S–INTEGER DYNAMICAL SYSTEMS: PERIODIC POINTS

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ABSTRACT. We associate via duality a dynamical system to each pair (R_S, ξ), where R_S is the ring of S–integers in an A–field k, and ξ is an element of $R_S \backslash \{0\}$. These dynamical systems include the circle doubling map, certain solenoidal and toral endomorphisms, full one– and two–sided shifts on prime power alphabets, and certain algebraic cellular automata.

In the arithmetic case, we show that for S finite the systems have properties close to hyperbolic systems: the growth rate of periodic points exists and the periodic points are uniformly distributed with respect to Haar measure. The dynamical zeta function is in general irrational however. For S infinite the systems exhibit a wide range of behaviour. Using Heath–Brown's work on the Artin conjecture, we exhibit examples in which S is infinite but the upper growth rate of periodic points is positive.

1. INTRODUCTION

Let α be a continuous map from a compact metric space $X = (X, d)$ onto itself. The points of period n under α comprise the set

$$F_n(\alpha) = \{x \in X \colon \alpha^n x = x\}.$$

Various dynamical properties of α may be studied via the sets F_n, and in this paper we shall be concerned with trying to understand how the cardinality of F_n grows in n, and how the points of F_n are distributed around the space X, for a special class of examples.

If F_n is finite for every $n \geq 1$, then following Smale [29] the numerical periodic point data may be assembled into a single (formal) function, the dynamical zeta function of α,

$$\zeta_\alpha(s) = \exp\left(\sum |F_n(\alpha)| \times \frac{s^n}{n}\right), \tag{1}$$

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and this is an invariant of topological conjugacy. If

$$\limsup_{n\to\infty} |F_n(\alpha)|^{1/n} = \frac{1}{R} < \infty, \tag{2}$$

then the series (1) converges in the disc $|s| < R$ and therefore defines a holomorphic function in that disc. In general there is no reason to expect the upper limit in (2) to be a limit, so we introduce upper and lower growth rates p^+ and p^- as follows:

$$p^+(\alpha) = \limsup_{n\to\infty} \frac{1}{n} \log |F_n(\alpha)| \quad \text{and} \quad p^-(\alpha) = \liminf_{n\to\infty} \frac{1}{n} \log |F_n(\alpha)|. \tag{3}$$

If X is a manifold and α is sufficiently hyperbolic (Axiom A), then by [20] the zeta function is rational, and the least real pole of ζ_α occurs at $\exp(-h_{top}(\alpha))$, where $h_{top}(\alpha)$ is the topological entropy of α. In this case, the upper and lower growth rates coincide and equal the topological entropy. In addition, the uniform measures on the set F_n (these measures are the *periodic point measures*) converge in the weak*–topology on X to an α–invariant measure positive on open sets [2].

Our purpose is to study in as uniform a fashion as possible a large class of examples of the following form. The compact set X is generally not a manifold, but is topologically either zero–dimensional or locally isometric to the Cartesian product of a manifold and a zero–dimensional set; arithmetically X is locally isometric to a product of balls in vector spaces over local fields and is therefore a natural generalization of a manifold. The action of α decomposes locally into analogues of eigendirections (analogous to the splitting into eigenspaces for the linear lift $\tilde{\alpha} : \mathbb{R}^n \to \mathbb{R}^n$ of a toral automorphism $\alpha : \mathbb{T}^n \to \mathbb{T}^n$). If all the directions are expanding or contracting, then we say α is (generalised) hyperbolic. Certain of these dynamical systems arise naturally as attractors in hyperbolic diffeomorphisms [3], and systems of this shape were also used in the early construction of expansive homeomorphisms [6], [10].

Within this class we consider various examples and examine the growth rate and distribution of the periodic points. The last two examples below are conditional on certain number–theoretical conjectures. In the case where X is locally of the form manifold$\times$Cantor set we obtain the following results.

(1) If there are only finitely many directions in which α is non–hyperbolic, then $p^+(\alpha) = p^-(\alpha) = h_{top}(\alpha)$ (Theorem 6.1). The zeta function is in general irrational (Example 8.3), and the periodic points are uniformly distributed (Theorem 7.1).

(2) There exist examples with infinitely many non–hyperbolic directions, and with $p^+(\alpha) = h_{top}(\alpha) > 0$ (Corollary 9.1). For these examples the maximal measure is in the weak closure of the set of periodic point measures, but the periodic points are probably not uniformly distributed.
(3) If an asymptotic version of Artin's conjecture holds, then there exist examples with infinitely many non–hyperbolic directions and with $p^+(\alpha) = p^-(\alpha) = h_{top}(\alpha)$, and therefore with uniformly distributed periodic points (Theorem 9.5).
(4) If there are infinitely many Mersenne primes, then there exist examples with infinitely many hyperbolic directions and with the property that the set of limit points of $\{\frac{1}{n}\log|F_n(\alpha)|\}$ comprises the infinite set $\{(1-\frac{1}{q})h_{top}(\alpha)\colon q \in \mathbb{N}\}$ (Theorem 9.2). There are sequences along which the periodic point measures converge weakly to the maximal measure, but the growth rate is not the topological entropy.

The examples above are arithmetical in nature, and the construction starts with an $\mathbb{A}$–field of zero characteristic (that is, an algebraic number field). Following Weil, [38], it is natural to try to treat the positive characteristic case simultaneously: here the space X is zero–dimensional and the problems seem much less tractable. However, an analogue of (1) above is shown for the upper growth rate (Theorem 6.2).

This paper has its early origins in the papers [18] by Lind where the rôle played by arithmetic hyperbolicity in the dynamical properties of compact group automorphisms was explicitly realized, and [19] by Lind and Ward where arithmetic hyperbolicity was used to calculate and "explain" Yuzvinskii's formula for the topological entropy of solenoidal automorphisms. Much of the material here appeared first in the thesis of the first author, [4], and full versions of proofs are available there.

2. The dynamical system associated to a set of places

Let k be an $\mathbb{A}$–field in the sense of Weil (that is, k is an algebraic extension of the rational field $\mathbb{Q}$ or of $\mathbb{F}_q(t)$ for some rational prime q). Let $P(k)$ denote the set of places of k. A place $w \in P$ is finite if w contains only non–archimedean valuations and is infinite otherwise (with one exception: for the case $\mathbb{F}_p(t)$ the place given by t^{-1} is regarded as being an infinite place despite giving rise to a non–archimedean valuation).

Example 2.1. For the case $k_0 = \mathbb{Q}$ or $k_0 = \mathbb{F}_q(t)$, the places are defined as follows.

THE RATIONALS $\mathbb{Q}$. The places of $\mathbb{Q}$ are in one–to–one correspondence with the set of rational primes $\{2, 3, 5, 7, \dots\}$ together with one additional place ∞ at infinity. The corresponding valuations are $|r|_\infty = |r|$ (the usual archimedean valuation), and for each p, $|r|_p = p^{-\operatorname{ord}_p(r)}$, where $\operatorname{ord}_p(r)$ is the (signed) multiplicity with which the rational prime p divides the the rational r.

THE FUNCTION FIELD $\mathbb{F}_q(t)$. For $\mathbb{F}_q(t)$ there are no archimedean places. For each monic irreducible polynomial $v(t) \in \mathbb{F}_q[t]$ there is a distinct place v, with corresponding valuation given by

$$|f|_v = q^{-\operatorname{ord}_v(f)\cdot\deg(v)},$$

where $\operatorname{ord}_v(f)$ is the signed multiplicity with which v divides the rational function f. There is one additional place given by $v(t) = t^{-1}$, and this place will be called an infinite place even though the corresponding valuation is non–archimedean.

Let k be a finite extension of k_0. A place $w \in P = P(k)$ is said to lie above a place v of $k_0 = \mathbb{Q}$ or $\mathbb{F}_q(t)$, denoted $w|v$, if $|\cdot|_w$ rectricted to the base field $k_0 \subset k$ coincides with $|\cdot|_v$. Denote by k_w the (metric) completion of k under the metric $d_w(x, y) = |x - y|_w$ on k. The *local degree* is defined by $d_w = [k_w : (k_0)_v]$. Choose a normalized valuation $|\cdot|_w$ corresponding to the place w to have

$$|x|_w = |x|_v^{d_w/d}$$

for each $x \in k_0\backslash\{0\}$, where $d = [k : k_0]$ is the global degree. With the above normalizations we have the Artin product formula [38], p. 75:

$$\prod_{w\in P(k)} |x|_w = 1 \tag{4}$$

for all $x \in k\backslash\{0\}$.

For each finite place w of k, the field k_w is a local field, and the maximal compact subring of k_w is

$$r_w = \{x \in k\colon |x|_w \le 1\}.$$

Elements of r_w are called w–adic integers in k_w. The group of units in the ring r_w is

$$r_w^* = \{x \in k\colon |x|_w = 1\}.$$

Let $P_\infty = P_\infty(k)$ denote the set of infinite places of k.

Definition 2.1. Let k be an $\mathbb{A}$–field. Given an element $\xi \in k^*$, and any set $S \subset P(k)\backslash P_\infty(k)$ with the property that $|\xi|_w \le 1$ for all $w \notin S \cup P_\infty$, define a dynamical system $(X, \alpha) = (X^{(k,S)}, \alpha^{(k,S,\xi)})$ as follows. The compact abelian group X is the dual group to the discrete countable group of S–integers R_S in k, defined by

$$R_S = \{x \in k\colon |x|_w \leq 1 \text{ for all } w \notin S \cup P_\infty(k)\}.$$

The continuous group endomorphism $\alpha : X \to X$ is dual to the monomorphism $\widehat{\alpha} : R_S \to R_S$ defined by $\widehat{\alpha}(x) = \xi x$.

Dynamical systems of the form $(X^{(k,S)}, \alpha^{(k,S,\xi)})$ are called S–integer dynamical systems. Following conventions from number theory, we shall divide these into two classes: *arithmetic* systems when k is a number field, and *geometric* when k has positive characteristic. To clarify this definition – and to show how these systems connect with previously studied ones – several examples follow.

Example 2.2. (1) Let $k = \mathbb{Q}$, $S = \emptyset$, and $\xi = 2$. Then

$$R_S = \{x \in \mathbb{Q}\colon |x|_p \leq 1 \text{ for all primes } p\} = \mathbb{Z},$$

so $X = \mathbb{T}$ and α is the circle doubling map.

(2) Let $k = \mathbb{Q}$, $S = \{2\}$, and $\xi = 2$. Then

$$R_S = \{x \in \mathbb{Q}\colon |x|_p \leq 1 \text{ for all primes } p \neq 2\} = \mathbb{Z}[\tfrac{1}{2}],$$

so X is the solenoid $\widehat{\mathbb{Z}[\frac{1}{2}]}$, and α is the automorphism of X dual to the automorphism $x \mapsto 2x$ of R_S. This is the natural invertible extension of the circle doubling map [5], Example (c) or [10], Section 2.

As pointed out in [3], Chapter 1 Example D, this dynamical system is topologically conjugate to the system (Y, β) defined as follows. Let $D = \{z \in \mathbb{C}\colon |z| \leq 1\}$ and $S^1 = \{z \in \mathbb{C}\colon |z| = 1\}$. Define a map $f : S^1 \times D \to S^1 \times D$ by

$$f(z, \omega) = (z^2, \tfrac{1}{2}z + \tfrac{1}{4}\omega).$$

Let $Y = \bigcap_{n\in\mathbb{N}} f^n(S^1 \times D)$ and let β be the map induced by f on Y. Then there is a homeomorphism $Y \to X$ that intertwines the maps β and α. For more details on this example and related "DE" (derived from expanding) examples, see [29], Section I.9; for a thorough and detailed treatment of this dyadic example see [11], Section 17.1.

