AUTOMORPHISMS OF $\mathbb{Z}^d$–SUBSHIFTS OF FINITE TYPE

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Revised December 13 1993

Abstract. Let (Σ, σ) be a $\mathbb{Z}^d$ subshift of finite type. Under a strong irreducibility condition (strong specification), we show that $\text{Aut}(\Sigma)$ contains any finite group. For $\mathbb{Z}^d$–subshifts of finite type without strong specification, examples show that topological mixing is not sufficient to give any finite group in the automorphism group in general: in particular, $\text{End}(\Sigma)$ may be an abelian semigroup. For an example of a topologically mixing $\mathbb{Z}^2$–subshift of finite type, the endomorphism semigroup and automorphism group are computed explicitly. This subshift has periodic–point permutations that do not extend to automorphisms.

1. Introduction

Let A be a finite set with $|A| > 1$, and for a finite set $E \subset \mathbb{Z}^d$ let $\pi_E : A^{\mathbb{Z}^d} \to A^E$ denote the restriction map, $\pi_E(\mathbf{x}) = \mathbf{x}|_E$, where $A^{\mathbb{Z}^d}$ is viewed as the space of maps $\mathbb{Z}^d \to A$ with the product topology. The group $\mathbb{Z}^d$ acts on $A^{\mathbb{Z}^d}$ via the shift σ, $\sigma_{\mathbf{n}}(\mathbf{x})_{\mathbf{m}} = \mathbf{x}_{\mathbf{n}+\mathbf{m}}$. A closed non–empty σ–invariant subset $\Sigma \subset A^{\mathbb{Z}^d}$ is a $\mathbb{Z}^d$*–subshift of finite type* if there is a finite set $E \subset \mathbb{Z}^d$ and a non–empty subset $P \subset A^E$ for which

$$\Sigma = \Sigma_{(P,E)} = \{\mathbf{x} \in A^{\mathbb{Z}^d} \mid \pi_{E+\mathbf{n}}(\mathbf{x}) \in P \text{ for every } \mathbf{n} \in \mathbb{Z}^d\}. \qquad (1.1)$$

Let σ also denote the restriction of the $\mathbb{Z}^d$ action σ to Σ (see Chapter 5 of the notes [Sc] for a discussion of this definition, and examples). The problem of determining for given data P and E whether $\Sigma_{(P,E)}$ is non–empty is excluded by our definition; this question is known to be undecidable in general for $d > 1$, because of the existence of $\mathbb{Z}^d$–subshifts of finite type without periodic points (see [B], [R]).

The *endomorphism semi–group* $\text{End}(\Sigma)$ of the $\mathbb{Z}^d$–subshift of finite type Σ is defined to be the semi–group of continuous surjective maps from Σ to Σ commuting with the action σ. The *automorphism group* $\text{Aut}(\Sigma)$ is the group of homeomorphisms of Σ commuting with σ. For $\mathbb{Z}$–subshifts of finite type it is well–known that

1991 *Mathematics Subject Classification.* 54H20.

The author gratefully acknowledges support from NSF grant DMS–91–03056 at the Ohio State University.

Aut(Σ) (and therefore End(Σ)) is very large: in particular, if Σ is a non–trivial topologically mixing $\mathbb{Z}$–subshift of finite type, then Aut(Σ) contains the free group on two generators (Theorem 2.4 of [BLR]) and the direct sum of every countable collection of finite groups (Theorem 2.3 of [BLR]). This kind of result was first given by Curtis, Hedlund and Lyndon who showed that the automorphism group of a full shift contains any finite group and contains a pair of involutions whose product has infinite order (Theorems 6.13 and 20.1 in [H2]). The paper [H1] contains a survey of their work.

The results below are an extension to the case of higher–dimensional subshifts of finite type: the discussion before Theorem 2.3 is an analogue of the Curtis–Hedlund–Lyndon theorem for the full shift on three symbols, and Theorem 2.3 is analogous to a weak version of Theorem 2.3 in [BLR], where sufficiently mixing subshifts of finite type are seen to behave like full shifts in this regard. Higher–dimensional subshifts of finite type differ from the one–dimensional case in that they may be topologically mixing without having strong specification. In Section 3 below we show that this allows some high–dimensional subshifts of finite type to have very few automorphisms.

Under a strong irreducibility condition (defined in Section 2) on a $\mathbb{Z}^d$–subshift of finite type, $d > 1$, namely *strong specification*, the automorphism group mimics the $\mathbb{Z}$ case in that the automorphism group is very large.

