S,\xi)}\right)$ and the lower limit is 0 μ_k^ρ–almost everywhere.

Remark 3. The subsequence with (13) does always exist though there does not seem to be a short proof of this fact: in the arithmetic case it follows from Zsigmondy's theorem [15] or the result in [9].

6. S–integer systems as isometric extensions

To prove Theorem 5, first notice that by (5) $H \subset S$, so there is a canonical embedding

$$R_H \hookrightarrow R_S. \tag{17}$$

Dual to the monomorphism (17) there is a surjective homomorphism $\pi : X^{(k,S)} \to X^{(k,H)}$ with $\pi\alpha^{(S)} = \alpha^{(H)}\pi$. This map realises $\alpha^{(H)}$ as a factor of $\alpha^{(S)}$: it remains to identify what $\alpha^{(H)}$ looks like and the action of $\alpha^{(S)}$ restricted to the fibre $Y = \pi^{-1}(1_{X^{(k,H)}})$.

If $H \cup P_\infty(k) \subset \{\nu \in P(k) \mid |\xi|_\nu \neq 1\}$, then by Corollary 4.2 of [3] the map $\alpha^{(H)}$ is hyperbolic (notice that $|\xi|_\omega$ for all ω above a given place ν' is determined by the value of $|\xi|_\nu$ for any one place ν above ν' except for the infinite places of an algebraic number field). If $H \cup P_\infty(k) \not\subset \{\nu \in P(k) \mid |\xi|_\nu \neq 1\}$ then there must be an infinite place ν for which $|\xi|_\nu = 1$, and then $\alpha^{(H)}$ is quasihyperbolic.

The action on the fibre is found as follows. The dual of the kernel of π is given by the co–kernel of $\hat{\pi} : R_H \to R_S$, so $\widehat{Y} \cong R_S/R_H$. Using the methods of [3] Section 3, one may show that $\widehat{(R_S/R_H)} \cong R_H^\perp \subset \widehat{R_S}$ and then that

$$Y \cong \widehat{(R_S/R_H)} \cong \prod_{\nu \in S\backslash H} r_\nu.$$

On each of the factors r_ν with $\nu \in S\backslash H$, $\alpha^{(S)}$ acts via multiplication by ξ, which is an isometry since $|\xi|_\nu = 1$ for $\nu \in S\backslash H$.

Example 1. To illustrate Theorem 5 some explicit examples follow.

(i) If $k = \mathbb{Q}$, $\xi = 2$, $S = \emptyset$, then $H = \emptyset$ so $\alpha^{(S)} = \alpha^{(H)}$ is the circle–doubling map and Y is trivial.

(ii) If $k = \mathbb{Q}$, $\xi = 2$, $S = \{3\}$, then $H = \emptyset$, so the hyperbolic base map $\alpha^{(H)}$ is the circle–doubling map. The fibre $Y = \mathbb{Z}_3$ (3–adic integers), and $\alpha^{(S)}$ restricted to Y is the isometry $x \mapsto 2x$ on $\mathbb{Z}_3$.

(iii) The general case when $k = \mathbb{Q}$ (that is, systems living on a one–dimensional solenoid) has the following structure. If $\xi = \frac{r}{s}$ in lowest terms, then H is the set of p–adic valuations corresponding to primes that divide rs. The action on the fibre is the isometry $x \mapsto \frac{r}{s} \cdot x$ on

$$Y = \prod_{\{p \in S | p \nmid rs\}} r_p.$$

(iv) A non–hyperbolic base map in the arithmetic case is given by Lind's example from [6], Section 3 (see also Example 2.2(5) and Example 6.1(1) in [3]). Let

$$\xi = \sqrt{2} - 1 + i\sqrt{2\sqrt{2} - 2},$$

$k = \mathbb{Q}(\xi)$, and $S = \emptyset$. Then $H = \emptyset$, $R_S = \mathbb{Z} + \xi\mathbb{Z} + \xi^2\mathbb{Z} + \xi^3\mathbb{Z}$, and $\alpha^{(H)} = \alpha^{(S)}$ is the quasihyperbolic automorphism of the 4–torus corresponding under duality to the integer matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & 2 & -4 \end{bmatrix}.$$

If S were non–empty, then H would still be empty, and Y would be a product over S of rings of integers on which $\alpha^{(S)}$ acts as an isometry.
(v) Let $k = \mathbb{F}_3(t)$, $\xi = \frac{1+t}{2+t}$, and $S = \{1+t, 2+t, 1+t^2\}$. Then

$$H = \{1+t, 2+t\},$$

and $\alpha^{(H)}$ is quasihyperbolic because of the infinite place where

$$\left| \frac{1+t}{2+t} \right|_{t^{-1}} = 1.$$

The fibre action is given by the isometry $x \mapsto \left(\frac{1+t}{2+t}\right) x$ on the compact ring $r_{(1+t^2)} \subset \mathbb{F}_3(t)_{(1+t^2)}$.