(3) Let $k = \mathbb{Q}$, $S = \{2, 3\}$, $\xi = \frac{3}{2}$. Then $R_S = \mathbb{Z}[\frac{1}{6}]$, and α is the map dual to multiplication by $\frac{3}{2}$ on R_S. This map has dense periodic points by [19], Section 3 and has topological entropy $\log 3$ by [19], Section 2.

(4) Let $k = \mathbb{Q}$, $S = \{2, 3, 5, 7, 11, \dots\}$, and $\xi = \frac{3}{2}$. Then $R_S = \mathbb{Q}$ and α is the automorphism of the full solenoid $\widehat{\mathbb{Q}}$ dual to multiplication by $\frac{3}{2}$ on $\mathbb{Q}$. This map has only one periodic point for any period by [19], Section 3, and has entropy $\log 3$ by [19], Section 2.

(5) Let ξ be an algebraic integer, $k = \mathbb{Q}(\xi)$ and $S = \emptyset$. Then R_S is the ring of algebraic integers in k. Taking $\xi = \sqrt{2} - 1 + i\sqrt{2\sqrt{2} - 2}$ gives a non–expansive quasihyperbolic automorphism of the 4–torus as pointed out in [16], Section 3. This example is examined more closely in Section 6.

(6) Let $k = \mathbb{F}_q(t)$, $S = \emptyset$, and $\xi = t$. Then $R_S = \mathbb{F}_q[t]$, and so $X = \widehat{R_S} = \prod_{i=0}^{\infty}\{0, 1, \dots, q-1\}$. The map α is therefore the full one–sided shift on q symbols.

(7) Let $k = \mathbb{F}_q(t)$, $S = \{t\}$, and $\xi = t$. Recall that the valuation corresponding to t is $|f|_t = q^{-\operatorname{ord}_t(f)}$, so $|t|_t = q^{-1}$. The ring of S–integers is

$$R_S = \{f \in \mathbb{F}_q(t)\colon |f|_w \le 1 \text{ for all } w \ne t, t^{-1}\} = \mathbb{F}_q[t^{\pm 1}].$$

The dual of R_S is then $\prod_{-\infty}^{\infty}\{0, 1, \dots, q-1\}$, and in this case α is the full two–sided shift on q symbols.

(8) Let $k = \mathbb{F}_q(t)$, $S = \{t\}$, and $\xi = 1 + t$. Then X is the two–sided shift space on q symbols, and α is the cellular automaton defined by

$$(\alpha(x))_k = x_k + x_{k+1} \bmod q.$$

(9) Let $k = \mathbb{F}_q(t)$, $S = \{t, 1+t\}$, and $\xi = 1+t$. Then α is the invertible extension of the cellular automaton in (8).

(10) Let α be an ergodic automorphism of a finite–dimensional torus. For each subset S of the rational primes let $\Gamma_S = \widehat{X} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{S}]$. Then α defines an endomorphism $\alpha_S : \widehat{\Gamma_S} \to \widehat{\Gamma_S}$. Each α_S has the same entropy as α by [19] (and is therefore measurably isomorphic to α), but they are all topologically distinct, so $\{\alpha_S\}$ forms an uncountable family of topological dynamical systems all measurably isomorphic to each other.

(11) Not all toral endomorphisms are S–integer dynamical systems. Let $\alpha_A : \mathbb{T}^n \to \mathbb{T}^n$ be the toral endomorphism corresponding to the integer matrix $A \in M_n(\mathbb{Z})$. Assume that the characteristic polynomial χ_A of A is irreducible, let λ have $\chi_A(\lambda) = 0$ and let $\mathbf{a} = (a_1, \dots, a_n)^t$ be a vector in $\mathbb{Q}(\lambda)^n$ with $A\mathbf{a} = \lambda\mathbf{a}$ with the property that $\mathfrak{a} = a_1 R_\lambda + \dots + a_n R_\lambda$ is an ideal in the ring $R_\lambda = \mathbb{Z}[\lambda]$. Two ideals determined in this way from the same matrix belong to the same ideal class by [31], Theorem 2.

Lemma 2.1. *The toral endomorphism α is topologically conjugate to the S–integer dynamical system given by $k = \mathbb{Q}(\lambda)$, $\xi = \lambda$, $S = \emptyset$ if and only if $\mathfrak{a}$ defines a trivial element in the ideal class group of R_λ.*

Proof. Let B be the companion matrix to the polynomial χ_A. Then there is an isomorphism from $\left(X^{(k,S)}, \alpha^{(k,\xi,S)}\right)$ to $(\mathbb{T}^n, \alpha_B)$. If $\mathfrak{a}$ defines a trivial element in the ideal class group of R_λ, then by [31], there is a matrix $S \in GL_n(\mathbb{Z})$ such that $A = SBS^{-1}$, so there is an isomorphism from $(\mathbb{T}^n, \alpha_B)$ to $(\mathbb{T}^n, \alpha_A)$.

Conversely, let θ be a topological conjugacy from $(\mathbb{T}^n, \alpha_A)$ to $\left(X^{(k,S)}, \alpha^{(k,\xi,S)}\right)$. Let H_1 denote the first Čech homology functor with coefficients in $\mathbb{T}$; H_1 sends any diagram of solenoids and endomorphisms to an isomorphic diagram by Lemma 6.3 in [12]. Then $H_1(\theta)$ defines an isomorphism from $(\mathbb{T}^n, \alpha_A)$ to $\left(X^{(k,S)}, \alpha^{(k,\xi,S)}\right)$; since $X^{(k,S)}$ is an n–dimensional torus, $\alpha^{(k,\xi,S)}$ corresponds to some matrix $C \in M_n(\mathbb{Z})$, and this isomorphism is given by a matrix $S \in GL_n(\mathbb{Z})$ with $A = SCS^{-1}$. It follows by [31] that $\mathfrak{a}$ defines a trivial element in the ideal class group of R_λ. □

3. Background on adele rings

In this section we assemble some basic facts about the ring R_S. For the case $S = \emptyset$ most of this is straightforward. At the opposite extreme, when S contains all finite places (so $R_S = k$), the adelic constructions of [38], Chapter IV show how to cover the group $X^{(k,S)}$. In the intermediate case, straightforward modifications of Weil's arguments are needed. The construction is also given in Tate's thesis, and we indicate below how to read off the results we shall need from this.

Fix an $\mathbb{A}$–field k and a set S of finite places of k.

Definition 3.1. The S–adele ring of k is the ring

$$k_{\mathbb{A}}(S) = \left\{ x = (x_\nu) \in \prod_{\nu \in S \cup P_\infty} k_\nu : |x_\nu|_\nu \leq 1 \text{ for all but finitely many } \nu \right\},$$

with the topology induced by the following property. For each finite set $S' \subset S$, the locally compact subring $k_{\mathbb{A}}^{S'} \subset k_{\mathbb{A}}(S)$ defined by

$$k_{\mathbb{A}}^{S'} = \prod_{\nu \in S' \cup P_\infty} k_\nu \times \prod_{\nu \in S \setminus S'} r_\nu$$

(with the product topology) is an open subring of $k_{\mathbb{A}}(S)$, and a fundamental system of open neighbourhoods of 0 in the additive group of $k_{\mathbb{A}}(S)$ is given by a fundamental system of neighbourhoods of 0 in any one of the subrings $k_{\mathbb{A}}^{S'}$.

Notice that $k_{\mathbb{A}}(S)$ is locally compact since each r_ν is compact.

Define a map $\Delta : R_S \to k_{\mathbb{A}}(S)$ by $\Delta(x) = (x, x, x, \dots)$. This map is a well–defined ring homomorphism: notice that for $\alpha \in R_S$, $|\alpha|_\nu \leq 1$ for all but finitely many ν by [38], Theorem III.1.3.

In [30], Tate introduces the notion of an *abstract restricted direct product*, under the hypothesis that P $(= S \cup P_\infty)$ is an arbitrary countable set of indices (places). Let $G_{\mathcal{P}}$ $(= k_\nu)$ be a locally compact abelian group for $\mathcal{P} \in P$, and for all but finitely many $\mathcal{P}$, let $H_{\mathcal{P}}$ $(= r_\nu)$ be an open compact subgroup of $G_{\mathcal{P}}$. The restricted direct product is defined as

$$G(P) = \left\{ g = (g_{\mathcal{P}}) \in \prod_{\mathcal{P} \in P} G_{\mathcal{P}} : g_{\mathcal{P}} \in H_{\mathcal{P}} \text{ for all but finitely many } \mathcal{P} \right\},$$

a locally compact abelian topological group. We topologise $G(P)$ by choosing a fundamental system of neighbourhoods of 1 in $G(P)$ of the form $N = \prod_{\mathcal{P} \in P} N_{\mathcal{P}}$, where each $N_{\mathcal{P}}$ is a neighbourhood of 1 in $G_{\mathcal{P}}$ and $N_{\mathcal{P}} = H_{\mathcal{P}}$ for all but finitely many $\mathcal{P}$, which accords with the topology in Definition 3.1.

The key results proved in Lemma 3.2.2 and Theorem 3.2.1 of [30] are the following.

(1) $\Delta(R_S)$ is discrete in $k_{\mathbb{A}}(S)$ and $k_{\mathbb{A}}(S)/\Delta(R_S)$ is compact,
(2) $R_S^{\perp} \cong R_S$, $\widehat{k_{\mathbb{A}}(S)} \cong k_{\mathbb{A}}(S)$ and so $k_{\mathbb{A}}(S)/\Delta(R_S) \cong \hat{R}_S$

where S is an arbitrary set of finite places of an $\mathbb{A}$–field k. We collect these remarks in the following Theorem, which is an extension of one of the “Main Theorems” in Chapter IV, Section 2 of [38] to arbitrary sets of places.

Theorem 3.1. *The map $\Delta : R_S \to k_{\mathbb{A}}(S)$ embeds R_S as a discrete cocompact subring in the S–adele ring of k. There is an isomorphism between the S–adele ring $k_{\mathbb{A}}(S)$ and itself, which induces an isomorphism between $\widehat{R_S}$ and $k_{\mathbb{A}}(S)/\Delta(R_S)$.*

Remark 1. The S–adele ring $k_{\mathbb{A}}(S)$ covering the dynamical system $(X^{(k,S)}, \alpha^{(k,S,\xi)})$ gives a complete local portrait of the hyperbolicity. A neighbourhood of the identity in $X^{(k,S)}$ is isometric to a neighbourhood of the identity in $k_{\mathbb{A}}(S)$. The map $\alpha^{(k,S,\xi)}$ under this isometry acts on each quasi–factor k_ν by multiplication, dilating the metric on that quasi–factor by $|\xi|_\nu$. If S is infinite, then the local action is an isometry on all but finitely many quasi–factors, making such systems very far from hyperbolic ones.

4. Entropy, Ergodicity and Expansiveness

The entropy of endomorphisms of solenoids is computed in [19], using the locally isometric adelic covering space and Bowen's work on topological entropy for uniformly continuous maps.

Theorem 4.1. *The topological entropy of the S–integer system $(X^{(k,S)}, \alpha^{(k,S,\xi)})$ is given by*

$$h(\alpha^{(k,S,\xi)}) = \sum_{w \in S \cup P_\infty(k)} \log^+ |\xi|_w \tag{5}$$

Proof. The entropy of maps dual to monomorphisms of number fields is computed in terms of valuations in [35] and [19]. For the geometric case, the method of proof used in [19] goes through, simply replacing $\mathbb{Q}$ by $\mathbb{F}_p(t)$. □

Recall the following standard criterion for ergodicity.