In the one–dimensional case Lind has shown that the set of possible entropies of automorphisms of a mixing subshift of finite type are dense in $[0, \infty)$ ([L]). Examples show that topologically mixing $\mathbb{Z}^d$–subshifts of finite type that do not have strong specification may have very few automorphisms, and may not have a dense set of possible entropies of automorphisms. It is conjectured that for $\mathbb{Z}^d$ subshifts with strong specification, each automorphism has either zero or infinite entropy.

It should be emphasised that the two kinds of subshifts of finite type discussed here – those with strong specification, and a zero–entropy example – lie at opposite ends of a spectrum of topological mixing properties, and we say nothing at all about the many interesting shifts in the middle of this spectrum.

Some of the results in Section 3 are obtained in [KS] from a different viewpoint, where the "rigidity" phenomenon in Ledrappier's example (and generalizations thereof), forcing topological conjugacies to be group homomorphisms, is exhibited for maps between certain Markov subgroups of $\{0,1\}^{\mathbb{Z}^2}$ and the Ledrappier shift (see Observation 4.1 in [KS]).

I am grateful to Mark Shereshevsky for pointing out an error in an earlier version, and to him and Jeff Steif for several discussions related to the issues dealt with in this note.

2. $\mathbb{Z}^d$–SUBSHIFTS WITH STRONG SPECIFICATION

For brevity we state and prove results in this section for the case $d = 2$; extension to the case $d > 2$ is straightforward.

Let d denote the Euclidean metric on $\mathbb{Z}^2$. A $\mathbb{Z}^2$–subshift of finite type Σ has *strong specification* if there is a constant M such that for any sets $R_1, R_2 \subset \mathbb{Z}^2$ with $d(R_1, R_2) \geq M$, and any words $\mathbf{x}_1 \in \pi_{R_1}(\Sigma)$ and $\mathbf{x}_2 \in \pi_{R_2}(\Sigma)$, there is an element $\mathbf{y} \in \Sigma$ with $\pi_{R_i}(\mathbf{y}) = \mathbf{x}_i$ for $i = 1, 2$.

A $\mathbb{Z}^2$–subshift of finite type Σ is *topologically mixing* if for any finite sets $R_1, R_2 \subset \mathbb{Z}^2$ there is a constant $M(R_1, R_2)$ such that for $d(R_1, R_2) \geq M(R_1, R_2)$, and for any words $\mathbf{x}_1 \in \pi_{R_1}(\Sigma)$ and $\mathbf{x}_2 \in \pi_{R_2}(\Sigma)$, there is an element $\mathbf{y} \in \Sigma$ with $\pi_{R_i}(\mathbf{y}) = \mathbf{x}_i$ for $i = 1, 2$.

Notice that we may exhibit many periodic points in a subshift of finite type with strong specification. Let $R = [0, n] \times [0, m] \cap \mathbb{Z}^2$, and choose $\mathbf{x} \in \pi_R(\Sigma)$ (the shift is certainly non–empty, since we may apply strong specification with R_1 and R_2 as singletons). Extend $\mathbf{x}$ to $\mathbf{x}^* \in \pi_{[0,n]\times\mathbb{Z}}(\Sigma)$, and apply strong specification to find $\mathbf{y}$ with $\pi_{[0,n]\times\mathbb{Z}}(\mathbf{y}) = \pi_{[n+M,2n+M]\times\mathbb{Z}}(\mathbf{y}) = \mathbf{x}^*$. Then the pattern $\pi_{[0,n+M]\times\mathbb{Z}}(\mathbf{y})$ may be concatenated to produce an element $\mathbf{y}^* \in \Sigma$ which is $\sigma_{(1,0)}$–periodic with period $(n+M)$. Now apply exactly the same argument to $\mathbf{z} = \pi_{\mathbb{Z}\times[0,m]}(\mathbf{y}^*)$ to produce $\mathbf{z}^*$ which is $\sigma_{(1,0)}$–periodic with period $(n+M)$, $\sigma_{(0,1)}$–periodic with period $(m+M)$, and has $\pi_R(\mathbf{z}^*) = \mathbf{x}$. We deduce that any finite word may be embedded in a periodic point of some period boundedly larger than the original word.

Before continuing, we should make clear that strong specification is enjoyed by many non–trivial subshifts of finite type.

Examples. The subshifts of finite type (1), (2) and (3) have strong specification; (4) does not.

