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Cluster algebras of finite mutation type via unfoldings

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We complete classification of mutation-finite cluster algebras by extending the technique derived by Fomin, Shapiro, and Thurston to skew-symmetrizable case. We show that for every mutation-finite skew-symmetrizable matrix a diagram characterizing the matrix admits an *unfolding* which embeds its mutation class to the mutation class of some mutation-finite skew-symmetric matrix. In particular, this establishes a correspondence between a large class of skew-symmetrizable mutation-finite cluster algebras and triangulated marked bordered surfaces.

1 Introduction

In the present paper, we continue investigation of cluster algebras of finite mutation type started in [5].

Cluster algebras were introduced by Fomin and Zelevinsky in the series of papers [9], [10], [2], [11]. Up to isomorphism, each cluster algebra is defined by a *skew-symmetrizable* $n \times n$ integer matrix called *exchange matrix*, where integer matrix B is skew-symmetrizable if there exists an integer diagonal $n \times n$ matrix D such that BD is skew-symmetric. Exchange matrices admit *mutations* (see 1). Collection of all exchange matrices of a cluster algebra form a *mutation class* of exchange matrices.

In [5], we classified all the *skew-symmetric* exchange matrices with finite mutation class. In this paper, we complete classification of finite mutation classes of exchange matrices by presenting an answer in full generality.

The method we use is based on the following two main tools. The first main tool is the technique of *block decompositions* introduced by Fomin, Shapiro, and Thurston in [4]. The results of [5] are primary based on application of this technique. We combine this technique with studying of *diagrams* associated to skew-symmetrizable matrices defined by Fomin and Zelevinsky in [10] by introducing *s-decomposable diagrams*. The second main tool is a counterpart of the *unfolding procedure* introduced by Lusztig in [14] for generalized Cartan matrices. Using the unfolding procedure, we assign to each diagram of a mutation-finite skew-symmetrizable matrix a mutation-finite quiver. Due to results of [4] and [5], this allows us to relate a large class of skew-symmetrizable mutation-finite matrices with 2-dimensional bordered marked surfaces.

We prove the following theorem (the precise definitions will be given in Sections 2 and 3).

Theorem 5.13. A skew-symmetrizable $n \times n$ matrix, $n \geq 3$, that is not skew-symmetric, has finite mutation class if and only if its diagram is either s-decomposable or mutation-equivalent to one of the seven types $\widetilde{G}_2$, F_4, $\widetilde{F}_4$, $G_2^{(*,+)}$, $G_2^{(*,*)}$, $F_4^{(*,+)}$, $F_4^{(*,*)}$ shown on Fig. 1.

□

Remark. The diagrams $G_2^{(*,+)}$, $G_2^{(*,*)}$, $F_4^{(*,+)}$, and $F_4^{(*,*)}$ are, actually, diagrams of extended affine root systems (see [15]). Each of them corresponds to two extended affine root systems: $G_2^{(*,+)}$ corresponds to root systems

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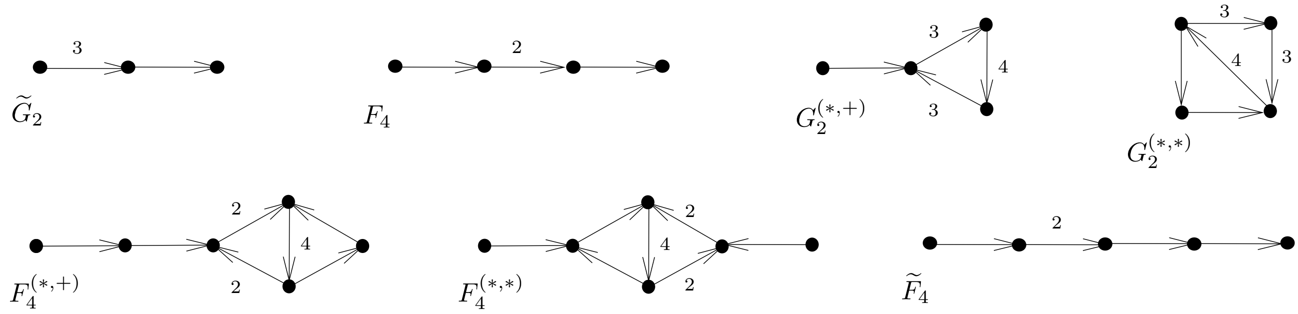


Fig. 1. Non-decomposable mutation-finite non-skew-symmetric diagrams of order at least 3

$G_2^{(1,3)}$ and $G_2^{(3,1)}$ (whose matrices are mutation-equivalent), $F_4^{(*,+)}$ corresponds to root systems $F_4^{(1,2)}$ and $F_4^{(2,1)}$ (whose matrices are also mutation-equivalent up to change of all signs), $G_2^{(*,*)}$ corresponds to root systems $G_2^{(1,1)}$ and $G_2^{(3,3)}$, and $F_4^{(*,*)}$ corresponds to root systems $F_4^{(1,1)}$ and $F_4^{(2,2)}$ (see Table 6 and [15, Table 1]).

We recall that mutation class of any 2×2 skew-symmetrizable matrix is finite.

Combined with results of [5], Theorem 5.13 completes the classification of mutation-finite skew-symmetrizable matrices.

Using Theorem 5.13, we prove the following theorem.

Theorem 6.1. Any s-decomposable diagram admits an unfolding to a diagram arising from ideal tagged triangulation of a marked bordered surface. Any mutation-finite matrix with non-decomposable diagram admits an unfolding to a mutation-finite skew-symmetric matrix. □

Tagged triangulations corresponding to unfoldings of skew-symmetrizable matrices with s-decomposable diagrams (constructed in Section 6.1) have special symmetry property: each of them contains a pair of edges representing the same isotopy class (one tagged plain and the other tagged notched, we call them *conjugate pair* of edges). In particular, we obtain a correspondence between s-decomposable diagrams and marked tagged triangulations:

Theorem 7.2. There is a one-to-one correspondence between s-decomposable skew-symmetrizable diagrams with fixed block decomposition and ideal tagged triangulations of marked bordered surfaces with fixed tuple of conjugate pairs of edges. □

In the correspondence above, one direction is provided by *local unfoldings* (see Section 6.1). The other direction is provided by *folding* (see Section 7) of some of conjugate pairs of edges: due to the existence of unfolding, this operation occurs to be well-defined. Under this correspondence, block-decomposable diagrams correspond to triangulations with no conjugate pairs chosen.

Note also that the correspondence above is invariant under mutations (resp., composite flips): if the triangulation $T(S)$ corresponds to a diagram S, then the triangulation $T(\mu_x(S))$ for a mutation $\mu_x(S)$ of a diagram S in the vertex x can be obtained by performing flips in all the edges of $T(S)$ corresponding to images of x under local unfolding.

As in the skew-symmetric case (cf. [5, Theorem 7.5]), consideration of minimal mutation-infinite diagrams gives rise to a polynomial-time algorithm to determine whether a large skew-symmetrizable matrix is mutation-finite:

Theorem 8.5. A skew-symmetrizable $n \times n$ matrix B, $n \geq 10$, has finite mutation class if and only if a mutation class of every principal 10×10 submatrix of B is finite.

□

The paper is organized as follows. In Section 2, we recall necessary definitions and basic facts on cluster algebras, exchange matrices, and their diagrams.

Section 3 is devoted to the technique of s-decomposable diagrams. We recall the basic facts from [4], and reformulate the results of [4] in the language of diagrams. Further, we introduce new blocks and prove several properties of block decompositions of diagrams. In particular, we show that s-decomposable diagrams are mutation-finite.

In Section 4 we give a definition of unfolding of skew-symmetrizable matrices introduced by A. Zelevinsky (personal communication), and extend it to a notion of unfolding of a diagram. This is the core construction of the paper. In general, an unfolding may not be unique. We construct a uniquely defined *local unfolding* for any s-decomposable diagram. Making use of this construction, we show that s-decomposable diagrams carry the same properties as block-decomposable quivers do.

Section 5 contains the proof of Theorem 5.13. In Section 6, we present a construction of unfolding for non-decomposable mutation-finite skew-symmetrizable matrices.

Section 7 is devoted to applications of the results of Section 6 to construction of relations between s-decomposable diagrams and triangulations of bordered surfaces.

Finally, in Section 8 we provide a polynomial-time algorithm which determines whether a skew-symmetrizable matrix has finite mutation class.

We would like to thank B. Keller who attracted our attention to foldings, and V. Fock, A. Goncharov, and S. Fomin for fruitful discussions and advices. We are especially grateful to A. Zelevinsky for introduction to unfoldings and numerous stimulating discussions leading to appearing of the present paper. The first author thanks the Max Planck Institute for Mathematics in Bonn for hospitality.

2 Cluster algebras, mutations, and diagrams

We briefly remind the definition of coefficient-free cluster algebra.

An integer $n \times n$ matrix B is called *skew-symmetrizable* if there exists an integer diagonal $n \times n$ matrix $D = diag(d_1, \dots, d_n)$, such that the product BD is a skew-symmetric matrix, i.e., $b_{ij}d_j = -b_{ji}d_i$.

A seed is a pair (f, B), where $f = \{f_1, \dots, f_n\}$ form a collection of algebraically independent rational functions of n variables $x_1, \dots, x_n$, and B is a skew-symmetrizable matrix.

The part f of seed (f, B) is called *cluster*, elements f_i are called *cluster variables*, and B is called *exchange matrix.*

Definition 2.1. For any k, $1 \le k \le n$ we define *the mutation* of seed (f, B) in direction k as a new seed (f', B') in the following way:

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}, & \text{otherwise.} \end{cases} \tag{1}$$

$$f'_i = \begin{cases} f_i, & \text{if } i \ne k; \\ \frac{\prod_{b_{ji}>0} f_j^{b_{ji}} + \prod_{b_{ji}<0} f_j^{-b_{ji}}}{f_i}, & \text{otherwise.} \end{cases} \tag{2}$$

□

We write $(f', B') = \mu_k((f, B))$. Notice that $\mu_k(\mu_k((f, B))) = (f, B)$. We say that two seeds are *mutation-equivalent* if one is obtained from the other by a sequence of seed mutations. Similarly we say that two clusters or two exchange matrices are *mutation-equivalent.*

Notice that exchange matrix mutation (1) depends only on the exchange matrix itself. The collection of all matrices mutation-equivalent to a given matrix B is called the *mutation class* of B.

For any skew-symmetrizable matrix B we define *initial seed* (x,B) as $(\{x_1, \dots, x_n\},B)$, B is the *initial exchange matrix*, $x = \{x_1, \dots, x_n\}$ is the *initial cluster.*

Cluster algebra $\mathfrak{A}(B)$ associated with the skew-symmetrizable $n \times n$ matrix B is a subalgebra of $\mathbb{Q}(x_1, \dots, x_n)$ generated by all cluster variables of the clusters mutation-equivalent to the initial seed (x, B).