Theorem 4.2. *If X is a compact metrizable abelian group and $T : X \to X$ is a surjective continuous endomorphism then Haar measure is ergodic for T if and only if the trivial character $\gamma \equiv 1$ is the only $\gamma \in \hat{X}$ satisfying $\gamma \circ T^n = \gamma$ for some $n > 0$.*

Proof. See [8], Theorem 1. □

Corollary 4.1. *Let $(X, \alpha) = (X^{(k,S)}, \alpha^{(k,S,\xi)})$ be an S–integer dynamical system. Then α is ergodic if and only if ξ is not a root of unity.*

It follows that in the geometric case α is ergodic if and only if $\xi \notin \mathbb{F}_p^*$.

Proof. The map α is non–ergodic if and only if there is a $r \in R_S \backslash \{0\}$ with $\xi^m r = r$ for some $m \neq 0$. This is possible in a field if and only if ξ is a unit root. □

Recall that a continuous map $\alpha : (X, d) \to (X, d)$ is *forwardly expansive* if there is a constant $\delta > 0$ such that for each pair $x \neq y \in X$ there is some $n \in \mathbb{N}$ with $d(\alpha^n x, \alpha^n y) > \delta$. A homeomorphism $\beta : (X, d) \to (X, d)$ is *expansive* if there is a constant $\delta > 0$ such that for each pair $x \neq y \in X$ there is some $n \in \mathbb{Z}$ with $d(\beta^n x, \beta^n y) > \delta$. Homeomorphisms can only be forwardly expansive on finite metric spaces by [5].

Theorem 4.3. *Let K be a non–discrete field complete with respect to a valuation $|\cdot|$, and let $\bar{K}$ denote the algebraic closure of K with the uniquely extended absolute value from K. Let E be a finite dimensional vector space over K, and let u be an automorphism of E. Then u is expansive if and only if $|\lambda| \neq 1$ for each eigenvalue λ of u in $\bar{K}$.*

Proof. See Eisenberg's paper [6], Theorem 3. □

Corollary 4.2. *Let $(X,\alpha)=(X^{(k,S)},\alpha^{(k,S,\xi)})$ be an S–integer dynamical system. Then α is expansive if and only if $S\cup P_\infty\subseteq\{\nu\le\infty : |\xi|_\nu\neq 1\}$.*

Proof. Recall that there is a local isometry between $k_{\mathbb{A}}(S)$ and X, so it is enough to check expansiveness of the lifted map on $k_{\mathbb{A}}(S)$. Here Eisenberg's criterion in Theorem 4.3 applies to each of the (finitely many) indicated quasifactors. □

Remark 2. Corollary 4.2 is a generalisation of [26], Proposition 7.2 where Schmidt considers k to be a number field and $S=\{\nu<\infty : |\xi|_\nu\neq 1\}$.

5. Periodic points

If $\beta:\mathbb{T}^d\to\mathbb{T}^d$ is a toral endomorphism, then the number of points of period n under β (if finite) is given by the cardinality of the kernel of (β^n-Id) on $\mathbb{T}^d$, and this in turn is equal to the determinant of the lifted map $(\tilde{\beta}^n-Id):\mathbb{R}^n\to\mathbb{R}^n$. The same relationship holds more generally, as may be seen using the module in a locally compact group, [38] p.73; we give a direct proof of the generalisation in order to explain the arithmetic consequence of ergodicity.

Let Γ be a discrete cocompact subgroup of a locally compact abelian group X. A *fundamental domain* F of X modulo Γ is a full (measurable) set of coset representatives of Γ in X. Denote by μ the Haar measure on X normalised to give $\mu(F)=1$. Let $\tilde{A}:X\to X$ be a continuous surjective mapping with $\tilde{A}(\Gamma)\subset\Gamma$, and let $A:X/\Gamma\to X/\Gamma$ be the induced map on the quotient space.

Lemma 5.1. *If* $\ker A$ *is discrete, then*

$$\mathrm{mod}_X(\tilde{A})=|\ker A|.$$

Proof. Since Γ is discrete in X, a fundamental domain F may be chosen so that there exists a neighbourhood $U(0_X)$ of the identity $0_X\in X$ with $U(0_X)\subset F$. The finiteness of $|\ker A|$ follows from the fact that X/Γ is compact. So for a sufficiently small neighbourhood $V(0_{X/\Gamma})$ of the identity $0_{X/\Gamma}\in X/\Gamma$,

$$A^{-1}V(0_{X/\Gamma})=\bigcup_{i=1,\dots,|\ker A|}V_i,$$

where each V_i is a neighbourhood of a point in the set $A^{-1}(0_{X/\Gamma})$ and their union is disjoint. Since A is measure–preserving, $\mu\left(A^{-1}V(0_{X/\Gamma})\right)=$

$\mu\left(V(0_{X/\Gamma})\right)$. Once again using the discreteness of Γ in X we have that X is locally isomorphic to X/Γ. This means that, assuming the neighbourhoods $U(0_X)$ and $V(0_{X/\Gamma})$ are small enough, $\pi|_{U(0_X)}$ is a homeomorphism between $U(0_X)$ and $V(0_{X/\Gamma})$. Thus we have

$$\begin{aligned}\mu\left(\tilde{A}U(0_X)\right) &= \mu\left(AV(0_{X/\Gamma})\right)\\ &= |\ker A|\mu\left(V(0_{X/\Gamma})\right)\\ &= |\ker A|\mu\left(U(0_X)\right)\end{aligned}$$

which proves the Lemma. Furthermore, since $U(0_X)\subset F$, $\mu(\tilde{A}F)=|\ker A|$. ☐

Lemma 5.2. *Let* $(X,\alpha)=(X^{(k,S)},\alpha^{(k,S,\xi)})$ *be an* S*–integer dynamical system. Then the number of points of period* $n\geq 1$ *is finite if* α *is ergodic, and*

$$|F_n(\alpha)|=\prod_{\nu\in S\cup P_\infty}|\xi^n-1|_\nu.$$

Proof. A fundamental domain of $k_{\mathbb{A}}(S)$ modulo k is a set

$$F=\begin{cases}[0,1)^d\times\prod_{\nu\in S}r_\nu & \text{if } k \text{ is a number field with } d=[k:\mathbb{Q}],\\ \prod_{\nu\in S\cup P_\infty}r_\nu & \text{otherwise.}\end{cases}$$

The set F is measurable. For each $\nu\in S\cup P_\infty$, let μ_ν denote a Haar measure on k_ν normalised to have $\mu_\nu(r_\nu)=1$ for all but finitely many ν. Then the product measure $\mu=\prod_{\nu\in S\cup P_\infty}\mu_\nu$ is well defined and is a Haar measure on $k_{\mathbb{A}}(S)$. Set $A=\alpha^n-I, X=k_{\mathbb{A}}(S)$ and $\Gamma=\Delta(R_S)$, then ergodicity implies that $\ker A$ is discrete in $\hat{R}_S$ and by Lemma 5.1 we have

$$|F_n(\alpha)|=|\ker(\alpha^n-1)|=\mu\left((\tilde{\alpha}^n-1)F\right)=\prod_{\nu\in S\cup P_\infty}|\xi^n-1|_\nu.$$

☐

6. Growth rates

Notice the following inequality for ergodic systems,

$$\limsup_{n\to\infty}\frac{1}{n}\log\prod_{\nu\in S\cup P_\infty}|\xi^n-1|_\nu\leq\sum_{\nu\in S\cup P_\infty}\log^+|\xi|_\nu=h(\alpha)$$

so that

$$p^-(\alpha)\leq p^+(\alpha)\leq h(\alpha). \tag{6}$$

In general, for a continuous map T of a metric space (X, d), we need expansiveness to deduce that $p^+(T) \le h(T)$. For these algebraic systems, we always have (6).

Lemma 6.1. [Baker's Theorem] *Let ξ be an algebraic number on the unit circle which is not a root of unity. Then*

$$|\xi^n - 1| > \frac{A}{n^B}$$

where A, B are constants independent of n.

From now on, we shall write $a(n) \ll b(n)$ if there is a constant c independent of n for which $a(n) < c \cdot b(n)$. Thus, the conclusion of Baker's theorem may be written in the form $|\xi^n - 1| \gg n^{-B}$.

A weaker (and earlier) result is sufficient to give convergence in the quasihyperbolic toral case by [17].

Lemma 6.2. [Gelfond's Lower Bound] *Let ξ be an algebraic number on the unit circle which is not a root of unity. Then given $\epsilon > 0$, there exists M such that $|\xi^n - 1| > e^{-\epsilon n}$ for all $n \ge M$.*

Proof. Both Baker's theorem and Gelfond's lower bound are on p.281 in [23]. □

Theorem 6.1. *Let $(X, \alpha) = (X^{(k,S)}, \alpha^{(k,S,\xi)})$ be an ergodic arithmetic S–integer dynamical system with S finite. Then the growth rate of the number of periodic points exists and is given by*

$$p^+(\alpha) = p^-(\alpha) = h(\alpha). \tag{7}$$

Proof. Note that

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \log |F_n(\alpha)| &= \lim_{n\to\infty} \frac{1}{n} \sum_{\nu \in S \cup P_\infty} \log |\xi^n - 1|_\nu \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{\nu \in P_\infty} \log |\xi^n - 1|_\nu + \sum_{\nu \in S : \xi \notin r_\nu^*} \log^+ |\xi|_\nu + \lim_{n\to\infty} D_n
\end{aligned}$$

where $D_n = \frac{1}{n} \sum_{\nu \in S : \xi \in r_\nu^*} \log |\xi^n - 1|_\nu$. We first handle the archimedean contribution. Suppose $|\xi| = 1$, and $\epsilon > 0$ is given. Then by Lemma 6.1 we have

$$\frac{1}{n} \log A - \frac{B}{n} \log n < \frac{1}{n} \log |\xi^n - 1| < \frac{1}{n} \log 2$$

So $\frac{1}{n} \log |\xi^n - 1| \to 0$ as $n \to \infty$. If $|\xi| < 1$ then clearly $\frac{1}{n} \log |\xi^n - 1| \to 0$ as $n \to \infty$. Finally, if $|\xi| > 1$ then

$$\frac{1}{n} \log |\xi^n - 1| = \frac{1}{n} \left(\log |\xi|^n + \log |1 - \xi^{-n}| \right) \to \log |\xi| \text{ as } n \to \infty.$$

Therefore for any algebraic number ξ,

$$\frac{1}{n}\log|\xi^n - 1| \to \log^+|\xi|$$

and hence,

$$\frac{1}{n}\sum_{\nu\in P_\infty}\log|\xi^n - 1|_\nu \to \sum_{\nu\in P_\infty}\log^+|\xi|_\nu.$$

We now show that $D_n \to 0$ by deriving the bound

$$n^{-1} \ll |\xi^n - 1|_\nu \leq 1 \tag{8}$$

for any finite place ν and $\xi \in r_\nu^*$. The upper bound is obvious. Suppose that the place ν of k lies above the place p of $\mathbb{Q}$ for some rational prime p. Using the Euclidean algorithm we may write $n = n_1(p-1)+r_1$ where $0 \leq r_1 < p-1$. Now, if $|\xi^n - 1|_\nu = 1$ then there is no ν–adic contribution in the quantity D_n, so we may suppose that $|\xi^n - 1|_\nu < 1$. Let $\not\leq_\nu$ denote the smallest field which contains $\mathbb{Q}$ and is both algebraically closed and complete with respect to $|.|_\nu$. The ν–adic logarithm $\log_\nu$ is defined as

$$\log_\nu(1+x) = \sum_{i=1}^{\infty}\frac{(-1)^{i+1}x^i}{i}$$

and converges for all $x \in \not\leq_\nu$ such that $|x|_\nu < 1$. Setting $x = \xi^n - 1$ we get

$$\log_\nu(\xi^n) = (\xi^n - 1) - \frac{(\xi^n-1)^2}{2} + \frac{(\xi^n-1)^3}{3} - \dots$$

and so $|\log_\nu(\xi^n)|_\nu \leq |\xi^n - 1|_\nu$. We always have $n^{-1} \ll |n\log_\nu(\xi)|_\nu = |\log_\nu(\xi^n)|_\nu$, so this establishes (8). Hence

$$\lim_{n\to\infty}\frac{1}{n}\log|F_n(\alpha)| = \sum_\nu \log^+|\xi|_\nu = h(\alpha).$$

$\square$

The case when $S = \emptyset$ and ξ has a conjugate on the unit circle but not a root of unity is an example of a *quasihyperbolic* toral endomorphism. As shown in Section 4 of Lind's paper on quasihyperbolic toral automorphisms [17] the existence of the limit

$$\lim_{n\to\infty}\frac{1}{n}\log|F_n(\alpha)| = h(\alpha)$$

is equivalent to the validity of Gelfond's lower bound, Lemma 6.2.