7. Appendix

There are three basic estimates used above. These may be extracted from proofs in [3]; we briefly prove them again here for completeness.

Lemma 5. *Let k be any $\mathbb{A}$–field, ξ not a unit root, and S any set of finite places for which $\xi \in R_S \backslash \{0\}$. Then*

$$\frac{1}{n} \log \prod_{\nu \in S \cup P_\infty(k); |\xi|_\nu \neq 1} |\xi^n - 1|_\nu \longrightarrow h_{top}\left(\alpha^{(k,S,\xi)}\right) = \sum_{\nu \in S \cup P_\infty(k)} \log^+ |\xi|_\nu > 0.$$

Proof. The convergence is clear: there can be only finitely many places for which $|\xi|_\nu \neq 1$, and at each of these $\frac{1}{n} \log |\xi^n - 1|_\nu \to \log^+ |\xi|_\nu$. In the arithmetic case the limit must be positive by Kronecker's theorem. In the geometric case, if $|\xi|_\nu \leq 1$ for all infinite ν, then $\xi \in \mathbb{F}_q(t)$ is of the form $\frac{c}{p(t)}$ for some constant c and polynomial $p(t) \in \mathbb{F}_q[t]$. Since $\xi \in R_S \backslash \{0\}$, there must be a $\nu \in S$ with $|\xi|_\nu > 1$ unless $p(t)$ is a constant, which is precluded by requiring that ξ not be a unit root. $\square$

Lemma 6. *Let k be an algebraic number field, ξ not a unit root, and T any finite set of places with $|\xi|_\nu = 1$ for $\nu \in T$. Then*

$$\frac{1}{n} \log \prod_{\nu \in T} |\xi^n - 1|_\nu \longrightarrow 0$$

as $n \to \infty$.

Proof. We follow the proof of Theorem 6.1 in [3]. If ν is Archimedean, then by Baker's Theorem (see [10], p.281) we have positive constants a,b with $|\xi^n - 1| > a/n^b$. It follows that

$$\frac{1}{n} \log |\xi^n - 1| \to 0 \tag{18}$$

as $n \to \infty$.

Assume therefore that ν is a finite place lying above the place p of $\mathbb{Q}$ with $|\xi|_\nu = 1$ and with $|\xi^n - 1|_\nu < 1$. Let Ω_ν be the usual completion of the algebraic closure of $\mathbb{Q}$ under ν; the ν–adic logarithm is defined by $\log_\nu(1+x) = \sum_{i=1}^{\infty} (-1)^{i+1} x^i / i$, convergent for all x with $|x|_\nu < 1$. Then

$$\log_\nu(\xi^n) = (\xi^n - 1) - \frac{(\xi^n - 1)^2}{2} + \frac{(\xi^n - 1)^3}{3} - \dots$$

and so $|\log_\nu(\xi^n)|_\nu \le |\xi^n - 1|_\nu$. Since we always have for some constant c

$$\frac{c}{n} \le |n \log_\nu(\xi)|_\nu = |\log_\nu(\xi^n)|_\nu,$$

this shows that

$$\frac{c}{n} \le |\xi^n - 1|_\nu \le 1 \tag{19}$$

for all n.

Since the set T is finite, (18) and (19) together show that

$$\frac{1}{n} \log \prod_{\nu \in T} |\xi^n - 1|_\nu \longrightarrow 0$$

as $n \to \infty$. □

Lemma 7. *Let k be a rational function field, ξ not a unit root, and T any finite set of places with $|\xi|_\nu = 1$ for $\nu \in T$. Then*

$$\lim_{n \to \infty; n \in \mathcal{P}} \frac{1}{n} \log \prod_{\nu \in T} |\xi^n - 1|_\nu = 0.$$

Proof. Following the proof of Theorem 6.2 in [3], split the set T into disjoint subsets $A = \{\nu \in T \mid |\xi - 1|_\nu = 1\}$ and $B = \{\nu \in T \mid |\xi - 1|_\nu < 1\}$. For each $\nu \in A$, write $\xi = a_0 + a_1 \pi + a_2 \pi^2 + \dots$ where $\pi \in k$ has $\mathrm{ord}_\nu(\pi) = 1/e$, where e is the index of ramification and the coefficients a_i come from the residue class field L. Let d be the multiplicative order of a_0 in L^*; $d \ge 2$ clearly. Then a simple calculation shows that $|\xi^n - 1|_\nu = 1$ if and only of d does not divide n. Since A is finite, we deduce that there is a finite set $\{d_1, \dots, d_m\}$ of integers each greater than or equal to one with the property that

$$\prod_{\nu \in T} |\xi^n - 1|_\nu = 1$$

whenever n is not divisible by any of $d_1, \dots, d_m$. We conclude that

$$\lim_{n\to\infty; n\in\mathcal{P}} \frac{1}{n} \log \prod_{\nu\in A} |\xi^n - 1|_\nu = 0. \tag{20}$$

