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BOUNDARY BEHAVIOUR OF SPECIAL COHOMOLOGY CLASSES ARISING FROM THE WEIL REPRESENTATION

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Abstract In our previous paper [J. Funke and J. Millson, Cycles with local coefficients for orthogonal groups and vector-valued Siegel modular forms, *American J. Math.* 128 (2006), 899–948], we established a correspondence between vector-valued holomorphic Siegel modular forms and cohomology with local coefficients for local symmetric spaces X attached to real orthogonal groups of type (p, q). This correspondence is realized using theta functions associated with explicitly constructed 'special' Schwartz forms. Furthermore, the theta functions give rise to generating series of certain 'special cycles' in X with coefficients.

In this paper, we study the boundary behaviour of these theta functions in the non-compact case and show that the theta functions extend to the Borel–Sere compactification $\overline{X}$ of X. However, for the $\mathbb{Q}$-split case for signature (p, p), we have to construct and consider a slightly larger compactification, the 'big' Borel–Serre compactification. The restriction to each face of $\overline{X}$ is again a theta series as in [J. Funke and J. Millson, loc. cit.], now for a smaller orthogonal group and a larger coefficient system.

As an application we establish in certain cases the cohomological non-vanishing of the special (co)cycles when passing to an appropriate finite cover of X. In particular, the (co)homology groups in question do not vanish. We deduce as a consequence a sharp non-vanishing theorem for L^2-cohomology.

Keywords: Weil representation; cohomology of locally symmetric spaces

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1. Introduction

The cohomology of arithmetic quotients $X = \Gamma\backslash D$ of a symmetric space D associated with a reductive Lie group G is of fundamental interest in number theory and for the field of automorphic forms. For dual reductive pairs, one can apply the 'geometric theta correspondence' (see below) obtained from the Weil representation to construct cohomology classes on locally symmetric spaces associated with these groups. One very attractive aspect of this method is that the classes obtained in this way often give rise to Poincaré dual forms for geometrically defined, 'special' cycles arising via the embedding $H \hookrightarrow G$ of suitable subgroups H.

Let V be a rational quadratic space of signature (p, q) with, for simplicity, even dimension m. Let $\underline{G} = \mathrm{SO}(V)$ and let $G = \underline{G}(\mathbb{R})_0 = \mathrm{SO}_0(V_{\mathbb{R}})$. Let $D_V = D = G/K$ be the symmetric space of G of dimension pq with K a maximal compact subgroup. We let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the associated Cartan decomposition of the Lie algebra of G.

Every partition λ of a non-negative integer ℓ' into at most n parts gives rise to a dominant weight λ of $\mathrm{GL}(n)$. We write $i(\lambda)$ for the number of non-zero entries of λ. We explicitly realize the corresponding irreducible representation of highest weight λ as the image $\mathbb{S}_\lambda(\mathbb{C}^n)$ of the Schur functor $\mathbb{S}_\lambda(\cdot)$ associated with λ applied to the tensor space $T^{\ell'}(\mathbb{C}^n)$. We can apply the same Schur functor to $T^{\ell'}(V_{\mathbb{C}})$ to obtain the space $\mathbb{S}_\lambda(V_{\mathbb{C}})$, and the harmonic ℓ'-tensors in $\mathbb{S}_\lambda(V_{\mathbb{C}})$ give the irreducible representation $\mathbb{S}_{[\lambda]}(V_{\mathbb{C}})$ for G with highest weight $\widetilde{\lambda}$ (under some restrictions). If $i(\lambda) \leqslant [\frac{m}{2}]$, then $\tilde{\lambda}$ has the same non-zero entries as λ (when $\tilde{\lambda}$ is expressed in coordinates relative to the standard basis $\{\epsilon_i\}$ of [7], Planche II and IV).

The Weil representation induces an action of $\mathrm{Sp}_n(\mathbb{R}) \times \mathrm{O}(V_{\mathbb{R}})$ on $\mathcal{S}(V_{\mathbb{R}}^n)$, the Schwartz functions on $V_{\mathbb{R}}^n$. The main point of our previous paper [11] is the construction of certain $(\mathfrak{g}, K)$-cocycles

$$\varphi^V_{nq,[\lambda]} \in \Big[\bigwedge\nolimits^{nq}(\mathfrak{p}^*_{\mathbb{C}}) \otimes \mathcal{S}(V^n_{\mathbb{R}}) \otimes \mathbb{S}_{[\lambda]}(V_{\mathbb{C}})\Big]^K$$

with values in $\mathcal{S}(V^n_{\mathbb{R}}) \otimes \mathbb{S}_{[\lambda]}(V_{\mathbb{C}})$. These classes generalize the work of Kudla and Millson (e.g. [22]) to the case of non-trivial coefficient systems $\mathbb{S}_{[\lambda]}(V_{\mathbb{C}})$. The cocycle $\varphi^V_{nq,[\lambda]}$ corresponds to a closed differential nq-form $\tilde{\varphi}^V_{nq,[\lambda]}$ on D with values in $\mathcal{S}(V^n_{\mathbb{R}}) \otimes \mathbb{S}_{[\lambda]}(V_{\mathbb{C}})$. For $\mathcal{L}$ a coset of a lattice in V^n, we define the theta distribution $\Theta_{\mathcal{L}} = \sum_{\ell \in \mathcal{L}} \delta_\ell$, where δ_ℓ is the delta measure concentrated at ℓ. It is obvious that $\Theta_{\mathcal{L}}$ is invariant under $\mathrm{Stab}(\mathcal{L}) \subset G$. Hence we can apply $\Theta_{\mathcal{L}}$ to $\tilde{\varphi}^V_{nq,[\lambda]}$ to obtain

$$\theta_{\varphi^V_{nq,[\lambda]}}(\mathcal{L}) = \langle \Theta_{\mathcal{L}}, \tilde{\varphi}^V_{nq,[\lambda]} \rangle,$$

which gives a closed nq-form on the finite volume quotient $X = \Gamma \backslash D$ with values in (the local system associated with) $\mathbb{S}_{[\lambda]}(V_{\mathbb{C}})$. Here $\Gamma \subseteq \mathrm{Stab}(\mathcal{L})$ is a congruence subgroup. Furthermore, it is shown in [11] that $\theta_{\varphi^V_{nq,[\lambda]}}$ also gives rise to a non-holomorphic vector-valued Siegel modular form for the representation $S_\lambda(\mathbb{C}^n) \otimes \det^{m/2}$ on the Siegel space $\mathbb{H}_n$. We may then use $\theta_{\varphi^V_{nq,[\lambda]}}$ as the integral kernel of a pairing of Siegel modular forms f with (closed) differential $(p-n)q$-forms η or nq-chains (cycles) C in X. The resulting pairing in f, η (or C), and (possibly different) Schwartz cocycles φ we call the *geometric theta correspondence*.

Special cycles Z_U arise from the embedding $G_U \hookrightarrow G$ of the stabilizer of a positive definite rational subspace $U \subset V$ of dimension n. Hence G_U is an orthogonal group of signature $(p-n, q)$. The special cycles Z_U for varying U give rise to a family of composite cycles Z_T parametrized by symmetric positive definite integral $n \times n$ matrices T. We obtain (by Poincaré duality) classes $[Z_T]$ in $H^{nq}(X, \mathbb{Z})$, and in [11] we explain how to attach $\mathbb{S}_{[\lambda]}(V_{\mathbb{C}})$-coefficients to the cycles to obtain classes

$$[Z_{T,[\lambda]}] \in \mathbb{S}_\lambda(\mathbb{C}^n)^* \otimes H^{nq}(X, \mathbb{S}_{[\lambda]}(V_{\mathbb{C}})). \tag{1.1}$$

Then the main result in [11] is that

$$[\theta_{\varphi^V_{nq,[\lambda]}}] = \sum_{T\geqslant 0} [Z_{T,[\lambda]}] e^{2\pi i \operatorname{tr}(T\tau)} \tag{1.2}$$

is a holomorphic vector-valued Siegel modular form with values in $H^{nq}(X, \mathbb{S}_{[\lambda]}(V_{\mathbb{C}}))$. Here $\tau \in \mathbb{H}_n$. (We omit the definition of $[Z_{T,[\lambda]}]$ for T semi-definite.) This result gives further justification to the term geometric theta correspondence.

Recently, it has been shown [2] that, for $2n < m - [m/2] - 1$, the geometric theta correspondence using the Schwartz cocycle $\varphi = \varphi^V_{nq,[\lambda]}$ induces on the adelic level an *isomorphism* from the appropriate space of (direct limits of) classical Siegel modular forms to $H^{n\times q}_{cusp}(X, \mathbb{S}_{[\lambda]}(V_{\mathbb{C}}))$, the subspace of $H^{nq}_{cusp}(X, \mathbb{S}_{[\lambda]}(V_{\mathbb{C}}))$ corresponding to the special refined Hodge type $n \times q$ (see [2]). In particular, for any congruence quotient, the cohomology groups $H^{n\times q}_{cusp}(X, \mathbb{S}_{[\lambda]}(V_{\mathbb{C}}))$ are spanned by the cuspidal projections of the Poincaré duals of special cycles. We note that for the case of hyperbolic space the space $H^{n\times q}_{cusp}(X, \mathbb{S}_{[\lambda]}(V_{\mathbb{C}}))$ coincides with the entire cuspidal cohomology group $H^{nq}_{cusp}(X, \mathbb{S}_{[\lambda]}(V_{\mathbb{C}}))$ and for the case of signature $(p, 2)$ it coincides with the usual cuspidal Hodge summand $H^{nq,nq}_{cusp}(X, \mathbb{S}_{[\lambda]}(V_{\mathbb{C}}))$. Thus, in the first case, the cuspidal projections of special cycles span the entire cuspidal cohomology and, in the second case, the span of cuspidal projections of the special cycles (in this case the special cycles are algebraic cycles) coincides with the span of the cuspidal projections of *all* algebraic cycles and furthermore this span is the entire cuspidal cohomology group of type (nq, nq). Thus, the results of [2] highlight the importance of the geometric theta correspondence which we analyse here.

It is therefore very natural to study $\theta_{\varphi^V_{nq,[\lambda]}}$ for non-compact X, and in particular to analyse its boundary behaviour. This is what we do in this paper.

We let $P = \underline{P}(\mathbb{R})_0$ be the connected component of the identity of the real points of a rational parabolic subgroup $\underline{P}$ in $\underline{G}$ stabilizing a flag $\mathbf{F}$ of totally isotropic rational subspaces in V. Conversely, for signature different to (p, p) all such flags give rise to a unique rational parabolic. Then the Borel–Serre compactification $\overline{X}$ compactifies X by adding to each rational P a face $e'(P)$, which is a nilmanifold bundle over a suitable quotient of the symmetric space associated with the semi-simple part of the Levi subgroup of P; see [5, 4]. This makes $\overline{X}$ into a manifold with corners.

However, for the $\mathbb{Q}$-split case in signature (p, p), the rational parabolics are not in 1–1 correspondence with the stabilizers of rational totally isotropic flags in V (but rather with so-called oriflammes). This turns out to be a critical issue for us. To remedy this we consider instead the spherical building of proper rational parabolic subgroups for the full (non-connected) orthogonal group $\mathrm{O}(p, p)$ instead. The space X does not change, but now isotropic flags do parametrize parabolics. The resulting compactification we call the *big* Borel–Serre compactification of X since it is (slightly) bigger. By abuse of notation we denote the big compactification also by $\overline{X}$. For an alternative construction of the big $\overline{X}$, we embed $X = X_{p,p}$ into a locally symmetric space $X_{p+1,p}$ for signature $(p+1, p)$ and then consider the closure of $X_{p,p}$ in $\overline{X}_{p+1,p}$.

To illustrate the big Borel–Serre compactification, we consider the split case for $\mathrm{SO}(2, 2)$, when $X = X_1 \times X_2$ is the product of two modular curves. Then the Borel–Serre compactification of X is the product of the two individual compactifications $\overline{X}_1 \times \overline{X}_2$ which adds to each cusp of the modular curves a circle S_1. Hence the corner at the cusp $(z_1, z_2) = (i\infty, i\infty)$ of X is given by a 2-torus T^2. Then the big Borel–Serre compactification of X blows up the corner to $T^2 \times \mathbb{R}_+$ with the new coordinate $\mathrm{Im}(z_1)/\mathrm{Im}(z_2) \in \mathbb{R}_+$ measuring the 'slope' by which one enters the corner from the interior. We explain the details of the big Borel–Serre compactification in § 10.

Let E be the largest element in the rational isotropic flag $\mathbf{F}$ with dimension ℓ corresponding to $\underline{P}$. Set $W = E^{\perp}/E$, which is naturally a quadratic space of signature $(p-\ell, q-\ell)$. Then a suitable arithmetic quotient X_W of the symmetric space D_W associated with W occurs as a factor in the base of the nilmanifold bundle $e'(P)$.

The main result of this paper is the following.

Theorem 1.1. (1) *The form $\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]})$ extends to a smooth differential form on the (big) Borel–Serre compactification $\overline{X}$ considered as a smooth manifold with corners. In fact, a stronger result holds. For each face $e'(P)$, there exists a neighbourhood U'_P in $\overline{X}$ such that the restriction of $\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]})$ to U'_P is the sum of a rapidly decreasing form and a special differential form in the sense of [16], p. 169, on U'_P.*

(2) *For a given face $e'(P)$, let $\tilde{r}_P$ be the restriction map from $\overline{X}$ to $e'(P)$. Then there exists a theta distribution $\widehat{\mathcal{L}}_W$ for W such that*

$$[\tilde{r}_P(\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]}))] = [\widetilde{\iota}_P(\theta_{\widehat{\mathcal{L}}_W}(\varphi^W_{n(q-\ell),[\ell\varpi_n+\lambda]}))].$$

Here $\widetilde{\iota}_P$ is an embedding

$$\widetilde{\iota}_P : H^{n(q-\ell)}(X_W, \mathbb{S}_{[\ell\varpi_n+\lambda]}(W_{\mathbb{C}})) \hookrightarrow H^{nq}(e'(P), \mathbb{S}_{[\lambda]}(V_{\mathbb{C}})),$$

where $\varpi_n = (1, \ldots, 1)$ is the n th fundamental weight for $\mathrm{GL}(n)$, so the Young diagram associated with $\ell\varpi_n$ is an n by ℓ rectangle.

In particular, $[\tilde{r}_P(\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]}))] = 0$ for $n > \min\left(p, \left[\frac{m}{2}\right]\right) - \ell$ (if $\ell \geqslant 2$) and $n > p-1$ or $n > m-2-i(\lambda)$ (if $\ell = 1$).

Loosely speaking, the restriction formula of Theorem 1.1 can be summarized by saying that the restriction of our theta series for $\mathrm{SO}(V)$ to a face of $\overline{X}$ is the theta series for $\mathrm{SO}(W)$ of the same type corresponding to an enlarged coefficient system given by placing an n by ℓ rectangle on the left of the Young diagram corresponding to λ to obtain a bigger Young diagram corresponding to $\ell\varpi_n + \lambda$. The theta series $\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]})$ is termwise moderately increasing, so the statement of the theorem is rather delicate. To capture the boundary behaviour we switch to a mixed model of the Weil representation.

As stated above, for the split $\mathrm{SO}(p,p)$-case, the differential form $\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]})$ does *not* extend to the usual Borel–Serre boundary.

Remark 1.2. We can also interpret our result in terms of weighted cohomology [16]. More precisely, let $r \leqslant \min(p,q)$ be the non-zero $\mathbb{Q}$-rank of $\underline{G}$. Then our forms $\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]})$ define classes in the cohomology for the weight profile associated with the weight $\widetilde{n\varpi_r}$ for $\underline{G}$. In particular, the forms are square integrable if

$$2n+1 < p+q-r.$$

Non-vanishing at the boundary and sharp non-vanishing of the L^2-cohomology. As an easy and direct application of Theorem 1.1 we obtain a non-vanishing result for the special (co)cycles.

Theorem 1.3. *Assume that the $\mathbb{Q}$-rank and the $\mathbb{R}$-rank of $\underline{G}$ coincide. Then for*

$$i(\lambda) \leqslant n \leqslant \begin{cases} \left[\dfrac{p-q}{2}\right] & \text{if } q \geqslant 2 \\ p-1-i(\lambda) & \text{if } q = 1, \end{cases}$$

there exists a finite cover X' of X such that

$$[\theta(\varphi^V_{nq,[\lambda]})] \neq 0.$$

Using (1.2) this gives $[Z_{T,[\lambda]}] \neq 0$ for infinitely many T. In particular, (for $n \neq 0$)

$$H^{nq}(X', \mathbb{S}_{[\lambda]}(V_{\mathbb{C}})) \neq 0.$$

Finally, for the case of trivial coefficients, $H^{nq}(X', \mathbb{C})$ is not spanned by classes given by invariant forms on D.

Combining this with Remark 1.2 we obtain the following corollary.

Corollary 1.4. *Under the hypotheses of Theorem 1.3 and $n < (p-1)/2$ if $q = 1$, both the L^2-cohomology group $H^{nq}_{(2)}(X', \mathbb{S}_{[\lambda]}(V_{\mathbb{C}}))$ and its image $H^{nq}(X', \mathbb{S}_{[\lambda]}(V_{\mathbb{C}}))^{(2)}$ in $H^{nq}(X', \mathbb{S}_{[\lambda]}(V_{\mathbb{C}}))$ do not vanish.*

The basic idea for the proof of Theorem 1.3 is to study the restriction to a face of $\overline{X}$ associated with a minimal rational parabolic subgroup. At such a face, the space W is positive definite, and hence the restriction becomes a positive definite theta series for which we establish non-vanishing.

There are numerous non-vanishing results in the literature, and we mention a few related ones. In the case of non-trivial coefficients for compact hyperbolic manifolds, Millson [27] proved the non-vanishing of the special cycles with coefficients in codimension n in the range $i(\lambda) \leqslant n \leqslant p - i(\lambda)$. Li [25] used the theta correspondence to establish non-vanishing for the cohomology of orthogonal groups, again in the compact (or L^2) case (without giving a geometric interpretation of the classes). Speh and Venkataramana [31] gave in general a criterion for the non-vanishing of certain modular symbols in terms of the compact dual. In contrast to our result, their non-vanishing occurs from classes defined by invariant forms on D.

From Section 9.4 and [32], § 8 it follows that for trivial coefficients all L^2 cohomology classes in degree less than q arise from invariant forms. Hence the lower bound q (known for $p+q \geqslant 8$ even; see [25]) obtained with $n = 1$ in Corollary 1.4 is sharp since the class $[\Theta(\varphi^V_q)]$ cannot be represented by an invariant form.

For non-trivial coefficients, the lower bound $i(\lambda)q$ obtained with $n = i(\lambda)$ (and in addition $2i(\lambda) < p - 1$ if $q = 1$) for non-vanishing of the L^2-cohomology is also sharp, as we now see. We have the following theorem.