(1) [THE FULL SHIFT] Let $E = \{(0, 0)\}$ and $P = A$. Then $\Sigma_{(P,E)}$ is the full shift on $|A|$ symbols, and this has strong specification with $M = 1$.
(2) [THE GOLDEN MEAN] Let $A = \{0, 1\}$, and define Σ by the rule that a "1" must be followed horizontally and vertically by a "0". This has strong specification with $M = 2$. It follows that it has positive entropy, though the exact value of this entropy is not known.
(3) [BURTON–STEIF EXAMPLE] Let $A = \{-L, -L+1, \ldots, -2, -1, 1, 2, \ldots, L-1, L\}$, and define Σ by the rule that horizontally and vertically adjacent positions must have the same sign unless they are both equal to ± 1. This subshift has strong specification with $M = 3$. Burton and Steif have shown that this shift has exactly two ergodic measures of maximal entropy if L is sufficiently large (see [BS], Theorem 1.17).
(4) [LEDRAPPIER'S EXAMPLE] Let $A = \{0, 1\}$, $E = \{(0, 0), (1, 0), (0, 1)\}$, and $P = \{(x_{(0,0)}, x_{(1,0)}, x_{(0,1)}) \mid x_{(0,0)} + x_{(1,0)} = x_{(0,1)} \bmod 2\}$. If $\mathbf{x} \in \Sigma_{(P,E)}$ then $x_{(0,0)} + x_{(2^n,0)} = x_{(0,2^n)}$ for all n, so $\Sigma_{(P,E)}$ does not have strong

specification. This example was introduced in [Le].

For brevity, let $\pi_n = \pi_{[0,n]\times[0,n]}$. For any subshift Σ, the *topological entropy* is defined to be

$$h(\Sigma) = \lim_{n\to\infty} \frac{1}{n^2} \log |\pi_n(\Sigma)|. \tag{2.1}$$

If μ is a σ–invariant Borel probability on Σ, the metric entropy $h_\mu(\Sigma)$ of Σ is defined as follows. For $\mathbf{x} \in \pi_R(\Sigma)$, let $[\mathbf{x}] = \{\mathbf{y} \in \Sigma \mid \pi_R(\mathbf{y}) = \mathbf{x}\}$ denote the cylinder set in Σ defined by $\mathbf{x}$. Then

$$h_\mu(\Sigma) = \lim_{n\to\infty} -\frac{1}{n^2} \sum_{\mathbf{x}\in\pi_n(\Sigma)} \mu([\mathbf{x}]) \log \mu([\mathbf{x}]). \tag{2.2}$$

Both of the limits (2.1) and (2.2) exist by subadditivity, and the variational principle for the pressure implies that

$$h(\Sigma) = \sup_\mu h_\mu(\Sigma), \tag{2.3}$$

where the supremum is taken over all invariant Borel probabilities (see [E]; a shorter proof of the variational principle for $\mathbb{N}^d$ actions is given in [M]).

Let $F_n(\Sigma) = \{\mathbf{x} \in \Sigma \mid \sigma_{\mathbf{m}}(\mathbf{x}) = \mathbf{x} \text{ for all } \mathbf{m} \in n\mathbb{Z} \times n\mathbb{Z}\}$ denote the set of points with period n both horizontally and vertically in Σ.

The following lemma is well–known; part (1) is in [BS] and part (2) is in [Sc].

Lemma 2.1. *Let Σ be a subshift of finite type with strong specification.*

(1) *The topological entropy of Σ is positive.*

(2) *The growth rate of periodic points equals the entropy,*

$$\lim_{n\to\infty} \frac{1}{n^2} \log |F_n(\Sigma)| = h(\Sigma).$$

(3) *Any weak limit of the sequence of uniform periodic point measures on sets of periodic points $F_n(\Sigma)$ is a measure of maximal entropy if it is shift–invariant.*

Proof. (1) Since $|A| > 1$, strong specification shows that there are words in Σ which allow one of two symbols to be seen at each point in $(M+1)\mathbb{Z} \times (M+1)\mathbb{Z}$. It follows that

$$|\pi_{n(M+1)}(\Sigma)| \geq 2^{n^2},$$

so $h(\Sigma) \geq \frac{1}{(M+1)^2} \log 2 > 0$.