Cluster algebra $\mathfrak{A}(B)$ is called *of finite type* if it contains only finitely many cluster variables. In other words, all clusters mutation-equivalent to initial cluster contain totally only finitely many distinct cluster variables.

In [10], Fomin and Zelevinsky proved a remarkable theorem that cluster algebras of finite type can be completely classified. More excitingly, this classification is parallel to the famous Cartan-Killing classification of simple Lie algebras.

Let B be an integer $n \times n$ matrix. Its *Cartan companion* $C(B)$ is the integer $n \times n$ matrix defined as follows:

$$C(B)_{ij} = \begin{cases} 2, & \text{if } i = j; \\ -|b_{ij}|, & \text{otherwise.} \end{cases}$$

Theorem 2.2 ([10])**.** There is a canonical bijection between the Cartan matrices of finite type and cluster algebras of finite type. Under this bijection, a Cartan matrix A of finite type corresponds to the cluster algebra $\mathfrak{A}(B)$, where B is an arbitrary skew-symmetrizable matrix with $C(B) = A$. □

The results by Fomin and Zelevinsky were further developed in [16] and [1], where the effective criteria for cluster algebras of finite type were given.

A cluster algebra of finite type has only finitely many distinct seeds. Therefore, any cluster algebra that has only finitely many cluster variables contains only finitely many distinct exchange matrices. Quite the contrary, the cluster algebra with finitely many exchange matrices is not necessarily of finite type.

Definition 2.3. A cluster algebra with only finitely many exchange matrices is called *of finite mutation type.* □

Example 2.4. The easiest example of infinite cluster algebra of finite mutation type is the algebra whose exchange matrix is

$$\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

This cluster algebra is not of finite type, however, mutation in any direction leads simply to sign change of exchange matrix. Therefore, the algebra is clearly of finite mutation type. □

Remark 2.5. Since the orbit of an exchange matrix depends on the exchange matrix only, we may speak about skew-symmetrizable matrices of finite mutation type. □

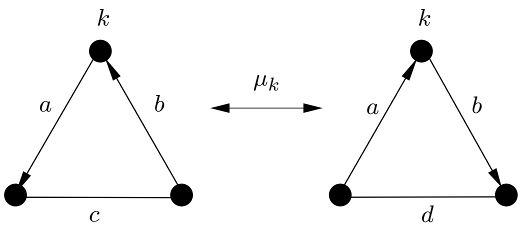
Therefore, Theorem 5.13 describes *all skew-symmetrizable integer matrices whose mutation class is finite.*

Following [10], we encode an $n \times n$ skew-symmetrizable integer matrix B by a finite simplicial 1-complex S with oriented weighted edges called *diagram*. The weights of a diagram are positive integers.

Vertices of S are labeled by $[1, \dots, n]$. If $b_{ij} > 0$, we join vertices i and j by an edge directed from i to j and assign to this edge weight $-b_{ij}b_{ji}$. Not every diagram corresponds to a skew-symmetrizable integer matrix: given a diagram S of a skew-symmetrizable integer matrix B, a product of weights along any chordless cycle of S is a perfect square (cf. [12, Exercise 2.1]).

Distinct matrices may have the same diagram. At the same time, it is easy to see that only finitely many matrices may correspond to the same diagram. All weights of a diagram of a skew-symmetric matrix are perfect squares. Conversely, if all weights of a diagram S are perfect squares, then there exists a skew-symmetric matrix B with diagram S.

As it is shown in [10], mutations of exchange matrices induce *mutations of diagrams.* If S is the diagram corresponding to matrix B, and B' is a mutation of B in direction k, then we call the diagram S' associated to B' a *mutation of S in direction k* and denote it by $\mu_k(S)$. A mutation in direction k changes weights of diagram in the way described in Figure 2 (see [10]).



$$\pm\sqrt{c} \pm \sqrt{d} = \sqrt{ab}$$

Fig. 2. Mutations of diagrams. The sign before $\sqrt{c}$ (resp., $\sqrt{d}$) is positive if the three vertices form an oriented cycle, and negative otherwise. Either c or d may vanish. If ab is equal to zero then neither value of c nor orientation of the corresponding edge does change.

For given diagram, the notion of *mutation class* is well-defined. We call a diagram (resp., matrix) *mutation-finite* if its mutation class is finite.

Remark 2.6. Note that the order of mutation class of a matrix may differ from the order of mutation class of corresponding diagram (see Example 2.7 below). However, mutation class of a matrix is finite if and only if a mutation class of the corresponding diagram is finite. □

Example 2.7. The mutation class of the following matrix

$$\begin{pmatrix} 0 & 2 & -4 \\ -1 & 0 & 2 \\ 1 & -1 & 0 \end{pmatrix}$$

consists of 6 matrices (up to simultaneous permutations of rows and columns). At the same time, the mutation class of the corresponding diagram contains 4 diagrams only.

□

Due to Remark 2.6, we can reduce the problem of classification of exchange matrices of finite mutation type to the following: *find all mutation-finite diagrams.*

The following criterion for a diagram to be mutation-finite is well-known. We present a short proof for the convenience of the reader.

Theorem 2.8. A connected diagram S of order at least 3 is mutation-finite if and only if any diagram in the mutation class of S contains no edges of weight greater than 4. □

Proof. The sufficiency is evident. To prove the necessity, it is sufficient to show that any connected diagram of order 3 containing an edge of weight at least 5 is mutation-infinite. For that we show that, in the assumptions above, there always exists a sequence of at most two mutations increasing the sum of the three weights (we call this sum *total weight*) and preserving the maximal weight.

Let S be a diagram of order 3 with weights (a,b,c), $a \ge b \ge c$, $a \ge 5$. If S is cyclically oriented (i.e., S is an oriented cycle), then mutating in the common vertex of edges with weights a and b we get a triple $(a, b, (\sqrt{ab}-\sqrt{c})^2)$, which has larger total weight since $a \ge b \ge c$ and $a \ge 5$ imply $(\sqrt{ab}-\sqrt{c})^2 > c$.

Now let S be not cyclically oriented. Applying one mutation (without changing weights) if needed, we may assume that the edges with weights a and b are oriented in the same way. Mutating in their common vertex, we get a triple $(a, b, (\sqrt{ab}+\sqrt{c})^2)$ which clearly has larger total weight than the initial triple did.

■

Remark 2.9. The case of *mutation-acyclic* diagrams was treated by Seven in [17]: it is proved there that mutation class of a mutation-finite diagram S contains a diagram without oriented cycles if and only if S is mutation equivalent to orientation of Dynkin (or extended Dynkin) diagram.

□

From now on, we use language of diagrams. The following notation will be used throughout the paper.

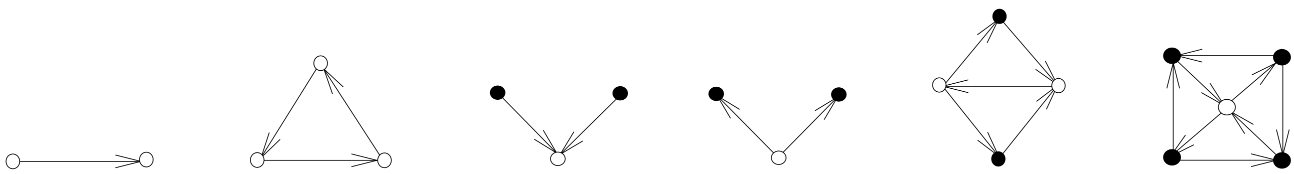
Let S be a diagram. A *subdiagram* $S_1 \subset S$ is a subcomplex of S. The *order* $|S|$ is the number of vertices of diagram S. If S_1 and S_2 are subdiagrams of diagram S, we denote by $\langle S_1, S_2 \rangle$ the subdiagram of S spanned by all the vertices of S_1 and S_2.

An edge is called *simple* if its weight is equal to one, and *multiple* otherwise.

3 Block decompositions of diagrams

First, we rephrase the definition 3.1 from [4] in terms of diagrams.

In [4], a *block* is a diagram isomorphic to one of the diagrams with black/white colored vertices shown on Fig. 3, or to a single vertex. Vertices marked in white are called *outlets*, we call the remaining ones *dead ends*. A connected diagram S is called *block-decomposable* if it can be obtained from a collection of blocks by identifying outlets of different blocks along some partial matching (matching of outlets of the same block is not allowed), where two simple edges with same endpoints and opposite directions cancel out, and two simple edges with same endpoints and same directions form an edge of weight 4. A non-connected diagram S is called block-decomposable either if S satisfies the definition above, or if S is a disjoint union of several mutually orthogonal diagrams satisfying the definition above. If S is not block-decomposable then we call S *non-decomposable*. Depending on a block, we call it *a block of type* I, II, III, IV, V, or simply *a block of n-th type*.



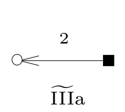
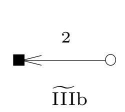
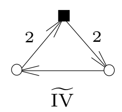
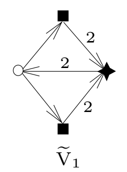
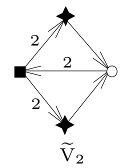
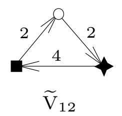
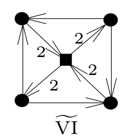
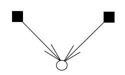
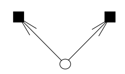
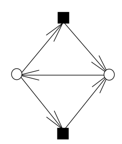
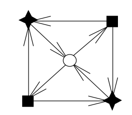
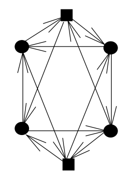
I II IIIa IIIb IV V

Fig. 3. Blocks. Outlets are colored in white, dead ends are black.

Block-decomposable diagrams are in one-to-one correspondence with adjacency matrices of arcs of ideal (tagged) triangulations of bordered two-dimensional surfaces with marked points (see [4, Section 13] for the detailed explanations). Mutations of block-decomposable diagrams correspond to flips of triangulations. In particular, this implies that mutation class of any block-decomposable diagram is finite, and any subdiagram of a block-decomposable one is block-decomposable too.

Clearly, adjacency matrices of arcs of ideal triangulations are skew-symmetric. To adopt the technique of blocks to general (skew-symmetrizable) case, we introduce new blocks of types $\widetilde{\mathrm{III}}$a, $\widetilde{\mathrm{III}}$b, $\widetilde{\mathrm{IV}}$, $\widetilde{\mathrm{V}}_1$, $\widetilde{\mathrm{V}}_2$, $\widetilde{\mathrm{V}}_{12}$, and $\widetilde{\mathrm{VI}}$ shown in Table 1.