Theorem 6.2. *Let $(X,\alpha)=(X^{(k,S)},\alpha^{(k,S,\xi)})$ be an ergodic geometric S–integer dynamical system with S finite. Then*

$$p^+(\alpha)=h(\alpha).$$

Proof. The periodic points formula in Lemma 5.2 gives

$$\frac{1}{n}\log|F_n(\alpha)|=\sum_{\nu\in S\cup P_\infty:\xi\notin r^*_\nu}\log^+|\xi|_\nu+\frac{1}{n}\sum_{\nu\in S'}\log|\xi^n-1|_\nu,$$

where S' is a subset of S for which $\xi\in r^*_\nu$ for all $\nu\in S'$. It is convenient to split S' into sets A and B defined by $A=\{\nu\in S':|\xi-1|_\nu=1\}$ and $B=\{\nu\in S':|\xi-1|_\nu<1\}$. For each $\nu_j\ \ (j=1,\dots,m)\in A$ we can associate integers $d_1,\dots,d_m\geq 2$ such that $|\xi^n-1|_{\nu_j}=1$ if and only if $d_j\not|n$.

Now consider the valuations $\nu_1,\dots,\nu_l\in B$. If $\nu\in B$ we may write

$$\xi=1+\sum_{i=1}^{\infty}a_i\pi^i,$$

where a_i and π are as above, and $|\xi-1|_\nu=p^{-s}$ where $s=\frac{1}{e}\min\{i:a_i\neq 0\}>0$ and $\mathrm{ord}_\nu(\pi)=\frac{1}{e}$. For each $\nu_j\in B$ label such s by s_j, the coefficients a_i by $a_i(j)$ and π by π_j. Then

$$\begin{aligned}\frac{1}{n}\sum_{\nu\in B}\log|\xi^n-1|_\nu &= \frac{1}{n}\sum_{\nu\in B}\log|\xi-1|_\nu+\frac{1}{n}\sum_{\nu\in B}\log|\xi^{n-1}+\cdots+\xi+1|_\nu\\ &= \frac{1}{n}\sum_{j=1}^{l}\log|\pi_j|_{\nu_j}^{s_j}+\frac{1}{n}\sum_{j=1}^{l}\log\left|n+\sum_{i=1}^{\infty}b_i(j){\pi_j}^i\right|_{\nu_j},\end{aligned}$$

for computable coefficients $b_i(j)\in r_{\nu_j}$ and $j=1,\dots,l$. This expression tends to zero if $p\not|n$. Hence

$$\frac{1}{n}\sum_{\nu\in S'}\log|\xi^n-1|_\nu\to 0\text{ as }n\to\infty$$

through the set

$$\{n\geq 1:p\not|n,d_j\not|n\text{ for }j=1,\dots,m\}.$$

It follows that $p^+(\alpha)=h(\alpha)$. □

Using periodic points we are able to give an arithmetical characterization of expansiveness subject to certain conditions.

Theorem 6.3. *Let $(X, \alpha) = (X^{(k,S)}, \alpha^{(k,S,\xi)})$ be an arithmetic S–integer dynamical system. If α is expansive, then $\lim_{n\to\infty} |F_n(\alpha)|/|F_{n+1}(\alpha)|$ exists. If S is finite, the converse holds also.*

Proof. Assume α is expansive. Then by Theorem 4.2 and the estimates in Lemma 6.1 and (8) we have

$$\lim_{n\to\infty} \frac{|F_n(\alpha)|}{|F_{n+1}(\alpha)|} = \lim_{n\to\infty} \prod_{\nu \in S\cup P_\infty} \left| \frac{\xi^n - 1}{\xi^{n+1} - 1} \right|_\nu = \exp\{-h(\alpha)\}.$$

For a partial converse, we assume that S is finite and prove that non–expansiveness implies that

$$|F_n(\alpha)|/|F_{n+1}(\alpha)|$$

can be made arbitrarily large as well as small. Since the dual of the group of points of period n is given by

$$\widehat{F_n(\alpha)} \cong R_S/(\xi^n - 1)R_S,$$

it is clear (for example, when $\xi = 1$) that in non–ergodic systems there exist values of n for which $|F_n(\alpha)|$ is infinite. Assume therefore that α is ergodic. We shall treat the archimedean and non–archimedean cases separately before determining their combined influence.

Firstly, suppose all the non–expansive behaviour is archimedean in nature. So the ratio $|F_n(\alpha)|/|F_{n+1}(\alpha)|$ contains a factor of the form

$$\prod_{j=1}^{m} \left| \frac{\xi_j^n - 1}{\xi_j^{n+1} - 1} \right|,$$

where $\xi_1, \dots, \xi_m$ are the conjugates of ξ which are not unit roots but are on the unit circle. For each j, write $\xi_j = e^{i\rho_j}$ where $\rho_j \in (0, 2\pi)$ is irrational. Then by Dirichlet's Theorem on simultaneous approximation [27] p.27, there exist infinitely many integers $l_1, \dots, l_m$ and infinitely many $n \in \mathbb{N}$ with

$$|n\rho_j + l_j| \ll \frac{1}{n^{1/m}} \text{ for } j = 1, \dots, m.$$

Denoting the set of all such n by A, we have

$$|\xi_j^n - 1| = |e^{i(n\rho_j + l_j)} - 1| \ll \frac{1}{n^{1/m}} \text{ for } \; j = 1, \dots, m \text{ and for all } n \in A.$$

Also observe that for any $1 \le j \le m$ and $n \in A$,

$$|\xi_j^{n+1} - 1| = |\xi_j^n - 1 + 1 - \xi_j^{-1}| \gg \left| \frac{1}{n^{1/m}} - |\xi_{\ j} - 1| \right| \ge |1 - |\xi_j - 1|| \ge 1,$$

and similarly $|\xi_j^{n-1}-1| \gg 1$. Thus for all $n \in A$ we have

$$\left|\frac{\xi_j^n - 1}{\xi_j^{n+1}-1}\right| \ll \frac{1}{n^{1/m}}$$

and

$$n^{1/m} \ll \left|\frac{\xi_j^{n-1}-1}{\xi_j^n - 1}\right|,$$

which implies that

$$\prod_{j=1}^m \left|\frac{\xi_j^n - 1}{\xi_j^{n+1}-1}\right|$$

is either arbitrarily small or arbitrarily large depending on whether n tends to infinity through A or $A-1=\{n-1 : n \in A\}$.

Now consider the case when the non–expansive behaviour is solely non–archimedean. The ratio $|F_n(\alpha)|/|F_{n+1}(\alpha)|$ contains a factor of the form

$$\prod_{j=1}^m \left|\frac{\xi^n - 1}{\xi^{n+1}-1}\right|_{\nu_j},$$

where $\nu_j|p_j$ and $|\xi|_{\nu_j} = 1$. The ν_j–adic expansion of ξ has a non–zero constant term, so there exist positive integers $d_1,\dots,d_m$ such that $|\xi^n-1|_{\nu_j}=1$ if and only if $d_j \not| n$. Define an infinite subset $B \subset \mathbb{N}$ by

$$B=\{d_1\cdots d_m(p_1\cdots p_m)^r : r \in \mathbb{N}\}.$$

Then for each $j=1,\dots,m$ and $n \in B$ we have,

$$|\xi^n-1|_{\nu_j} \le |n|_{p_j} = \frac{1}{p_j^r} \text{ for all } r \in \mathbb{N} \text{ and } |\xi^{n+1}-1|_{\nu_j} = |\xi^{n-1}-1|_{\nu_j} = 1.$$

Hence for all $r \in \mathbb{N}$

$$\prod_{j=1}^m \left|\frac{\xi^n - 1}{\xi^{n+1}-1}\right|_{\nu_j} \le \prod_{j=1}^m |n|_{p_j} = \frac{1}{(p_1\cdots p_m)^r}$$

and

$$(p_1\cdots p_m)^r \le \prod_{j=1}^m \left|\frac{\xi^{n-1} - 1}{\xi^{n}-1}\right|_{\nu_j}.$$

So once again $|F_n(\alpha)|/|F_{n+1}(\alpha)|$ fails to converge.

Finally, suppose non–expansiveness is a hybrid of archimedean and non–archimedean contributions. Then the ratio $|F_n(\alpha)|/|F_{n+1}(\alpha)|$ contains a factor of the form

$$\prod_{j=1}^{m_1} \left|\frac{\xi_j^n - 1}{\xi_j^{n+1}-1}\right| \prod_{j=1}^{m_2} \left|\frac{\xi^n - 1}{\xi^{n+1}-1}\right|_{\nu_j},$$

where $\xi_1, \dots, \xi_{m_1}$ are those conjugates of ξ which are not unit roots but are on the unit circle and $|\xi|_{\nu_j} = 1$ for $j = 1, \dots, m_2$.

This time we can afford to be more generous with the definition of $B \subset \mathbb{N}$: let

$$B = \{rd_1 \cdots d_{m_2} : r \in \mathbb{N}\},$$

where the d_j's are those positive integers for which $|\xi^n - 1|_{\nu_j} = 1$ if and only if $d_j \not| n$.

Since $n'\rho_j$ is irrational for each $j = 1, \dots, m_1$ and for all $n' \in B$, applying Dirichlet's Theorem again, there exist infinitely many integers $l_1, \dots, l_{m_1}$ and infinitely many $n \in \mathbb{N}$ with

$$|nn'\rho_j + l_j| \ll \frac{1}{n^{1/m_1}} \text{ for } j = 1, \dots, m_1.$$

Hence, denoting the set of all such n by A and defining an infinite subset $E \subset \mathbb{N}$ by

$$E = \{n_1 n_2 : n_1 \in A, n_2 \in B\},$$

we have,

$$|\xi_j^n - 1| = |e^{i(n_1 n_2 \rho_j + l_j)} - 1| \ll \frac{1}{n^{1/m_1}} \text{ for } j = 1, \dots, m_1 \text{ and for all } n \in E.$$

As before, both $|\xi_j^{n+1} - 1|$ and $|\xi_j^{n-1} - 1|$ are bounded below by some positive constant for $j = 1, \dots, m_1$ and for all $n \in E$. Thus for all $n \in E$,

$$\prod_{j=1}^{m_1} \left| \frac{\xi_j^n - 1}{\xi_j^{n+1} - 1} \right| \prod_{j=1}^{m_2} \left| \frac{\xi_j^n - 1}{\xi_j^{n+1} - 1} \right| \ll \frac{1}{n}$$

and

$$n \ll \prod_{j=1}^{m_1} \left| \frac{\xi_j^{n-1} - 1}{\xi_j^n - 1} \right| \prod_{j=1}^{m_2} \left| \frac{\xi_j^{n-1} - 1}{\xi_j^n - 1} \right|.$$

Hence $\lim_{n\to\infty} |F_n(\alpha)|/|F_{n+1}(\alpha)|$ does not exist. □

Example 6.1. To illustrate Theorem 6.3 and the different ways in which expansiveness can break down, consider the following examples. The first three fit the hypotheses of Theorem 6.3, while the fourth is a simple geometric example whose behaviour suggests that Theorem 6.3 may extend to that setting.