The set B is also finite; let $B = \{\nu_1, \dots, \nu_\ell\}$. For each $j \in \{1, 2, \dots, \ell\}$ write

$$\xi = 1 + \sum_{i=1}^{\infty} a_i \pi_j^i$$

with a_i and π_j as above, and $|\xi - 1|_\nu = p^{-s_j}$ where $s_j = \frac{1}{e}\min\{i \mid a_i \neq 0\}$. Then

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^{\ell} \log|\xi^n - 1|_{\nu_j} &= \frac{1}{n}\sum_{j=1}^{\ell} \log|\xi - 1|_{\nu_j} + \frac{1}{n}\sum_{j=1}^{\ell} \log|\xi^{n-1} + \xi^{n-2} + \dots + \xi + 1|_{\nu_j} \\
&= \frac{1}{n}\sum_{j=1}^{\ell} \log|\pi_j|_{\nu_j}^{s_j} + \frac{1}{n}\sum_{j=1}^{\ell} \log\left| n + \sum_{i=1}^{\infty} b_i(j)\pi_j{}^i \right|_{\nu_j}
\end{aligned}$$

for coefficients $b_i(j) \in k_{\nu_j}$ with $|b_i(j)|_{\nu_j} \leq 1$ for all i and j. This expression converges to zero so long as p, the characteristic of k, does not divide n. We deduce that

$$\lim_{n\to\infty; n\in\mathcal{P}} \frac{1}{n} \log \prod_{\nu\in A} |\xi^n - 1|_\nu = 0,$$

which together with (20) gives the result. □

References

[1] R. Bowen and O.E. Lanford III, *Zeta functions of restrictions of the shift transformation*, Global Analysis, Proceedings of Symposia in Pure Mathematics, vol. 14, American Mathematical Society, Providence, R.I., 1970, pp. 43–49.

[2] J.W.S. Cassels, *Local Fields*, Cambridge University Press, Cambridge, 1986.

[3] V. Chothi, G. Everest, and T. Ward, *S-integer dynamical systems: periodic points*, Journal für die Riene und. Angew. **489** (1997), 99–132.

[4] D.R. Heath-Brown, *Artin's conjecture for primitive roots*, Quart. J. Math. Oxford **37** (1986), 27–38.

[5] R. Lidl and H. Niederreiter, *Introduction to finite fields and their applications*, Cambridge University Press, Cambridge, 1986.

[6] D.A. Lind, *Ergodic group automorphisms and specification*, Springer Lecture Notes in Math. **729** (1978), 93–103.

[7] ———, *Dynamical properties of quasihyperbolic toral automorphisms*, Ergodic Theory and Dynamical Systems **2** (1982), 49–68.

[8] K. Mahler, *Eine arithmetische Eigenschaft der Taylor-Koeffizienten rationaler Funktionen*, Proceedings k. akad. wet. Amsterdam **38** (1935), 50–60.

[9] C. Methfessel, *Rekurrente Folgen mit lokal gleichmäßig beschränkter Primteileranzahl*, Acta Arithmetica **70** (1995), 1–7.

[10] P. Philippon and M. Waldschmidt, *Lower bounds for linear forms in logarithms*, New advances in Transcendence Theory (A. Baker, ed.), Cambridge University Press, Cambridge, 1988, pp. 280–312.

[11] P. Ribenboim, *The Fibonacci numbers and the Arctic Ocean*, Proceedings of the Second Gaussian Symp. Conf. A: Mathematics and Theoretical Physics (München 1994) (M. Behara, R. Fritsch, and R.G. Lintz, eds.), W. de Gruyter, Berlin, 1995, pp. 41–83.

[12] P. Walters, *An Introduction to Ergodic Theory*, Springer-Verlag, New York, 1982.

[13] T. Ward, *An uncountable family of group automorphisms, and a typical member*, Bulletin of the London Math. Soc. **29** (1997), 577–584.

[14] A. Weil, *Basic Number Theory*, third ed., Springer, New York, 1974.

[15] K. Zsigmondy, *Zur Thoerie der Potenzreste*, Monatshefte Math. **3** (1892), 265–284.

School of Mathematics, University of East Anglia, Norwich NR4 7TJ, U.K.
E-mail address: t.ward@uea.ac.uk