Table 1. New blocks and their local unfoldings (see Sections 4, 6.1). Vertex x_i and the set E_i are marked in the same way.

New blocks							
Unfoldings							

Again, outlets are marked white. We keep the way of gluing (this remains well-defined since any edge with two outlets as ends is simple). More precisely, gluing of two edges of weight one will result in either empty edge (in case of distinct orientations) or an edge with weight 4.

Definition 3.1. A diagram is *s-decomposable* if it can be glued from blocks (both old and new). □

We keep the term "block-decomposable" for s-decomposable diagrams corresponding to skew-symmetric matrices.

Our aim is to prove that s-decomposable diagrams satisfy the same properties as block-decomposable ones do. In particular, in Theorem 3.5 we show that the set of s-decomposable diagrams is invariant under mutations (which implies that they are mutation-finite). In the next section we prove that any subdiagram of s-decomposable diagrams is s-decomposable (see Corollary 4.10).

Let S be an s-decomposable diagram with fixed decomposition (we denote this by S_{dec}). We say that $x \in S_{\mathrm{dec}}$ is an *outlet* if x is contained in exactly one block, and x is an outlet in that block. Further, suppose that for some $y \in S_{\mathrm{dec}}$ the diagram $\mu_y(S)$ is s-decomposable. Then a block decomposition $\mu_y(S)_{\mathrm{dec}}$ of $\mu_y(S)$ is *y-good* if all outlets of S_{dec} (probably, except y itself) are outlets of $\mu_y(S)_{\mathrm{dec}}$.

If S is s-decomposable and a decomposition is fixed, we define $N_x(S_{\mathrm{dec}})$ to be the union of all blocks containing x. Note that $N_x(S_{\mathrm{dec}})$ may not be a subdiagram of S.

Lemma 3.2. Let S_{dec} coincide with $N_x(S_{\mathrm{dec}})$ (i.e. S_{dec} is composed of blocks B_1 and B_2, B_2 may be empty), $x \in S$, where $x \in \mathsf{B}_1 \cap \mathsf{B}_2$ if $\mathsf{B}_2 \neq \emptyset$. Then there exists an x-good block decomposition of $\mu_x(S)$. □

Proof is straightforward: we need to examine 49 diagrams of gluings of two blocks.

Example 3.3. We illustrate the proof of lemma 3.2 on one example shown on Fig. 4, left. Here B_1 is of type II, and B_2 is of type $\widetilde{\mathrm{IV}}$. Outlets of S_{dec} are y_1, y_2, and y_3.

Then $\mu_x(S)$ has a block decomposition $\mu_x(S)_{\mathrm{dec}}$ shown on Fig. 4, right. Clearly, the vertices y_1, y_2, and y_3 are outlets of $\mu_x(S)_{\mathrm{dec}}$, so the decomposition is x-good.

□

Lemma 3.4. Suppose $N_x(S_{\mathrm{dec}}) = \langle \mathsf{B}_1, \mathsf{B}_2 \rangle$, B_2 may be empty. Let x_1, x_2 be outlets of $N_x(S_{\mathrm{dec}})$ $(x_1, x_2 \neq x)$. Suppose also that S_{dec} consists of $N_x(S_{\mathrm{dec}})$ and a block B, where x_1 and x_2 are outlets of B. Then $\mu_x(S)$ is s-decomposable with block B, i.e.

$$\langle \mu_x(N_x(S_{\mathrm{dec}})), \mathsf{B} \rangle = \mu_x(\langle N_x(S_{\mathrm{dec}})), \mathsf{B} \rangle)$$

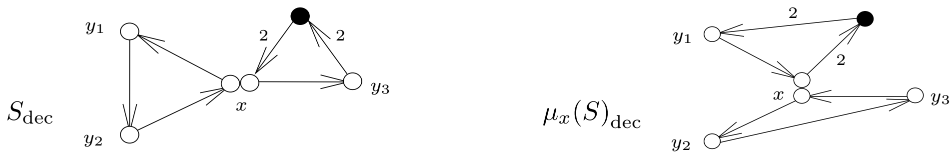


Fig. 4.

□

The s-decomposability immediately follows from Lemma 3.2. The equality follows from the definition of mutation, see Fig. 2.

As a corollary, we get the following theorem.

Theorem 3.5. Let S be s-decomposable. Then any mutation of S is s-decomposable. □

Proof follows from Lemma 3.4. Indeed, given decomposition of S and $x \in S$, μ_x affects only $N_x(S_{\text{dec}})$ and blocks with at least two points in common with $N_x(S_{\text{dec}})$. According to Lemma 3.2, $\mu_x(N_x(S_{\text{dec}}))$ admits x-good decomposition. By Lemma 3.4, we can construct a decomposition of $\mu_x(S)$ by attaching to x-good decomposition of $\mu_x(N_x(S_{\text{dec}}))$ the same blocks as in S_{dec} in the same way.

Corollary 3.6. All s-decomposable diagrams are mutation-finite. □

Remark 3.7. As one can notice, the block $\widetilde{\text{VI}}$ has no outlets. However, it is essential: its mutation class consists of 4 diagrams, 3 of them are s-decomposable (without making use of block $\widetilde{\text{VI}}$), and the fourth one is block $\widetilde{\text{VI}}$ itself (which cannot be decomposed in any other way). □

4 Unfoldings of matrices and diagrams

Let B be an indecomposable $n \times n$ skew-symmetrizable integer matrix, and let BD be a skew-symmetric matrix, where $D = (d_i)$ is diagonal integer matrix with positive diagonal entries. Notice that for any matrix $\mu_i(B)$ the matrix $\mu_i(B)D$ will be skew-symmetric.

We use the following definition of unfolding of a skew-symmetrizable matrix (communicated to us by A. Zelevinsky).

Suppose that we have chosen disjoint index sets $E_1, \dots, E_n$ with $|E_i| = d_i$. Denote $m = \sum\limits_{i=1}^{n} d_i$. Suppose also that we choose a skew-symmetric integer matrix C of size $m \times m$ with rows and columns indexed by the union of all E_i, such that

(1) the sum of entries in each column of each $E_i \times E_j$ block of C equals b_{ij};

(2) if $b_{ij} \geq 0$ then the $E_i \times E_j$ block of C has all entries non-negative.

Define a *composite mutation* $\widehat{\mu}_i = \prod_{\hat{\imath} \in E_i} \mu_{\hat{\imath}}$ on C. This mutation is well-defined, since all the mutations $\mu_{\hat{\imath}}$, $\hat{\imath} \in E_i$, for given i commute.

We say that C is an *unfolding* for B if C satisfies assertions (1) and (2) above, and for any sequence of iterated mutations $\mu_{k_1} \dots \mu_{k_m}(B)$ the matrix $C' = \widehat{\mu}_{k_1} \dots \widehat{\mu}_{k_m}(C)$ satisfies assertions (1) and (2) with respect to $B' = \mu_{k_1} \dots \mu_{k_m}(B)$.

Example 4.1. The matrix C below is an unfolding for the matrix B. Here $d_1 = 1$, $d_2 = 2$, $E_1 = \{1\}$, $E_2 = \{2, 3\}$.

$$B = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

□

Example 4.2. The matrices B and C below satisfy the assertions (1) and (2) of the definition of the unfolding. Here $d_1 = 2$, $d_2 = 1$, $d_3 = 2$, $E_1 = \{1, 2\}$, $E_2 = \{3\}$, $E_3 = \{4, 5\}$.

$$B = \begin{pmatrix} 0 & 2 & -2 \\ -1 & 0 & 1 \\ 2 & -2 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 \\ -1 & -1 & 0 & 1 & 1 \\ 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \end{pmatrix}$$

However, the matrix C is not an unfolding for the matrix B. Indeed, after mutation μ_2 of B (resp, μ_3 of C), the assertion (2) does not hold for block $E_1 \times E_3$ of $\mu_3(C)$. $\square$

If C is an unfolding of a skew-symmetrizable integer matrix B, it is natural to define an *unfolding of a diagram* of B as a diagram of C. In general, we say that a diagram $\widehat{S}$ is an unfolding of a diagram S if there exist matrices B and C with diagrams S and $\widehat{S}$ respectively, and C is an unfolding of B. This definition is equivalent to the following one.

Definition 4.3. Let S be a diagram with vertices $x_1, \dots, x_n$, and let $d_1, \dots, d_n$ be positive integers. Let $\widehat{S}$ be a connected skew-symmetric diagram with vertices $x_{\hat{\imath}}$ indexed by sets E_i of order d_i, such that for each $i, j \in [1 \dots n]$ the following holds:

(A) there are no edges joining vertices inside E_i and E_j;

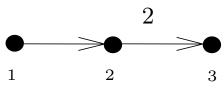
(B) for all $\hat{\imath} \in E_i$ the sum of weights of all edges joining $x_{\hat{\imath}}$ with E_j is the same, and all the arrows are oriented simultaneously either from E_i to E_j or from E_j to E_i;

(C) the product of total weight of edges joining $x_{\hat{\imath}}$ with E_j and total weight of edges joining $x_{\hat{\jmath}}$ with E_i equals the weight of $x_i x_j$.

Define a *composite mutation* $\widehat{\mu}_i = \prod_{\hat{\imath} \in E_i} \mu_{\hat{\imath}}$ on $\widehat{S}$. As in the case of matrices, the mutation is well-defined. We say that $\widehat{S}$ is an *unfolding* of S if for any sequence of iterated mutations $\mu_{i_1} \dots \mu_{i_k}$ a pair of diagrams $(\mu_{i_1} \dots \mu_{i_k} S, \widehat{\mu}_{i_1} \dots \widehat{\mu}_{i_k} \widehat{S}\,)$ satisfies the same conditions as the pair $(S, \widehat{S})$ does, i.e. for each $i, j \leq n$ the assumptions (A), (B) and (C) hold.

$\square$

The following example shows that an unfolding of a diagram may not be unique.

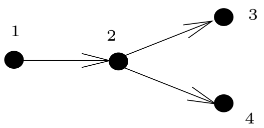
Example 4.4. Diagram  corresponds to two matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix}$$

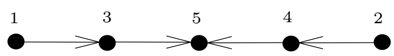
with unfoldings, respectively,

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 \end{pmatrix}$$

It is easy to see that these two unfoldings correspond, respectively, to diagrams



and



$\square$

Lemma 4.5. The diagrams in the second row of Table 1 are unfoldings of the corresponding blocks shown in the first row of the table.