(2) It is clear that

$$\begin{aligned} |\pi_n(\Sigma)| \leq |F_n(\Sigma)| &\leq |\pi_{[-M,n+M]\times[-M,n+M]}(\Sigma)| \\ &\leq |\pi_n(\Sigma)| \times |A|^{4Mn+4M^2}. \end{aligned} \tag{2.4}$$

It follows that the growth rate of periodic points exists, and

$$\lim_{n\to\infty} \frac{1}{n^2} \log |F_n(\Sigma)| = h(\Sigma).$$

(3) Let μ_n denote the uniform measure on $F_n(\Sigma)$, and let μ denote the shift–invariant weak limit as $j \to \infty$ of μ_{n_j}.

We claim that μ is a maximal measure. Fix some n and consider finite words $\mathbf{x}, \mathbf{y} \in \pi_n(\Sigma)$. Then $|\mu([\mathbf{x}]) - \mu([\mathbf{y}])|$ is equal to

$$\lim_{k\to\infty} \frac{1}{|F_{n_k}(\Sigma)|} \big| |\{\mathbf{z} \in F_{n_k}(\Sigma) \mid \pi_n(\mathbf{z}) = \mathbf{x}\}| - |\{\mathbf{w} \in F_{n_k}(\Sigma) \mid \pi_n(\mathbf{w}) = \mathbf{y}\}| \big|.$$

Now by strong specification with constant M, it is clear that

$$\big| |\{\mathbf{z} \in F_{n_k}(\Sigma) \mid \pi_n(\mathbf{z}) = \mathbf{x}\}| - |\{\mathbf{w} \in F_{n_k}(\Sigma) \mid \pi_n(\mathbf{w}) = \mathbf{y}\}| \big| \leq |A|^{(n_j+2M)^2},$$

so that $|\mu([\mathbf{x}]) - \mu([\mathbf{y}])| = 0$. It follows that, for any fixed n,

$$-\frac{1}{n^2} \sum_{\mathbf{x}\in\pi_n(\Sigma)} \mu([\mathbf{x}]) \log \mu([\mathbf{x}]) = \frac{1}{n^2} \log |\pi_n(\Sigma)|,$$

so that $h_\mu(\Sigma) = h(\Sigma)$. □

If the subshift of finite type Σ has a unique measure of maximal entropy then Lemma 2.1 implies that any invariant measure obtained as the weak limit of periodic point measures is ergodic. In general this is not the case (see [BS]).

We now show that a subshift of finite type with strong specification has many automorphisms. The method is exactly that originated by Hedlund and used in [BLR], that of markers acting on data.

Definition 2.2. Let $S \subset R \subset \mathbb{Z}^2$ be subsets of $\mathbb{Z}^2$. A *marker* (for $S \subset R$) is a word $M \in \pi_{R\backslash S}(\Sigma)$ and a set $\mathcal{D} \subset \{\pi_S(\mathbf{x}) \mid \pi_{R\backslash S}(\mathbf{x}) = M\}$ with the following *trivial overlaps* property: if $\mathbf{x} \in \Sigma$ has $\pi_S(\mathbf{x}) \in \mathcal{D}$ and $\pi_{R\backslash S}(\mathbf{x}) = M$, and there is an $\mathbf{n} \in \mathbb{Z}^2\backslash\{(0,0)\}$ with $\pi_{(R\backslash S)+\mathbf{n}}(\mathbf{x}) = \pi_{R\backslash S}(\mathbf{x})$, and $\pi_{S+\mathbf{n}}(\mathbf{x}) \in \mathcal{D}, \pi_S(\mathbf{x}) \in \mathcal{D}$, then $R + \mathbf{n} \cap R = \emptyset$.

Given a marker $(M, \mathcal{D})$, the full symmetry group $\mathrm{Sym}(\mathcal{D})$ embeds into $\mathrm{Aut}(\Sigma)$: for each $\tau \in \mathrm{Sym}(\mathcal{D})$, define $\alpha_\tau \in \mathrm{Aut}(\Sigma)$ as follows. If $\mathbf{x} \in \Sigma$ has $\pi_{(R\backslash S)+\mathbf{n}}(\mathbf{x}) = M$ and $\pi_{S+\mathbf{n}}(\mathbf{x}) = D \in \mathcal{D}$, then $\alpha_\tau(\mathbf{x}) = \mathbf{y}$, where $\mathbf{y}$ is the unique element of Σ with $\pi_{\mathbb{Z}^2\backslash(R+\mathbf{n})}(\mathbf{y}) = \pi_{\mathbb{Z}^2\backslash(R+\mathbf{n})}(\mathbf{x})$, $\pi_{(R\backslash S)+\mathbf{n}}(\mathbf{y}) = M$, and $\pi_{S+\mathbf{n}}(\mathbf{y}) = \tau(D) \in \mathcal{D}$. That is, the map α_τ acts by applying τ to words from $\mathcal{D}$ (the data) that are "marked" by M. The trivial overlaps property ensures that the data, markers and the other parts of words are kept separated.