(1) Let α be the S–integer dynamical system corresponding to

$$S = \emptyset, \ \xi = \sqrt{2} - 1 + i\sqrt{2\sqrt{2} - 2}, \text{ and } k = \mathbb{Q}(\xi).$$

Then α is isomorphic to the automorphism of the 4–torus given by the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & 2 & -4 \end{bmatrix}.$$

As pointed out in [16] this automorphism is ergodic but not expansive. It is non–expansive because of non–hyperbolicity at an archimedean place.

(2) Let β be the S–integer dynamical system corresponding to

$$S = \emptyset,\ \xi = \tfrac{1}{2}\left(1+\sqrt{5}\right),\ \text{and } k = \mathbb{Q}(\xi).$$

Then β is isomorphic to the expansive automorphism of the 2–torus with matrix

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

(3) Let γ be the S–integer dynamical system corresponding to

$$S = \{3,5\},\ \xi = 2,\ \text{and } k = \mathbb{Q}.$$

This is non–expansive because of non–hyperbolicity at two non–archimedean places.

(4) Let δ be the S–integer dynamical system corresponding to

$$S = \{t-1\},\ \xi = t,\ \text{and } k = \mathbb{F}_2(t).$$

As shown in [4], Example 4.5 (see also Example 8.5 below), $F_n(\delta) = 2^{n-2^{\mathrm{ord}_2(n)}}$. It follows that

$$|F_{2^k-1}(\delta)|/|F_{2^k}(\delta)| = 2^{2^k-2},$$

and

$$|F_{2^k}(\delta)|/|F_{2^k+1}(\delta)| = 2^{-2^k}.$$

This non–expansive geometric example therefore also has the property that the ratio of fixed points fails to converge.

The table shows the ratio $|F_n|/|F_{n+1}|$ for $1 \leq n \leq 22$, indicating the erratic behaviour in the non–expansive maps α, γ and δ and the convergence of the ratio for β.

n	$\lvert F_n(\alpha)\rvert/\lvert F_{n+1}(\alpha)\rvert$	$\lvert F_n(\beta)\rvert/\lvert F_{n+1}(\beta)\rvert$	$\lvert F_n(\gamma)\rvert/\lvert F_{n+1}(\gamma)\rvert$	$\lvert F_n(\delta)\rvert/\lvert F_{n+1}(\delta)\rvert$
1	0.12500	1.00000	1.00000	1.00000
2	0.16327	0.25000	0.14286	0.25000
3	0.38281	0.80000	7.00000	4.0000
4	1.58025	0.45455	0.03226	0.06250
5	0.20663	0.68750	4.42857	1.00000
6	0.03070	0.55172	0.05512	4.00000
7	0.12724	0.64444	7.47059	64.00000
8	0.25855	0.59211	0.03327	0.00391
9	0.62327	0.62810	1.49853	1.00000
10	1177.17958	0.60804	0.16659	0.25000
11	0.00004	0.62187	22.49451	4.00000
12	0.07789	0.61420	0.01111	0.06250
13	0.18507	0.61950	1.49991	1.00000
14	0.37744	0.61657	0.16666	0.25000
15	1.65035	0.61859	7.49989	16384.00000
16	0.18696	0.61747	0.03333	0.00002
17	0.03182	0.61825	13.49995	1.00000
18	0.12857	0.61782	0.01852	0.12500
19	0.26083	0.61812	37.49996	8.00000
20	0.63352	0.61795	0.00667	0.06250
21	287.58483	0.61807	1.50000	1.00000
22	0.00015	0.61800	0.16667	0.25000

Periodic point ratios for α, β, γ and δ.

7. Distribution of Periodic Points

Let μ_n denote the Haar measure on the subgroup of points with period n in a dynamical system (X, β) where β is an endomorphism of a compact abelian group X. If X is a torus and β is ergodic, then μ_n converges weakly to Haar measure by [17]. If the action is expansive, then an analogous result holds for actions of $\mathbb{Z}^d$, [37]. Both these results depend on a positive exponential growth rate of the number of periodic points. In an alternative proof for ergodic toral automorphisms, Waddington [34] showed that $|F_n(\alpha)| \to \infty$ implies that $\mu_n \to \mu =$ Haar measure without any rate assumption. This result does not extend to solenoids as shown by the following.

Example 7.1. Let $\Gamma = \mathbb{Z} \times \mathbb{Q}$, and let $\beta : \widehat{\Gamma} \to \widehat{\Gamma}$ be the map dual to $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Then $F_n(\beta) = 2^n - 1 \to \infty$, but the measures μ_n converge to the measure $\mu_{\mathbb{T}} \times \delta_0$ on $\mathbb{T} \times \Sigma = \widehat{\Gamma}$.

It is easy to see that Waddington's result does carry over to S–integer systems. Note, however, that the simple argument below does not apply to the toral case: for example, the toral automorphism dual to $A = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}$ is the S–integer system given by $k = \mathbb{Q}(\xi)$, $S = \emptyset$, $\xi = 2 + \sqrt{5}$, while the automorphism dual to B$=\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$ is not an S–integer system since B corresponds to a non–trivial element in the ideal class group of the splitting field of the characteristic polynomial of A and B; see [14] or the discussion in Example 2.2(11).

Theorem 7.1. *Let $(X, \alpha) = (X^{(k,S)}, \alpha^{(k,S,\xi)})$ be an S–integer dynamical system with $|F_n(\alpha)| \to \infty$ as $n \to \infty$. Then the periodic points are uniformly distributed with respect to Haar measure on X.*

Proof. We need to prove that for any non–trivial character $r \in R_S = \widehat{X}$, $\hat{\mu_n}(r) = 0$ for sufficiently large n. Assume not: there is such an r and a sequence $n_j \to \infty$ for which $r \in (\xi^{n_j} - 1)\, R_S$ for all j. This implies that

$$\infty > \prod_{\nu \in S \cup P_\infty} |r|_\nu = |R_S / r \cdot R_S| \geq |R_S / (\xi^{n_j} - 1) R_S| = |F_{n_j}(\alpha)|$$

for all j, which is impossible since the number of periodic points goes to infinity. □

8. Zeta Functions

Lemma 8.1. *Let X be a compact, connected group (necessarily abelian) and let α be an expansive automorphism of X Then ζ_α is rational.*

Proof. By Theorem 6.1 in [12], X is isomorphic to

$$Y_{H(A)} = \left\{ x = \{x_i\}_{-\infty}^{\infty} \in (\mathbb{T}^n)^{\mathbb{Z}} : (x_i, x_{i+1}) \in H(A) \text{ for all } i \in \mathbb{Z} \right\},$$

where $H(A) \subset \mathbb{T}^n \times \mathbb{T}^n$ is defined by

$$H(A) = \tau \left(\{ (y, Ay) : y \in \mathbb{R}^n \} \right)$$

for some $n \geq 1$, $A \in GL(n, \mathbb{Q})$ and τ is the quotient map

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{T}^n.$$

The isomorphism carries α to T^A, the shift on $Y_{H(A)}$. The group $Y_{H(A)}$ is a *generalised solenoidal group* as studied by Lawton in [15]. Let d be the least positive integer for which dA has integer entries. Then the number of periodic points is given by

$$|F_v(\alpha)| = d^v \prod |\lambda_i^v - 1|$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of dA. Expanding the finite product shows that the zeta function is rational. □

Example 8.1. If $A = [3/2]$ then the number of points of period n under T^A is $(3^n - 2^n)$.

Lemma 8.2. *Let X be a compact, zero–dimensional topological group and let α be an expansive automorphism. Then ζ_α is rational if α is ergodic.*

Proof. By Theorem 1(ii) in [13], (X, α) is homeomorphic to $(F, \psi) \times (G^{\mathbb{Z}}, \sigma)$ where F is a finite group, ψ is an automorphism, G is a finite group and σ is the shift. For $n \geq 1$,

$$|F_n(\alpha)| = |F_n(\psi \times \sigma)| = |F_n(\psi)| \cdot |G|^n,$$

which is finite. So the zeta function is given by

$$\zeta_\alpha(z) = \exp\left(\sum_{n=1}^{\infty} \frac{|F_n(\alpha)|}{n} z^n\right) = \exp\left(\sum_{n=1}^{\infty} \frac{|F_n(\psi)| \cdot |G|^n}{n} z^n\right).$$

Now α is ergodic if and only if $F = \{e\}$, in which case $|F_n(\psi)| = 1$ and the zeta function has the form $\frac{1}{1-|G|z}$. □

Remark 3. The dynamical zeta function does characterise ergodicity in the setting of Lemma 8.2 (assuming the group X is infinite): α is ergodic if and only if the only pole of ζ_α in the closed unit disc is at $\exp(-h_{top}(\alpha))$.

Example 8.2. Let α be the expansive automorphism of $\widehat{\mathbb{Z}[\frac{1}{6}]}$ dual to $\times\frac{2}{3}$ on $\mathbb{Z}[\frac{1}{6}]$. The entropy of α is $\log 3$ and for each $n \geq 1$,

$$\begin{aligned} |F_n(\alpha)| &= \left|\left(\frac{2}{3}\right)^n - 1\right|_\infty \left|\left(\frac{2}{3}\right)^n - 1\right|_2 \left|\left(\frac{2}{3}\right)^n - 1\right|_3 \\ &= 3^n - 2^n. \end{aligned}$$

The zeta function is given by

$$\zeta_\alpha(z) = \frac{1 - 2z}{1 - 3z}.$$

Example 8.3. Let α be the endomorphism of $\widehat{\mathbb{Z}[\frac{1}{30}]}$ dual to $\times\frac{3}{2}$ on $\mathbb{Z}[\frac{1}{30}]$. By Theorem 4.2 α is non–expansive (since $|\frac{3}{2}|_5 = 1$). The number of points of period n is given by

$$\begin{aligned} |F_n(\alpha)| &= \left|\left(\frac{3}{2}\right)^n - 1\right|_\infty \left|\left(\frac{3}{2}\right)^n - 1\right|_2 \left|\left(\frac{3}{2}\right)^n - 1\right|_3 \left|\left(\frac{3}{2}\right)^n - 1\right|_5 \\ &= (3^n - 2^n)\,|3^n - 2^n|_5\,, \end{aligned}$$

the first few values of which are

$$1, 1, 19, 13, 211, 133, 2059, 1261, 19171, 2321, 175099, \dots$$

By Theorem 6.1 the logarithmic growth rate of this sequence is equal to $\log 3$, the entropy of α. We claim that ζ_α is irrational and we shall use Theorem 8.1, the so–called Hadamard Quotient Theorem, to prove this.