$\square$

The proof consists of an elementary straightforward verification. We call the unfoldings of blocks shown in the second row of Table 1 *local unfoldings*. They can be characterized as follows:

Definition 4.6. An unfolding is *local* if for any outlet x_i of the initial skew-symmetrizable diagram, the corresponding integer d_i is equal to one. $\square$

This allows us to define for each s-decomposable diagram S with fixed decomposition S_{dec} a skew-symmetric diagram (denote it by $\tau(S_{\mathrm{dec}})$) by gluing of unfoldings of corresponding blocks. Since all the local unfoldings of blocks are skew-symmetric blocks, $\tau(S_{\mathrm{dec}})$ is block-decomposable diagram. In other words, we may understand τ as a map from block decompositions of s-decomposable diagrams to block decompositions of block-decomposable ones. Our current goal is to prove Theorem 4.9 which states that $\tau(S_{\mathrm{dec}})$ is an unfolding for S.

Lemma 4.7. Let $S_{\rm dec}$ coincide with $N_x(S_{\rm dec})$ (i.e. $S_{\rm dec}$ is composed of blocks B_1 and B_2, B_2 may be empty), $x \in S$, where $x \in \mathsf{B}_1 \cap \mathsf{B}_2$ if $\mathsf{B}_2 \neq \emptyset$. Suppose also that $S_{\rm dec}$ is different from ones shown on Fig. 5. Then there exists an x-good decomposition of $\mu_x(S)$, such that

$$\tau(\mu_x(S)_{\rm dec}) = \widehat{\mu}_x(\tau(S_{\rm dec}))$$

□

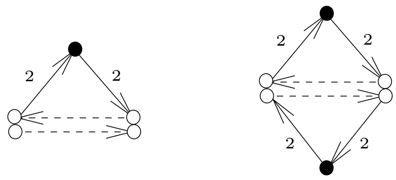


Fig. 5. Exceptional s-decomposable diagrams. Dotted edges are the edges of blocks disappearing in the diagram.

The proof considers the same cases as in the proof of Lemma 3.2 (in fact, this consideration includes proof of Lemma 3.2 as a partial case).

Combining Lemmas 3.4 and 4.7, we get the following lemma.

Lemma 4.8. Let S be s-decomposable, and $x \in S_{\rm dec}$. If $S_{\rm dec}$ is different from ones shown on Fig. 5, then there exists a decomposition of $\mu_x(S)$, such that

$$\tau(\mu_x(S)_{\rm dec}) = \widehat{\mu}_x(\tau(S_{\rm dec}))$$

□

As a corollary, we obtain the unfolding theorem for diagrams.

Theorem 4.9. Every s-decomposable diagram has a block-decomposable unfolding.

□

Proof. For diagrams that are not mutation-equivalent to ones shown on Fig. 5 the statement follows from Lemma 4.8 (note that these two diagrams have no outlets, so they do not affect other mutation classes). Now consider the two mutation classes represented by the diagrams shown on Fig. 5.

The left diagram has another block decomposition: it can be glued from two blocks of type $\widetilde{\rm III}$. Starting from this decomposition, we get an unfolding according to Lemma 4.8.

Mutation class of the right diagram from Fig. 5 consists of three diagrams. Unfoldings are shown in Table 2. All of them are block-decomposable: they can be glued either from two blocks of type IV (diagrams on the left and on the right), or from four blocks of type II (the one in the middle).

■

Lemma 4.10. Subdiagram of s-decomposable diagram is s-decomposable. □

To prove the lemma, it is sufficient to show a way to substitute any block B with a vertex x removed by some s-decomposable diagram such that all outlets remain outlets. The choice of substitutions is shown in Table 3.

Remark 4.11. Lemma 4.10 can be considered as a corollary of Theorem 4.9. More precisely, Theorem 4.9 gives a geometric interpretation of Table 3. It is known that any subdiagram $S \setminus x$ of block-decomposable diagram S is block-decomposable: to obtain the corresponding triangulation of a bordered surface we need to cut the triangulation for S along the edge corresponding to x. It is easy to check that if a block $\widehat{\mathsf{B}}$ is an unfolding of a block B, and $x \in \mathsf{B}$, then removing all the vertices of type $\widehat{x}$ from $\widehat{\mathsf{B}}$ we are always left with a union of several blocks, such that initial symmetries of the block $\widehat{\mathsf{B}}$ are preserved. In other words, unfolding $\widehat{\mathsf{B}}$ of block B with $\widehat{x}$ removed can be "folded back".

□

Table 2. Exceptional s-decomposable diagrams and their unfoldings

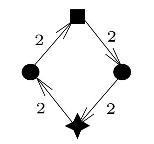
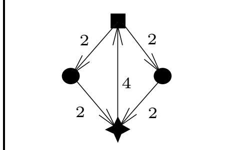
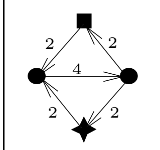
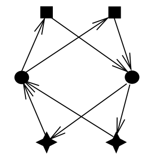
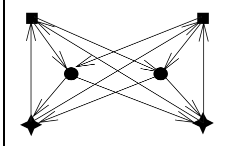
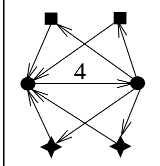
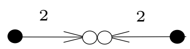
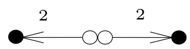
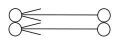
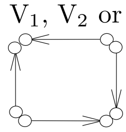
Diagrams			
Unfoldings			

Table 3. Block decompositions of blocks with one vertex removed.

Block B	$\widetilde{\mathrm{IV}}$	$\widetilde{\mathrm{V}}_1$	$\widetilde{\mathrm{V}}_2$	$\widetilde{\mathrm{V}}_{12}$	$\widetilde{\mathrm{VI}}$
Decomposition of $\mathsf{B} \setminus x$	$\widetilde{\mathrm{III}}$ or I	$\widetilde{\mathrm{IV}}$, III or 	$\widetilde{\mathrm{IV}}$, III or 	III or 	$\widetilde{\mathrm{V}}_1$, $\widetilde{\mathrm{V}}_2$ or 

5 Classification of mutation-finite diagrams and matrices

Our proof of Theorem 5.13 follows the proof of Theorem 6.1 from [5].

First, we define *minimal non-decomposable diagram* as a diagram which is not s-decomposable, but any its subdiagram is s-decomposable. According to Corollary 4.10, a non-decomposable diagram of order n is minimal if and only if any its subdiagram of order $n-1$ is s-decomposable.

Then we prove the following generalization of [5, Theorem 5.2].

Theorem 5.1. Any minimal non-decomposable diagram contains at most 7 vertices.

□

The proof follows the proof of [5, Theorem 5.2]. The only difference is now we need to consider more types of blocks. All essential tools remain the same. The complete list of refinements is contained in [6, Appendix A].

The further program is the same as in skew-symmetric case (see [5]).

Theorem 5.2. The only minimal non-decomposable mutation-finite diagrams with at least three vertices are ones mutation-equivalent to one of the four diagrams E_6, X_6, $\widetilde{G}_2$ and F_4 shown on Figure 6.

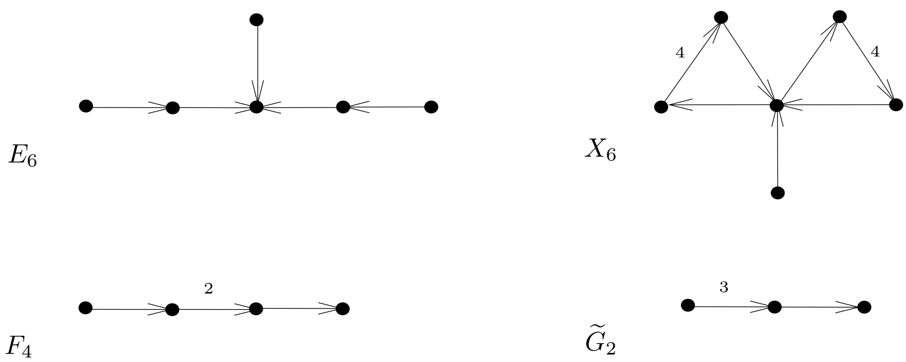


Fig. 6. Minimal non-decomposable mutation-finite diagrams of order at least three

□

Remark 5.3. Amongst diagrams of order two, there is exactly one non-decomposable diagram (called G_2) admitting an unfolding to a block-decomposable diagram (this diagram and corresponding unfolding D_4 are shown on Figure 7). Moreover, G_2 is a unique non-decomposable diagram of order 2 that can be a subdiagram of a mutation-finite diagram. Due to this fact, we may think G_2 to be minimal non-decomposable instead of $\widetilde{G}_2$ (every mutation of which contains G_2).

□

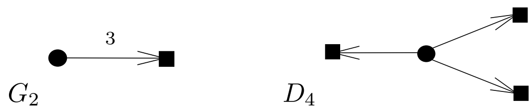


Fig. 7. G_2 is a unique non-decomposable diagram of order two admitting an unfolding to a block-decomposable diagram (which is D_4).

Proof of Theorem 5.2. It is easy to see that the four diagrams shown on Figure 6 are mutation-finite and non-decomposable (E_6 and X_6 are discussed in [5]). To prove the theorem, it is sufficient to show that all other mutation-finite diagrams on at most 7 vertices either are s-decomposable, or contain subdiagrams which are mutation-equivalent to one of $\widetilde{G}_2$, F_4, E_6 or X_6. Due to Remark 5.3, instead of looking for subdiagrams mutation-equivalent to $\widetilde{G}_2$ it is enough to find an edge of weight 3.

Let S be a minimal non-decomposable mutation-finite diagram. By Theorem 5.1, $|S| \leq 7$. Since the mutation class of S is finite, weights of edges of S do not exceed 4. The number of diagrams on at most 7 vertices with bounded multiplicities of edges is finite. We use a computer [8] to list all diagrams, choose mutation-finite ones, and check which of them are s-decomposable. The check is organized as in the proof of Theorem 5.11 from [5].

As a result, besides skew-symmetric diagrams, we get 7 mutation classes of non-decomposable mutation-finite diagrams of order at least two: 1 of order three, 3 of order four, 1 of order five, and 2 of order six. All these diagrams are shown on Figure 1. Furthermore, a short straightforward check (using `Java` applet [13]) shows that any diagram which is mutation-equivalent to any of these 7 ones contains either an edge of weight 3 (and a subdiagram mutation-equivalent to $\widetilde{G}_2$) or a subdiagram mutation-equivalent to F_4. The minimality is evident.

■

Corollary 5.4. Every non-decomposable mutation-finite diagram contains an edge of weight 3 or subdiagram mutation-equivalent to one of F_4, E_6 and X_6.