As an illustration, consider the full shift on three symbols, $\Sigma = \Sigma_{(A,\{(0,0)\})}$ where $A = \{0, 1, 2\}$. Let $R = [0, n] \times [0, n]$ and $S = [1, n-1] \times [1, n-1]$. Let $\mathcal{D} = \{0, 1\}^S$ and let M be the element of $\{0, 1, 2\}^{R\backslash S}$ given by 2's in every position. The pair $(M, \mathcal{D})$ form a marker for $S \subset R$, and the above construction embeds a copy of the symmetry group on 2^{n^2} symbols into $\mathrm{Aut}(\Sigma)$.

Theorem 2.3. *If Σ is a $\mathbb{Z}^2$–subshift of finite type with strong specification, then* $\mathrm{Aut}(\Sigma)$ *contains any finite group.*

Proof. We will embed a symmetric group of arbitrary size into $\mathrm{Aut}(\Sigma)$. First notice that $\Sigma_1 = \pi_{\mathbb{Z}\times\{0\}}(\Sigma)$ is a one–dimensional mixing subshift of finite type. By [BLR], Lemma 2.2, we may find in Σ_1 a word M of length ℓ (how long will be chosen later), with the property that M can only overlap itself trivially. Build a marker M^* in Σ as follows. In positions $[-\ell-n,-n]\times\{0\}$ and $[\ell+n,n]\times\{0\}$, place the word M, where $2n+1<\ell$. By strong specification, there is a number L with the property that we may place further copies of the word M in positions $[-\ell-n,-n]\times\{\pm(kL+(k-1))\}$ and $[\ell+n,n]\times\{\pm(kL+(k-1))\}$ for every $k=1,2,\dots$. The partially–defined word so produced (call it M_1) has the property that M_1 cannot overlap any translate by (a,b) of M_1 for $|a|\le\ell$ and $|b|\ge L$.

Enumerate the (finite) collection of translates (a,b) with $|a|\le\ell$ and $|b|<L$ in some order, $(a_1,b_1),(a_2,b_2),\dots,(a_j,b_j)$. Consider (a_1,b_1): if the partial word M_1 translated by (a_1,b_1) can overlap M_1, then add to M_1 a further copy of M in the position $[\ell+n,n]\times\{0\}+(a_1,b_1)$ if $a\ge 0$, and in the position $[-\ell-n,-n]\times\{0\}+(a_1,b_1)$ if $a<0$. Call the enlarged partial word so produced M_2. Apply the same process with (a_2,b_2) to M_2 to produce M_3 and so on. Let $M^*=M_{k+1}$ be the final partial word produced: M^* then has the property that the only translate of M^* that is compatible with M^* is a translate (a,b) with $|a|>\ell$. That is, M^* acts as a marker for the strip $[-n,n]\times\mathbb{Z}$. Now assume ℓ has been chosen large enough to have $n>L+r$, where L is the strong specification constant for Σ. It follows that we may find in the strip $[-n,n]\times\mathbb{Z}$ marked by M^* a square of side $2r-1$ with many (how many depending on r) allowed words with fixed boundary. Any permutation of these words defines an element of $\mathrm{Aut}(\Sigma)$ by applying that permutation whenever the marker M^* is seen.

Thus, we may embed copies of $\mathrm{Sym}(N)$ with N arbitrarily large into $\mathrm{Aut}(\Sigma)$.□

3. A zero entropy example: the Ledrappier shift

In this section we consider an example of a topologically mixing $\mathbb{Z}^2$–subshift of finite type with zero entropy introduced as a measurable dynamical system by Ledrappier ([Le]), and studied in that context in [W1], [Sh1].

Let

$$\Sigma=\{\mathbf{x}\in\{0,1\}^{\mathbb{Z}^2}\mid x_{(n,m+1)}=x_{(n,m)}+x_{(n+1,m)}\} \tag{3.1}$$

in which the addition is performed mod 2, and let $\theta:\Sigma\to\Sigma$ be an endomorphism of (Σ,σ).