Theorem 1.5. *Suppose* $0 < i(\lambda) < [p/2]$ *and* $k < i(\lambda)q$. *Then*

$$H^k\left(X', \mathbb{S}_{[\lambda]}(V_{\mathbb{C}})\right)^{(2)} = H^k_{(2)}(X', \mathbb{S}_{[\lambda]}(V_{\mathbb{C}})) = 0.$$

This indeed implies the sharpness of our L^2-non-vanishing result since we have $i(\lambda) < [\frac{p}{2}]$ in the non-vanishing range of Corollary 1.4.

Vanishing at the boundary and the extension of the cohomological theta lift. We first describe the general main motivation for our work. From (1.1) and (1.2) we see that the theta series $\theta_{\mathcal{L}}(\varphi_{nq,0})$ (for simplicity, we only consider trivial coefficients for the moment) gives rise to a map

$$\Lambda_{nq} : H_c^{(p-n)q}(X, \mathbb{C}) \to M^{(n)}_{m/2}(\Gamma') \tag{1.3}$$

from the cohomology with compact supports to the space of holomorphic Siegel modular forms of degree n of weight $m/2$. We are interested in extending the lift (1.3) to other cohomology groups of the space X which capture its boundary. The present paper should be considered in this context, and is central to our efforts. This programme is in particular motivated by the work of Hirzebruch and Zagier [18], which is the $\mathbb{Q}$-rank 1 case for signature $(2, 2)$ when X is a Hilbert modular surface, and the cycles in question are the famous Hirzebruch–Zagier curves ($n = 1$).

Whenever the restriction of $\theta_{\mathcal{L}}(\varphi_{nq,0})$ to ∂X is cohomologically trivial, then such an extension of the lift exists – at least in principle. Namely, in this case, one can utilize a mapping cone construction to modify $\theta_{\mathcal{L}}(\varphi_{nq,0})$ to represent a class in the compactly supported cohomology of X. However, there is a further problem beyond the homological triviality of the restriction, that is explicitly constructing suitable primitives for the restriction (again using the theta correspondence). Once this second problem is solved one obtains an extension of Λ_{nq} to the full cohomology of X.

We have already carried this out in several instances. First and foremost, the restriction vanishes in the Hirzebruch–Zagier case, and on the basis of this, we give in [12] a new treatment and extension of the results in [18] using the theta correspondence. The $\mathbb{Q}$-rank 2 case when X is the product of two modular curves is of course highly interesting as well. Now the boundary faces in the big Borel–Serre compactification are no longer isolated, and in addition some subtle analytic complications arise when constructing the primitives at the boundary. We will consider this case in the near future.

The case which resembles the Hirzebruch–Zagier one most closely is the one for Picard modular surfaces (quotients of $\mathrm{U}(2, 1)$; the results of this paper generalize to unitary groups). Cogdell [8] considered this case in the spirit of Hirzebruch and Zagier. We will consider this case from our point of view in a subsequent paper.

Another case is $\mathrm{SO}(2, 1)$, when X is a modular curve, and the cycles are geodesics. For *non-trivial* coefficients, the restriction to the boundary vanishes. This case is particularly attractive since one can interpret our classes as (co)homology classes for even powers of the universal elliptic curve. We discussed this case in detail in [13].

Finally, we mention that [10] gives an introductory survey of the results obtained in this paper.

2. Basic notation

2.1. Orthogonal symmetric spaces

Let V be a rational vector space of dimension $m = p + q$ and let $(\ ,\ )$ be a non-degenerate symmetric bilinear form on V with signature (p, q). We fix a standard orthogonal basis $e_1, \dots, e_p, e_{p+1}, \dots, e_m$ of $V_{\mathbb{R}}$ such that $(e_\alpha, e_\alpha) = 1$ for $1 \leqslant \alpha \leqslant p$ and $(e_\mu, e_\mu) = -1$ for $p + 1 \leqslant \mu \leqslant m$. (We will use 'early' Greek letters to denote indices between 1 and p, and 'late' ones for indices between $p + 1$ and m.) With respect to this basis the matrix of the bilinear form is given by the matrix $I_{p,q} = \begin{pmatrix} 1_p & \\ & -1_q \end{pmatrix}$.

We let $\underline{G} = \mathrm{SO}(V)$ viewed as an algebraic group over $\mathbb{Q}$. We let $G := \underline{G}(\mathbb{R})_0$ be the connected component of the identity of $\underline{G}(\mathbb{R})$ so $G \simeq \mathrm{SO}_0(p, q)$. We let K be the maximal compact subgroup of G stabilizing $\mathrm{span}\{e_\alpha; 1 \leqslant \alpha \leqslant p\}$. Thus $K \simeq \mathrm{SO}(p) \times \mathrm{SO}(q)$. Let $D = G/K$ be the symmetric space of dimension pq associated with G. We realize D as the space of negative q-planes in $V_{\mathbb{R}}$:

$$D \simeq \{z \subset V_{\mathbb{R}} : \dim z = q;\ (\ ,\ )|_z < 0\}. \tag{2.1}$$

Thus $z_0 = \mathrm{span}\{e_\mu; p + 1 \leqslant \mu \leqslant m\}$ is the base point of D. Furthermore, we can also interpret D as the space of minimal majorants for $(\ ,\ )$. That is, $z \in D$ defines a majorant $(\ ,\ )_z$ by $(x, x)_z = -(x, x)$ if $x \in z$ and $(x, x)_z = (x, x)$ if $x \in z^\perp$. We write $(\ ,\ )_0$ for the majorant associated with the base point z_0.

The Cartan involution θ_0 of G corresponding to the base point z_0 is obtained by conjugation with the matrix $I_{p,q}$. We will systematically abuse notation below and write $\theta_0(v)$ for the action of the linear transformation of V with matrix $I_{p,q}$ relative to the above basis acting on $v \in V$. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ be that of K. We obtain the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{2.2}$$

where

$$\mathfrak{p} = \mathrm{span}\{X_{\alpha\mu} := e_\alpha \wedge e_\mu; 1 \leqslant \alpha \leqslant p, p + 1 \leqslant \mu \leqslant m\}. \tag{2.3}$$

Here $w \wedge w' \in \bigwedge^2 V_{\mathbb{R}}$ is identified with an element of $\mathfrak{g}$ via

$$(w \wedge w')(v) = (w, v)w' - (w', v)w. \tag{2.4}$$

We let $\{\omega_{\alpha\mu}\}$ be the dual basis of $\mathfrak{p}^*$ corresponding to $\{X_{\alpha\mu}\}$. Finally note that we can identify $\mathfrak{p}$ with the tangent space $T_{z_0}(D)$ at the base point z_0 of D.

We let r be the Witt rank of V, i.e., the dimension of a maximal totally isotropic subspace of V over $\mathbb{Q}$ and assume that $r > 0$. Let F be a totally isotropic subspace of V of dimension ℓ. Then we can describe the ℓ-dimensional isotropic subspace $\theta_0(F)$ as follows. For U a subspace of V, let $U^\perp$ (resp., $U^{\perp_0}$) be the orthogonal complement of U for the form $(\ ,\ )$ (resp., $(\ ,\ )_0$). Then $\theta_0(F) = (F^\perp)^{\perp_0}$. We fix a maximal totally isotropic subspace E_r and choose a basis $u_1, u_2, \dots, u_r$ of E_r. Let $E'_r = \theta_0(E_r)$. We pick a basis $u'_r, \dots, u'_1$ of E'_r such that $(u_i, u'_j) = \delta_{ij}$. More generally, we let

$$E_\ell := \mathrm{span}\{u_1, \dots, u_\ell\}, \tag{2.5}$$

and we call E_ℓ a *standard* totally isotropic subspace. Furthermore, we set $E'_\ell = \theta_0(E_\ell) = \operatorname{span}(u'_\ell, \dots, u'_1)$. Note that E'_ℓ can be naturally identified with the dual space of E_ℓ. We can assume that with respect to the standard basis of $V_{\mathbb{R}}$ we have $e_\alpha = \frac{1}{\sqrt{2}}(u_\alpha + u'_\alpha)$ and $e_{m+1-\alpha} = \frac{1}{\sqrt{2}}(u_\alpha - u'_\alpha)$ for $\alpha = 1, \dots, \ell$. We let

$$W_\ell = E_\ell^\perp / E_\ell, \tag{2.6}$$

and note that W_ℓ is a non-degenerate space of signature $(p-\ell, q-\ell)$. We can realize W_ℓ as a subspace of V through

$$W_\ell = (E_\ell \oplus E'_\ell)^\perp, \tag{2.7}$$

where the orthogonal complement is with respect to either $(\ ,\ )$ or $(\ ,\ )_0$. This gives

$$V = E_\ell \oplus W_\ell \oplus E'_\ell, \tag{2.8}$$

a θ_0-invariant Witt splitting for V. Note that with these choices, θ_0 restricts to a Cartan involution for $\mathrm{O}(W_\ell)$. We obtain a Witt basis $u_1, \dots, u_\ell, e_{\ell+1}, \dots, e_{m-\ell}, u'_\ell, \dots, u'_1$ for $V_{\mathbb{R}}$. We will denote coordinates with respect to the Witt basis with y_i and coordinates with respect to the standard basis with x_i.

We often drop the subscript ℓ and just write E, E', and W.

2.2. Parabolic subgroups

We describe the rational parabolic subgroups of G.

2.2.1. Isotropic flags and parabolic subgroups. We let $\mathbf{F}$ be a flag of totally isotropic subspaces $F_1 \subset F_2 \subset \cdots \subset F_k$ of V over $\mathbb{Q}$. Then we let $\underline{P} = \underline{P}_{\mathbf{F}}$ be the parabolic subgroup of $\underline{G}$ stabilizing the flag $\mathbf{F}$:

$$\underline{P}_{\mathbf{F}} = \{g \in \underline{G};\ gF_i = F_i\}, \tag{2.9}$$

and write $P = P_{\mathbf{F}} = (\underline{P}_{\mathbf{F}}(\mathbb{R}))_0$ for the resulting rational parabolic in G. The first fundamental fact is given in the following lemma.

Lemma 2.1. *Assume that V is not a rational $\mathbb{Q}$-split space of signature (p,p). Then the assignment $\mathbf{F} \mapsto \underline{P}_{\mathbf{F}}$ defines a bijection between the rational totally isotropic flags in V and rational parabolic subgroups in G. Furthermore, under this map isotropic subspaces give rise to maximal parabolics.*

In this situation, we can assume by conjugation that the flag $\mathbf{F}$ consists of standard totally isotropic subspaces E_i (2.5) and call such parabolics *standard* $\mathbb{Q}$-parabolics.

However, if V is a rational $\mathbb{Q}$-split space of signature (p,p), then the map from totally isotropic flags to parabolics is surjective but not 1–1. We need a incidence relation between totally isotropic subspaces more involved than inclusion to describe parabolic subgroups which gives rise to a configuration called oriflammes; see e.g. [14], chapter 11.

Definition 2.2 (oriflammes). We define the incidence relation $\sim$ on non-zero totally isotropic subspaces of V of dimension different to $p-1$ by $F_1 \sim F_2$ if either

(i) $F_1 \subset F_2$ or $F_2 \subset F_1$, or

(ii) if $\dim F_1 = \dim F_2 = p$, then $F_1 \cap F_2$ has dimension $p-1$.

Then an oriflamme is a collection of such subspaces in which any two members are incident.

One then has the following lemma (see e.g. [1, 14], and also Example 10.6).

Lemma 2.3. *Assume that V is a rational $\mathbb{Q}$-split space of signature (p,p). Then the rational parabolic subgroups in G are in 1–1 correspondence with the rational oriflammes in V by taking the stabilizer of the oriflamme. Concretely:*

(1) *The maximal parabolics are attached to totally isotropic subspaces of dimension different to $p-1$. The totally isotropic subspaces of dimension $p-1$ do not give rise to a maximal parabolic.*

(2) *All totally isotropic flags which do not include a constituent of dimension $p-1$ give rise to different standard parabolic subgroups.*

(3) *Let F_{p-1} be a totally isotropic space of dimension $p-1$ and $\mathbf{F} = F_1 \subset F_2 \subset \cdots F_k \subset F_{p-1}$ be a totally isotropic flag. Since $F_{p-1}^{\perp}/F_{p-1}$ is naturally a $\mathbb{Q}$-split space of signature $(1,1)$ there are exactly two totally isotropic spaces $F_{p,1}$, $F_{p,2}$ of (maximal) dimension p which contain F_{p-1}. Then the three flags $\mathbf{F}$, $\mathbf{F} \subset F_{p,1}$, $\mathbf{F} \subset F_{p,2}$ are fixed by the same parabolic in G. This parabolic fixes the oriflamme $(F_1, F_2, \ldots, F_k, F_{p,1}, F_{p,2})$.*

Let $E_+ = E_p = \operatorname{span}(u_1, \ldots, u_{p-1}, u_p)$ and $E_- = \operatorname{span}(u_1, \ldots u_{p-1}, u'_p)$. Then we define the standard $\mathbb{Q}$-parabolics to be the ones given by fixing a suboriflamme of the maximal oriflamme $(E_1, E_2, \ldots, E_{n-2}, E_+, E_-)$. We discuss the case where V is a rational $\mathbb{Q}$-split space of signature (p,p) in more detail in § 10.

2.2.2. The Langlands decomposition. We let $\underline{N_P}$ be the unipotent radical of $\underline{P}$. It acts trivially on all quotients of the flag. We let $\underline{L_P} = \underline{N_P}\backslash\underline{P}$ and let $\underline{S_P}$ be the split centre of $\underline{L_P}$ over $\mathbb{Q}$. Note that $\underline{S_P}$ acts by scalars on each quotient. Let $\underline{M_P} = \cap_{\chi \in X(\underline{L_P})} \operatorname{Ker}(\chi^2)$. We let $N = N_P$ and $L = L_P$ be their respective real points in G, and as before we set $M = M_P = (\underline{M_P}(\mathbb{R}))_0$, and $A = A_{\underline{P}} = (\underline{S_P}(\mathbb{R}))_0$. We can realize $\underline{L_P}$ (and also $\underline{S_P}, \underline{M_P}$) as θ_0-stable subgroups of $\underline{P}$:

$$\underline{L_P} = \underline{P} \cap \theta_0(\underline{P}). \tag{2.10}$$

Then $\underline{M_P}$ is the semi-simple part of the centralizer of $\underline{S_P}$ in $\underline{P}$. We will often drop the subscripts $\mathbf{F}$, $\underline{P}$, and P. We obtain the (rational) Langlands decomposition of P:

$$P = NAM \simeq N \times A \times M, \tag{2.11}$$

and we write $\mathfrak{n}$, $\mathfrak{a}$, and $\mathfrak{m}$ for their respective Lie algebras. The map $P \to N \times A \times M$ is equivariant with the P-action defined by

$$n'a'm'(n,a,m) = (n'Ad(a'm')(n), a'a, m'm). \tag{2.12}$$

2.2.3. The Levi. We let $\mathbf{F}$ be a standard rational totally isotropic flag $0 = E_0 \subset E_{i_1} \subset \cdots \subset E_{i_k} = E_\ell = E$ and assume that the last (biggest) totally isotropic space in the flag $\mathbf{F}$ is equal to E_ℓ for some ℓ. The reader will make the necessary adjustments when considering an oriflamme in the $\mathbb{Q}$-split $\mathrm{SO}(p,p)$-case.

Let $U_{i_j} = \mathrm{span}(u_{i_{j-1}+1}, \dots, u_{i_j})$ be the orthogonal complement of $E_{i_{j-1}}$ in E_{i_j} with respect to $(\ ,\ )_0$ and U'_{i_j} be the orthogonal complement of E'_{i_j} in $E'_{i_{j+1}}$ and let $W = W_\ell = (E_\ell \oplus E'_\ell)^\perp$. We obtain a refinement of the Witt decomposition of V such that the subspaces U_{i_j}, U'_{i_s}, and W are mutually orthogonal for $(\ ,\ )_0$ and defined over $\mathbb{Q}$:

$$V = \left(\bigoplus_{i_j=1}^{k} U_{i_j} \right) \oplus W \oplus \left(\bigoplus_{i_j=1}^{k} U'_{i_j} \right). \tag{2.13}$$

Then $\underline{L}_P$ is the subgroup of $\underline{P}$ that stabilizes each of the subspaces in the above decomposition of V. In what follows we will describe matrices in block form relative to the above direct sum decomposition of V. We first note that we naturally have $\mathrm{O}(W) \times \mathrm{GL}(E) \subset \mathrm{O}(V)$ via

$$\left\{ \begin{pmatrix} g & & \\ & h & \\ & & \tilde{g} \end{pmatrix}; h \in \mathrm{O}(W), g \in \mathrm{GL}(E) \right\}, \tag{2.14}$$

where $\tilde{g} = Jg^*J$, $g^* = {}^t g^{-1}$, and $J = \begin{pmatrix} & & 1 \\ & \cdots & \\ 1 & & \end{pmatrix}$. In particular, we can view the corresponding Lie algebras $\mathfrak{o}(W_\mathbb{R})$ and $\mathfrak{gl}(E_\mathbb{R})$ as subalgebras of $\mathfrak{g}$. Namely,

$$\mathfrak{o}(W_\mathbb{R}) \simeq \mathrm{span}\{e_i \wedge e_j;\ \ell < i < j \leqslant m - \ell\}, \tag{2.15}$$

$$\mathfrak{gl}(E_\mathbb{R}) \simeq \mathrm{span}\{u'_i \wedge u_j;\ i, j \leqslant \ell\}, \tag{2.16}$$

via $\mathfrak{g} \simeq \bigwedge^2 V_\mathbb{R}$. We see that

$$\underline{L} \simeq \left\{ \begin{pmatrix} g & & \\ & h & \\ & & \tilde{g} \end{pmatrix}; h \in \mathrm{SO}(W), g = \mathrm{diag}(g_1, \dots, g_k) \in \prod_{j=1}^{k} \mathrm{GL}(U_{i_j}), \right\}. \tag{2.17}$$

We now consider the isotropic flag $\mathbf{F}$ in V as a flag $\mathbf{F}(E)$ of subspaces inside E. We let $\underline{P}'$ be the parabolic subgroup of $\mathrm{GL}(E)$ stabilizing $\mathbf{F}(E)$. Then for the real points $P' = (\underline{P}'(\mathbb{R}))_0$, we have

$$P' = N_{P'} A M_{P'}, \tag{2.18}$$

with unipotent radical $N_{P'}$ and Levi factor

$$M_{P'} = \prod_{j=1}^{k} \mathrm{SL}(U_{i_j}(\mathbb{R})). \tag{2.19}$$

Here A is as above, viewed as a subgroup of $\mathrm{GL}_+(E_\mathbb{R})$. Furthermore, we can view $\underline{P}'$ and its subgroups naturally as subgroups of $\underline{P}$ via the embedding of $\mathrm{GL}(E)$ into $\mathrm{O}(V)$ given by (2.14). We obtain

$$M \simeq \mathrm{SO}_0(W_{\mathbb{R}}) \times M_{P'}. \tag{2.20}$$

We also define

$$\mathfrak{p}_M = \mathfrak{p} \cap \mathfrak{m} = \mathfrak{p}_W \oplus \mathfrak{p}_E, \tag{2.21}$$

where $\mathfrak{p}_E = \mathfrak{sl}(E) \cap \mathfrak{p}$ and

$$\mathfrak{p}_W = \mathfrak{o}_W \cap \mathfrak{p} = \mathrm{span}\{X_{\alpha\mu} = e_\alpha \wedge e_\mu;\, \ell+1 \leqslant \alpha \leqslant p, p+1 \leqslant \mu \leqslant m-\ell\}. \tag{2.22}$$