□

Remark 5.5. As it follows from computations made in the proof of Theorem 5.1, any non-decomposable mutation-finite diagram of order 7 is skew-symmetric. In other words, for any non-decomposable diagram S of order 7 containing an edge of weight 2 or 3, and any diagram S' containing S as a subdiagram, S' is mutation-infinite. We will use this to show that there are no other non-decomposable diagrams except ones listed above.

The same computations show that any mutation-finite diagram containing an edge of weight 3 is of order at most 4. Clearly, all such diagrams are non-decomposable (since no block contains an edge of weight 3).

□

Theorem 5.6. A connected non-decomposable mutation-finite diagram of order greater than 2 is mutation-equivalent to one of the eleven diagrams E_6, E_7, E_8, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, X_6, X_7, $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$ shown on Figure 8, or to one of the seven diagrams $\widetilde{G}_2$, F_4, $\widetilde{F}_4$, $G_2^{(*,+)}$, $G_2^{(*,*)}$, $F_4^{(*,+)}$, $F_4^{(*,*)}$ shown on Figure 1.

□

As we have already shown (see the proof of Theorem 5.2), all these diagrams have finite mutation class and are non-decomposable (for skew-symmetric ones see [5]). We need to prove completeness of the list.

The following two lemmas are evident.

Lemma 5.7 ([5], Lemma 6.4)**.** Let S_1 be a proper subdiagram of S, let S_0 be a diagram mutation-equivalent to S_1. Then there exists a diagram S' which is mutation-equivalent to S and contains S_0.

□

Lemma 5.8 ([5], Lemma 6.2)**.** Let S be a non-decomposable diagram of order $d \geq 7$ with finite mutation class. Then S contains a non-decomposable mutation-finite subdiagram S_1 of order $d-1$.

□

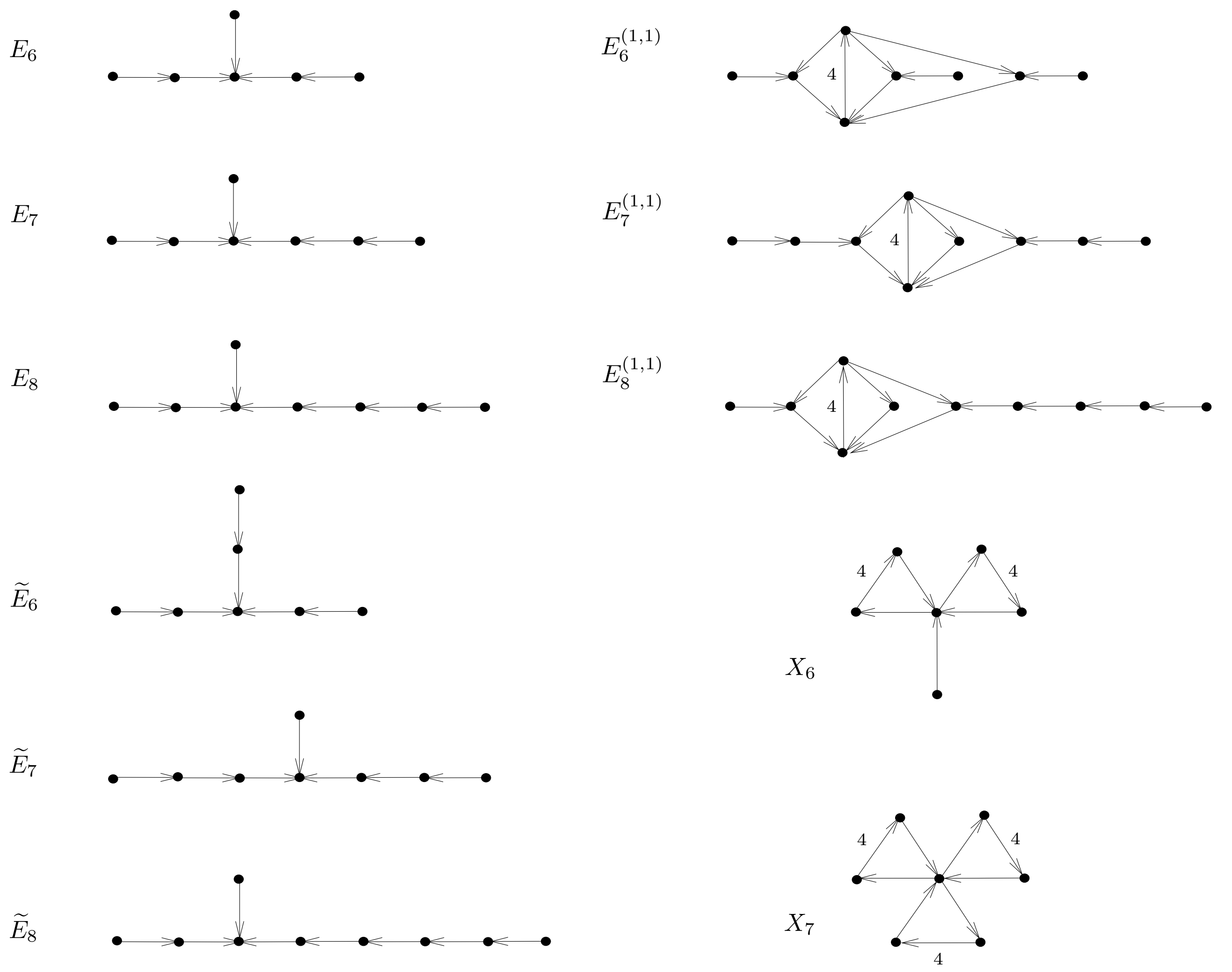


Fig. 8. Non-decomposable mutation-finite skew-symmetric diagrams of order at least 3

Corollary 5.9. Suppose that for some $d \geq 7$ there are no non-decomposable mutation-finite diagrams of order d. Then order of any non-decomposable mutation-finite diagram does not exceed $d-1$. $\square$

Proof of Theorem 5.6. In the proof of Theorem 5.1 we listed all non-decomposable mutation-finite diagrams of order at most 7. Now we want to show that all non-decomposable mutation-finite diagrams of order at least 8 (in fact, at least 7, see Remark 5.5) are skew-symmetric.

Suppose that S is a non-decomposable mutation-finite diagram of order at least 8, and S is not skew-symmetric. Then S contains a minimal non-decomposable mutation-finite subdiagram S_1 which is mutation-equivalent to a diagram of one of the four types shown on Fig. 6 (Theorem 5.1). If S_1 is mutation-equivalent to $\widetilde{G}_2$ or F_4 then, taking any connected subdiagram $S' \subset S$ of order 7 we see that S' is mutation-infinite, which implies that S is mutation-infinite, too. Therefore, S_1 is mutation-equivalent to E_6 or X_6.

Notice that any connected subdiagram $S' \subset S$ of order 7 containing S_1 is skew-symmetric (otherwise S' is mutation-infinite due to Remark 5.5), so it is mutation-equivalent to one of E_7, X_7, and $\widetilde{E}_6$. According to Lemma 5.7, we may assume that S' coincides with E_7, X_7, or $\widetilde{E}_6$.

Suppose that $|S| = 8$, and consider the unique vertex $x \in S \setminus S'$. If x is joined with some vertex of S_1, then $S_2 = \langle S_1, x \rangle$ is of order 7, so S_2 is skew-symmetric. This implies that the only edge which breaks skew-symmetry of S is one joining x with $S' \setminus S_1$. Therefore, this edge cannot be contained in any cycle: otherwise S is not skew-symmetrizable. In particular, x is not joined with any vertex of S_1.

In X_7 and $\widetilde{E}_6$ every vertex is contained in some X_6 or E_6 respectively, so there is no way to add a vertex to X_7 or $\widetilde{E}_6$ to get a mutation-finite diagram that is not skew-symmetric. In E_7 there is a unique vertex not contained in E_6. Attaching to that vertex an edge of weight 2 or 4 we get mutation-infinite diagrams [13] (weight 3 is prohibited by Remark 5.5). Thus, all non-decomposable mutation-finite diagrams of order 8 are skew-symmetric.

Now we proceed in the same way for diagrams of order 9. Any such non-decomposable mutation-finite diagram is mutation-equivalent to one (denote it by S) containing E_6 or X_6. As it was proved, any connected subdiagram of S of order 8 containing E_6 or X_6 is skew-symmetric, so, performing some mutations, we can assume that S contains S' equal to one of $E_6^{(1,1)}$, $\widetilde{E}_7$, E_8, and the remaining vertex of S is not joined with any of E_7 and $\widetilde{E}_6$ contained in S'. Again, any vertex of $E_6^{(1,1)}$ and $\widetilde{E}_7$ belongs to some $\widetilde{E}_6$ or E_7, and there is a unique vertex of E_8 not contained in E_7. Attaching to that vertex an edge of weight 2 or 4 we get mutation-infinite diagrams, so all non-decomposable diagrams of order 9 are skew-symmetric.

We repeat the same procedure for diagrams of order 10 without any new results (here we attach a node to $\widetilde{E}_8$, while any vertex of $E_7^{(1,1)}$ belongs to some $\widetilde{E}_7$), and then for diagrams of order 11 (here any vertex of $E_8^{(1,1)}$ belongs to some $\widetilde{E}_8$). Finally, we see that there are no non-decomposable diagrams of order 11. In view of Corollary 5.9, this completes the proof.

■

Now we will reformulate the result of this section in terms of matrices. We recall two evident statements about exchange matrices and their diagrams.

Lemma 5.10. Diagram of mutation-finite matrix is mutation-finite.

□

Lemma 5.11. Any diagram is represented only by a finite number of skew-symmetrizable matrices.

□

Combining Lemmas 5.10 and 5.11, we get the following lemma.

Lemma 5.12. A skew-symmetrizable matrix is mutation-finite if and only if its diagram is mutation-finite.

□

As an immediate corollary of Lemma 5.12 and Theorem 5.6, we obtain the following theorem.

Theorem 5.13. A skew-symmetrizable $n \times n$ matrix, $n \geq 3$, that is not skew-symmetric, has finite mutation class if and only if its diagram is either s-decomposable or mutation-equivalent to one of the seven types $\widetilde{G}_2$, F_4, $\widetilde{F}_4$, $G_2^{(*,+)}$, $G_2^{(*,*)}$, $F_4^{(*,+)}$, $F_4^{(*,*)}$ shown on Fig. 1. □

6 Unfoldings of mutation-finite matrices and diagrams

In this section we complete the construction of unfoldings for all mutation-finite diagrams, and specify the corresponding matrices. We also construct unfoldings for all mutation-finite matrices with non-decomposable diagrams.