It will be convenient to allow θ to be composed with the shift maps in either domain or range without altering the notation; the maps below should therefore be understood modulo σ. Let G denote the group $\{0,1\}$. By (3.1) we may without loss of generality view θ as a map from $G^{\mathbb{Z}}$ to $G^{\mathbb{Z}}$ (the line of co–ordinates in Σ given by the $(1,0)$ direction determines all the remaining positions (a,b) with $b \geq 0$; since θ is continuous it can only depend on finitely many positions with negative coordinates in the $(0,1)$ direction, finally the commutation with the shift allows us to view θ as a function on the $(1,0)$ line of co–ordinates). Moreover θ is given by a surjective sliding code,

$$\phi : G^{\mathbb{Z}} \to G \tag{3.2}$$

where

$$\theta(\mathbf{x})_k = \phi(\sigma_{(k,0)}(\mathbf{x})) \tag{3.3}$$

for all $k \in \mathbb{Z}$. Since θ is continuous, ϕ can only depend on finitely many coordinates, so we may assume that ϕ is a map from G^N to G for some N.

The condition that θ commute with $\sigma_{(1,0)}$ is implicit in the reduction to $\phi : G^N \to G$; the condition that θ commute with $\sigma_{(0,1)}$ is equivalent to requiring that

$$\phi(a_1 + a_2, a_2 + a_3, \dots, a_N + a_{N+1}) = \phi(a_1, \dots, a_N) + \phi(a_2, \dots, a_{N+1}) \tag{3.4}$$

for every $a_1, \dots, a_{N+1}$ in G. Assume now that ϕ does depend on the first and last variables (that is, choose N minimal given ϕ), and consider $P(a_2, \dots, a_{N+1}) = \phi(a_2, a_3, \dots, a_{N+1}) + \phi(a_2, a_3, \dots, a_{N+1} + 1)$. By (3.4),

$$P(a_1 + a_2, a_2 + a_3, \dots, a_N + a_{N+1}) = P(a_2, \dots, a_{N+1}) \tag{3.5}$$

for all $a_1, \dots, a_{N+1}$ in G. Applying the rule (3.5) repeatedly, we have

$$\begin{aligned} P(a_1 + a_2, a_2 + a_3, \dots, a_N + a_{N+1}) &= P(a_0 + a_1 + a_2, a_1 + a_3, \dots, a_{N-1} + a_{N+1}) \\ &= P(a_{-1} + a_0 + a_1 + a_2, a_0 + a_2 + a_3, \dots, \\ &\qquad a_{N-2} + a_{N-1} + a_N + a_{N+1}) \\ &= \dots . \end{aligned}$$

After N steps, we can choose the values of the introduced variables $a_{-N}, \dots, a_1$ to deduce that $P(a_2, \dots, a_{N+1}) = P(0, \dots, 0)$. It follows that $\phi(a_2, a_3, \dots, a_{N+1}) + \phi(a_2, a_3, \dots, a_{N+1} + 1)$ is a constant; by minimality of N we deduce that

$$\phi(a_2, a_3, \dots, a_{N+1} + 1) = \phi(a_2, a_3, \dots, a_{N+1}) + 1. \tag{3.6}$$

The same argument may be applied to each variable in turn, the only difference being that we may not assume that ϕ depends on the other variables until we reach the first.

We have proved the following lemma.

Lemma 3.1. *If ϕ is a surjective sliding block code of the form (3.3) defining an element of* $\mathrm{End}(\Sigma)$ *then*

$$\phi(a_1, \dots, a_N) = \sum_{j \in J} a_j \tag{3.7}$$

where $\{1, N\} \subset J \subset \{1, \dots, N\}$.

Lemma 3.1 also follows from Observation 4.1 in [KS], which shows that homeomorphisms commuting with the shift from any Markov subgroup of $\{0,1\}^{\mathbb{Z}^2}$ onto Σ must preserve the group structures.

It follows that the endomorphisms of Σ are homomorphisms of the group structure of Σ. The dual of Σ is the module $\mathbb{F}_2[x^{\pm 1}, y^{\pm 1}]/\langle 1+x+y \rangle \cong \mathbb{F}_2[x^{\pm 1}, (1+x)^{\pm 1}]$. If θ is given by the block map $\phi = \phi_J$ then $\widehat{\theta}$ is, up to multiplication by powers of x and $(1+x)$, given by multiplication by $h_J(x) = \sum_{j \in J} x^j$.