2.2.4. Roots. We let $\underline{S}$ be the maximal $\mathbb{Q}$-split torus of $\underline{G}$ given by

$$\underline{S} = \left\{ a(t_1, \dots, t_r) := \begin{pmatrix} \mathrm{diag}(t_1, \dots, t_r) & & \\ & 1 & \\ & & \mathrm{diag}(t_r^{-1}, \dots, t_1^{-1}) \end{pmatrix} \right\}. \tag{2.23}$$

Note that $(\underline{S}(\mathbb{R}))_0 = A_{P_0}$, where P_0 is the minimal parabolic contained in all standard parabolics. We write $\mathbf{t} = (t_1, \dots, t_r)$ and $\widetilde{\mathbf{t}} = \mathbf{t}J = (t_r, \dots, t_1)$. Note that $a(0, \dots, 0, 1, 0, \dots, 0) = \exp(u_i' \wedge u_i)$. The set of simple rational roots for $\underline{G}$ with respect to $\underline{S}$ is given by $\Delta = \Delta(\underline{S}, \underline{G}) = \{\alpha_1, \dots, \alpha_r\}$, where

$$\alpha_i(a) = t_i t_{i+1}^{-1}, \quad (1 \leqslant i \leqslant r-1) \tag{2.24}$$

$$\alpha_r(a) = \begin{cases} t_r & \text{if } W_r \neq 0 \\ t_{r-1} t_r & \text{if } W_r = 0. \end{cases} \tag{2.25}$$

We write $\Phi(P, A_{\underline{P}})$ for the positive roots of P with respect to $A_{\underline{P}}$ and $\Delta(P, A_{\underline{P}})$ for the simple roots of P with respect to $A_{\underline{P}}$, which are those $\alpha \in \Delta$ which act non-trivially on $\underline{S_P}$. We let $\underline{Q}$ be the standard maximal parabolic stabilizing the totally isotropic rational subspace E_ℓ of dimension $\ell \leqslant r$. We have $A_{\underline{Q}} = \{a(t, \dots, t, 1, \dots 1)\}$ and $\Delta(Q, A_{\underline{Q}}) = \{\alpha_\ell\}$ except in the $\mathbb{Q}$-split case for $\mathrm{SO}(p,p)$ and where Q stabilizes E_- in which case $A_{\underline{Q}} = \{a(t, \dots, t, t^{-1})\}$ and $\Delta(Q, A_{\underline{Q}}) = \{\alpha_{p-1}\}$. For general P, we have

$$\Delta(P, A_{\underline{P}}) = \{\alpha_{i_1}, \dots, \alpha_{i_k}\}; \tag{2.26}$$

the reader will make the necessary adjustments in the $\mathbb{Q}$-split case for $\mathrm{SO}(p,p)$.

2.2.5. The nilradical. With P and P' as before, we can naturally view $\underline{N_{P'}} \subset \mathrm{SL}(E)$ as a subgroup of $\underline{N_P}$. We then have a semi-direct product decomposition

$$\underline{N_P} = \underline{N_{P'}} \ltimes \underline{N_Q}, \tag{2.27}$$

where $\underline{Q}$ is, as above, the maximal parabolic containing $\underline{P}$. Furthermore, we let $\underline{Z_Q}$ be the centre of $\underline{N_Q} \subseteq \underline{N_P}$. It is given by

$$\underline{Z_Q} = \left\{ z(b) := \begin{pmatrix} 1 & & b \\ & 1 & \\ & & 1 \end{pmatrix};\, J^t b J = -b \right\}. \tag{2.28}$$

Then for the coset space $\underline{N_P}/(\underline{N_{P'}} \ltimes \underline{Z_Q})$, we have

$$\underline{N_P}/(\underline{N_{P'}} \ltimes \underline{Z_Q}) \simeq \underline{N_Q}/\underline{Z_Q} \simeq W \otimes E \tag{2.29}$$

as vector spaces. Explicitly, the basis of E gives rise to an isomorphism $W \otimes E \simeq W^\ell$. Then for $(w_1, \dots, w_\ell) \in W^\ell$, the corresponding coset is represented by

$$n(w_1, \dots, w_\ell) := \begin{pmatrix} I_\ell & (\cdot, w_1) & & & -w_1^2 \\ & \vdots & & \cdot & \\ & (\cdot, w_\ell) & -w_\ell^2 & & \\ & I_W & -w_\ell & \dots & -w_1 \\ & & & I_\ell & \end{pmatrix}. \tag{2.30}$$

Here we write $w_i^2 = \frac{1}{2}(w_i, w_i)$ for short. On the Lie algebra level, we let $\mathfrak{z}_Q$ be the centre of $\mathfrak{n}_Q \subseteq \mathfrak{n}_P$, whence corresponding to (2.28), we have

$$\mathfrak{z}_Q \simeq \bigwedge^2 E_\mathbb{R}. \tag{2.31}$$

We let $\mathfrak{n}_{P'}$ be the Lie algebra of $N_{P'}$; thus $\mathfrak{n}_{P'} \subset E'_\mathbb{R} \wedge E_\mathbb{R} = \mathfrak{gl}(E_\mathbb{R})$. We can realize, corresponding to (2.30), $W_\mathbb{R} \otimes E_\mathbb{R}$ as a subspace of $\mathfrak{n}$. Namely, we obtain an embedding

$$W_\mathbb{R} \otimes E_\mathbb{R} \hookrightarrow \mathfrak{n}, \tag{2.32}$$

$$w \otimes u \to w \wedge u =: \mathfrak{n}_u(w), \tag{2.33}$$

and we denote this subspace by $\mathfrak{n}_W$, which we frequently identify with $W_\mathbb{R} \otimes E_\mathbb{R}$. Furthermore, this embedding is $\mathfrak{o}(W_\mathbb{R}) \oplus \mathfrak{gl}(E_\mathbb{R})$-equivariant, i.e.,

$$[X, \mathfrak{n}_u(w)] = \mathfrak{n}_u(Xw) \qquad [Y, \mathfrak{n}_u(w)] = \mathfrak{n}_{Yu}(w) \tag{2.34}$$

for $X \in \mathfrak{o}(W_\mathbb{R})$ and $Y \in \mathfrak{gl}(E_\mathbb{R})$. We easily see that

$$\exp(\mathfrak{n}_{u_i}(w)) = n(0, \dots, w, \dots, 0). \tag{2.35}$$

A standard basis of $\mathfrak{n}_W$ is given by

$$X_{\alpha i} := \mathfrak{n}_{u_i}(e_\alpha) = e_\alpha \wedge u_i, \qquad X_{\mu i} := \mathfrak{n}_{u_i}(e_\mu) = e_\mu \wedge u_i \tag{2.36}$$

with $1 \leqslant i \leqslant \ell$, $\ell + 1 \leqslant \alpha \leqslant p$, and $p + 1 \leqslant \mu \leqslant m - \ell$. The dual space $\mathfrak{n}_W^*$ we can identify with $W_\mathbb{R} \otimes E'_\mathbb{R}$, and we denote the elements of the corresponding dual basis by $\nu_{\alpha i} = e_\alpha \wedge u'_i$ and $\nu_{\mu i} = -e_\mu \wedge u'_i$.

Summarizing, we obtain the following lemma.

Lemma 2.4. *We have a direct sum decomposition (of vector spaces)*

$$\mathfrak{n}_P = \mathfrak{n}_{P'} \oplus \mathfrak{n}_W \oplus \mathfrak{z}_Q.$$

Furthermore, the adjoint action of $\mathfrak{o}(W_\mathbb{R}) \oplus \mathfrak{gl}(E_\mathbb{R})$ *on* $\mathfrak{n}_P$ *induces an action on the space* $\mathfrak{n}_P/(\mathfrak{n}_{P'} \oplus \mathfrak{z}_Q) \simeq \mathfrak{n}_W$ *such that*

$$\mathfrak{n}_W \simeq W_\mathbb{R} \otimes E_\mathbb{R}$$

as $\mathfrak{o}(W_\mathbb{R}) \oplus \mathfrak{gl}(E_\mathbb{R})$*-representations.*

2.3. The Maurer–Cartan forms and horospherical coordinates

The Langlands decomposition of P gives rise to the (rational) horospherical coordinates on D associated with P through

$$\begin{gathered}\sigma = \sigma_P : N \times A \times D_{\underline{P}} \longrightarrow D, \\ \sigma(n, a, m) = n\,a\,m z_0.\end{gathered} \tag{2.37}$$

Here $D_{\underline{P}} = M_P/K_P$ is the boundary symmetric space associated with $\underline{P}$ with $K_P = M \cap K$. We note that D_P factors into a product of symmetric spaces for special linear groups and one orthogonal factor, the symmetric space D_W associated with $\mathrm{SO}(W)$. We call D_W the *orthogonal factor* in the boundary symmetric space D_P. We have

$$D_P = D_W \times \prod_{j=1}^{k} D_{U_{i_j}}, \tag{2.38}$$

where $D_{U_{i_j}}$ denotes the symmetric space associated with $\mathrm{SL}(U_{i_j})$.

We now describe the basic cotangent vectors $\omega_{\alpha\mu} = (e_\alpha \wedge e_\mu)^* \in \mathfrak{p}^* \simeq T^*_{z_0}(D)$ in NAM coordinates. We extend σ to $N \times A \times M \times K \longrightarrow G$ with $\sigma(n, a, m, k) = namk$, and this induces an isomorphism between the left-invariant forms on NAM (which we identify with $\mathfrak{n}^* \oplus \mathfrak{a}^* \oplus \mathfrak{p}_M^*$) and the horizontal left-invariant forms on G (which we identify with $\mathfrak{p}^*$). Thus we have an isomorphism

$$\sigma^* : \mathfrak{p}^* \longrightarrow \mathfrak{n}^* \oplus \mathfrak{a}^* \oplus \mathfrak{p}_M^*. \tag{2.39}$$

Lemma 2.5. *Let $1 \leqslant i \leqslant \ell$. For the preimage under σ^* of the elements in $\mathfrak{n}_W^*$ coming from $W_+ \otimes E$, we have*

$$\sigma^*\, \omega_{\alpha m+1-i} = -\frac{1}{\sqrt{2}} \nu_{\alpha i}, \tag{2.40}$$

where $\ell + 1 \leqslant \alpha \leqslant p$. Furthermore, for the ones coming from $W_- \otimes E$, we have

$$\sigma^* \omega_{i\mu} = \frac{1}{\sqrt{2}} \nu_{\mu i}, \tag{2.41}$$

where $p + 1 \leqslant \mu \leqslant m + 1 - \ell$. On $\mathfrak{p}_M^*$, the map σ^* is the identity. In particular, for $\ell + 1 \leqslant \alpha \leqslant p$ and any $\mu \geqslant p + 1$, we have

$$\sigma^* \omega_{\alpha\mu} \in \mathfrak{p}_W^* \oplus \mathfrak{n}_W^*. \tag{2.42}$$

The remaining elements of $\mathfrak{p}^*$ are of the form $\omega_{i\mu}$ with $p + 1 \leqslant \mu \leqslant m + 1 - \ell$. These elements are mapped under σ^* to $\mathfrak{z}_Q^* \oplus \mathfrak{n}_{P'}^* \oplus \mathfrak{a}^* \oplus \mathfrak{p}_E^* \subset \bigwedge^2 E_{\mathbb{R}}^* \oplus \mathfrak{gl}(E_{\mathbb{R}})^*$.

2.4. Borel–Serre compactification

We now briefly describe the Borel–Serre compactification of D and of $X = \Gamma \backslash D$. For a more detailed discussion see also the last section where we discuss the $\mathbb{Q}$-split case for $\mathrm{SO}(p, p)$ in detail. In that situation the Borel–Serre compactification is *not* the right compactification for our purposes, and we need to work with a slightly larger compactification.

We follow [4], III.9. We first partially compactify the symmetric space D. For any rational parabolic $\underline{P}$, we define the boundary component

$$e(\underline{P}) = N_P \times D_{\underline{P}} \simeq P/A_{\underline{P}}K_{\underline{P}}. \tag{2.43}$$

Then as a set the (rational) Borel–Serre enlargement $\overline{D}^{BS} = \overline{D}$ is given by

$$\overline{D} = D \cup \coprod_{\underline{P}} e(\underline{P}), \tag{2.44}$$

where $\underline{P}$ runs over all rational parabolic subgroups of $\underline{G}$. As for the topology of $\overline{D}$, we first note that D and $e(\underline{P})$ have the natural topology. Furthermore, a sequence of $y_j = \sigma_P(n_j, a_j, z_j) \in D$ in horospherical coordinates of D converges to a point $(n, z) \in e(\underline{P})$ if and only if $n_j \to n$, $z_j \to z$ and $\alpha(a_j) \to \infty$ for all roots $\alpha \in \Phi(\underline{P}, A_{\underline{P}})$. For convergence within boundary components, see [4], III.9.

With this, $\overline{D}$ has a canonical structure of a real analytic manifold with corners. Moreover, the action of $\underline{G}(\mathbb{Q})$ extends smoothly to $\overline{D}$. The action of $g = kp = kman \in KMAN = G$ on $e(\underline{P})$ is given by

$$g \cdot (n', z') = k \cdot (Ad(am)(nn'), mz') \in e(Ad(k)\underline{P}) = e(Ad(g)\underline{P}) \tag{2.45}$$

with $k \cdot (n', z') = (Ad(k)n, Ad(k)mK_{Ad(k)\underline{P}}) \in e(Ad(k)\underline{P})$. Finally,

$$\overline{X} := \Gamma\backslash\overline{D} \tag{2.46}$$

is the Borel–Serre compactification of $X = \Gamma\backslash D$ to a manifold with corners. If $\underline{P}_1, \dots, \underline{P}_k$ is a set of representatives of Γ-conjugacy classes of rational parabolic subgroups of $\underline{G}$, then

$$\Gamma\backslash\overline{D} = \Gamma\backslash D \cup \coprod_{i=1}^{k} \Gamma_{P_i}\backslash e\,(\underline{P})_i, \tag{2.47}$$

with $\Gamma_{P_i} = \Gamma \cap P_i$. We will write $e'(\underline{P}) = \Gamma_{\underline{P}}\backslash e(\underline{P})$. We write Γ_M for the image of Γ_P under the quotient map $P \to P/N$. Furthermore, Γ_P acts on $E_{\mathbb{R}}^{\perp}/E_{\mathbb{R}}$, and we denote this transformation group by Γ_W. Note that Γ_M and Γ_W when viewed as subgroups of P contain $\Gamma \cap M$ and $\Gamma \cap \mathrm{SO}_0(W_{\mathbb{R}})$ respectively as subgroups of finite index.

We now describe Siegel sets. For $t \in \mathbb{R}_+$, let

$$A_{P,t} = \{a \in A_P;\, \alpha(a) > t \text{ for all } \alpha \in \Delta(P, A_{\underline{P}})\}, \tag{2.48}$$

and for bounded sets $U \subset N_P$ and $V \subset D_P$, we define the Siegel set

$$\mathfrak{S}_{P,U,t,V} = U \times A_{P,t} \times V \subset N_P \times A_P \times D_P. \tag{2.49}$$

Note that for t sufficiently large, two Siegel sets for different parabolic subgroups are disjoint. Furthermore, if $P_1, \dots, P_k$ are representatives of the $\underline{G}(\mathbb{Q})$-conjugacy classes of rational parabolic subgroups of G, then there are Siegel sets $\mathcal{S}_i$ associated with P_i such that the union $\bigcup \pi(\mathcal{S}_i)$ is a fundamental set for Γ. Here π denotes the projection $\pi : D \to \Gamma\backslash D$.

3. A review of representation theory for general linear and orthogonal groups

In this section, we will briefly review the construction of the irreducible finite dimensional (polynomial) representations of $\mathrm{GL}(\mathbb{C}^n)$ and $\mathrm{O}(V)$. Here, in this section, we assume that V is an orthogonal complex space of dimension m. Basic references are [9], §§ 4.2 and 6.1 and [15], §§ 9.3.1–9.3.4, to which we refer the reader for details.

3.1. Representations of $\mathbf{GL}_n(\mathbb{C})$

Let $\lambda = (b_1, \dots, b_n)$ be a partition of ℓ' with the b_i arranged in decreasing order. We will use $D(\lambda)$ to denote the associated Young diagram. We identify the partition λ with the dominant weight λ for $GL(n)$ in the usual way. A standard filling λ of the Young diagram $D(\lambda)$ with the elements of the set $[\ell'] = \{1, 2, \cdots, \ell'\}$ is an assignment of each of the numbers in $[\ell']$ to a box of $D(\lambda)$ such that the entries in each row strictly increase when read from left to right and the entries in each column strictly increase when read from top to bottom. A Young diagram equipped with a standard filling will be also called a standard tableau.

We let $s_{t(\lambda)}$ be the idempotent in the group algebra of the symmetric group $S_{\ell'}$ associated with a standard tableau with ℓ' boxes corresponding to a standard filling $t(\lambda)$ of a Young diagram $D(\lambda)$. Note that $S_{\ell'}$ acts on the space of ℓ'-tensors $T^{\ell'}(\mathbb{C}^n)$ in the natural fashion on the tensor factors. Therefore $s(t(\lambda))$ gives rise to a projection operator in $\mathrm{End}(T^{\ell'}(\mathbb{C}^n))$, which we also denote by $s_{t(\lambda)}$. We write

$$\mathbb{S}_{t(\lambda)}(\mathbb{C}^n) = s_{t(\lambda)}(T^{\ell'}(\mathbb{C}^n)). \tag{3.1}$$

We have a direct sum decomposition

$$T^{\ell'}(\mathbb{C}^n) = \bigoplus_{\lambda} \bigoplus_{t(\lambda)} \mathbb{S}_{t(\lambda)}(\mathbb{C}^n), \tag{3.2}$$

where λ runs over all partitions of ℓ' and $t(\lambda)$ over all standard fillings of $D(\lambda)$. This gives the decomposition of $T^{\ell'}(\mathbb{C}^n)$ into irreducible constituents, i.e. for every standard filling $t(\lambda)$, the $\mathrm{GL}(\mathbb{C}^n)$-module $\mathbb{S}_{t(\lambda)}(\mathbb{C}^n)$ is irreducible with highest weight λ. In particular, $\mathbb{S}_{t(\lambda)}(\mathbb{C}^n)$ and $\mathbb{S}_{t'(\lambda)}(\mathbb{C}^n)$ are isomorphic for two different standard fillings $t(\lambda)$ and $t'(\lambda)$. We denote this isomorphism class by $\mathbb{S}_{\lambda}(\mathbb{C}^n)$ (or if we do not want to specify the standard filling).