First, we consider mutation-finite matrices admitting local unfoldings. As it is shown in Section 4, this leads to a block-decomposable unfolding for every s-decomposable diagram. All these unfoldings appear to be block-decomposable. Next, we show examples of non-local unfoldings for matrices with s-decomposable diagrams. Finally, we present unfoldings for all mutation-finite matrices with non-decomposable diagrams. These unfoldings are also mutation-finite but have (usually) non-decomposable diagrams. In particular, we obtain the following generalization of the results of Section 4.

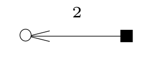
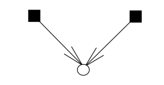
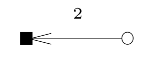
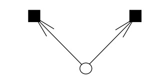
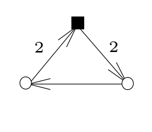
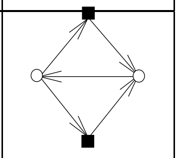
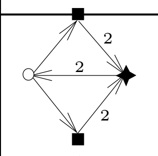
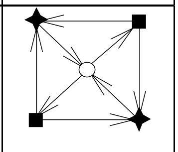
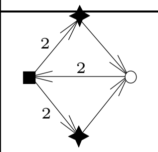
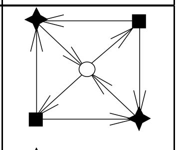
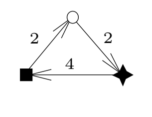
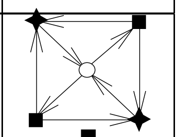
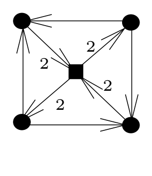
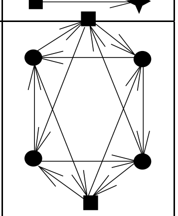
Theorem 6.1. Any s-decomposable diagram admits an unfolding to a diagram arising from ideal tagged triangulation of a marked bordered surface. Any mutation-finite matrix with non-decomposable diagram admits an unfolding to a mutation-finite skew-symmetric matrix. □

6.1 Local unfoldings

In Section 4 we constructed a local unfolding for every s-decomposable diagram. Let us describe the choice of matrices B and C corresponding to a diagram S and its local unfolding $\widehat{S}$ respectively.

These matrices can be easily reconstructed by looking at the local unfoldings of blocks, see Table 4. To each edge of weight 4 we assign a skew-symmetric submatrix. To each new block we assign a submatrix in such a (unique) way that for each outlet x_i the number d_i is a unit. In terms of matrix elements, this means that for any outlet x_i and entry $b_{ij} \neq -b_{ji}$ the inequality $|b_{ij}| < |b_{ji}|$ holds if and only if $i < j$. The local unfoldings of blocks are diagrams of unfoldings of these matrices with coprime numbers d_i.

Table 4. Local unfoldings of blocks

Block	Diagram	Matrix	Unfolding	Diagram unfolding
$\widetilde{\mathrm{III}}\mathrm{a}$		$\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	
$\widetilde{\mathrm{III}}\mathrm{b}$		$\begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}$	
$\widetilde{\mathrm{IV}}$		$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 2 & -2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$	
$\widetilde{\mathrm{V}}_1$		$\begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 2 & -2 & 0 & -2 \\ -1 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & -1 & -1 & 1 \\ -1 & 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 1 & 0 \end{pmatrix}$	
$\widetilde{\mathrm{V}}_2$		$\begin{pmatrix} 0 & 2 & -2 & 2 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & -1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$	
$\widetilde{\mathrm{V}}_{12}$		$\begin{pmatrix} 0 & 2 & -2 \\ -1 & 0 & 1 \\ 2 & -2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \\ -1 & -1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \end{pmatrix}$	
$\widetilde{\mathrm{VI}}$		$\begin{pmatrix} 0 & 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 0 & 1 \\ 2 & -2 & 2 & -2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 1 & -1 & -1 \\ -1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ -1 & 0 & -1 & 0 & 1 & 1 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 \end{pmatrix}$	

Now we take any block decomposition of a diagram S, assign to each block S_j a matrix B_j defined above (for skew-symmetric blocks the matrix is uniquely defined), and then glue all them in a natural way to obtain matrix B with diagram S. In terms of matrices "gluing" is equivalent to summation of matrices, composed of B_j at corresponding place and zeros outside. Since $d_i = 1$ for any outlet x_i, after gluing we still have $|b_{ij}| < |b_{ji}|$ if and only if $i < j$ and $b_{ij} \neq -b_{ji}$.

To obtain an unfolding C of B we take unfoldings C_j of all matrices B_j and glue them along outlets. Again, this procedure is well-defined since for every outlet x_i the number d_i is equal to one.

Example 6.2. Consider a diagram S shown on Fig. 9, left. It has a block decomposition shown in the middle of the figure.

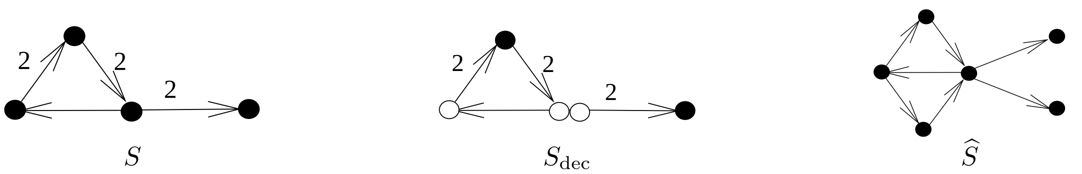


Fig. 9. Diagram S with block decomposition S_{dec} and unfolding $\widehat{S}$

Let S_1 and S_2 be blocks of type $\widetilde{\mathrm{IV}}$ and $\widetilde{\mathrm{III}}\mathrm{b}$ respectively. Then the corresponding matrices are

$$B_1 = \begin{pmatrix} 0 & 1 & -1 \\ -2 & 0 & 2 \\ 1 & -1 & 0 \end{pmatrix} \qquad \text{and} \qquad B_2 = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix},$$

so we can write down the matrix

$$B = \begin{pmatrix} 0 & 1 & -1 & 0 \\ -2 & 0 & 2 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & -2 & 0 \end{pmatrix}$$

corresponding to diagram S. Unfoldings of B_1 and B_2 are

$$C_1 = \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & -1 & -1 & 0 \end{pmatrix} \qquad \text{and} \qquad C_2 = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Gluing them together, we obtain an unfolding C of B,

$$C = \begin{pmatrix} 0 & 1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

The diagram $\widehat{S}$ of C is shown on Fig. 9 on the right. □

Remark 6.3. By construction, diagrams of all the unfoldings described in the section are block-decomposable. This proves the first statement of Theorem 6.1.

□

6.2 Matrices with s-decomposable diagrams

Now consider arbitrary skew-symmetrizable matrix B with s-decomposable diagram S. Let $x_1, \dots, x_n$ be vertices of S. We can assume numbers $d_1, \dots, d_n$ to be coprime (otherwise, divide all of them by the common divisor). Take any block decomposition of S.

Lemma 6.4. For any two blocks S_1 and S_2 and any outlets $x_i \in S_1$ and $x_j \in S_2$ the numbers d_i and d_j are equal.

□

Proof. Looking at the list of blocks, it is easy to see that for any block S' and any matrix B' representing this block all outlets in S' have the same numbers d'_i, where d'_i are entries of diagonal matrix D' skew-symmetrizing B'. Further, for any x_i the number d_i is a product of d'_i and some number $d(S')$ which is the same for all vertices of S'. Thus, any two outlets in one block of S_{dec} have the same d_i. Now we are left to observe that for any outlets $x_i, x_j \in S_{\text{dec}}$ there exists a sequence of outlets $x_{i_1} = x_i, x_{i_2}, \dots, x_{i_k} = x_j$, such that any two consecutive entries belong to one block.

■

Given S, B, and block decomposition of S, Lemma refequal allows us to define the *weight* of S_{dec} as the number $w = d_i$ for any outlet x_i of any block. We call by a *regular part* of S_{dec} a union of blocks represented either by skew-symmetric matrices, or by matrices admitting a local unfolding. Regular part may not be connected, and every connected component of regular part always admits a local unfolding. Blocks of regular part are called *regular blocks*. The union of blocks admitting no local unfolding is called *irregular part* of S_{dec}. Blocks of this part are *irregular blocks*.

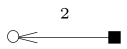
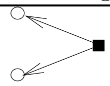
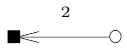
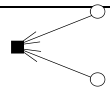
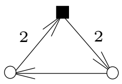
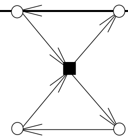
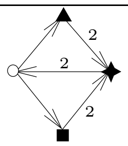
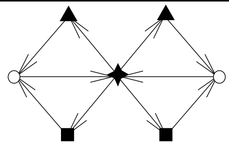
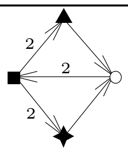
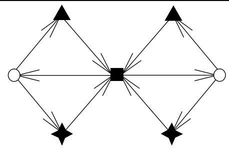
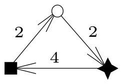
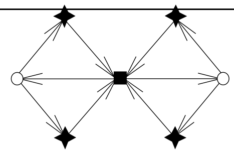
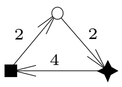
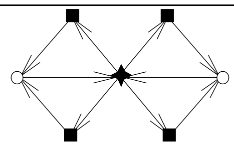
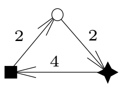
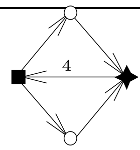
Lemma 6.5. Either $w = 1$ and B admits a local unfolding, or $w = 2$.

□

Proof. If $w = 1$ then we are in assumptions of previous section, so B admits a local unfolding. Now suppose that $w > 1$. Looking at the list of blocks (see Table 1), we see that w is at most two times larger than the minimal value of d_i. Moreover, all d_i are powers of two. In view of GCD equal to one, this implies that the minimal value is also one, so $w = 2$.

■

Table 5. Unfoldings of irregular blocks

Block number	Diagram	Matrix	Unfolding	Diagram unfolding
$\widetilde{\mathrm{III}}$a		$\begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}$	
$\widetilde{\mathrm{III}}$b		$\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	
$\widetilde{\mathrm{IV}}$		$\begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 2 \\ 1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 & 0 \end{pmatrix}$	
$\widetilde{\mathrm{V}}_1$		$\begin{pmatrix} 0 & 1 & -2 & 1 \\ -1 & 0 & 2 & 0 \\ 1 & -1 & 0 & -1 \\ -1 & 0 & 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	
$\widetilde{\mathrm{V}}_2$		$\begin{pmatrix} 0 & 1 & -1 & 1 \\ -2 & 0 & 1 & 0 \\ 2 & -1 & 0 & -1 \\ -2 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	
$\widetilde{\mathrm{V}}_{12}$		$\begin{pmatrix} 0 & 1 & -1 \\ -2 & 0 & 1 \\ 4 & -2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$	
$\widetilde{\mathrm{V}}_{12}$		$\begin{pmatrix} 0 & 2 & -4 \\ -1 & 0 & 2 \\ 1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & 0 \end{pmatrix}$	
$\widetilde{\mathrm{V}}_{12}$		$\begin{pmatrix} 0 & 1 & -2 \\ -2 & 0 & 2 \\ 2 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 & -2 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 2 & -1 & -1 & 0 \end{pmatrix}$	

Now we construct unfoldings for all matrices representing irregular blocks. The proof of the following lemma is straightforward.