We now turn to the problem of identifying the automorphisms of Σ. The endomorphism defined by the block map $\phi = \phi_J$ according to (3.7) will be invertible if

$$\sum_{j \in J} a_{j+k} = 0 \quad \text{for all } k \in \mathbb{Z} \tag{3.8}$$

implies that some power of $\sigma_{(0,1)}$ applied to $\mathbf{a}$ gives the constant string

$$(\dots, 0, 0, 0, \dots).$$

Theorem 3.2. *The endomorphisms of Σ modulo the shift are identified with a quotient of the semigroup $\mathbb{Z}[x]$ under multiplication; the automorphisms of Σ modulo the shift correspond to (the image of) polynomials of the form $(1+x)^n$. It follows that* $\mathrm{Aut}(\Sigma) \cong \{\sigma_{\mathbf{n}} \mid \mathbf{n} \in \mathbb{Z}^2\} \cong \mathbb{Z}^2$ *is trivial.*

Proof. The first assertion is contained in the remarks above. Let h be the polynomial corresponding to the endomorphism; without loss of generality we may assume that h has constant term 1.

I am grateful to Klaus Schmidt for showing me the following argument. Write $m_h(g) = hg$ for multiplication by h on the module $\mathbb{F}_2[x^{\pm 1}, y^{\pm 1}]/\langle 1+x+y \rangle$. It is clear that m_h is injective if and only if $h \notin \langle 1+x+y \rangle$, so assume that. We claim first that m_h is bijective if and only if

$$V(\langle h \rangle) \cap V(\langle 1+x+y \rangle) = \emptyset, \tag{3.9}$$

where the varieties are the sets of common zeros over k^*, where k is an algebraic closure of $\mathbb{F}_2$. (We cannot include 0 as a permitted value for x or y because they are invertible elements of the ring of Laurent polynomials).

To prove the claim, notice that (3.9) is equivalent by the Nullstellensatz to having polynomials a and b with $1 = ah + b(1+x+y)$. It follows that (3.9) implies that m_h is bijective; conversely, if m_h is bijective then there is a polynomial a for which $ha - 1 \in \langle 1 + x + y \rangle$.

We may therefore assume that if m_h is bijective, then (3.9) holds. We claim that this means that $h(x, 1+x)$ can only vanish on the point 1. If $h(a, 1+a) = 0$ for some $a \neq 1$, then (3.9) is contradicted. It follows that $h(x, 1+x)$ must be some power of $(1+x)$.

That is, if h corresponds to an automorphism, then $h(x, 1+x) = (1+x)^n = y^n$ in the module, so the only automorphisms are the shifts themselves. □

Corollary 3.3. *Any automorphism of* Σ *has entropy an integer multiple of* $\log 2$.

Remarks. (1) The set of points with period 3 under both $\sigma_{(0,1)}$ and $\sigma_{(1,0)}$ has exactly four elements: it is clear that there is no automorphism of Σ extending some permutations of these four points. This can also occur for $\mathbb{Z}$–subshifts of finite type (see [KRW]), though it is highly non–trivial to see there.
(2) The automorphism group is abelian in contrast to the case $d = 1$ (see [R1],[R2]).

4. Questions

We close with some questions about the entropy behaviour of automorphisms of topologically mixing subshifts of finite type. Without the assumption of topological mixing, there are degenerate examples of the following form: let (Σ, σ) be a $\mathbb{Z}$–subshift of finite type, and define a $\mathbb{Z}^d \times \mathbb{Z}$–subshift of finite type (Σ^*, σ^*) by $\Sigma^* = \Sigma$, $\sigma^*_{(0,1)} = \sigma$, and $\sigma^*_{(\mathbf{n},0)} = \text{id}$. Then $\text{Aut}(\Sigma^*) = \text{Aut}(\Sigma)$ so this $\mathbb{Z}^d \times \mathbb{Z}$–subshift of finite type inherits the automorphism group of a $\mathbb{Z}$–subshift of finite type.
(1) Which topologically mixing $\mathbb{Z}^2$–subshift of finite type Σ have trivial automorphism group $\text{Aut}(\Sigma) \cong \mathbb{Z}^2$? Subshifts of finite type on different groups may have trivial automorphism group (see [W2]).
(2) Is there a strong specification $\mathbb{Z}^2$–subshift of finite type with an automorphism of finite positive entropy? (It is clear that without some specification property stronger than mixing this has a trivial answer: the direct product of a full shift with one of the topologically mixing zero entropy examples above has positive entropy and has an automorphism of finite positive entropy.) See [Sh2] for some interesting results in this direction. A very special case has been solved: in [W3] it is shown that an ergodic group automorphism commuting with some $\mathbb{Z}^d$–action by automorphisms with completely positive entropy must have infinite entropy if $d > 1$. This shows, for instance, that a cellular automaton on the two–dimensional full shift $\{0, 1\}^{\mathbb{Z}^2}$ cannot have finite positive entropy if its block–map representation is permutive on every coordinate in its support.
(3) Given $\varepsilon > 0$, is there a topologically mixing $\mathbb{Z}^2$–subshift of finite type with an automorphism of entropy less than ε? Less than $\varepsilon \times h_0$, where h_0 is the smaller