Explicitly, we let A be the standard filling of a Young diagram $D(A)$ corresponding to the partition λ with less than or equal to n rows and ℓ' boxes obtained by filling the rows in order beginning at the top with $1, 2, \dots, \ell'$. We let $R(A)$ be the subgroup of $S_{\ell'}$ which preserves the rows of A and $C(A)$ be the subgroup that preserves the columns of A. We define elements r_A and c_A in the group ring of $S_{\ell'}$ by

$$r_A = \sum_{s \in R(A)} s \quad \text{and} \quad c_A = \sum_{s \in C(A)} \mathrm{sgn}(s) s. \tag{3.3}$$

Let $h(A)$ be the product of the hook lengths of the boxes in $D(A)$; see [9], p. 50. Then the idempotent s_A is given by

$$s_A = \frac{1}{h(A)} c_A r_A. \tag{3.4}$$

We will also need the 'dual' idempotent s_A^* given by $s_A^* = \frac{1}{h(A)} r_A c_A$. We let $\varepsilon_1, \dots, \varepsilon_n$ denote the standard basis of $\mathbb{C}^n$ and $\theta_1, \dots, \theta_n \in (\mathbb{C}^n)^*$ be its dual basis. We set

$$\varepsilon_A = \varepsilon_1^{b_1} \otimes \cdots \otimes \varepsilon_n^{b_n} \tag{3.5}$$

and let θ_A be the corresponding element in $T^{\ell'}(\mathbb{C}^n)^*$. Then $s_A(\varepsilon_A)$ is a highest weight vector in $\mathbb{S}_A(\mathbb{C}^n)$; see [15], § 9.3.1. We have the following lemma.

Lemma 3.1. *Let $|R(A)|$ be the order of $R(A)$. Then*

$$s_A^* \theta_A(s_A \varepsilon_A) = \frac{|R(A)|}{h(A)}.$$

Proof. We compute

$$s_A^* \theta_A(s_A \varepsilon_A) = \theta_A(s_A^2 \varepsilon_A) = \theta_A(s_A \varepsilon_A) = \frac{|R(A)|}{h(A)} \theta_A(c_A \varepsilon_A) = \frac{|R(A)|}{h(A)} \theta_A(\varepsilon_A).$$

The last equation holds because $\theta_A(q\varepsilon_A) = 0$ for any non-trivial q in the column group of A as the reader will easily verify. We have used $r_A \varepsilon_A = |R(A)| \varepsilon_A$ (since all elements of $R(A)$ fix ε_A) and $s_A = \frac{1}{h(A)} c_A r_A$. □

3.2. Enlarging the Young diagram

We let $B = B_{n,\ell}$ be the standard tableau with underlying shape $D(B)$ an n by ℓ rectangle with the standard filling obtained by putting 1 to ℓ in the first row, $\ell + 1$ to 2ℓ in the second row etc. Then $D(B)$ is the Young diagram corresponding to the dominant weight $\ell \varpi_n$. Here $\varpi_n = (1, 1, \dots, 1)$ is the nth fundamental weight for $\mathrm{GL}(n)$. We note that we have $\varepsilon_B = \epsilon_1^\ell \otimes \cdots \otimes \epsilon_n^\ell$ and $\theta_B = \theta_1^\ell \otimes \cdots \otimes \theta_n^\ell$.

Lemma 3.2. *The space $s_B T^{n\ell}(\mathbb{C}^n)$ is one dimensional, and*

$$s_B T^{n\ell}(\mathbb{C}^n) = \mathbb{C} s_B \varepsilon_B$$

as $\mathrm{GL}(n, \mathbb{C})$-modules. Correspondingly, $s_B^* T^{n\ell}(\mathbb{C}^n)^*$ is one dimensional and

$$s_B^* T^{n\ell}(\mathbb{C}^n)^* = \mathbb{C} s_B^* \theta_B.$$

In particular,

$$s_B^* T^{n\ell}(\mathbb{C}^n)^* \cong \left(\bigwedge^n (\mathbb{C}^n)^* \right)^{\otimes \ell}.$$

We let A be the standard filling of the Young diagram $D(\lambda)$ as above. Then $B|A$ denotes the standard tableau with underlying shape $D(B|A)$ given by making the shape of A abut B (on the right), using the above filling for B and filling A in the standard way (as above) with $n\ell+1$ to $n\ell+\ell'$. For example, if

$$B = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array} \quad \text{and} \quad A = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}, \quad \text{then} \quad B|A = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 10 & 11 & 12 \\ \hline 4 & 5 & 6 & 13 & 14 \\ \cline{1-5} 7 & 8 & 9 \\ \cline{1-3} \end{array}$$

We have an idempotent $s_{B|A}$ in the group ring of $S_{n\ell+\ell'}$ and $\varepsilon_{B|A} \in T^{n\ell+\ell'}(\mathbb{C}^n)$, which give rise to a highest weight vector $s_{B|A}\varepsilon_{B|A}$ in $s_{B|A}(T^{n\ell+\ell'}(\mathbb{C}^n))$. Note that

$$\varepsilon_{B|A} = \varepsilon_B \otimes \varepsilon_A. \tag{3.6}$$

Lemma 3.3. *There is a positive number $c(A,B)$ such that*

$$s_B\varepsilon_B \otimes s_A\varepsilon_A = c(A,B)s_{B|A}\varepsilon_{B|A}.$$

Proof. Since the Young diagrams $D(B)$ and $D(A)$ are abutted along their vertical borders, we see that

$$c_{B|A} = (c_B \otimes 1_{\ell'}) \circ (1_{n\ell} \otimes c_A) = (1_{n\ell} \otimes c_A) \circ (c_B \otimes 1_{\ell'}). \tag{3.7}$$

Also $r(C)\varepsilon_C = |R(C)|\varepsilon_C$. Then we easily compute (using (3.6) and (3.7))

$$s_B\varepsilon_B \otimes s_A\varepsilon_A = \frac{h(B|A)}{h(B)h(A)} \frac{|R(B)||R(A)|}{|R(B|A)|} s_{B|A}\varepsilon_{B|A}. \quad \square$$

Corollary 3.4. *Under the identification of $T^{n\ell}(\mathbb{C}^n) \otimes T^{\ell'}(\mathbb{C}^n) \to T^{n\ell+\ell'}(\mathbb{C}^n)$ given by tensor multiplication, we have the equality of maps*

$$s_B \otimes s_A = s_{B|A}.$$

That is,

$$\mathbb{S}_B(\mathbb{C}^n) \otimes \mathbb{S}_A(\mathbb{C}^n) = \mathbb{S}_{B|A}(\mathbb{C}^n)$$

as (physical) subspaces of $T^{n\ell+\ell'}(\mathbb{C}^n)$. The same statements hold for the dual space $\mathbb{S}^*_{B|A}(\mathbb{C}^{n\ell+\ell'})^*$ etc.

Proof. Since $\mathbb{S}_B(\mathbb{C}^n)$ is one dimensional, the tensor product $\mathbb{S}_B(\mathbb{C}^n) \otimes \mathbb{S}_A(\mathbb{C}^n)$ defines an irreducible representation for $\mathrm{GL}_n(\mathbb{C}^n)$ (under the tensor multiplication map $T^{n\ell}(\mathbb{C}^n) \otimes T^{\ell'}(\mathbb{C}^n)$ inside $T^{n\ell+\ell'}(\mathbb{C}^n)$). But by Lemma 3.3 it has non-zero intersection with the irreducible $\mathrm{GL}_n(\mathbb{C})$-representation $\mathbb{S}_{B|A}(\mathbb{C}^n)$ inside $T^{n\ell+\ell'}(\mathbb{C}^n)$. Hence the two subspaces coincide. $\square$

3.3. Representations of $\mathrm{O}(V)$

We extend the bilinear form $(\ ,\ )$ on V to $T^{\ell'}(V)$ as the ℓ'-fold tensor product and note that the action of $S_{\ell'}$ on $T^{\ell'}(V)$ is by isometries. We let $V^{[\ell']}$ be the space of harmonic ℓ'-tensors (which are those ℓ'-tensors which are annihilated by all contractions with the form $(\ ,\ )$). We let $\mathcal{H}$ be the orthogonal projection $\mathcal{H} : T^{\ell'}(V) \to V^{[\ell']}$ onto the harmonic

ℓ'-tensors of V. Note that $V^{[\ell']}$ is invariant under the action of $S_{\ell'}$. We then define for λ as above the harmonic Schur functor $\mathbb{S}_{[t(\lambda)]}(V)$ by

$$\mathbb{S}_{[t(\lambda)]}(V) = \mathcal{H}\mathbb{S}_{t(\lambda)}(V). \tag{3.8}$$

If the sum of the lengths of the first two columns of $D(\lambda)$ is at most m, then $\mathbb{S}_{[t(\lambda)]}(V_{\mathbb{C}})$ is a non-zero irreducible representation for $\mathrm{O}(V_{\mathbb{C}})$; see [9] § 19.5. Otherwise, it vanishes. Of course, for different fillings $t(\lambda)$ of $D(\lambda)$, these representations are all isomorphic and we write $\mathbb{S}_{[\lambda]}(V)$ for the isomorphism class. Furthermore, it is also irreducible when restricted to G unless m is even and $i(\lambda) = \frac{m}{2}$, in which case it splits into two irreducible representations. If $i(\lambda) \leqslant [\frac{m}{2}]$, then the corresponding highest weight $\tilde{\lambda}$ for the representation $\mathbb{S}_{[\lambda]}(V)$ of G has the same non-zero entries as λ.

4. The Weil representation

We review different models of the Weil representation. In this section, V denotes a real quadratic space of signature (p, q) and dimension m.

We let V' be a real symplectic space of dimension $2n$. We denote by $G' = \mathrm{Mp}(n, \mathbb{R})$ the metaplectic cover of the symplectic group $\mathrm{Sp}(V') = \mathrm{Sp}(n, \mathbb{R})$ and let $\mathfrak{g}'$ be its Lie algebra. We let K' be the inverse image of the standard maximal compact $\mathrm{U}(n) \subset \mathrm{Sp}(n, \mathbb{R})$ under the covering map $\mathrm{Mp}(n, \mathbb{R}) \to \mathrm{Sp}(n, \mathbb{R})$. Note that K' admits a character $\det^{1/2}$, i.e., its square descends to the determinant character of $\mathrm{U}(n)$. The embedding of $\mathrm{U}(n)$ into $\mathrm{Sp}(n, \mathbb{R})$ is given by $A + iB \mapsto \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right)$. We write $\mathcal{W}_{n,V}$ for (an abstract model of) the K'-finite vectors of the restriction of the Weil representation of $\mathrm{Mp}(V' \otimes V)$ to $\mathrm{Mp}(n, \mathbb{R}) \times \mathrm{O}(V)$ associated with the additive character $t \mapsto e^{2\pi i t}$.

4.1. The Schrödinger model

We let V_1' be a Lagrangian subspace of V'. Then $V \otimes V_1'$ is a Lagrangian subspace of $V \otimes V'$ (which is naturally a symplectic space of dimension $2nm$). The Schrödinger model of the Weil representation consists of the space of (complex-valued) Schwartz functions on the Lagrangian subspace $V_1' \otimes V \simeq V^n$. We write $\mathcal{S}(V^n)$ for the space of Schwartz functions on V^n and write $\omega = \omega_{n,V}$ for the action.

The Siegel parabolic $P' = M'N'$ has Levi factor

$$M' = \left\{ m'(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} ; a \in \mathrm{GL}(n, \mathbb{R}) \right\} \tag{4.1}$$

and unipotent radical

$$N' = \left\{ n'(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} ; b \in \mathrm{Sym}_n(\mathbb{R}) \right\}. \tag{4.2}$$

It is well-known that we can embed P' into $\mathrm{Mp}(n,\mathbb{R})$, and the action of P' on $S(V^n)$ is given by

$$\omega\left(m'(a)\right)\varphi(\mathbf{x}) = (\det a)^{m/2}\,\varphi(\mathbf{x}a) \quad (\det a > 0), \tag{4.3}$$

$$\omega\left(n'(b)\right)\varphi(\mathbf{x}) = e^{\pi i tr(b(\mathbf{x},\mathbf{x}))}\varphi(\mathbf{x}) \tag{4.4}$$

with $\mathbf{x} = (x_1, \ldots, x_n) \in V^n$. The orthogonal group G acts on $\mathcal{S}(V^n)$ via

$$\omega(g)\varphi(\mathbf{x}) = \varphi(g^{-1}\mathbf{x}), \tag{4.5}$$

which commutes with the action G'. The standard Gaussian is given by

$$\varphi_0(\mathbf{x}) = e^{-\pi tr\,(\mathbf{x},\mathbf{x})_{z_0}} \in \mathcal{S}\left(V^n\right)^K. \tag{4.6}$$

Here $(\mathbf{x},\mathbf{x})$ is the inner product matrix $(x_i, x_j)_{ij}$.

We let $S(V^n)$ be the space of K'-finite vectors inside the space of Schwartz functions on V^n. It consists of those Schwartz functions of the form $p(\mathbf{x})\varphi_0(\mathbf{x})$, where p is a polynomial function on V^n.

4.2. The mixed model and local restriction for the Weil representation

We let P be a standard parabolic of G stabilizing a totally isotropic flag in V with $E = E_\ell$ the largest constituent of the flag and the associated Witt decomposition $V = E \oplus W \oplus E'$.

We describe a different model for the Weil representation, the so-called mixed model. Furthermore, we will define a 'local' restriction $r_P^{\mathcal{W}}$ from $\mathcal{S}(V^n)$ to the space of Schwartz functions $\mathcal{S}(W^n)$ for W, a subspace of signature $(p-\ell, q-\ell)$.

4.2.1. The mixed model. We let $E = E_\ell$ be one of the standard totally isotropic subspaces of V; see (2.5). As before, we identify the dual space of E with E'. Accordingly, we write $\mathbf{x} = \begin{pmatrix} u \\ \mathbf{x}_W \\ u' \end{pmatrix}$ for $\mathbf{x} \in V^n$, where $u \in E^n$, $u' \in (E')^n$, and $\mathbf{x}_W \in W^n$. We then have an isomorphism of two models of the Weil representation given by

$$\begin{aligned} \mathcal{S}(V^n) &\longrightarrow \mathcal{S}((E')^n) \otimes S(W^n) \otimes \mathcal{S}((E')^n) \\ \varphi &\longmapsto \widehat{\varphi} \end{aligned} \tag{4.7}$$

given by the partial Fourier transform operator

$$\widehat{\varphi}\begin{pmatrix} \xi \\ \mathbf{x}_W \\ u' \end{pmatrix} = \int_{E^n} \varphi \begin{pmatrix} u \\ \mathbf{x}_W \\ u' \end{pmatrix} e^{-2\pi i tr(u,\xi)} du \tag{4.8}$$

with $\xi \in (E')^n$, $\mathbf{x}_W \in W^n$, $u' \in (E')^n$. We need some formulas relating the action of ω in the two models. A straightforward calculation gives the following lemma.

Lemma 4.1. *Let $\xi \in (E')^n$, $\mathbf{x}_W \in W^n$, $u' \in (E')^n$.*

(i) *Let $n \in N_Q$ and write $n\,(u')_W$ for the image of $n(u')$ under the orthogonal projection onto W. Then*

$$\widehat{n\varphi}({}^t(\xi, \mathbf{x}_W, u')) = e\left(\mathrm{tr}(n(\mathbf{x}_W + u'), \xi)\right)\widehat{\varphi}({}^t(\xi, \mathbf{x}_W + n\,(u')_W, u')).$$

(ii) *For* $g \in \mathrm{SL}(E) \subset G$ *(in particular,* $g \in N_{P'}$ *or* $g \in M_{P'}$*) we have*

$$\widehat{g\varphi}({}^t(\xi, \mathbf{x}_W, u')) = \widehat{\varphi}({}^t(\widetilde{g}\xi, \mathbf{x}_W, \widetilde{g}^{-1}u'))$$

with $\widetilde{g} = Jg^*J$ *and* $g^* = {}^tg^{-1}$.

(iii) *For* $\mathbf{t} = (t_1, \dots, t_\ell)$*, set* $\widetilde{\mathbf{t}} = \mathbf{t}J = (t_\ell, \dots, t_1)$ *and* $|\mathbf{t}| = t_1 \cdot t_2 \cdots t_\ell$*. Then*

$$\widehat{a(\mathbf{t})\varphi}({}^t(\xi, \mathbf{x}_W, u')) = |\mathbf{t}|^n\, \widehat{\varphi}({}^t(\widetilde{\mathbf{t}}\xi, \mathbf{x}_W, \widetilde{\mathbf{t}}u')).$$

(iv) *For* $h \in \mathrm{SO}_0(W) \subset M$*, we have*

$$\widehat{h\varphi}({}^t(\xi, \mathbf{x}_W, u')) = \widehat{\varphi}({}^t(\xi, h^{-1}\mathbf{x}_W, u')).$$

(v) *For* $m'(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & {}^ta^{-1} \end{smallmatrix}\right) \in M' \subset Sp(n, \mathbb{R})$ *with* $a \in GL_n^+(\mathbb{R})$*,*

$$(\widehat{m'(a)\varphi})({}^t(\xi, \mathbf{x}_W, u')) = (\det a)^{\frac{m}{2}-\ell}\, \widehat{\varphi}({}^t(\xi a^*, \mathbf{x}_W a, u'a)).$$

(vi) *For* $n'(b) = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \in N' \subset \mathrm{Sp}(n, \mathbb{R})$ *with* $b \in \mathrm{Sym}_n(\mathbb{R})$*,*

$$(\widehat{n'(b)\varphi})({}^t(\xi, \mathbf{x}_W, u')) = e\left(tr\left(b\frac{(\mathbf{x}_W, \mathbf{x}_W)}{2}\right)\right) \widehat{\varphi}({}^t(\xi - u'b, \mathbf{x}_W, u')).$$

We obtain the following proposition.