Lemma 6.6. The third column of Table 5 contains all possible matrices representing irregular blocks. Matrices in the fourth column are unfoldings of ones on the left.

□

From now on we can assume $w = 2$. We will use matrices from Table 5 together with local unfoldings (see Table 4) as a construction set for the following procedure. In the case the matrix has s-decomposable diagram containing only regular blocks and irregular blocks of types $\widetilde{\mathrm{V}}_{12}$ (listed in the last row of Table 5) and $\widetilde{\mathrm{III}}$, the procedure gives rise to an unfolding. We will generalize this construction and prove the existence of unfoldings in [7] using a geometric description in terms of triangulations of underlying orbifolds.

We describe the procedure in terms of diagrams, then it can be easily translated to the language of matrices.

First, for each connected component S' of regular part we take its local unfolding $\widehat{S'}$. Then we take two copies of $\widehat{S'}$ and paint one of them in black, and the other in red. Now, looking at the list of unfoldings of irregular blocks (Table 5) one can note the following two properties: in all but one block there is exactly one vertex x_i with $d_i = 1$ (the exception is the last one, where unfolding contains two such vertices x_i and y_i), and the unfolding consists of two similar blocks (of type I, II, or IV) glued along x_i (or x_i and y_i). In other words, blocks contained in the unfolding of irregular part form pairs.

Therefore, we can do the following. For each irregular block S'' we take the corresponding unfolding $\widehat{S}''$ from Table 5, and paint one half of it (which is a skew-symmetric block) in black, and the other in red (we are interested in the color of outlets only, so the vertices x_i and y_i may remain uncolored). Now for every irregular block S'' and every outlet $x \in S''$, glue the unfolding $\widehat{S}''$ to red copy of the regular part of S_{dec} along red copy of $\widehat{x}$, and to black copy of the regular part of S_{dec} along black copy of $\widehat{x}$. In this way we get a diagram $\widehat{S}$. Performing the same operations with corresponding matrices, we obtain a matrix C.

Example 6.7. We show an example of a non-local unfolding provided by the construction above. Consider a diagram S shown on Fig. 10, left, with block decomposition shown at the center of the figure.

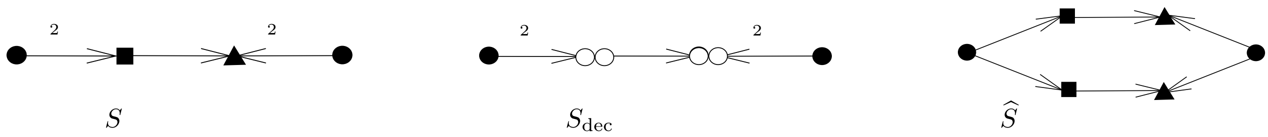


Fig. 10. Diagram S with block decomposition S_{dec} and non-local unfolding $\widehat{S}$

Let both blocks S_1 and S_3 of type $\widetilde{\mathrm{III}}$ be irregular. Then the corresponding matrices are

$$B_1 = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \qquad \text{and} \qquad B_3 = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$$

The regular part B_2 with diagram S_2 consists of skew-symmetric matrix

$$B_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The matrix B representing S will look like

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Unfoldings of B_1 and B_3 are

$$C_1 = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad C_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

Gluing two copies of regular part with C_1 and C_3, we obtain the matrix

$$C = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

The diagram $\widehat{S}$ of C is shown on Fig. 10 on the right. A direct verification by checking all mutations in the complete mutation class shows that C is an unfolding of B. □

6.3 Matrices with non-decomposable diagrams

According to Theorem 5.13, the number of mutation-finite matrices with non-decomposable diagrams is finite, and the number of mutation classes is small. In Table 6 we present unfoldings for all matrices with non-decomposable mutation-finite diagrams. The straightforward proof makes use of Keller's `Java` applet [13] and elementary `C++` code [8].

Remark 6.8. As we can see from Table 6, all the unfoldings constructed are mutation-finite. Together with Remark 6.3, this completes the proof of Theorem 6.1.

□

Table 6. Unfoldings of matrices with non-decomposable mutation-finite diagrams

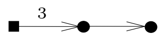
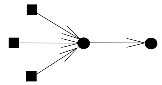
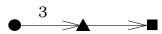
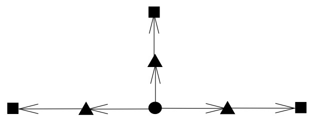
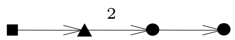
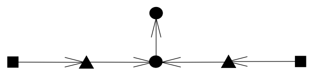
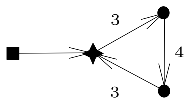
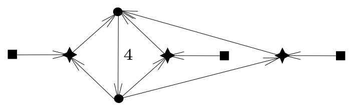
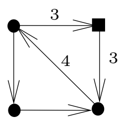
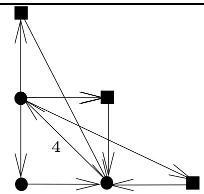
Diagram		Matrix	Unfolding	Diagram unfolding	Mutation class of the unfolding
$\widetilde{G}_2$		$\begin{pmatrix} 0 & 3 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$		block-decomposable
$\widetilde{G}_2$		$\begin{pmatrix} 0 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$		$\widetilde{E}_6$
F_4		$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$		E_6
$G_2^{(*,+)}$ ($G_2^{(1,3)}$ or $G_2^{(3,1)}$)		$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 3 & -3 \\ 0 & -1 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & -2 & 0 \end{pmatrix}$		$E_6^{(1,1)}$
$G_2^{(*,*)}$ ($G_2^{(3,3)}$)		$\begin{pmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & -1 & 0 \\ -2 & 1 & 0 & 1 \\ 3 & 0 & -3 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 2 & -1 & -1 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$		block-decomposable

Table 6. Cont.

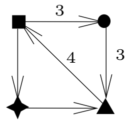
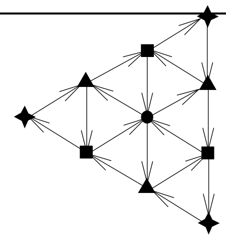
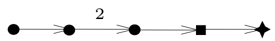
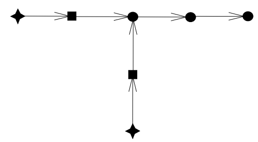
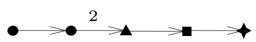
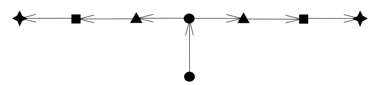
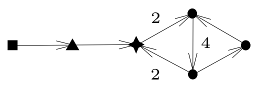
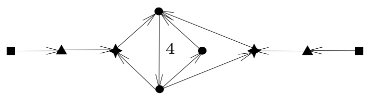
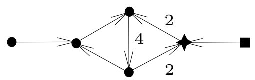
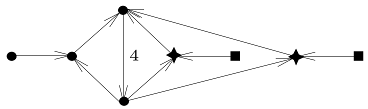
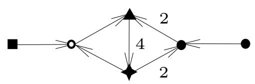
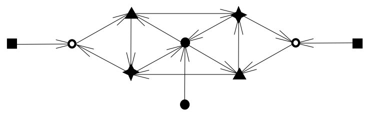
Diagram		Matrix	Unfolding	Diagram unfolding	Mutation class of the unfolding
$G_2^{(*,*)}$ $(G_2^{(1,1)})$		$\begin{pmatrix} 0 & -1 & 2 & -3 \\ 1 & 0 & -1 & 0 \\ -2 & 1 & 0 & 3 \\ 1 & 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \end{pmatrix}$		$E_8^{(1,1)}$
$\widetilde{F}_4$		$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$		$\widetilde{E}_6$
$\widetilde{F}_4$		$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$		$\widetilde{E}_7$

Table 6. Cont.

Diagram		Matrix	Unfolding	Diagram unfolding	Mutation class of the unfolding
$F_4^{(*,+)}$ $(F_4^{(1,2)}$ or $F_4^{(2,1)})$		$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & -2 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$		$E_7^{(1,1)}$
$F_4^{(*,*)}$ $(F_4^{(2,2)})$		$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 2 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$		$E_6^{(1,1)}$
$F_4^{(*,*)}$ $(F_4^{(1,1)})$		$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 2 & -2 & 0 \\ 0 & 1 & -2 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$		$E_8^{(1,1)}$

7 Triangulations of bordered surfaces and s-decomposable diagrams

In this section we discuss relations between s-decomposable diagrams and triangulations of bordered surfaces. In Section 6, we have shown that for any s-decomposable diagram S there is a matrix admitting an unfolding with a block-decomposable diagram $\widehat{S}$. Abusing notation, we will call the original matrix B (resp., diagram S) *folding* of C (resp, $\widehat{S}$). Every time we use notion of folding we keep in mind a fixed unfolding. Further, if a vertex x of s-decomposable diagram S corresponds to vertices $x_1, \dots, x_k$ of its unfolding $\widehat{S}$ we say that x is a *folding* of $x_1, \dots, x_k$, and mutation of S in the vertex x is called the *folding* of the composite mutation $\widehat{\mu}_x$, which is a k-tuple of corresponding mutations of $\widehat{S}$ in vertices $x_1, \dots, x_k$.

As we mentioned above block-decomposable diagrams are in one-to-one correspondence with adjacency matrices of arcs of ideal tagged triangulations of bordered two-dimensional surfaces with marked points. Below we identify diagram of unfolding (with fixed block decomposition) and the corresponding triangulation. We refer to [4] for background on tagged triangulations.

New blocks of types $\widetilde{\text{III}} - \widetilde{\text{VI}}$ admit local unfoldings into block-decomposable diagrams shown in Table 4. These unfoldings are in one-to-one correspondence with the triangulations shown on Figure 7. The last one is a tagged triangulation of a sphere (the exterior is also a triangle). The others are tagged triangulations of a disk.

Remark 7.1. The triangulation corresponding to the local unfolding of block $\widetilde{\text{VI}}$ has no decomposition into surfaces representing blocks of type $\text{I} - \text{V}$, and thus, does not correspond to any block decomposition of the unfolding diagram. Therefore, this triangulation occurs to be an exclusion from the theory derived in [4].