of the entropies of the generators?
(4) If so, is there a single topologically mixing $\mathbb{Z}^2$–subshift of finite type Σ with the property that $\mathrm{Aut}(\Sigma)$ has elements with arbitrarily small positive entropy?
(5) Given a finite group F, is there a zero entropy topologically mixing $\mathbb{Z}^d$–subshift of finite type whose automorphism group contains an isomorphic copy of F?
(6) If so, is there a single zero entropy topologically mixing $\mathbb{Z}^d$–subshift of finite type whose automorphism group contains any finite group?

References

[B] R. Berger, *The undecidability of the Domino problem*, Mem. Amer. Math. Soc. **66** (1966).

[BLR] M. Boyle, D. Lind and D. Rudolph, *The automorphism group of a shift of finite type*, Trans. Amer. Math. Soc. **306** (1988), 71–114.

[BS] R. Burton and J. Steif, *Nonuniqueness of measures of maximal entropy for subshifts of finite type*, Ergod. Th. & Dyn. Sys., to appear.

[E] S. A. Elsanousi, *A variational principle for the pressure of a continuous Z^2–action on a compact metric space*, Amer. J. Math. **99** (1977), 77–106.

[H1] G.A. Hedlund, *Transformations commuting with the shift*, Topological Dynamics, Benjamin, New York, 1968, pp. 259–289.

[H2] G.A. Hedlund, *Endomorphisms and automorphisms of the shift dynamical system*, Math. Sys. Theory **3** (1969), 320–375.

[KRW] K. H. Kim, F. W. Roush and J. B. Wagoner, *Automorphisms of the dimension group and gyration numbers*, J. Amer. Math. Soc. **5** (1992), 191–212.

[KS] B. Kitchens and K. Schmidt, *Markov subgroups of* $(Z/2Z)^{Z^2}$, Cont. Math **135** (1992), 265–283.

[Le] F. Ledrappier, *Un champ markovien peut être d'entropie nulle et melangeant*, Comptes Rendus Acad. Sci. Paris **Ser. A, 287** (1978), 561–562.

[L] D. A. Lind, *Entropies of automorphisms of a topological Markov shift*, Proc. Amer. Math. Soc. **99** (1987), 589–595.

[M] M. Misiurewicz, *A short proof of the variational principle for a Z_+^n–action on a compact space*, Astérisque **49** (1977), 61–74.

[R] R. Robinson, *Undecidability and nonperiodicity for tilings of the plane*, Invent. Math. **12** (1971), 177-209.

[R1] J. P. Ryan, *The shift and commutivity*, Math. Sys. Th. **6** (1971), 82–85.

[R2] J. P. Ryan, *The shift and commutivity II*, Math. Sys. Th. **8** (1975), 249–250.

[Sc] K. Schmidt, *Algebraic ideas in ergodic theory*, C.B.M.S. Reg. Conf. Ser. in Math. **76** (1990).

[Sh1] M. A. Shereshevsky, *On the classification of some two–dimensional Markov shifts with group structure*, Ergod. Th. & Dyn. Sys. **12** (1992), 823–833.

[Sh2] M. A. Shereshevsky, *Expansiveness, entropy and polynomial growth for groups acting on subshifts by automorphisms*, Indag. Mathem. **4** (1993), 203–210.

[W1] T. Ward, *An algebraic obstruction to isomorphism of Markov shifts with group alphabets*, Bull. London Math. Soc. **25** (1993), 240–246.

[W2] T.Ward, *Endomorphisms and automorphisms of group subshifts of finite type: non–abelian examples*, Preprint (1993).

[W3] T. Ward, *Entropy bounds for automorphisms commuting with algebraic actions*, Preprint (1993).

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