Proposition 4.2. *Let* $\varphi \in \mathcal{S}(V^n)$*. Then the restriction of* $\widehat{\varphi}$ *to* W^n*,*

$$\varphi \mapsto \widehat{\varphi}|_{W^n},$$

defines a $G' \times MN$ *intertwiner from* $\mathcal{S}(V^n)$ *to* $\mathcal{S}(W^n)$*. Here, we identify* W *with* $E^\perp/E$ *to define the action of* MN *on* W*. In particular,* N *and* $M_{P'}$ *(see (2.20)) act trivially on* $\mathcal{S}(W^n)$*.*

4.2.2. Weil representation restriction.

Definition 4.3. Let $\varphi \in \mathcal{S}(V^n)$ and let P be the parabolic as before. We define the 'local' restriction $r_P^{\mathcal{W}}(\varphi) \in \mathcal{S}(W^n)$ with respect to P for the Schrödinger model of the Weil representation $\mathcal{W}$ by

$$r_P^{\mathcal{W}}(\varphi) = \widehat{\varphi}|_{W^n}.$$

We now describe this restriction on a certain class of Schwartz functions on V^n. For $\mathbf{x} = (x_1, \dots, x_n) \in V^n$, we write $\begin{pmatrix} x_{1j} \\ \vdots \\ x_{mj} \end{pmatrix}$ for the standard coordinates of x_j. We define a family of commuting differential operators on $\mathcal{S}(V^n)$ by

$$\mathcal{H}_{rj} = \left(x_{rj} - \frac{1}{2\pi}\frac{\partial}{\partial x_{rj}}\right), \tag{4.9}$$

where $1 \leqslant r \leqslant m$ and $1 \leqslant j \leqslant n$. Define a polynomial $\widetilde{H}_k$ by

$$\widetilde{H}_k(x) = (2\pi)^{-k/2}\, H_k(\sqrt{2\pi}x), \tag{4.10}$$

where $H_k(x) = (-1)^k\, e^{x^2} \frac{d^k}{dx^k} e^{-x^2}$ is the k th Hermite polynomial. Then

$$\mathcal{H}_{rj}^k \varphi_0(\mathbf{x}) = \widetilde{H}_k(x_{rj})\varphi_0(\mathbf{x}), \tag{4.11}$$

where $\varphi_0(\mathbf{x})$ is the standard Gaussian; see (4.6). We let $\Delta \in M_{m\times n}(\mathbb{Z}) = (\delta_{rj})$ be an integral matrix with non-negative entries and split Δ into $\Delta_+ \in M_{p\times n}(\mathbb{Z})$ and $\Delta_- \in M_{q\times n}(\mathbb{Z})$ into its 'positive' and 'negative' parts, where Δ_+ consists of the first p rows of Δ and Δ_- of the last q. (Recall that $m = p + q$.) We define operators

$$\mathcal{H}_\Delta = \prod_{\substack{1\leqslant r\leqslant m\\ 1\leqslant j\leqslant n}} \mathcal{H}_{rj}^{\delta_{rj}}, \quad \mathcal{H}_{\Delta_+} = \prod_{\substack{1\leqslant \alpha\leqslant p\\ 1\leqslant j\leqslant n}} \mathcal{H}_{\alpha j}^{\delta_{\alpha j}}, \quad \mathcal{H}_{\Delta_-} = \prod_{\substack{p+1\leqslant \mu\leqslant m\\ 1\leqslant j\leqslant n}} \mathcal{H}_{\mu j}^{\delta_{\mu j}},$$

so $\mathcal{H}_\Delta = \mathcal{H}_{\Delta_+}\mathcal{H}_{\Delta_-}$. Here again we make use of our convention to use early Greek letters for the 'positive' indices of V and late ones for the 'negative' indices.

Definition 4.4. For Δ as above, we define the Schwartz function φ_Δ by

$$\varphi_\Delta(\mathbf{x}) = \mathcal{H}_\Delta\varphi_0(\mathbf{x}) = \prod_{\substack{1\leqslant \alpha\leqslant p\\ p+1\leqslant \mu\leqslant m\\ 1\leqslant j\leqslant n}} \widetilde{H}_{\delta_{\alpha j}}(x_{\alpha j})\widetilde{H}_{\delta_{\mu j}}(x_{\mu j})\varphi_0(\mathbf{x}).$$

Similarly, we set $\varphi_{\Delta_+}(\mathbf{x}) = \mathcal{H}_{\Delta_+}\varphi_0(\mathbf{x})$.

We now describe φ_Δ^V in the mixed model. The superscript V emphasizes that the Schwartz function is associated with the space V. We begin with some auxiliary considerations. The following fact will be crucial for us.

Lemma 4.5. *For a Schwartz function $f \in \mathcal{S}(\mathbb{R})$, let $\widehat{f}(\xi) = \int_\mathbb{R} f(y)e^{-2\pi i y\xi}dy$ be its Fourier transform. Let $g_k(y) = \widetilde{H}_k(-\frac{y}{\sqrt{2}})e^{-\pi y^2}$. Then*

$$\widehat{g_k}(\xi) = \left(-\sqrt{2}i\xi\right)^k e^{-\pi\xi^2}.$$

Proof. We use induction and differentiate the equation $\widehat{(\widehat{g_k})}(-y) = \widetilde{H}_k(\frac{y}{\sqrt{2}})e^{-\pi y^2}$. The assertion follows from the recursion $\widetilde{H}_{k+1}(y) = 2y\widetilde{H}_k(y) - \frac{1}{2\pi}\widetilde{H}_k'(y)$, which is immediate from the definition of $\widetilde{H}_k$. The claim also follows easily from [23], (4.11.4). □

Remark 4.6. Recall that on the other hand $\widetilde{H}_k(y)e^{-\pi y^2}$ is an eigenfunction under the Fourier transform with eigenvalue $(-i)^k$; see [23], (4.12.3). This fact underlies the automorphic properties of the theta series associated with the special forms $\varphi_{nq,[\lambda]}$.

The Gaussian is given in standard coordinates by $\varphi_0^V(\mathbf{x}) = \exp(-\pi\sum_{j=1}^n\sum_{i=1}^m x_{ij}^2)$. In Witt coordinates, we have $x_{rj} = \frac{1}{\sqrt{2}}(y_{rj} - y_{(m-r)j})$ and $x_{(m-r)j} = \frac{1}{\sqrt{2}}(y_{rj} + y_{(m-r)j})$; thus $x_{rj}^2 + x_{(m-r),j}^2 = y_{rj}^2 + y_{(m-r)j}^2$ for $r \leqslant \ell$. Thus

$$\varphi_0^V\begin{pmatrix} u\\ \mathbf{x}_W\\ u'\end{pmatrix} = \exp\left(-\pi\sum_{j=1}^n\sum_{r=1}^\ell(y_{rj}^2 + y_{(m-r)j}^2)\right)\varphi_0^W(\mathbf{x}_W). \tag{4.12}$$

We write, slightly abusing notation,

$$\varphi_0^E(u,u') := \varphi_0^V \begin{pmatrix} u \\ 0 \\ u' \end{pmatrix} = \exp\left(-\pi \sum_{j=1}^{n} \sum_{r=1}^{\ell} (y_{rj}^2 + y_{(m-r)j}^2) \right). \tag{4.13}$$

We let Δ' be the truncated matrix of size $(m-2\ell)\times n$ given by eliminating the first and the last ℓ rows from Δ. We let Δ'' be the matrix of these eliminated rows. Note that $\mathcal{H}_{\Delta'}$ now defines an operator on $\mathcal{S}(W^n)$ and $\mathcal{H}_{\Delta''}$ on $\mathcal{S}((E\oplus E')^n)$. We also obtain matrices Δ'_+ of size $(p-\ell)\times n$ and Δ'_- of size $(q-\ell)\times n$ by eliminating the first ℓ and the last ℓ rows from Δ_+ and Δ_- respectively. Similarly we obtain Δ''_+ and Δ''_-. We define $\varphi^W_{\Delta'_{(+)}}(\mathbf{x}) = \mathcal{H}_{\Delta'_{(+)}}\varphi_0^W(\mathbf{x})$ and $\varphi^E_{\Delta''_{(+)}}(\mathbf{x}) = \mathcal{H}_{\Delta''_{(+)}}\varphi_0^E(\mathbf{x})$ as before.

With this notation we obtain the following lemma.

Lemma 4.7.

$$\text{(i)}\ \ \widehat{\varphi^V_\Delta} \begin{pmatrix} \xi \\ \mathbf{x}_W \\ u' \end{pmatrix} = \varphi^W_{\Delta'}(\mathbf{x}_W)\widehat{\varphi^E_{\Delta''}}(\xi,u').$$

$$\text{(ii)}\qquad r_P^{\mathcal{W}}\left(\varphi^V_\Delta\right)(\mathbf{x}_W) = \varphi^W_{\Delta'}(\mathbf{x}_W)\widehat{\varphi^E_{\Delta''}}(0,0).$$

In our applications all entries of Δ_- will be zero, so $\Delta=\Delta_+$ (by abuse of the notation).

Lemma 4.8.

$$\widehat{\varphi^E_{\Delta''_+}}(\xi,0) = \left(\prod_{j=1}^{n} \prod_{\alpha=1}^{\ell} (-\sqrt{2}i\xi_{\alpha j})^{\delta_{\alpha j}} \right) \varphi_0^E(\xi,0).$$

In particular, if in addition all entries of Δ''_+ vanish, then

$$\widehat{\varphi^V_{\Delta_+}} \begin{pmatrix} \xi \\ \mathbf{x}_W \\ 0 \end{pmatrix} = \varphi^W_{\Delta'_+}(\mathbf{x}_W)\varphi_0^E(\xi,0).$$

Proof. This follows from applying Lemma 4.5. □

We conclude the following proposition.

Proposition 4.9. (i) *Assume that one of the entries of Δ''_+ is non-zero; then*

$$r_P^{\mathcal{W}}(\varphi^V_{\Delta_+}) = 0.$$

(ii) *If all of the entries of Δ''_+ vanish, then*

$$r_P^{\mathcal{W}}\left(\varphi^V_{\Delta_+}\right) = \varphi^W_{\Delta'_+}.$$

Remark 4.10. Analogous results hold for $r_P^{\mathcal{W}}(\varphi_{\Delta_-}^V)$. However, a general formula for the restriction of $r_P^{\mathcal{W}}(\varphi_{\Delta}^V)$ is more complicated (and is not needed in this paper).

4.3. The Fock model

It will be convenient to also consider the Fock model $\mathcal{F} = \mathcal{F}_{n,V}$ of the Weil representation. For more details on $\mathcal{F}_{n,V}$, see the appendix of [11].

There is an intertwining map $\iota : S(V^n) \to \mathcal{P}(\mathbb{C}^{n(p+q)})$ from the K'-finite Schwartz functions to the infinitesimal Fock model of the Weil representation acting on the space of complex polynomials $\mathcal{P}(\mathbb{C}^{n(p+q)})$ in $n(p+q)$ variables such that $\iota(\varphi_0) = 1$. We denote the variables in $\mathcal{P}(\mathbb{C}^{n(p+q)})$ by $z_{\alpha i}$ $(1 \leqslant \alpha \leqslant p)$ and $z_{\mu i}$ $(p+1 \leqslant \mu \leqslant p+q)$ with $i = 1, \ldots, n$. Moreover, the intertwining map ι satisfies

$$\iota \left(x_{\alpha i} - \frac{1}{2\pi} \frac{\partial}{\partial x_{\alpha i}} \right) \iota^{-1} = \frac{1}{2\pi i} z_{\alpha i}, \quad \iota \left(x_{\mu j} - \frac{1}{2\pi} \frac{\partial}{\partial x_{\mu j}} \right) \iota^{-1} = -\frac{1}{2\pi i} z_{\mu j}.$$

By a slight abuse of notation, we use the same symbol for corresponding objects in the Schrödinger and Fock models. In the Fock model, φ_{Δ}^V looks as follows.

Lemma 4.11.

$$\varphi_{\Delta}^V = \prod_{\substack{1 \leqslant \alpha \leqslant p \\ p+1 \leqslant \mu \leqslant m \\ 1 \leqslant j \leqslant n}} \left(\frac{1}{2\pi i} z_{\alpha j} \right)^{\delta_{\alpha j}} \left(-\frac{1}{2\pi i} z_{\mu j} \right)^{\delta_{\mu j}}.$$

Proposition 4.9 translates to the following proposition.

Proposition 4.12. *If one of the entries of Δ''_+ is non-zero, then*

$$r_P^{\mathcal{W}}(\varphi_{\Delta_+}^V) = 0.$$

If all of the entries of Δ''_+ vanish, then

$$r_P^{\mathcal{W}}(\varphi_{\Delta_+}^V) = \prod_{\substack{\ell+1 \leqslant \alpha \leqslant p \\ 1 \leqslant j \leqslant n}} \left(\frac{1}{2\pi i} z_{\alpha j} \right)^{\delta_{\alpha j}}.$$

5. Differential graded algebras associated with the Weil representation

In this section, we construct certain differential graded algebras $C_V^\bullet$ and $A_P^\bullet$ and define a local restriction map r_P from $C_V^\bullet$ to $A_P^\bullet$. Again V will denote a non-degenerate real quadratic space of dimension m and signature (p, q).

5.1. Relative Lie algebra complexes

For the convenience of the reader, we briefly review some basic facts about relative Lie algebra complexes; see, e.g., [6]. For this subsection, we deviate from the notation of the paper and let $\mathfrak{g}$ be any real Lie algebra $\mathfrak{g}$ and let $\mathfrak{k}$ be any subalgebra. We let (U, π) be a representation of $\mathfrak{g}$. We set

$$C^q(\mathfrak{g}, \mathfrak{k}; U) = \left[\mathrm{Hom}\left(\bigwedge^q(\mathfrak{g}/\mathfrak{k}), U\right)\right]^{\mathfrak{k}} \simeq \left[\bigwedge^q(\mathfrak{g}/\mathfrak{k})^* \otimes U\right]^{\mathfrak{k}}, \tag{5.1}$$

where the action of $\mathfrak{k}$ on $\bigwedge^q(\mathfrak{g}/\mathfrak{k})$ is induced by the adjoint representation. Then in the setting of $[\bigwedge^q(\mathfrak{g}/\mathfrak{k})^* \otimes U]^{\mathfrak{k}}$, the differential d is given by

$$d = \sum_i A(\omega_i) \otimes \pi(X_i) + \frac{1}{2}\sum_i A(\omega_i)\mathrm{ad}^*(X_i) \otimes 1. \tag{5.2}$$

Here $A(\omega_i)$ denotes the left multiplication with ω_i in $\bigwedge^\bullet(\mathfrak{g}/\mathfrak{k})^*$, and $\mathrm{ad}^*(X)$ is the dual of the adjoint action on $\bigwedge^\bullet(\mathfrak{g}/\mathfrak{k})^*$, that is, $(\mathrm{ad}^*(X)(\alpha))(Y_1, \dots, Y_q) = \sum_{i=1}^q \alpha(Y_1, \dots, [Y_i, X], \dots, Y_q)$. We easily see the following lemma.

Lemma 5.1. *Consider two relative Lie algebra complexes $C^\bullet(\mathfrak{g}, \mathfrak{k}; U)$ and $C^\bullet(\mathfrak{g}', \mathfrak{k}'; U')$. Then given*

(i) $\rho : \mathfrak{g} \to \mathfrak{g}'$, *a Lie algebra homomorphism such that* $\rho(\mathfrak{k}) \subseteq \mathfrak{k}'$,

(ii) $T : U' \to U$, *an intertwining map with respect to* ρ *(i.e.,* $T(\rho(X) \cdot u') = X \cdot T(u')$ *for* $X \in \mathfrak{g}$*),*

we obtain a natural map of complexes

$$C^\bullet(\mathfrak{g}', \mathfrak{k}'; U') \to C^\bullet(\mathfrak{g}, \mathfrak{k}; U)$$

given by

$$\varphi \mapsto T \circ \varphi \circ \rho.$$

On realizing φ as an element $\left[\bigwedge^q(\mathfrak{g}'/\mathfrak{k}')^* \otimes U'\right]^{\mathfrak{k}'}$, the map is given by

$$\varphi \mapsto (\rho^* \otimes T)(\varphi),$$

where $\rho^* : (\mathfrak{g}'/\mathfrak{k}')^* \to (\mathfrak{g}/\mathfrak{k})^*$ is the dual map.

Now let G be any real Lie group with Lie algebra $\mathfrak{g}$ and let K be a closed connected subgroup of G (not necessarily compact) with Lie algebra $\mathfrak{k}$. For U a smooth G-module, we let $\mathcal{A}^q(G/K; U)$ be the U-valued differential q-forms on G/K (with the usual exterior differentiation). The G-action on $\mathcal{A}^q(G/K; U)$ is given by

$$(g \circ w)_x(X) = g(\omega_{g^{-1}\cdot x}(g^{-1} \cdot X)), \tag{5.3}$$

for $\omega \in \mathcal{A}^q(G/K; U)$, $x \in G/K$, and $X \in T_x^q(G/K)$. Then evaluation at the base point of G/K gives rise to an isomorphism of complexes

$$\mathcal{A}^\bullet(G/K; U)^G \simeq C^\bullet(\mathfrak{g}, \mathfrak{k}; U) \tag{5.4}$$

of the G-invariant forms on G/K with $C^\bullet(\mathfrak{g}, \mathfrak{k}; U)$.

5.2. The differential graded algebra $C_V^\bullet$

We begin this section by defining a differential graded (but not graded commutative) algebra $C_V^\bullet$. The complex $C_V^\bullet$ is essentially the relative Lie algebra complex for $\mathrm{O}(V)$

with values in $\mathcal{W}_{n,V}$ tensored with the tensor algebra of $V_{\mathbb{C}}$ and twisted by some factors associated with $\mathbb{C}^n$. To be precise, it is the complex given by

$$C_V^{j,r,k} = \left[T^j(U)\left[-\frac{p-q}{2}\right] \otimes T^k(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,V} \otimes \bigwedge^r \mathfrak{p}_{\mathbb{C}}^* \otimes T^k(V_{\mathbb{C}})\right]^{K' \times K \times S_k}$$
$$\simeq \left[T^j(U)\left[-\frac{p-q}{2}\right] \otimes T^k(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,V} \otimes \mathcal{A}^r(D) \otimes T^k(V_{\mathbb{C}})\right]^{K' \times G \times S_k}.$$

Here j, r, k are non-negative integers and $\mathcal{A}^r(D)$ denotes the space of complex-valued differential r-forms on D. We let $U = \bigwedge^n (\mathbb{C}^n)^*$, and we define the action of K' on $T^j(U)[\frac{p-q}{2}]$ by requiring K' to act through the character $\det^{-j-\frac{p-q}{2}}$ on $T^j(U)$. Thus K' acts by algebra homomorphisms shifted by the character $\det^{-\frac{p-q}{2}}$. We will usually drop the $[\frac{p-q}{2}]$ in what follows. The differential is the usual relative Lie algebra differential for the action of $\mathrm{O}(V)$. The group K' acts on the first three factors, while the maximal compact subgroup $K_V = K$ of $\mathrm{SO}_0(V)$ fixing the base point z_0 acts on the last three factors. Finally, the symmetric group S_k acts on the second and the last factor.

We now give the complex $C_V^\bullet$ an associative multiplication. In order to give the complex the structure of a graded algebra we choose, as a model for the Weil representation that has an algebra structure, the Fock model $\mathcal{F}_{n,V}$, the multiplication law is multiplication of polynomials. However, it is important to observe that K' does not act on $\mathcal{F}_{n,V}$ by algebra homomorphisms (but rather by homomorphisms twisted by the character $\det^{\frac{p-q}{2}}$). Now the vector space underlying $C_V^\bullet$ is a subspace (of invariants under a group action) of a tensor product of graded algebras. Thus it remains to prove that the group acts by homomorphisms of the product multiplication.

Lemma 5.2. *The group $K' \times K \times S_k$ acts by algebra homomorphisms on the tensor product of algebras $T^\bullet(U) \otimes T^\bullet(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,V} \otimes \bigwedge^\bullet \mathfrak{p}_{\mathbb{C}}^* \otimes T^\bullet(V_{\mathbb{C}})$.*

Proof. The statement is obvious except possibly for the action of the group K'. The group K' acts on the algebra $\mathcal{F}_{n,V}$ by algebra homomorphisms twisted by the character $\det^{\frac{p-q}{2}}$. It acts on the tensor product $T^\bullet(U)$ by algebra homomorphisms twisted by the inverse character $\det^{-\frac{p-q}{2}}$; see e.g. [11] Lemma A.1. The two twists cancel on the tensor product and we find that K' acts by algebra homomorphisms. □

Sometimes it is more convenient to view an element $\varphi \in C_V^{j,r,k}$ as an element in

$$\left[\mathrm{Hom}\left(T^k(\mathbb{C}^n); T^j(U) \otimes \mathcal{W}_{n,V} \otimes \bigwedge^r \mathfrak{p}_{\mathbb{C}}^* \otimes T^k(V_{\mathbb{C}})\right)\right]^{K' \times K \times S_k}. \tag{5.5}$$

For $w \in T^k(\mathbb{C}^n)$, we write $\varphi(w)$ for its value in $T^j(U) \otimes \mathcal{W}_{n,V} \otimes \bigwedge^r \mathfrak{p}_{\mathbb{C}}^* \otimes T^k(V_{\mathbb{C}})$.