Theorem 8.1. *Let $\mathbb{F}$ be a field of characteristic zero and (a'_n) a sequence of elements of a subring R of $\mathbb{F}$ which is finitely generated over $\mathbb{Z}$. Let $\sum b_n X^n$ and $\sum c_n X^n$ be formal series over $\mathbb{F}$ representing rational functions. Denote by J the set of integers $n \geq 0$ such that $b_n \neq 0$. Suppose that $a'_n = c_n/b_n$ for all $n \in J$. Then there is a sequence (a_n) with $a_n = a'_n$ for $n \in J$, such that the series $\sum a_n X^n$ represents a rational function.*

Proof. This was proved by van der Poorten; see [33] and the lecture notes of [25] for a proof, and [32] for a general discussion. □

Proposition 8.1. *The number of values that a recurrence sequence can take on infinitely often is bounded by some integer that depends only on the poles of its generating rational function.*

Proof. See [21], Proposition 2. □

Returning to Example 8.3 suppose, for a contradicton, that ζ_α is rational. Then by differentiating ζ_α, $\sum_{n=1}^{\infty} |F_n(\alpha)| z^n$ is also rational. The sequence defined by $a_n = 3^n - 2^n$ is a recurrence sequence since it satisfies the linear, homogeneous recurrence relation

$$a_{n+2} = 5a_{n+1} - 6a_n,$$

together with the initial conditions $a_0 = 0, a_1 = 1$. Hence $\sum_{n=1}^{\infty} a_n z^n$ represents the rational function

$$\frac{z}{1 - 5z + 6z^2}.$$

By Theorem 8.1, with $|F_n(\alpha)| \neq 0$,

$$\sum_{n=1}^{\infty} \frac{z^n}{|3^n - 2^n|_5}$$

is a rational function $P(z)/Q(z)$ and hence $b_n = |3^n - 2^n|_5^{-1}$ is a recurrence sequence. The Taylor series coefficients are given by

$$b_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 5^{1+\mathrm{ord}_5(n)} & \text{if } n \text{ is even.} \end{cases}$$

By the above Proposition the number of values that b_n can take on infinitely often is bounded by some integer depending on the roots of $Q(z)$. However, the set $\{1, 5, 5^2, \dots\}$ is infinite, giving a contradiction. Hence ζ_α is irrational.

Example 8.4. Let $k = \mathbb{F}_p(t)$ and $S = \{t\}$. Define α to be the endomorphism of $\hat{R}_S = \widehat{\mathbb{F}[t^{\pm 1}]}$ dual to multiplication by t on $\mathbb{F}_p[t^{\pm 1}]$. The entropy of α is

$$h(\alpha) = \sum_{\nu \leq \infty} \log^+ |t|_\nu = \log p$$

and the number of periodic points is given by

$$\begin{aligned} |F_n(\alpha)| &= |t^n - 1|_\infty |t^n - 1|_t \\ &= p^n \end{aligned}$$

Alternatively, we may note that $\hat{R}_S \cong \widehat{\oplus_{\mathbb{Z}} \mathbb{F}_p} \cong \mathbb{F}_p{}^{\mathbb{Z}}$ and that α is the one–sided shift action on p symbols. Thus the entropy and the number of periodic points are as expected. Clearly $\zeta_\alpha(z) = \frac{1}{1-pz}$ in accordance with Remark 3 before Example 8.2.

Example 8.5. Let $k = \mathbb{F}_p(t)$ and $S = \{t-1\}$. Define α to be the endomorphism of $\hat{R}_S = \widehat{\mathbb{F}_p[t][\frac{1}{t-1}]}$ dual to multiplication by t on $\mathbb{F}_p[t][\frac{1}{t-1}]$. The entropy of α is once again $\log p$ and the number of periodic points is given by applying the binomial theorem to $t^n = (1 + t - 1)^n$:

$$\begin{aligned} |F_n(\alpha)| &= |t^n - 1|_\infty |(1 + t - 1)^n - 1|_{t-1} \\ &= p^{n - p^{\mathrm{ord}_p(n)}}. \end{aligned}$$

Suppose, if possible, that ζ_α is rational, so $\sum_{n=1}^{\infty} |F_n(\alpha)| z^n$ is also rational. We already know that $\sum_{n=1}^{\infty} p^n z^n = \frac{1}{1-pz}$ is rational. Proceeding in the manner previously described we see that

$$\frac{p^n}{|F_n(\alpha)|} = p^{p^{\mathrm{ord}_p(n)}}$$

is a recurrence sequence in $\mathbb{Z}$, and by Theorem 8.1

$$\sum_{n=1}^{\infty} p^{p^{\mathrm{ord}_p(n)}} z^n$$

is a rational function. However the sequence $p^{p^{\mathrm{ord}_p(n)}}$ has an infinite number of values that it takes on infinitely often, namely $\{p, p^p, p^{p^2}, \dots\}$. This contradicts Proposition 8.1 and therefore implies that ζ_α is irrational, and so α is non–expansive.

Furthermore, writing $n = qp^{\mathrm{ord}_p(n)}$ where $p \not| q$, we have

$$\begin{aligned} |F_n(\alpha)| &= |t^n - 1|_\infty |t^q - 1|_{t-1} \\ &= p^n p^{-p^{\mathrm{ord}_p(n)}} \text{ since } p \not| q \\ &= p^{n\left(1-\frac{1}{q}\right)}. \end{aligned}$$

So for a sequence $n_j \to \infty$ with $n_j/p^{\mathrm{ord}_p(n_j)} = q$ for a fixed q,with $p \not| q$,

$$\lim_{\mathrm{ord}_p(n_j)\to\infty} \frac{1}{n_j} \log |F_{n_j}(\alpha)| = \left(1 - \frac{1}{q}\right) \log p.$$

Also, $p^+(\alpha) = h(\alpha)$ is obtained by letting $n \to \infty$ through numbers coprime with p. Hence the set of limit points of $\left\{\frac{1}{n}\log|F_n(\alpha)|\right\}_{n=1}^{\infty}$ is

$$\left\{\left(1 - \frac{1}{q}\right) h(\alpha) : q \in \mathbb{N}, p \not| q\right\} \cup \{h(\alpha)\}.$$

Remark 4. A non–constructive proof that many of these dynamical systems have irrational zeta function is given in [36].

9. Examples

Example 9.1. Let $k = \mathbb{Q}$ and $S = \emptyset$, so $R_S = \mathbb{Z}$. Let α be the toral endomorphism dual to multiplication by 2 on $\mathbb{Z}$. The periodic points formula gives $|F_n(\alpha)| = 2^n - 1$ for all $n \geq 1$ and clearly

$$\frac{1}{n} \log |2^n - 1| \to \log 2,$$

which is equal to the entropy $h(\alpha; \mathbb{T})$. The zeta function is $\zeta_\alpha(z) = \frac{1-z}{1-2z}$.

Example 9.2. Let $k = \mathbb{Q}$ and S consist of all the finite places of $\mathbb{Q}$, so $R_S = \mathbb{Q}$. If α is as above, acting on the full solenoid $\hat{\mathbb{Q}}$, then $|F_n(\alpha)| = \prod_{p\leq\infty} |2^n - 1|_p = 1$ for all $n \geq 1$ and $h(\alpha; \hat{\mathbb{Q}}) = \log 2$.

Example 9.3. Let k be an $\mathbb{A}$–field and $S = \emptyset$, then $R_S = O_k$ the ring of algebraic integers in k. If $k = \mathbb{Q}(\sqrt{2})$ and $\alpha : \hat{O}_k \to \hat{O}_k$ is dual to multiplication by $\sqrt{2}$ on $O_k = \mathbb{Z}[\sqrt{2}]$, then the elements of P_∞ are the valuations induced by the embeddings of $k \to \mathbb{R}$ namely $\sqrt{2} \mapsto \sqrt{2}$ and $\sqrt{2} \mapsto -\sqrt{2}$. Then

$$\begin{aligned} |F_n(\alpha)| &= \left|\left(\sqrt{2}\right)^n - 1\right| \left|\left(-\sqrt{2}\right)^n - 1\right| \text{ for all } n \geq 1 \\ &= \begin{cases} 2^n - 1 & \text{if } n \text{ is odd,} \\ \left(2^{n/2} - 1\right)^2 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Also,

$$\begin{aligned} h(\alpha; \hat{O}_k) &= \sum_{\nu} \log^+ \left|\sqrt{2}\right|_\nu \\ &= \sum_{\nu \in P_\infty} \log^+ \left|\sqrt{2}\right|_\nu \\ &= \log\left|\sqrt{2}\right| + \log\left|-\sqrt{2}\right| \\ &= \log 2. \end{aligned}$$

The system $\left(\widehat{\mathbb{Z}[\sqrt{2}]}, \widehat{\times\sqrt{2}}\right)$ is isomorphic to the endomorphism of $\mathbb{T}^2$ determined by the matrix $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$.

Example 9.4. Let $k = \mathbb{F}_p(t), S = \{t\}$ and $\xi = t + 1$. Then $\hat{R}_S$ is the two–sided shift space on p symbols, and α is the non–expansive cellular automaton defined by

$$(\alpha(x))_m = x_m + x_{m+1} (\mathrm{mod} p) \text{ for all } x_m \in \mathbb{F}_p, m \in \mathbb{Z}.$$

The entropy of α is $\log p$ and

$$\begin{aligned} |F_n(\alpha)| &= \left|(t+1)^n - 1\right|_\infty \left|(t+1)^n - 1\right|_t \\ &= p^n \left| t^n + \binom{n}{1} t^{n-1} + \ldots + \binom{n}{n-1} t \right|_t . \end{aligned}$$

We claim that the set of limit points of $\left\{\frac{1}{n} \log |F_n(\alpha)|\right\}_{n=1}^\infty$ is

$$\left\{ \left(1 - \frac{1}{q}\right) h(\alpha) : q \in \mathbb{N}, p \not| q \right\} \cup \{h(\alpha)\}.$$

This is seen as follows: write $n = qp^{\mathrm{ord}_p(n)}$ where $p \not| q$ then

$$\begin{aligned} |F_n(\alpha)| &= |(t+1)^n - 1|_\infty |(t+1)^q - 1|_t \\ &= p^n p^{-p^{\mathrm{ord}_p(n)}} \text{ since } p \not| q \\ &= p^{n\left(1-\frac{1}{q}\right)}. \end{aligned}$$

So for a sequence $n_j \to \infty$ with $n_j / p^{\mathrm{ord}_p(n_j)} = q$ for a fixed $q, p \not| q$,

$$\lim_{\mathrm{ord}_p(n_j)\to\infty} \frac{1}{n_j} \log |F_{n_j}(\alpha)| = \left(1 - \frac{1}{q}\right) \log p.$$

Also, $p^+(\alpha) = h(\alpha)$ is obtained by letting $n \to \infty$ through the numbers which are coprime to p.

We shall now restrict our attention to the case $k = \mathbb{Q}$ and S an infinite set; (X, α) will always denote the corresponding S–integer dynamical system. We seek examples where $p^-(\alpha)$ and $p^+(\alpha)$ can be computed. Such dynamical systems will be highly non–expansive, so examples where $p^+(\alpha) = h(\alpha)$ (as in the expansive case) will be particularly striking.

Example 9.5. Let k be an algebraic number field and let S comprise all but a finite set F of non–archimedean places of k. Using the Artin product formula (4) we have

$$\begin{aligned} \frac{1}{n} \log |F_n(\alpha)| &= \frac{1}{n} \sum_{\nu \in P_\infty} \log |\xi^n - 1|_\nu - \frac{1}{n} \sum_{\nu \in F \cup P_\infty} \log |\xi^n - 1|_\nu \\ &\to 0, \end{aligned}$$

since $|\xi|_\nu \leq 1$ for all $\nu \in F$ and the estimate (8) in Section 6 holds. Hence $p^+(\alpha) = p^-(\alpha) = 0$.

Example 9.6. Let $k = \mathbb{Q}$ and fix $\xi = 2$. Define S to be the set of primes for which the corresponding Mersenne number $M_p = 2^p - 1$ is prime. There are heuristic arguments giving strong evidence that M_p is prime for infinitely many values of p. Denote the elements of S by $p_1 < p_2 < \dots$.

Theorem 9.1. *If there are infinitely many Mersenne primes then $p^-(\alpha) = 0$ and $p^+(\alpha) = h(\alpha)$ where S is as above, and $\xi = 2$.*

Proof. Define a sequence n_m such that $M_{n_m} = p_m$ for all $m > 0$, that is, the indices giving the Mersenne primes. Then

$$\begin{aligned}
\frac{1}{n_m}\log|F_{n_m}(\alpha)| &= \frac{1}{n_m}\log(2^{n_m}-1) + \frac{1}{n_m}\sum_{M_p\in S}\log|2^{n_m}-1|_{M_p} \\
&= \frac{1}{n_m}\log(2^{n_m}-1) + \frac{1}{n_m}\log|2^{n_m}-1|_{p_m} \\
&= \frac{1}{n_m}\log(2^{n_m}-1) - \frac{\log(2^{n_m}-1)}{n_m} \\
&= 0.
\end{aligned}$$

Hence $p^-(\alpha) = 0$.