In fact, similarly to block $\widetilde{\text{VI}}$, its local unfolding diagram has no outlets, so it cannot be used in any construction of further diagrams. This is the reason the authors of [4] have made no use of that diagram as a block. For completeness of our theory, it is convenient to define the local unfolding of block $\widetilde{\text{VI}}$ (see Table 1 or 4) to be a skew-symmetric block of type VI.

□

Note that any such local unfolding (except the last one) corresponds to the triangulation with two edges inside a digon (or monogon) representing the same isotopy class: one tagged plain and the other tagged notched. Let us call such pair of edges *conjugate*. Conjugate pair of edges represents two vertices of the unfolding diagram whose folding in s-decomposable diagram is exactly one vertex. Mutation of the folding vertex corresponds to the flips of the both edges from the conjugate pair. These flips do commute, and as a result we obtain again a triangulation where the corresponding edges form a conjugate pair.

Similar to the notion of composite mutation for an unfolding diagram, we define a *composite flip* of a triangulation corresponding to an unfolding diagram as a collection of flips in all edges representing vertices whose folding is the same vertex. An example of a composite flip is a sequence of two flips in conjugate edges. Note that individual flips in a composite flip always mutually commute.

Given an s-decomposable diagram S_{dec} with fixed block decomposition (different from block $\widetilde{\text{VI}}$), the considerations above allow us to construct a unique tagged triangulation of a marked bordered surface with chosen tuple of conjugate pairs. This surface (with triangulation) can be obtained by gluing of surfaces corresponding to local unfoldings of blocks of S_{dec}, and we mark every conjugate pair that corresponds to one vertex in S. This construction is invariant under mutations of S: mutating S, the corresponding triangulation can be obtained from the initial one by corresponding composite flips.

Conversely, looking at tagged triangulations containing conjugate pairs (different from block VI), one can easily see that every conjugate pair lies either inside a digon, or inside a monogon. Recalling the definition of block-decomposable diagram, this implies that the first case corresponds to blocks of types III and IV, and the latter corresponds to blocks of type V (in this case there is another conjugate pair inside the same monogon). In other words, every such triangulation with arbitrary chosen tuple of conjugated pairs of edges can be obtained via local unfolding from some s-decomposable skew-symmetrizable diagram.

Furthermore, every such triangulation with chosen conjugate pairs may come from a unique s-decomposable diagram (with fixed block decomposition) only. Indeed, given a triangulation, there is a unique way to distribute triangles, digons and monogons amongst blocks, which implies uniqueness of block decomposition of folding.

The case of block $\widetilde{\text{VI}}$ can be easily treated separately. Folding one of the three conjugate pairs of the triangulation corresponding to block VI leads to the diagram of block $\widetilde{\text{VI}}$.

Summarizing the discussion above, we come to the following statement.

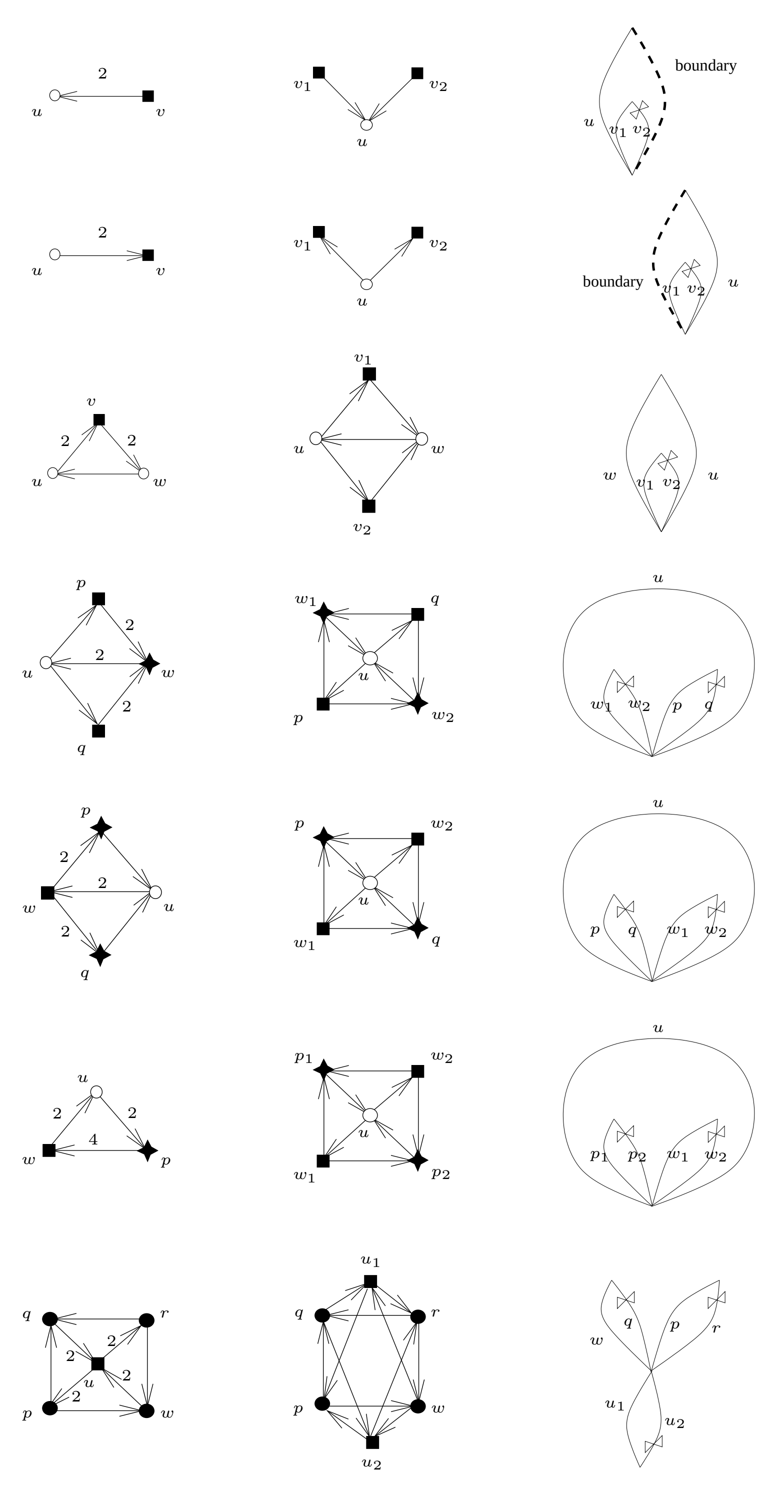
Theorem 7.2. There is a one-to-one correspondence between s-decomposable skew-symmetrizable diagrams with fixed block decomposition and ideal tagged triangulations of marked bordered surfaces with fixed tuple of conjugate pairs of edges.

□

The correspondence above is invariant under mutations: mutating a skew-symmetrizable diagram, the corresponding triangulation can be obtained from the initial one by corresponding composite flips.

Table 7. Triangulations of blocks corresponding to local unfoldings

Diagram	Unfolding	Triangulation



8 Minimal non-decomposable diagrams

In this section we provide a polynomial-time criterion for a diagram to be mutation-finite by proving Theorem 8.3. The considerations are identical to ones used in [5, Section 7].

Definition 8.1. A *minimal mutation-infinite diagram* S is a diagram that

- has infinite mutation class;

- any proper subdiagram of S is mutation-finite.

□

Any minimal mutation-infinite diagram is connected. Notice that the property to be minimal mutation-infinite is not mutation invariant. Note also that minimal mutation-infinite diagram of order at least 4 does not contain edges of multiplicity greater than 4.

We will deduce the criterion from the following lemma.

Lemma 8.2. Any minimal mutation-infinite diagram contains at most 10 vertices.

□

Proof. Let S be a minimal mutation-infinite diagram.

First, we prove a weaker statement, i.e. we show that $|S| \leq 11$. In fact, this bound follows immediately from Theorems 5.1 and 5.6. Indeed, either all the proper subdiagrams of S are block-decomposable, or S contains a proper mutation-finite non-decomposable subdiagram of order $|S|-1$ (we can assume that this diagram is connected: if it is not connected but non-decomposable, it contains a non-decomposable connected component S_0, and any connected subdiagram of S of order $|S|-1$ containing S_0 is non-decomposable). In the former case $|S| \leq 7$ according to Theorem 5.1 (again, we emphasize that we did not require S to be mutation-finite in the assumptions of Theorem 5.1). In the latter case $|S|-1 \leq 10$ due to Theorem 5.6, which proves inequality $|S| \leq 11$.

Now suppose that $|S| = 11$. Then S contains a proper finite mutational non-decomposable subdiagram S' of order 10. According to Theorem 5.6, S' is mutation-equivalent to $E_{10}^{(1,1)}$. The mutation class of $E_{10}^{(1,1)}$ consists of 5739 diagrams, which can be easily computed using Keller's `Java` applet [13]. In other words, we see that S contains one of 5739 diagrams of order 10 as a proper subdiagram.

Hence, we can list all minimal mutation-infinite diagrams of order 11 in the following way. To each of 5739 diagrams above we add one vertex in all possible ways (we can do that since the weight of edge is bounded by 4; the sources codes can be found in [8]). For every obtained diagram we check whether all its proper subdiagrams of order 10 (and, therefore, all the others) are mutation-finite. However, the resulting set of the procedure above is empty: every obtained diagram has at least one mutation-infinite subdiagram of order 10, so it is not minimal.

■

As a corollary of Lemma 8.2, we get the criterion for a diagram to be mutation-finite.

Theorem 8.3. A diagram S of order at least 10 is mutation-finite if and only if all subdiagrams of S of order 10 are mutation-finite.

□

Proof. According to Definition 8.1, every mutation-infinite diagram contains some minimal mutation-infinite diagram as a subdiagram. Thus, a diagram is mutation-finite if and only if it does not contain any minimal mutation-infinite subdiagram. By Lemma 8.2, this holds if and only if all subdiagrams of order at most 10 are mutation-finite. Since a subdiagram of a mutation-finite diagram is also mutation-finite, the latter condition, in its turn, holds if and only if all subdiagrams of order 10 are mutation-finite, which completes the proof.

■

Remark 8.4. The bound in Lemma 8.2 is sharp: as it was mentioned in [5], there exist skew-symmetric minimal mutation-infinite diagrams of order 10. □

Reformulating Theorem 8.3 in terms of matrices, we obtain the following result.

Theorem 8.5. A skew-symmetrizable $n \times n$ matrix B, $n \geq 10$, has finite mutation class if and only if a mutation class of every principal 10×10 submatrix of B is finite.

□

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