By Schur–Weyl theory (see [9], Lecture 6), we have the decomposition

$$T^k(\mathbb{C}^n)^* \simeq \bigoplus_\lambda s_{t(\lambda)}(T^k(\mathbb{C}^n)^*) \otimes V_\lambda^*. \tag{5.6}$$

Here the sum is over the Young diagrams λ with k boxes and no more than n rows, $t(\lambda)$ is a chosen standard filling of λ for each λ and V_λ is the irreducible representation of S_k

corresponding to λ. We also have the corresponding decomposition

$$T^k(V_{\mathbb{C}}) \simeq \bigoplus_{\mu} s_{t'(\mu)}(T^k(V_{\mathbb{C}})) \otimes V_\mu. \tag{5.7}$$

Combining the two decompositions we obtain

$$C_V^{j,r,k} \simeq \bigoplus_{\lambda,\mu} \left[T^j(U) \otimes \mathbb{S}_{t(\lambda)}\,(\mathbb{C}^n)^* \otimes V_\lambda^* \otimes \mathcal{W}_{n,V} \otimes \bigwedge\nolimits^r \mathfrak{p}_{\mathbb{C}}^* \otimes \mathbb{S}_{t'(\mu)}(V_{\mathbb{C}}) \otimes V_\mu\right]^{K'\times K\times S_k}. \tag{5.8}$$

Noting that

$$(V_\lambda^* \otimes V_\mu)^{S_k} \simeq \begin{cases} 0 & \text{if } \lambda \neq \mu \\ \mathbb{C} & \text{if } \lambda = \mu, \end{cases} \tag{5.9}$$

we obtain the following lemma.

Lemma 5.3.

$$C_V^{j,r,k} \simeq \bigoplus_{\lambda} \left[T^j(U) \otimes \mathbb{S}_{t(\lambda)}(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,V} \otimes \bigwedge\nolimits^r \mathfrak{p}_{\mathbb{C}}^* \otimes \mathbb{S}_{t(\lambda)}(V_{\mathbb{C}})\right]^{K'\times K}.$$

We have assumed (as we may do) that the fillings $t(\lambda)$ and $t'(\lambda)$ are the same. For the summands in the lemma we write $C_V^{j,r,t(\lambda)}$ (or just $C_V^{j,r,\lambda}$ if we do not want to specify the filling). The application of the Schur functor $\mathbb{S}^*_{t(\lambda)}(\cdot)$ on $T^k\,(\mathbb{C}^n)^*$, or equivalently applying $\mathbb{S}_{t(\lambda)}(\cdot)$ on $T^k(V_{\mathbb{C}})$, gives rise to a projection map

$$\pi_{t(\lambda)} : C_V^{j,r,k} \longrightarrow C_V^{j,r,t(\lambda)}. \tag{5.10}$$

That is,

$$\begin{aligned} \pi_{t(\lambda)} &= 1_U \otimes s_{t(\lambda)\,(\mathbb{C}^n)^*} \otimes 1_{\mathcal{W}_{n,V}} \otimes 1_{\mathfrak{p}^*} \otimes 1_V \\ &= 1_U \otimes 1_{\mathbb{C}^n} \otimes 1_{\mathcal{W}_{n,V}} \otimes 1_{\mathfrak{p}^*} \otimes s_{t(\lambda)\,V}. \end{aligned} \tag{5.11}$$

Here we use subscripts to indicate which spaces the respective identity transformations 1 operate on. We apply the harmonic projection $\mathcal{H}_V$ (see (3.8)) on the last factor to obtain $\mathbb{S}_{[t(\lambda)]}(V_{\mathbb{C}})$, and we obtain a complex $C_V^{\bullet,[t(\lambda)]}$ (or $C_V^{\bullet,[\lambda]}$) and a projection map

$$\pi_{[t(\lambda)]} : C_V^{j,r,k} \longrightarrow C^{j,r,[t(\lambda)]}. \tag{5.12}$$

That is,

$$\pi_{[t(\lambda)]} = 1_U \otimes 1_{\mathbb{C}^n} \otimes 1_{\mathcal{W}_{n,V}} \otimes 1_{\mathfrak{p}^*} \otimes s_{[t(\lambda)]\,V}. \tag{5.13}$$

Remark 5.4. We can interpret an element $\varphi \in C_V^{j,r,k}$ as a $K' \times K \times S_k$-invariant homomorphism from $T^k(\mathbb{C}^n)$ to $T^j(U) \otimes \mathcal{W}_{n,V} \otimes \bigwedge^r \mathfrak{p}_{\mathbb{C}}^* \otimes T^k(V_{\mathbb{C}})$; see (5.5). In this setting, we can interpret $\pi_{t(\lambda)}\varphi$ as the restriction of the homomorphism φ to the $\mathbb{S}_{t(\lambda)}(\mathbb{C}^n)$. From this point of view, Lemma 5.3 states that the homomorphism $\pi_{t(\lambda)}\varphi$ on $\mathbb{S}_{t(\lambda)}(\mathbb{C}^n)$ automatically takes values in $\mathcal{W}_{n,V} \otimes \bigwedge^r \mathfrak{p}_{\mathbb{C}}^* \otimes \mathbb{S}_{t(\lambda)}(V_{\mathbb{C}})$.

5.3. The face differential graded algebra $A_P^\bullet$ and the map r_P

In this section we assume that P is the stabilizer of a standard flag $E_{i_1} \subset E_{i_2} \subset \cdots \subset E_{i_k} = E_\ell = E$ and N_P is the unipotent radical of P. We let Q be the stabilizer of E. We will now construct a differential graded algebra $A_P^\bullet$, which is the relative Lie algebra version of a differential graded subalgebra of the de Rham complex of the face $e(P)$ of the Borel–Serre enlargement of D. We will continue with the notation of § 2.

We define the differential graded algebra $A_P^\bullet$ associated with the face $e(P)$ of the Borel–Serre boundary corresponding to P by

$$\begin{aligned} A_P^{j,r,k} &= \left[T^j(U) \otimes T^k\,(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,W} \otimes \bigwedge^r\,(\mathfrak{n} \oplus \mathfrak{p}_M)^*_{\mathbb{C}} \otimes T^k(V_{\mathbb{C}})\right]^{K' \times K_P \times S_k} \\ &\simeq \left[T^j(U) \otimes T^k\,(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,W} \otimes \mathcal{A}^r(e(P)) \otimes T^k(V_{\mathbb{C}})\right]^{K' \times NM \times S_k}. \end{aligned} \tag{5.14}$$

Furthermore, we define $A_P^{\bullet;\lambda}$ and $A_P^{\bullet;[\lambda]}$ as for $C_V^\bullet$.

Definition 5.5. The 'local' restriction map of de Rham algebras with coefficients

$$r_P : C_V^\bullet \to A_P^\bullet$$

of de Rham algebras with coefficients is given by

$$1 \otimes 1 \otimes r_P^{\mathcal{W}} \otimes \iota^* \otimes 1.$$

Here $\iota : \mathfrak{n} \oplus \mathfrak{m} \hookrightarrow \mathfrak{g}$ is the underlying Lie algebra homomorphism, and the map from the coefficients of $C_V^\bullet$ to the coefficients of $A_P^\bullet$ is given by the tensor product

$$1 \otimes 1 \otimes r_P^{\mathcal{W}} \otimes 1,$$

where $r_P^{\mathcal{W}} : \mathcal{W}_{n,V} \to \mathcal{W}_{n,W}$ is the restriction map of the Weil representation (see Definition 4.3). By Lemma 5.1 we therefore see that r_P is a map of complexes. We note that $r_P^{\mathcal{W}}$ is not a ring homomorphism so r_P is not a map of algebras. Since r_P commutes with the action of the symmetric group S_k, we obtain maps $C_V^{\bullet;\lambda} \to A_P^{\bullet;\lambda}$ and $C_V^{\bullet;[\lambda]} \to A_P^{\bullet;[\lambda]}$ as well, which we also denote by r_P.

Note that the induced map $\iota^* : (\mathfrak{g}/\mathfrak{k})^* \simeq \mathfrak{p}^* \to ((\mathfrak{n} \oplus \mathfrak{m})/\mathfrak{k}_M)^* \simeq (\mathfrak{n} \oplus \mathfrak{p}_M)^*$ is the composition of the isomorphism $\sigma^* : \mathfrak{p}^* \longrightarrow (\mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{p}_M)^*$ (see (2.39)), with the restriction $(\mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{p}_M)^* \to (\mathfrak{n} \oplus \mathfrak{p}_M)^*$.

Finally observe that on the level of homogeneous spaces, the map r_P arises by realizing $e(P)$ as the orbit of the base point z_0 under the group NM. So in this setting, we are no longer thinking of $e(P)$ as being at the boundary of D; we have pushed $e(P)$ far inside D.

6. Aspects of nilpotent Lie algebra cohomology and the map ι_P

6.1. An explicit constituent in the Lie algebra cohomology of $\mathfrak{n}_P$

In what follows, we discuss some aspects of the Lie algebra cohomology of the nilpotent Lie algabra $\mathfrak{n}_P$ which we need later. Some parts that we develop here could have been deduced from the general work of Kostant [19]. However, our concern here is proving that certain explicit cocycles are (non-)zero rather than computing the cohomology itself.

As before, we let P be a standard parabolic subgroup of G. Recall that we have the decomposition of vector spaces $\mathfrak{n}_P = \mathfrak{n}_{P'} \oplus \mathfrak{n}_Q$, where Q is the maximal parabolic containing P. For the two-step nilpotent algebra $\mathfrak{n}_Q$, we have the central extension $\mathfrak{z}_Q \to \mathfrak{n}_Q \to \mathfrak{n}_W$ with $\mathfrak{z}_Q \simeq \bigwedge^2 E$ and $\mathfrak{n}_W \simeq W \otimes E$. On the other hand, $\mathfrak{n}_{P'}$ is a nilpotent subgroup of $\mathfrak{sl}(E) \subset E' \otimes E$.

We assume for the following subsections that V, W, $\mathfrak{n}_P$ etc are defined over $\mathbb{C}$. We let

$$\mathcal{C}^{\bullet,\ell'} = \bigwedge\nolimits^{\bullet}(\mathfrak{n}_P^*) \otimes T^{\ell'}(V)$$

be the complex for the nilpotent cohomology with coefficients in $T^{\ell'}(V)$ and define analogously $\mathcal{C}^{r,\lambda} = \mathcal{C}^{r,A}$ and $\mathcal{C}^{r,[\lambda]} = \mathcal{C}^{r,[A]}$ for $\mathbb{S}_\lambda(V)$ and $\mathbb{S}_{[\lambda]}(V)$ respectively.

We are interested in certain cohomology classes arising from $\bigwedge^r \mathfrak{n}_W^*$. By Lemma 2.4, $\mathfrak{n}_W^* \simeq W \otimes E'$ as $\mathrm{O}(W) \times \mathrm{GL}(E)$-modules. Furthermore (see e.g. [9], p. 80),

$$\bigwedge\nolimits^r(\mathfrak{n}_W^*) \simeq \bigwedge\nolimits^r \left(W \otimes E'\right) \simeq \bigoplus_\mu \mathbb{S}_\mu(W) \otimes \mathbb{S}_{\mu'}(E'), \tag{6.1}$$

as $\mathrm{O}(W) \times \mathrm{GL}(E)$-modules. Here the sum extends over all partitions μ of r with at most $\dim W = m - 2\ell$ rows and at most $\dim E = \ell$ columns, and μ' denotes the conjugate partition of μ.

We will be mainly interested in the case $r = n\ell$. Then we can take $\mu = \ell\varpi_n = (\ell, \ell, \dots, \ell)$, and so $\mu' = n\varpi_\ell = (n, n, \dots, n)$ and $\mathbb{S}_{\mu'}(E') = \left(\bigwedge^\ell E'\right)^{\otimes n} \simeq \mathbb{C}$ is the trivial (one-dimensional) $\mathrm{SL}(E)$-module. We obtain

$$\mathbb{S}_B(W) \otimes \mathbb{S}_{B'}(E') \simeq \left[\bigwedge\nolimits^{n\ell}(W \otimes E')\right]^{\mathrm{SL}(E)} \simeq \left[\bigwedge\nolimits^{n\ell} (\mathfrak{n}_W^*)_{\mathbb{C}}\right]^{\mathrm{SL}(E)} \tag{6.2}$$

as $\mathrm{O}(W) \times \mathrm{SL}(E)$-modules. Here $B = B_{n,\ell}$ is the filling of the Young diagram associated with μ described in § 3.2.

To realize this isomorphism, we define a $\mathrm{GL}(W) \times \mathrm{GL}(E)$ intertwining map

$$\tau_{r,\ell'} : T^r(W) \otimes T^{\ell'}(W) \otimes T^r(E') \to \bigwedge\nolimits^r(W \otimes E') \otimes T^{\ell'}(V) \subset \mathcal{C}^{r,\ell'} \tag{6.3}$$

given by

$$\tau_{r,\ell'}((w_1 \otimes \cdots \otimes w_r) \otimes \bar{\mathbf{w}} \otimes (v_1' \otimes \cdots \otimes v_r')) = (w_1 \otimes v_1') \wedge \cdots \wedge (w_r \otimes v_r') \otimes \bar{\mathbf{w}},$$

where $\bar{\mathbf{w}} \in T^{\ell'}(W)$. We also write τ_r for $\tau_{r,0}$. We immediately see the following lemma.

Lemma 6.1. *The map $\tau_{r,\ell'}$ is $\mathrm{O}(W) \times \mathrm{SL}(E) \times S_{r+\ell'} \times S_r$-equivariant. Here the action of the symmetric group $S_{r+\ell'}$ (resp., S_r) is on the tensor factors involving W (resp., E').*

For $r = n\ell$, the map $\tau_{n\ell}$ realizes the isomorphism (6.2). Furthermore, we have the following lemma.

Lemma 6.2. *Let $\mathbf{w} \in T^{n\ell+\ell'}(W)$ and $\mathbf{v}' \in T^{n\ell}(E')$. Then*

$$\tau_{n\ell,\ell'}(s_{B|A}(\mathbf{w}) \otimes \mathbf{v}') \in (\mathcal{C}^{n\ell,A})^{\mathrm{SL}(E)}.$$

We view from now on $\tau_{n\ell,\ell'}$ as a map of $T^{n\ell+\ell'}(W)$ by setting

$$\tau_{n\ell,\ell'}(\mathbf{w}) := \tau_{n\ell,\ell'}(\mathbf{w} \otimes (u'_1 \otimes \cdots \otimes u'_\ell)^n).$$

We let $V^{[k]}$ ($W^{[k]}$) be the space of harmonic k-tensors in V (W), i.e., the tensors which are annihilated by all the contractions C_{ij}. We let $\mathcal{E}^k(V) \subset T^k(V)$ be the orthogonal complement of the harmonic tensors. Thus $\mathcal{E}^k(V)$ is the sum of the images of the insertion maps $E_{ij}(g_V^*) : T^{k-2}(V) \to T^k(V), 1 \leqslant i < j \leqslant k$, with the metric g_V^* of V. Similarly, we define $\mathcal{E}^k(W) \subset T^k(W)$. Note that $\mathbb{S}_{[\lambda]}(W) \subset \mathbb{S}_{[\lambda]}(V)$. However note that if $\bar{\mathbf{w}} \in T^{\ell'}(W)$ is a non-zero tensor in the orthogonal complement of $T^{[\ell']}(W)$ (i.e., spanned by tensors in the image of the inclusion with the metric for $\mathcal{E}^{\ell'}(W)$), then $\bar{\mathbf{w}}$ does *not* necessarily lie in the orthogonal complement in $T^{[\ell']}(V)$ (since the metric of V is different).

Proposition 6.3. *Let B again be the given filling of the Young diagram associated with $\ell\omega_n$ and A be a filling for λ.*

(i) *Let $\mathbf{w} \in \mathbb{S}_{B|A}(W)$. Then $\tau_{n\ell,\ell'}(\mathbf{w})$ defines a cocycle in $\mathcal{C}^{n\ell,\ell'}$. More precisely, we obtain a map*

$$\mathbb{S}_{B|A}(W) \to H^{n\ell}\left(\mathfrak{n}_P, \mathbb{S}_A(V)\right)^{\mathrm{SL}(E)}.$$

(ii) *Let $n \leqslant \left[\frac{\dim W}{2}\right]$ and let $\mathbf{w} \in \mathbb{S}_{[B|A]}(W)$. Then the cohomology class*

$$[\tau_{n\ell,\ell'}(\mathbf{w})] \in H^{n\ell}\left(\mathfrak{n}_P, \mathbb{S}_{[A]}(V)\right)^{\mathrm{SL}(E)}$$

does not vanish. Thus we obtain an embedding

$$\mathbb{S}_{[B|A]}(W) \hookrightarrow H^{n\ell}\left(\mathfrak{n}_P, \mathbb{S}_{[A]}(V)\right)^{\mathrm{SL}(E)}.$$

(iii) *Let $\mathbf{w} \in \mathbb{S}_{B|A}(W) \cap \mathcal{E}^{n\ell+\ell'}(W)$ be in the orthogonal complement of $\mathbb{S}_{[B|A]}(W)$ inside $\mathbb{S}_{B|A}(W)$. Then*

$$[\pi_{[A]} \circ \tau_{n\ell,\ell'}(\mathbf{w})] = 0$$

in $H^{n\ell}(\mathfrak{n}_P, \mathbb{S}_{[A]}(V))$. Here $\pi_{[A]}$ is the natural projection from $H^\bullet(\mathfrak{n}_P, \mathbb{S}_A(V))$ to $H^\bullet(\mathfrak{n}_P, \mathbb{S}_{[A]}(V))$ induced by the orthogonal projection $\mathbb{S}_A(V) \to \mathbb{S}_{[A]}(V)$. In particular, for $\mathbf{w} \in \mathbb{S}_{B|A}(W)$, we have

$$[\pi_{[A]} \circ \tau_{n\ell,\ell'}(\mathbf{w})] = [\tau_{n\ell,\ell'}(\pi_{[B|A]}(\mathbf{w}))].$$

The next section will be concerned with proving this proposition.