In order to compute the upper growth rate, define a sequence n_l^* by $(n_l^*, n_m) = 1$ for all $l, m > 0$. For example, we could set $n_l^* = 11^l$ because 11 is coprime with the Mersenne indices $2, 3, 5, 7, 13, \dots$ for all $l > 0$. We claim that

$$2^{n_l^*} \equiv 1 \bmod M_{n_m} \text{ for some } l, m > 0 \Rightarrow n_m | n_l^*.$$

Write $n_l^* = \alpha(M_{n_m} - 1) + \beta$ where $0 \leq \beta < M_{n_m} - 1$. By Fermat's Little Theorem,

$$2^{n_l^*} \equiv 1 \bmod M_{n_m} \Rightarrow 2^\beta \equiv 1 \bmod M_{n_m}.$$

Note that the order of $2 \bmod M_{n_m}$ is n_m, so $n_m|\beta$. Also $M_{n_m} - 1 = 2^{n_m} - 2 \equiv 0 \bmod n_m$ by Fermat again. Therefore $n_m | M_{n_m} - 1, n_m|\beta$ and so $n_m|n_l^*$. Hence the claim is proved and so by construction of n_l^* we deduce that

$$\frac{1}{n_l^*}\log|F_{n_l^*}(\alpha)| = \frac{1}{n_l^*}\log|2^{n_l^*} - 1|_\infty \to \log 2.$$

□

More can be said.

Theorem 9.2. *If there are infinitely many Mersenne primes, then the set of limit points of* $\left\{\frac{1}{n}\log|F_n(\alpha)|\right\}_{n=1}^{\infty}$ *is*

$$\left\{\left(1 - \frac{1}{q}\right)h(\alpha) : q \in \mathbb{N}\right\} \cup \{h(\alpha)\}.$$

Proof. We once again use the sequence of Mersenne indices n_m mentioned above. Observe that $2^{n_m} \equiv 1 \bmod M_{n_m}$ implies that $2^{qn_m} \equiv 1 \bmod M_{n_m}$, so

$$|2^{qn_m} - 1|_{M_{n_m}} \leq \frac{1}{M_{n_m}}.$$

We claim that equality holds for all n_m sufficiently large. To see this note that the order of 2 mod $M_{n_m}^2$ is $n_m M_{n_m}$ since

$$(2^{n_m})^{M_{n_m}} = (1 + M_{n_m})^{M_{n_m}} \equiv 1 \bmod M_{n_m}^2.$$

So if n_m is greater than the exponent of the largest Mersenne prime factor of q, say $M(q)$, then $|2^{qn_m} - 1|_{M_{n_m}} = \frac{1}{M_{n_m}}$. Thus we have

$$\begin{aligned} \frac{1}{qn_m} \log |F_{qn_m}(\alpha)| &= \frac{1}{qn_m} \log |2^{qn_m} - 1|_\infty + \frac{1}{qn_m} \sum_{M_p \in S} \log |2^{qn_m} - 1|_{M_p} \\ &= \frac{1}{qn_m} \log(2^{qn_m} - 1) + \frac{1}{qn_m} \log\left(\frac{1}{M_{n_m}}\right) \text{ for all } n_m > M(q) \\ &\to \left(1 - \frac{1}{q}\right) \log 2 \text{ as } m \to \infty. \end{aligned}$$

To prove that no other limit points exist, write $n = ab$ and let $a \to \infty$ through the Mersenne indices. We have two possibilities. Firstly, if $(a, b) = 1$ for a, b sufficiently large then

$$\begin{aligned} \frac{1}{n} \log |F_n(\alpha)| &= \frac{1}{ab} \log(2^{ab} - 1) + \frac{1}{ab} \log(\frac{1}{2^a - 1}) \text{ for all } a, b \text{ sufficiently large} \\ &\to \log 2 \text{ as } a, b \to \infty. \end{aligned}$$

Otherwise $b \to \infty$ through the set $\{qa^j : q, j \in \mathbb{N}\}$. Hence

$$\begin{aligned} \frac{1}{n} \log |F_n(\alpha)| &= \frac{1}{qa^{j+1}} \log\left(2^{qa^{j+1}} - 1\right) + \frac{1}{qa^{j+1}} \sum_{M_p \in S} \log \left|2^{qa^{j+1}} - 1\right|_{M_p} \\ &= \frac{1}{qa^{j+1}} \log\left(2^{qa^{j+1}} - 1\right) + \frac{1}{qa^{j+1}} \log\left(\frac{1}{(2^a - 1)^{j+1}}\right) \text{ for all } a > M(q) \\ &\to \left(1 - \frac{1}{q}\right) \log 2 \text{ as } a \to \infty. \end{aligned}$$

□

For this example the zeta function would then have an infinite number of isolated singularities at $\frac{1}{2}, 1, 2^{\frac{1}{2}}, 2^{\frac{2}{3}}, 2^{\frac{3}{4}}, \ldots$. Also, assuming the conjecture holds, there are sequences along which the periodic point measures converge weakly to Haar measure on $\widehat{\mathbb{Z}[\frac{1}{3 \cdot 7 \cdot 31 \cdot 127 \cdots}]}$.

Using results by Heath–Brown, it is possible to construct (see Corollary 9.1) a different example which has S infinite and p^+ positive, this time without assuming any conjectures. An alternative approach to showing that there must be such examples is described in [36].

Example 9.7. Let $k = \mathbb{Q}$ and suppose ξ is a non–zero integer. Recall that ξ is said to be a *primitive root* modulo a prime p if and only if the residue classes modulo p of $\xi, \xi^2, \dots, \xi^{p-1} \equiv 1$ are all distinct. The number of primitive roots modulo p is $\phi(p-1)$, where ϕ is the Euler function. For example, 2 is not a primitive root modulo 7 since $2^3 \equiv 1 (\mathrm{mod}\ 7)$. In 1927 Artin made the following conjecture: if a is neither a square nor -1, then there exist infinitely many primes such that a is a primitive root modulo p. So, if we choose $\xi \in \mathbb{Z}$ to be neither a square nor -1 and define S to be the set of places $|.|_p$ for which ξ is a primitive root modulo p, then Artin's conjecture implies that S is infinite. Let α be the endomorphism of $\hat{R}_S$ dual to multiplication by ξ on R_S.

Theorem 9.3. *If Artin's conjecture holds for ξ then $p^+(\alpha) = h(\alpha)$.*

Proof. Since $|\xi^n - 1|_p = 1$ if and only if $p - 1 \not| n$ for each $p \in S$, we have

$$\frac{1}{n} \log |F_n(\alpha)| = \frac{1}{n} \log |\xi^n - 1|_\infty + \frac{1}{n} \sum_{p \in S : p-1|n} \log |\xi^n - 1|_p.$$

So by letting $n \to \infty$ through all the prime numbers, we get

$$\limsup_{n\to\infty} \frac{1}{n} \log |F_n(\alpha)| = \log |\xi| = h(\alpha).$$

□

Theorem 9.4. *There are infinitely many primes p with either* 2 *or* 3 *or* 5 *as a primitive root.*

Proof. See Heath–Brown's paper [9] in which he proves that, with the exception of at most two primes the following is true: for each prime q there are infinitely many primes p with q a primitive root modulo p. □

Corollary 9.1. *There exist non–expansive systems $(\hat{R}_S, \alpha)$ with S infinite such that $p^+(\alpha) = h(\alpha) > 0$.*

These dynamical systems have the remarkable property that on the one hand they mimic hyperbolic behaviour ($p^+(\alpha) = h(\alpha)$), while on the other they have infinitely many directions in which they behave as isometries.

Conjecture 9.1. If Artin's conjecture holds for ξ then $p^-(\alpha) = p^+(\alpha) = h(\alpha)$.

We would guess that if S is sufficiently sparse but still infinite, then $p^+(\alpha) = p^-(\alpha) = h(\alpha)$. Whilst the primitive root approach gives us some control over the n's for which $p|\xi^n - 1$, it seems difficult to control the size of $\mathrm{ord}_p(\xi^n - 1)$. Indeed, for this example we have the following bound,

$$\inf_p \left\{ \frac{\mathrm{ord}_p(\xi^{p-1} - 1)}{p} \right\} \ll \frac{1}{n} \sum_{p \in S : p-1|n} \log|\xi^n - 1|_p < 0.$$

The research into Wieferich primes and the Fermat quotient suggest that

$$\mathrm{ord}_p(\xi^{p-1} - 1) = 2$$

for infinitely many p, which would imply that

$$\liminf_{p \to \infty} \left\{ \frac{\mathrm{ord}_p(\xi^{p-1} - 1)}{p} \right\} = 0,$$

so for these examples, $p^-(\alpha) = p^+(\alpha) = h(\alpha)$ is expected. This circle of conjectures is discussed in [24], Chapter 3.

Example 9.8. The following example, though conjectural, raises hopes for the existence of dynamical systems satisfying

$$p^+(\alpha) = p^-(\alpha) = h(\alpha) > 0,$$

with S infinite. It follows a suggestion from Heath-Brown, for which the authors are grateful.

Definition 9.1. For $m > 1$ an integer g is called a *primitive root* modulo m if the least positive integer t satisfying

$$g^t \equiv 1 (\mathrm{mod}\, m)$$

is $\phi(m)$.

Let S be the set of places $|.|_p$ such that ξ is a primitive root modulo p^2, for some integer $\xi \neq 0, \pm 1$. The following assertions are equivalent as pointed out in Chapter 2 of [24]:

(1) ξ is a primitive root modulo p and $\xi^{p-1} \not\equiv 1 (\mathrm{mod}\, p^2)$;
(2) ξ is a primitive root modulo p^2;
(3) for every $m \geq 2$, ξ is a primitive root modulo p^m.

Along with Artin's conjecture, it seems reasonable to conjecture that S is an infinite set.

Theorem 9.5. *If ξ is a primitive root modulo p^2 for infinitely many primes p, then*

$$p^+(\alpha) = p^-(\alpha) = h(\alpha). \tag{9}$$

Proof. First observe that, given $l \geq 1$,

$$p^l | \xi^n - 1 \text{ if and only if } \phi(p^l) | n.$$

This is an easy consequence of Euler's Theorem: $(\xi, p) = 1$ implies that $\xi^{\phi(p^l)} \equiv 1 (\text{mod } p^l)$. So,

$$\frac{1}{n} \log |F_n(\alpha)| = \frac{1}{n} \log |\xi^n - 1|_\infty - \frac{1}{n} \sum_{p \in S : p^{l-1}(p-1)|n, p^l(p-1) \nmid n} l \log p.$$

For terms with $l \geq 2$, $p^{l-1}|n$, whence

$$\sum_{p \in S : p^{l-1}|n} (l-1) \log p \leq \log n.$$

So, for $l \geq 2$,

$$\frac{1}{n} \sum_{p \in S : p^{l-1}|n} l \log p \leq \frac{2}{n} \sum_{p \in S : p^{l-1}|n} (l-1) \log p \leq \frac{2}{n} \log n. \tag{10}$$

For the case $l = 1$ we need an estimate for $\sum_{p \in S : p-1|n} \log p$. This is achieved as follows: if $p-1|n$ then $p \leq n+1$ and $\log p = O(\log n)$. The number of possible p is at most $d(n)$ (the number of divisors of n), and from standard results in number theory $d(n) = O(n^c)$ for any $c > 0$. Thus, choosing $0 < c < 1$,

$$\frac{1}{n} \sum_{p \in S : p-1|n} \log p = O(n^{c-1} \log n). \tag{11}$$

From (10) and (11) we deduce that

$$\lim_{n \to \infty} \frac{1}{n} \log |F_n(\alpha)| = \log |\xi|,$$

and so (9) is established. □

Note that for both of these Artin systems, the conjectures imply that the periodic points are uniformly distributed.