6.2. Proof of Proposition 6.3

We give V the positive definite Hermitian metric coming from the majorant $(\ ,\ )_0$. This induces positive definite metrics on $\bigwedge^2 E$, $W \otimes E$, and $E' \otimes E$ and hence an admissible metric on the entire Lie algebra complex $\mathcal{C}^{\bullet,\ell'}$, which we also denote by $(\ ,\ )_0$. Using $(\ ,\ )_0$ we obtain an adjoint d^* to the differential d on $\mathcal{C}^\bullet$. We then have the finite dimensional analogues of Hodge theory. Namely, we define the Laplacian $\Delta = dd^* + d^*d$ and say that a form in $\mathcal{C}^\bullet$ is harmonic if it is in the kernel of Delta. It is immediate that

$$\ker \Delta = \ker d \cap \ker d^*.$$

We let $\mathcal{H}^{r,\ell'}$ be the harmonic forms of degree r, given by the intersection $\ker \Delta \cap \mathcal{C}^{r,\ell'}$. In particular, we have the Hodge decomposition.

Lemma 6.4. *The space* $\mathcal{C}^{r,\bullet}$ *is the orthogonal direct sum of the exact forms* Imd*, the coexact forms* Imd^* *and the harmonic forms. Furthermore, the map* $\mathcal{H}^{r,\bullet} \to H^{r,}(\mathfrak{n}_P, \bullet)$ *is an isomorphism.*

The Lie algebra complex $\mathcal{C}^{\bullet,\ell'}$ is in fact triple graded via

$$\mathcal{C}^{r,s,t,\ell'} := \bigwedge^r (W \otimes E') \otimes \bigwedge^s \left(\bigwedge^2 E'\right) \otimes \bigwedge^t \mathfrak{n}_{P'}^* \otimes T^{\ell'}(V)$$

and we define analogously $\mathcal{C}^{r,s,t,\lambda}$ and $\mathcal{C}^{r,s,t,[\lambda]}$ for $U = \mathbb{S}_\lambda(V)$ and $U = \mathbb{S}_{[\lambda]}(V)$ respectively. Here again we have used the form $(\ ,\ )$ to identify $W^* \simeq W$ and $E^* \simeq E'$.

We now give explicit formulas for the Lie algebra differential d and its adjoint d^* on $\mathcal{C}$. We omit the proofs. We write $d = d_\mathfrak{n} + d_V$ with a 'Lie algebra part' $d_\mathfrak{n}$ and a 'coefficient' part d_V. That is,

$$d_\mathfrak{n} = d_{\mathfrak{n}_Q} + d_{\mathfrak{n}_{P'}} \quad \text{and} \quad d_V = d_V^W + d_V^E + d_V^{\mathfrak{n}_{P'}} \tag{6.4}$$

with

$$d_{\mathfrak{n}_Q} = \frac{1}{2}\sum_{\alpha,i} A(e_\alpha \otimes u_i')ad^*(e_\alpha \wedge u_i') + \frac{1}{2}\sum_{1\leqslant i<j\leqslant \ell} A(u_i' \wedge u_j')ad^*(u_i \wedge u_j) \tag{6.5}$$

and

$$d_V^W = \sum_{\alpha,i} A(e_\alpha \otimes u_i') \otimes \rho(e_\alpha \wedge u_i) \quad \text{and} \quad d_V^E = \sum_{1\leqslant i<j\leqslant \ell} A(u_i' \wedge u_j') \otimes \rho(u_i \wedge u_j).$$

Here ρ denotes the action of $\mathfrak{n}_P$ on the coefficient system $T^{\ell'}(V)$. Finally, $d_{\mathfrak{n}_{P'}} + d_V^{\mathfrak{n}_{P'}}$ is the part of the differential arising from $\mathfrak{n}_{P'}$. (We do not need it more precisely.)

Since $[\mathfrak{n}_W, \mathfrak{n}_W] \subseteq \mathfrak{z}_Q$, we first note that $d_{\mathfrak{n}_Q}$ has triple degree $(2, -1, 0)$. In particular, all elements of degree $(r, 0, t)$ are $d_{\mathfrak{n}_Q}$-closed. Accordingly, $d_{\mathfrak{n}_Q}$ is determined by its values on $\mathcal{C}^{0,s,0,\bullet}$. In fact, it suffices to consider $s = 1$.

Lemma 6.5. *Let* $v_1', v_2' \in E'$ *and* $\mathbf{v} \in T^{\ell'}(V)$*. Then*

$$d_{\mathfrak{n}_Q}((v_1' \wedge v_2') \otimes \mathbf{v}) = -\tau_2(E_{1,2}(g_W^*) \otimes (v_1' \otimes v_2')) \otimes \mathbf{v}.$$

It suffices to compute the dual $d_{\mathfrak{n}_Q}^*$ on basic forms.

Lemma 6.6.

$$d^*_{\mathfrak{n}_Q}((w_1 \otimes v'_1) \wedge \cdots \wedge (w_k \otimes v'_k) \otimes \mathbf{v})$$
$$= \sum_{i<j} \{(-1)^{i+j}(w_i, w_j)(w_1 \otimes v'_1) \wedge \cdots \wedge \widehat{(w_i \otimes v'_i)} \wedge \cdots \wedge \widehat{(w_j \otimes v'_j)} \wedge \cdots \wedge (w_k \otimes v'_k)$$
$$\otimes (v'_i \wedge v'_j)\} \otimes \mathbf{v}.$$

For the $\mathfrak{n}_{P'}$-contribution, we have the following lemma.

Lemma 6.7. *The differential* $d_{\mathfrak{n}_{P'}} + d_V^{\mathfrak{n}_{P'}}$ *has triple degree* $(0, 0, 1)$. *The adjoint action of* $\mathfrak{n}_{P'} \subset \mathfrak{sl}(E)$ *on* $\mathfrak{n}_Q = (W \otimes E) \oplus \bigwedge^2 E$ *arises from the natural action of* $\mathfrak{sl}(E)$ *on* E. *Hence* $d_{\mathfrak{n}_{P'}} + d_V^{\mathfrak{n}_{P'}}$ *vanishes on* $(\mathcal{C}^{r,s,0,\ell'})^{\mathrm{SL}(E)}$. *In particular,*

$$(d_{\mathfrak{n}_{P'}} + d_V^{\mathfrak{n}_{P'}})\tau_{n\ell,\ell'}(s_{B|A}(\mathbf{w})) = 0$$

for $\mathbf{w} \in T^{n\ell+\ell'}(W)$. *Finally, the dual* $d^*_{\mathfrak{n}_{P'}} + (d_V^{\mathfrak{n}_{P'}})^*$ *vanishes on* $\mathcal{C}^{r,s,0,\ell'}$.

We now turn our attention to d_V and d_V^*. It suffices to consider the case $\ell' = 1$.

Lemma 6.8. (i) *Let* $\mathbf{w} \in T^k(W)$, $w \in W$, *and* $\mathbf{v}' \in T^k(E')$. *Then*

$$d_V^W\left(\tau_{k,1}(\mathbf{w} \otimes w \otimes \mathbf{v}')\right) = \sum_{i=1}^{\ell} \tau_{k+1}\left((w \otimes \mathbf{w}) \otimes (u'_i \otimes \mathbf{v}')\right) \otimes u_i.$$

(ii) *Let* $\mathbf{w} \in T^k(W)$, $\mathbf{v}' \in T^k(E')$, *and* $u' \in E'$. *Then*

$$d_V^W\left(\tau_k(\mathbf{w} \otimes \mathbf{v}') \otimes u'\right) = -\tau_{k+1,1}\left(E_{1,k+1}(g_W^*)(\mathbf{w}) \otimes (u' \otimes \mathbf{v}')\right).$$

Lemma 6.9. *Let* $\mathbf{w} \in T^k(W)$, $w \in W$, *and* $\mathbf{v}' \in T^k(E')$. *Then*

$$d_V^*\left(\tau_{k,1}(\mathbf{w} \otimes w \otimes \mathbf{v}')\right)$$
$$= \sum_{i=1,\dots,k} (-1)^{i-1}(w_i, w)((w_1 \otimes v'_1) \wedge \cdots \wedge \widehat{(w_i \otimes v'_i)} \wedge \cdots \wedge (w_k \otimes v'_k)) \otimes v'_k.$$

We obtain as a consequence of Lemmas 6.6, 6.9 and 6.7 the following proposition.

Proposition 6.10. *Let* $\mathbf{w} \in W^{[k+\ell']}$ *be a harmonic* $(k + \ell')$*-tensor. Then for any* $\mathbf{v}' \in T^k(E')$, *we have*

$$d^*\tau_{k,\ell'}(\mathbf{w} \otimes \mathbf{v}') = 0.$$

We are now ready to prove Proposition 6.3. For (i), first note that the action of $\sigma \in S_{\ell'}$ on the coefficients $T^{\ell'}(V)$ commutes with the differentiation d, that is, $d \circ (1 \otimes \sigma \otimes 1) = (1 \otimes \sigma \otimes 1) \circ d$. Furthermore, in the first factor $T^{n\ell}(W)$, $\tau_{n\ell,\ell'}$ factors through c_B, the column anti-symmetrizer for Young tableau B, that is,

$\tau_{n\ell,\ell'} \circ (c_B \otimes 1) = \tau_{n\ell,\ell'}$. Combining this with Lemma 6.1 gives $\tau_{n\ell,\ell'} \circ (c_{B|A}) = (1 \otimes c_A) \circ \tau_{n\ell,\ell'}$ on $T^{n\ell+\ell'}(W)$. Therefore it suffices to show that $\tau_{n\ell,\ell'}(r_{B|A}(\mathbf{w}))$ is closed. Indeed, we have

$$d(\tau_{n\ell,\ell'}(s_{B|A}(\mathbf{w}))) = d((1 \otimes c_A) \circ \tau_{n\ell,\ell'}(r_{B|A}(\mathbf{w}))) = (1 \otimes c_A) \circ d(\tau_{n\ell,\ell'}(r_{B|A}(\mathbf{w}))).$$

Furthermore, it suffices to establish closedness for $n = 1$. Indeed, if the Young diagram A arises from the partition $(\ell'_1, \ell'_2, \dots, \ell'_n)$ of ℓ', we write $\mathbf{w} = \mathbf{w}_1 \otimes \cdots \otimes \mathbf{w}_n \in T^{n\ell}(W)$ with $\mathbf{w}_i \in T^{\ell}(W)$ and $\bar{\mathbf{w}} = \bar{\mathbf{w}}_1 \otimes \cdots \otimes \bar{\mathbf{w}}_n$ with $\bar{\mathbf{w}}_i \in T^{\ell'_i}(W)$. We then have a natural product decomposition

$$\tau_{n\ell,\ell'}(\mathbf{w} \otimes \bar{\mathbf{w}}) = \tau_{\ell,\ell'_1}(\mathbf{w}_1 \otimes \bar{\mathbf{w}}_1) \wedge \cdots \wedge \tau_{\ell,\ell'_n}(\mathbf{w}_n \otimes \bar{\mathbf{w}}_2), \tag{6.6}$$

for which d acts as a graded derivation. Also note that $d_{\mathfrak{n}_Q}$ vanishes on the image of $\tau_{n\ell,\ell'}$ and by Lemma 6.7 so does the $\mathfrak{n}_{P'}$-contribution. Now for $n = 1$, using Lemma 6.8(i), we see that applying d_V to $\tau_{\ell,\ell'}(\mathbf{w})$ with $\mathbf{w} \in \mathrm{Sym}^{\ell+\ell'}(W)$ gives rise to a map

$$\begin{aligned} \mathrm{Sym}^{\ell+\ell'}(W) &\to \bigoplus_{i=1}^{\ell} \bigwedge^{\ell+1}(W \otimes E') \otimes (E'_i \otimes T^{\ell'-1}(W)) \\ &= \bigoplus_{i=1}^{\ell} \bigoplus_{C} \mathbb{S}_C(W) \otimes \mathbb{S}_{C'}(E') \otimes (E'_i \otimes T^{\ell'-1}(W)). \end{aligned} \tag{6.7}$$

Here $E'_i = \mathbb{C}u'_i$, and the sum extends over all Young diagrams C of size $\ell + 1$, which have at least two rows (otherwise the dual diagram C' would have at least $\ell + 1$ rows, which is impossible as $\dim E' = \ell$). By the Littlewood–Richardson rule we now see that in the decomposition of $\mathbb{S}_C(W) \otimes T^{\ell'-1}(W)$ into irreducibles, only Young diagrams with at least two rows can occur. Hence $\mathrm{Sym}^{\ell+\ell'}(W)$ does not occur on the right hand side of (6.7), and the map vanishes identically. This proves Proposition 6.3(i).

Proposition 6.3(ii) now follows immediately from Proposition 6.10 and Lemma 6.4.

For (iii), it suffices to show that for any $\mathbf{w} \in T^{n\ell+\ell-2}$, the form $\pi_{[A]} \circ \tau_{n\ell,\ell'}(s_{B|A}(E_{i,j}(g^*_W)(\mathbf{w})))$ is exact. For this, it suffices to show that $\tau_{n\ell,\ell'}(r_{B|A}(E_{i,j}(g^*_W)(\mathbf{w})))$ is exact up to terms involving the inclusion of the metric g^*_V into the coefficient system. The product decomposition (6.6) reduces the claim to the cases of $n = 1$ (if the metric g^*_W occurs in one factor for (6.6)) or $n = 2$ (if g^*_W occurs in two factors). It is not too hard but a bit tedious to explicitly construct primitives for these cases. We omit this.

6.3. The map ι_P

We now assume again that all objects are defined over $\mathbb{R}$. We construct a map $\iota_P : C^\bullet_W \hookrightarrow A^\bullet_P$ of complexes.

We let U, U' be two representations of G and $T : U' \to U$ be a G-intertwiner. We let $\mathcal{C}^\bullet(\mathfrak{n}_P, U) = (\bigwedge^\bullet \mathfrak{n}_P^*) \otimes U$ be the complex computing the nilpotent cohomology $\mathcal{H}^s(\mathfrak{n}_P, U)$, and we let $\mathcal{C}^\bullet_{\mathrm{closed}}(\mathfrak{n}_P, U)$ be the subspace of cocycles in $\mathcal{C}^\bullet(\mathfrak{n}_P, U)$.

Lemma 6.11. *Define a map*

$$\eta^{r,s}: \left[\bigwedge^r(\mathfrak{p}_M^*) \otimes \left(\left(\bigwedge^s \mathfrak{n}_P^*\right) \otimes U'\right)\right]^{K_P} \to \left[\bigwedge^{r+s}(\mathfrak{p}_M^* \oplus \mathfrak{n}_P^*) \otimes U\right]^{K_P}$$

by

$$\eta^{r,s}(\omega^{(r)} \otimes (\omega^{(s)} \otimes u')) = (\omega^{(r)} \wedge \omega^{(s)}) \otimes T(u').$$

Then $\eta^{r,s}$ induces a map of relative Lie algebra complexes

$$\eta : \mathcal{C}^\bullet(\mathfrak{m}, \mathfrak{k}_P; \mathcal{C}^s_{closed}(\mathfrak{n}_P, U')) \longrightarrow \mathcal{C}^{\bullet+s}(\mathfrak{p}, \mathfrak{k}_P; U)$$

and the induced map in cohomology factors through $H^\bullet\left(\mathfrak{m}, \mathfrak{k}_P; H^s(\mathfrak{n}_P, U')\right)$.

Proof. This essentially is [17], Lemma 2.6 (see also [29], § 2), together with the standard spectral sequences argument in this context. Note that Harder actually considers instead of cocycles in $\mathcal{C}(\mathfrak{n}_P, U')$ the nilpotent cohomology group $\mathcal{H}^s(\mathfrak{n}_P, U')$ realized as a subspace in $\mathcal{C}(\mathfrak{n}_P, U')$ by harmonic forms as discussed in § 6. □

Definition 6.12. We define the map ι_P on $C_W^{j,r,k}$ as follows. In fact, it is defined on the underlying tensor spaces without taking the group invariants. First we set ι_P to be zero if $k < n\ell$. If $k \geqslant n\ell$ we split the two tensor factors:

$$T^k(\mathbb{C}^n)^* = T^{n\ell}(\mathbb{C}^n)^* \otimes T^{k-n\ell}(\mathbb{C}^n)^* \quad \text{and} \quad T^k(W_\mathbb{C}) = T^{n\ell}(W_\mathbb{C}) \otimes T^{k-n\ell}(W_\mathbb{C}).$$

We define ι_P on tensors which are decomposable relative to these two splittings. We let $u_1 = \theta_1 \wedge \cdots \wedge \theta_n$ be the standard generator of $U = \bigwedge^n(\mathbb{C}^n)^*$ (with the twisted K'-action). Let $u_1^j \otimes x \otimes f \otimes \omega \otimes w$ be a single tensor component of an element in $C_W^{j,r,k}$ and assume that $k \geqslant n\ell$. Assume that x and w are decomposable, that is

$$x = x_1 \otimes x_2 \in T^{n\ell}(\mathbb{C}^n)^* \otimes T^{k-n\ell}(\mathbb{C}^n)^* \quad \text{and} \quad w = w_1 \otimes w_2 \in T^{n\ell}(W_\mathbb{C}) \otimes T^{k-n\ell}(W_\mathbb{C}).$$

Then we define

$$\begin{aligned}
&\iota_P(u_1^j \otimes x \otimes f \otimes \omega \otimes w) \\
&\quad = (-1)^{n\ell(\frac{(q-\ell)(n-1)}{2}+1)} \eta^{r,n\ell}((u_1^j \otimes s_B^*(x_1)) \otimes x_2 \otimes f \otimes \omega \otimes \tau_{n\ell}(w_1) \otimes w_2) \\
&\quad\ \in T^{j+\ell}(U) \otimes T^{k-n\ell}(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,W} \otimes \bigwedge^r(\mathfrak{p}_W^*)_\mathbb{C} \otimes \bigwedge^{n\ell}(\mathfrak{n}_W^*)_\mathbb{C} \otimes T^{k-n\ell}(W_\mathbb{C}).
\end{aligned}$$

Note here that by Lemma 3.2, we see that $\mathbb{S}_B(\mathbb{C}^n)^* = s_B^* T^{n\ell}(\mathbb{C}^n)^* \simeq T^\ell(U)[0]$ and therefore $u_1^j \otimes s_B^*(x_1)$ lies in $T^{j+\ell}(U)[-\frac{p-q}{2}]$ and is zero if and only if $s_B^*(x_1) = 0$.

Proposition 6.13. *ι_P is a map of complexes*

$$\iota_P : C_W^{j,r,k} \to A_P^{j+\ell,r+n\ell,k-n\ell}.$$

Proof. In view of Lemma 6.11, it suffices to show that the map on $C_W^{j,r,k}$ to

$$\mathcal{C}^r\left(\mathfrak{m}, \mathfrak{k}_P; \mathcal{C}^{n\ell}(\mathfrak{n}_P, T^{k-n\ell}(W_\mathbb{C})) \otimes T^{j+\ell}(U)\left[-\frac{p-q}{2}\right] \otimes \mathcal{W}_{n,W}\right) \tag{6.8}$$

induced by

$$u_1^j \otimes x \otimes f \otimes \omega \otimes w \mapsto (u_1^j \otimes s_B^*(x_1)) \otimes x_2 \otimes f \otimes \omega \otimes \tau_{n\ell}(w_1) \otimes w_2 \tag{6.9}$$

gives a cocycle for the nilpotent $\mathfrak{n}_P$-complex. Going through the proof of Proposition 6.3(i), we see that the composition of the $\mathfrak{n}_P$-differential with (6.9) factors when viewed as a map on $T^k(W_{\mathbb{C}})$ through representations $\mathbb{S}_C(W_{\mathbb{C}})$ with C having at least $n+1$ rows. But now by Lemma 5.3 such representations do not occur in $C_W^{j,r,k}$. □

The reader can easily check from the definition that ι_P satisfies the following properties.