10. Mahler Measures and Entropy

In this section we indicate a possible way to find a uniform entropy formalism for the geometric and arithmetic cases simultaneously.

Let $F(x) \in \mathbb{Z}[x]$ denote a non–zero polynomial with rational integer coefficients. There are several ways to measure the height of $F(x)$. The definition proposed by Mahler has proved to be important,

$$m(F) = \int_0^1 \log |F(e^{2\pi i \alpha})| d\alpha. \tag{12}$$

Suppose $F(x)$ has the factorisation

$$F(x) = a\prod_i (x-\alpha_i), \quad a \in \mathbb{Z}, \quad \alpha_i \in \mathbb{C}. \tag{13}$$

Then an alternative form for (12) is

$$m(F) = \log|a| + \sum_i \log^+|\alpha_i|. \tag{14}$$

The proof of this is an immediate application of Jensen's formula

$$\int_0^1 \log|e^{2\pi i\alpha} - \alpha| = \log^+|\alpha|. \tag{15}$$

So $m(F)$ is the non–negative logarithm of an algebraic number. Given the nature of the logarithm, we might as well assume that F is irreducible and that $F(0) \neq 0$.

Theorem 10.1. [Kronecker] *If $\alpha_1, \dots, \alpha_n$ are the roots of the polynomial $P(x) = x^n + c_1x^{n-1} + \cdots + c_n$, where $c_1, \dots, c_n$ are integers with $P(0) \neq 0$, and if all the roots lie inside the closed unit disc, then they must all be roots of unity.*

Corollary 10.1. *If $\alpha_{\mathcal{R}_1/<f>}$ is a $\mathbb{Z}$-action of a compact abelian group X_M, then*
$h(\alpha_{\mathcal{R}_1/<f>}) = 0$ if and only if f is cyclotomic.

Suppose $F(x)$ is a non–zero irreducible polynomial with integral coefficients such that $F(0) \neq 0$, and let $F(x)$ have the factorisation as in (13). We first establish a third definition of $m(F)$ equivalent to (12) and (14), which shows that $m(F)$ is locally the sum of an archimedean component and p–adic components for each rational prime p. The key step will be to prove a p–adic analogue of Jensen's formula (15). We shall call these components *local measures* and define them as Shnirelman integrals. This corrects an error in [7], where the Haar integral was used to define the local measure.

Let k be a splitting field for F and let ν denote any valuation of k which lies above the valuation p of $\mathbb{Q}$. The field $\not\leq_\nu$ is defined as the smallest field extension of $\mathbb{Q}$ which is both complete and algebraically closed with respect to $|.|_\nu$. For each i we define local measures by

$$m_\nu(\alpha_i) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log|\zeta^j - \alpha_i|_\nu, \tag{16}$$

where ζ is a primitive n-th root of unity inside $\not\leq_\nu$.

This definition of the local measure is a slight specialization of the Shnirelman integral, introduced in 1938 [28] as a p–adic analogue of the line integral (in general, for the case $v|p$, the condition $p \not| n$ has to be imposed to guarantee convergence; because we are dealing with a special class of functions this condition is not needed here).

If $\nu|\infty$ then it follows, after applying Baker's result (Lemma 6.1) in Section 6, that

$$m_\nu(\alpha_i) = \log^+ |\alpha_i|_\nu. \tag{17}$$

The same is clearly true if ν is non–archimedean by the inequality (8) of Section 6. One may think of (17) as a ν–adic analogue of Jensen's formula. This simple fact allows the $\log |a|$ term in (14) to be recognised as a sum over non–archimedean contributions (see Lemma 10.1). So the third representation of $m(F)$ is

$$m(F) = \sum_i \sum_\nu d_\nu m_\nu(\alpha_i), \tag{18}$$

where $d_\nu = [k_\nu : \mathbb{Q}_p]$ are the local degrees making the Artin product formula work. This local–to–global treatment of $m(F)$ is analogous to the local–to–global treatment of the topological entropy of a solenoidal endomorphism [19], the local entropies being precisely local Mahler measures. So the ν–adic decomposition of $h(\alpha; \hat{R}_S)$ for the geometric dynamical system $(\hat{R}_S, \alpha)$ is suggestive of a local–to–global geometric Mahler measure theory.

We now extend from this arithmetic setting to $\mathbb{A}$–fields. As usual, let k_0 denote either $\mathbb{Q}$ or $\mathbb{F}_p(t)$ and let r_∞ denote the ring of algebraic integers in k. Choose $F(x) \in r_\infty[x]$ to be a non–zero irreducible element with $F(0) \neq 0$. Suppose F splits in some finite extension $k : k_0$ of degree d and has the factorisation

$$F(x) = a \prod_i (x - \alpha_i), \quad a \in r_\infty, \quad \alpha_i \in k.$$

Let ν denote any valuation of k extending the valuation ω of k_0. In the geometric case, it will be once again convenient to denote the distinguished place corresponding to the polynomial t^{-1} by ∞. Define, as in (16) and (18),

$$m_\nu(\alpha_i) = \lim_{n\to\infty}{}' \frac{1}{n} \sum_{j=1}^{n} \log |\zeta^j - \alpha_i|_\nu = \log^+ |\alpha_i|_\nu, \tag{19}$$

where ζ is a primitive n-th root of unity inside $\not\leq_\nu$, and

$$m_{\mathbb{A}}(F) = \sum_i \sum_\nu d_\nu m_\nu(\alpha_i), \tag{20}$$

where $d_\nu = [k_\nu : (k_0)_\omega]$. The notation $\lim'$ in (19) indicates no restriction on n in the arithmetic case, but in the geometric case we need to use an analogue of the classical Shnirelman integral by imposing the condition M_ν does not divide n for some number $M_\nu \neq 1$ dependent on ν. The number M_ν can be computed from the proof of Theorem 6.2.

Lemma 10.1.

$$m_{\mathbb{A}}(F) = \log|a|_\infty + \sum_i \sum_{\nu|\infty} d_\nu m_\nu(\alpha_i).$$

Proof. The method employed is as in Section 6 of [19]. Write $\prod_i(x - \alpha_i) = x^d + b_1x^{d-1} + \cdots + b_d$ where $b_1, \dots, b_d \in k_0$ and the lowest common multiple of these coefficients is a. Then for any place of k_0 with $\omega \neq \infty$,

$$|a|_\omega = \min\{|b_1|_\omega^{-1}, \dots, |b_d|_\omega^{-1}, 1\}.$$

Suppose the roots $\alpha_1, \dots, \alpha_d$ lie in a finite extension $k_v : (k_0)_\omega$ of degree d_ν, and order them so that

$$|\alpha_1|_\nu \geq |\alpha_2|_\nu \geq \cdots \geq |\alpha_m|_\nu > 1 \geq |\alpha_{m+1}|_\nu \geq \cdots \geq |\alpha_d|_\nu.$$

We shall prove that

$$\log|a|_\omega^{-1} = \sum_i d_\nu m_\nu(\alpha_i);$$

it follows that $\log|a|_\infty$ is just the sum over the non–archimedean contributions in (20).

If $|\alpha_i|_\nu \leq 1$ for all i, then $|a|_\omega = 1$ and $\sum_i d_\nu m_\nu(\alpha_i) = 0$. Thus we may assume that $|\alpha_1|_\nu > 1$. Then we have

$$\begin{aligned} |b_m|_\omega &= \left|\sum_{i_1<\cdots<i_m} \alpha_{i_1}\cdots\alpha_{i_m}\right|_\nu \\ &= |\alpha_1\cdots\alpha_m + \text{smaller terms}|_\nu \\ &= |\alpha_1\cdots\alpha_m|_\nu, \end{aligned}$$

and by a similar calculation $|b_i|_\omega \leq |b_m|_\omega$ for all $i = 1, \dots, d$. So

$$|a|_\omega = \min\{|b_1|_\omega^{-1}, \dots, |b_d|_\omega^{-1}\} = \prod_{|\alpha_i|_\nu>1} |\alpha_i|_\nu^{-1}$$

and

$$\log|a|_\omega^{-1} = \sum_{|\alpha_i|_\nu>1} \log|\alpha_i|_\nu = \sum_i d_\nu m_\nu(\alpha_i),$$

completes the proof. $\square$

Call $m_{\mathbb{A}}(F)$ the Mahler measure of F associated to an $\mathbb{A}$–field. Note the exact analogy between (17) and (19), also between (18) and (20). Using Lemma 10.1 we can rewrite the definition of $m_{\mathbb{A}}(F)$ as

$$m_{\mathbb{A}}(F) = \frac{1}{d}\sum_{\nu|\infty} \lim_{n\to\infty}{}' \frac{1}{n}\sum_{j=1}^{n} \log |F(\zeta^j)|_\nu. \tag{21}$$

So if $k_0 = \mathbb{Q}$, $F(x)$ has integer coefficients and (21) collapses to the arithmetic Mahler measure (12).

Theorem 10.2. *The generalised Mahler measure $m_{\mathbb{A}}(F) = 0$ if and only if F is a division polynomial.*

Proof. We have already seen that the arithmetic case follows from Kronecker's result (Theorem 10.1), and, using exactly the same argument in the geometric case, we must have $\alpha_i \in \mathbb{F}_p^*$ for each i and $a \in \mathbb{F}_p^*$. □

Notice how $\mathbb{F}_p^*$ plays the role of the roots of unity as in Kronecker's Theorem. The term *division polynomial* was first used in [7], where it was proved that the elliptic Mahler measure vanishes if and only if the roots of $F(x)$ (an integral polynomial) are the x–coordinates of torsion (or division) points of the underlying elliptic curve. The results in this section allow us to make the following connection between topological entropy and Mahler measure.

Theorem 10.3. *Let $(\hat{R}_S, \alpha)$ be an S–integer dynamical system and let $F(x)$ be the polynomial with coefficients in r_∞, obtained by multiplying the minimum polynomial of ξ by the lowest common multiple of the denominators of its coefficients. Then*

(i): $h(\alpha; \hat{R}_S) = m_{\mathbb{A}}(F)$,
(ii): $h(\tilde{\alpha}; k_\nu) = m_\nu(\xi)$ *for each place ν of k.*
(iii): $h(\alpha; \hat{R}_S) = 0$ *if and only if F is a division polynomial.*

11. Remarks and Problems

(1) The arithmetic dynamical systems studied here are perhaps less special than might at first appear (within the general setting of actions dual to monomorphisms of algebraic number fields). While it is not the case that every finitely generated subring R of an algebraic number field k containing 1 and with k as its quotient field must be a ring of S–integers, this is almost the case. What is true is that any such subring R will be a subring of finite index in its integral closure R', and R' is a full ring of S–integers in k. Conversely, any such subring is a finitely generated ring of algebraic numbers. For details, see [22]. Without the assumption that the subgroup be a subring, for the case $k = \mathbb{Q}$ the

results of Beaumont and Zuckerman in [1] show that the subgroups are (up to isomorphism) S–integral with the exception of one case.
(2) Is there a reason for the curious similarity between Example 8.5 and Theorem 9.2?
(3) As pointed out in Example 2.2(2), certain expansive S–integer systems are known to arise as attractors in hyperbolic diffeomorphisms. Are there non–hyperbolic diffeomorphisms $f : M \to M$ with the property that f restricted to $\bigcap_{n\in\mathbb{N}} f^n(M)$ is conjugate to a non–expansive S–integer system?

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