Lemma 6.14. (1) *ι_P is a $[T(U) \otimes \mathcal{W}_{n,W} \otimes \bigwedge \mathfrak{p}_W^*]^{K' \times K_W}$-module homomorphism. That is,*

$$\iota_P(\varphi^W_{j',r',0} \cdot \varphi^W_{j,r,k}) = \varphi^W_{j',r',0} \cdot \iota_P(\varphi^W_{j,r,k})$$

for $\varphi^W_{j',r',0} \in C_W^{j',r',0}$ and $\varphi^W_{j,r,k} \in C_W^{j,r,k}$.

(2) *$\iota_P(\varphi^W_{j,r,k})$ is zero if $k < n\ell$.*

(3) *Suppose that $\varphi^W_{j,r,k} \in C_W^{j,r,k}$ with $k \geqslant n\ell$ and $\varphi^W_{j',r',\ell'} \in C_W^{j',r',\ell'}$. Then*

$$\iota_P(\varphi^W_{j,r,k} \cdot \varphi^W_{j',r',\ell'}) = \iota_P(\varphi^W_{j,r,k}) \cdot \varphi^W_{j',r',\ell'}.$$

(4) *Let $x \in T^{n\ell}(\mathbb{C}^n)^*$ and $w \in T^{n\ell}(W_{\mathbb{C}})$. Then*

$$\iota_P(1_U \otimes x \otimes 1_{\mathcal{F}} \otimes 1_{\mathfrak{p}_W^*} \otimes w) = x(\varepsilon_B)(u_\ell \otimes 1_{\mathbb{C}^n} \otimes 1_{\mathcal{F}} \otimes 1_{\mathfrak{p}_W^*} \otimes \tau_{n\ell}(w) \otimes 1_{T(V_{\mathbb{C}})}).$$

Proposition 6.15. *Let $k = n\ell + \ell'$ as above. Let λ be a dominant weight of $\mathrm{GL}_n(\mathbb{C})$, and we let A be a standard filling of the associated Young diagram $D(\lambda)$. We let $B|A$ be the associated filling for the weight $\ell\varpi_n + \lambda$; see § 3.*

(i) *Then the preimage of $A_P^{j+\ell,r+n\ell,A}$ under ι_P lies in $C_W^{j,r,B|A}$; i.e.,*

$$\iota_P^{-1}(A_P^{j+\ell,r+n\ell,A}) = C_W^{j,r,B|A}.$$

Moreover, if $\iota_P(\varphi') = \varphi$ for $\varphi' \in C_W^{j,r,n\ell+\ell'}$ and $\varphi \in A_P^{j+\ell,r+n\ell,\ell'}$, then

$$\pi_A(\varphi) = \iota_P(\pi_{B|A}(\varphi')).$$

Here $\pi_{B|A}$ is the projection from $C_W^{j,r,n\ell+\ell'}$ to $C_W^{j,r,B|A}$ (see (5.10)), and π_A is that from $A_P^{j+\ell,r+n\ell,\ell'}$ to $A_P^{j+\ell,r+n\ell,A}$.

(ii) *Let $\varphi \in A_P^{j+\ell,r+n\ell,[A]}$ be a closed form such that $\iota_P(\varphi') = \varphi$ for some $\varphi' \in C_W^{j,r,B|A}$. Let $\pi_{[B|A]}$ be the projection from $C_W^{j,r,B|A}$ to $C_W^{j,r,[B|A]}$. Then the cohomology class $[\varphi]$ satisfies*

$$[\varphi] = [\iota_P(\pi_{[B|A]}(\varphi'))].$$

Proof. (i) We first observe that ι_P is invariant under s_B in the $T^{n\ell}(W)$-factor and also $s(B^*)$-invariant in the $T^{n\ell}(\mathbb{C}^n)^*$-factor, that is,

$$\begin{aligned}\iota_P &= \iota_P \circ (1_U \otimes 1_{T^{n\ell}(\mathbb{C}^n)^*} \otimes 1_{T^{\ell'}(\mathbb{C}^n)^*} \otimes 1_{\mathcal{W}} \otimes 1_{\mathfrak{p}_W^*} \otimes s_B \otimes 1_{T^{\ell'}(W)}) \\ &= \iota_P \circ (1_U \otimes s(B^*) \otimes 1_{T^{\ell'}(\mathbb{C}^n)^*} \otimes 1_{\mathcal{W}} \otimes 1_{\mathfrak{p}_W^*} \otimes 1_{T^{n\ell}(W)} \otimes 1_{T^{\ell'}(W)}).\end{aligned}$$

Taking the $S_{\ell'}$-invariance into account, we see that ι_P maps

$$\left[T^j(U) \otimes \mathbb{S}_B(\mathbb{C}^n)^* \otimes \mathbb{S}_A(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,W} \otimes \bigwedge\nolimits^r(\mathfrak{p}_W^*)_{\mathbb{C}} \otimes \mathbb{S}_B(W_{\mathbb{C}}) \otimes \mathbb{S}_A(W_{\mathbb{C}})\right]^{K' \times K_W} \tag{6.10}$$

to $A^{j+2\ell,r+n\ell,A}$. But now we have the following lemma.

Lemma 6.16.

$$\begin{aligned}&\left[T^j(U) \otimes \mathbb{S}_B(\mathbb{C}^n)^* \otimes \mathbb{S}_A(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,W} \otimes \bigwedge\nolimits^r(\mathfrak{p}_W^*) \otimes \mathbb{S}_B(W_{\mathbb{C}}) \otimes \mathbb{S}_A(W_{\mathbb{C}})\right]^{K' \times K_W} \\ &\quad = C_W^{j,r,B|A}.\end{aligned} \tag{6.11}$$

Proof. In (6.11), we first observe that $\mathbb{S}_B(\mathbb{C}^n)^* \otimes \mathbb{S}_A(\mathbb{C}^n)^* = \mathbb{S}_{B|A}(\mathbb{C}^n)^*$ as subspaces of $T^{n\ell+\ell'}(\mathbb{C}^n)$; see Corollary 3.4. But then by Schur–Weyl theory (see Lemma 5.3 or Remark 5.4), we can now replace $\mathbb{S}_B(W_{\mathbb{C}}) \otimes \mathbb{S}_A(W_{\mathbb{C}})$ with its subspace $\mathbb{S}_{B|A}(W_{\mathbb{C}})$ in (6.11), that is, the left hand side in (6.11) is equal to $C_W^{j,r,B|A}$. □

From this we obtain Proposition 6.15(i). Proposition 6.15(ii) follows from Proposition 6.3(iii) and Lemma 6.11. □

7. Special Schwartz forms

Again V will denote a real quadratic space of dimension m and signature (p, q).

7.1. Construction of the special Schwartz forms

We recall the construction in [11] of the special Schwartz forms $\varphi_{nq,\ell'}$, $\varphi_{nq,\lambda}$, and $\varphi_{nq,[\lambda]}$, which define cocycles in $C_V^{\bullet;\ell'}$, $C_V^{\bullet;\lambda}$, and $C_V^{\bullet;[\lambda]}$ respectively. It will be more convenient to use the model $C_V^{\bullet}$ consisting of homomorphisms on $T^{\ell'}(\mathbb{C}^n)$ (and its subspaces $\mathbb{S}_{t(\lambda)}(\mathbb{C}^n)$); see (5.5) and Remark 5.4. We will initially use the Schrödinger model $\mathcal{S}(V^n)$.

In [11], we construct for $n \leqslant p$ a family of Schwartz forms $\varphi_{nq,\ell'}$ on V^n such that $\varphi_{nq,\ell'} \in C_V^{q,nq,\ell'}$. So

$$\begin{aligned}\varphi_{nq,\ell'} &\in [\mathrm{Hom}(T^{\ell'}(\mathbb{C}^n), T^q(U) \otimes \mathcal{S}(V^n) \otimes \mathcal{A}^{nq}(D) \otimes T^{\ell'}(V_{\mathbb{C}}))]^{K' \times G \times S_{\ell'}} \\ &\simeq [\mathrm{Hom}(T^{\ell'}(\mathbb{C}^n), T^q(U) \otimes \mathcal{S}(V^n) \otimes \bigwedge\nolimits^{nq}(\mathfrak{p}_{\mathbb{C}}^*) \otimes T^{\ell'}(V_{\mathbb{C}}))]^{K' \times K \times S_{\ell'}}.\end{aligned} \tag{7.1}$$

These Schwartz forms are generalizations of the Schwartz forms considered by Kudla and Millson [20–22]. Under the isomorphism in (7.1), the standard Gaussian $\varphi_0(\mathbf{x}) = 1 \otimes e^{-\pi\, tr\,(\mathbf{x},\mathbf{x})_{z_0}} \in \left[T^0(U) \otimes \mathcal{S}(V^n)\right]^{K' \times K}$ corresponds to

$$\varphi_0(\mathbf{x}, z) = 1 \otimes e^{-\pi\, tr\,(\mathbf{x},\mathbf{x})_z} \in [T^0(U) \otimes \mathcal{S}(V^n) \otimes C^{\infty}(D)]^{K' \times G}.$$

Definition 7.1. Let $n \leqslant p$. The form $\varphi_{nq,0}$ with trivial coefficients is given by applying the operator

$$\mathcal{D} = \frac{1}{2^{nq/2}} A(u_1^q) \otimes \prod_{i=1}^{n} \prod_{\mu=p+1}^{p+q} \left[\sum_{\alpha=1}^{p} \left(x_{\alpha i} - \frac{1}{2\pi} \frac{\partial}{\partial x_{\alpha i}} \right) \otimes A(\omega_{\alpha\mu}) \right]$$

to φ_0:

$$\varphi_{nq,0} = \mathcal{D}(\varphi_0) \in C_V^{q,nq,0} = \left[T^q(U) \otimes \mathcal{S}(V^n) \otimes \bigwedge\nolimits^{nq}(\mathfrak{p}_{\mathbb{C}}^*) \right]^{K' \times K}.$$

Here as before $A(\cdot)$ denotes left multiplication and u_1 is the generator of $U = \bigwedge^n(\mathbb{C}^n)^*$. Furthermore, Theorem 3.1 of [20] implies that $\varphi_{nq,0}$ is indeed K'-invariant.

For $T(V_{\mathbb{C}})$, we define another K-invariant differential operator $\mathcal{D}_i'$ which acts on

$$\mathcal{S}(V^n) \otimes \bigwedge\nolimits^{\bullet}(\mathfrak{p}_{\mathbb{C}}^*) \otimes T(V_{\mathbb{C}}) \tag{7.2}$$

by

$$\mathcal{D}_i' = \frac{1}{2} \sum_{\alpha=1}^{p} \left(x_{\alpha i} - \frac{1}{2\pi} \frac{\partial}{\partial x_{\alpha i}} \right) \otimes 1 \otimes A(e_\alpha). \tag{7.3}$$

Let $I = (i_1, \dots, i_{\ell'}) \in \{1, \dots, n\}^{\ell'}$ be a multi-index of length ℓ' and write

$$\varepsilon_I = \varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_{\ell'}} \tag{7.4}$$

for the corresponding standard basis element of $T^{\ell'}(\mathbb{C}^n)$. Then for $\varepsilon_I \in T^{\ell'}(\mathbb{C}^n)$, we define an operator by

$$\mathcal{T}_{\ell'}(\varepsilon_I) = \mathcal{D}_{i_1}' \circ \cdots \circ \mathcal{D}_{i_{\ell'}}' \tag{7.5}$$

extending $\mathcal{T}_{\ell'}$ linearly to $T^{\ell'}(\mathbb{C}^n)$.

Definition 7.2. Define

$$\varphi_{nq,\ell'} \in C_V^{q,nq,\ell'} = \operatorname{Hom}_{\mathbb{C}}\left(T^{\ell'}(\mathbb{C}^n), T^q(U) \otimes \mathcal{S}(V^n) \otimes \bigwedge\nolimits^{nq}(\mathfrak{p}_{\mathbb{C}}^*) \otimes T^{\ell'}(V_{\mathbb{C}}) \right)^{K' \otimes K \otimes S_{\ell'}}$$

by

$$\varphi_{nq,\ell'}(w) = \mathcal{T}_{\ell'}(w) \varphi_{nq,0}$$

for $w \in T^{\ell'}(\mathbb{C}^n)$. We put $\varphi_{nq,\ell'} = 0$ for $\ell' < 0$. Here the $S_{\ell'}$-invariance of $\varphi_{nq,\ell'}$ is shown in Proposition 5.2 in [11], while the K'-invariance is Theorem 5.6 in [11].

Using the projections $\pi_{t(\lambda)}$ and $\pi_{[t(\lambda)]}$ (see (5.10) and (5.12)), we can therefore give the following definitions.

Definition 7.3. For any standard filling $t(\lambda)$ of $D(\lambda)$, we define

$$\varphi_{nq,t(\lambda)} = \pi_{t(\lambda)} \varphi_{nq,\ell'} \in C_V^{q,nq,\lambda},$$
$$\varphi_{nq,[t(\lambda)]} = \pi_{[t(\lambda)]} \varphi_{nq,\ell'} \in C_V^{q,nq,[\lambda]}.$$

We write $\varphi_{nq,\lambda}$ and $\varphi_{nq,[\lambda]}$ if we do not want to specify the standard filling.

Proposition 7.4 (Theorem 5.7 [11]). *The form $\varphi_{nq,\ell'}$ is closed. That is, for $w \in T^{\ell'}(\mathbb{C}^n)$ and $\mathbf{x} \in V^n$, the differential form*

$$\varphi_{nq,\ell'}(w)(\mathbf{x}) \in \left[A^{nq}\left(D; T^{\ell'}(V_{\mathbb{C}})\right)\right]^G$$

is closed.

7.2. Explicit formulas

We give more explicit formulas for $\varphi_{nq,\ell'}$ in the various models of the Weil representation.

7.2.1. The Schrödinger model. We introduce multi-indices $\underline{\alpha_i} = (\alpha_{i1}, \dots, \alpha_{iq})$ of length q (typically) with $1 \leqslant i \leqslant n$ and $\underline{\beta} = (\beta_1, \dots, \beta_{\ell'})$ of length ℓ' (typically) with values in $\{1, \dots, p\}$ (typically). Note that we suppressed their length from the notation. We also write $\alpha = (\alpha_{ij})$ for the $n \times q$ matrix of indices. With I as above, we then define

$$\begin{aligned} \omega_{\underline{\alpha_i}} &= \omega_{\alpha_{i1}p+1} \wedge \cdots \wedge \omega_{\alpha_{iq}p+q} \\ \omega_\alpha &= \omega_{\underline{\alpha_1}} \wedge \cdots \wedge \omega_{\underline{\alpha_n}} \\ \mathcal{H}_{\underline{\alpha_i}} &= \mathcal{H}_{\alpha_{i1}i} \circ \cdots \circ \mathcal{H}_{\alpha_{iq}i}, \\ \mathcal{H}_\alpha &= \mathcal{H}_{\underline{\alpha_1}} \circ \cdots \circ \mathcal{H}_{\underline{\alpha_n}} \\ \mathcal{H}_{\underline{\beta},I} &= \mathcal{H}_{\beta_1 i_1} \circ \cdots \circ \mathcal{H}_{\beta_{\ell'} i_{\ell'}} \\ e_{\underline{\beta}} &= e_{\beta_1} \otimes \cdots \otimes e_{\beta_{\ell'}}. \end{aligned} \tag{7.6}$$

Let $1 \leqslant \gamma \leqslant p$ and $1 \leqslant j \leqslant n$. For I, α, and $\underline{\beta}$ fixed, let

$$\delta_{\gamma j} = \#\{k;\, \alpha_{kj} = \gamma\} + \#\{k;\, (\beta_k, i_k) = (\gamma, j)\}. \tag{7.7}$$

This defines a $p \times n$ matrix $\Delta_{\alpha,\underline{\beta},I} = \Delta_{\alpha,\underline{\beta},I;+}$ and Schwartz functions $\varphi_{\Delta_{\alpha,\underline{\beta},I}}$ as in Definition 4.4.

Lemma 7.5. *The Schwartz form $\varphi_{nq,\ell'}(\varepsilon_I)$ is given by*

$$\varphi_{nq,\ell'}(\varepsilon_I) = \frac{1}{2^{nq/2+\ell'}} \sum_{\alpha,\underline{\beta}} u_1^q \otimes \varphi_{\Delta_{\alpha,\underline{\beta},I}} \otimes \omega_\alpha \otimes e_{\underline{\beta}}.$$

Proof. With the above notation we have

$$\begin{aligned} \varphi_{nq,\ell'}(\varepsilon_I) &= \frac{1}{2^{nq/2+\ell'}} \sum_{\substack{\underline{\alpha_1},\dots,\underline{\alpha_n} \\ \underline{\beta}}} u_1^q \otimes ((\mathcal{H}_{\underline{\alpha_1}} \circ \cdots \circ \mathcal{H}_{\underline{\alpha_n}} \circ \mathcal{H}_{\underline{\beta},I})\varphi_0) \otimes (\omega_{\underline{\alpha_1}} \wedge \cdots \wedge \omega_{\underline{\alpha_q}}) \otimes e_{\underline{\beta}} \\ &= \frac{1}{2^{nq/2+\ell'}} \sum_{\alpha,\underline{\beta}} u_1^q \otimes (\mathcal{H}_\alpha \circ \mathcal{H}_{\underline{\beta},I})\varphi_0 \otimes \omega_\alpha \otimes e_{\underline{\beta}}. \end{aligned} \tag{7.8}$$

But now we easily see that

$$(\mathcal{H}_\alpha \circ \mathcal{H}_{\underline{\beta},I})\varphi_0(\mathbf{x}) = \prod_{\gamma=1}^{p}\prod_{j=1}^{n} \widetilde{H}_{\delta_{\gamma,j}}(x_{\gamma j})\varphi_0(\mathbf{x}), \tag{7.9}$$

which gives the assertion. □

7.2.2. The mixed model. We now describe the Schwartz form $\varphi_{nq,\ell'}$ in the mixed model. We describe this in terms of the individual components $\varphi_{\Delta_{\alpha,\underline{\beta},I}}$ described in the Schrödinger model. From Lemma 4.7, Lemma 4.8, and Proposition 4.9 we see the following lemma.

Lemma 7.6.

$$\widehat{\varphi^V_{\Delta_{\alpha,\underline{\beta},I}}}\begin{pmatrix}\xi\\ \mathbf{x}_W\\ u'\end{pmatrix} = \varphi^W_{\Delta'_{\alpha,\underline{\beta},I}}(\mathbf{x}_W)\widehat{\varphi^E_{\Delta''_{\alpha,\underline{\beta},I}}}(\xi,u').$$

Note that $\varphi^W_{\Delta'_{\alpha,\underline{\beta},I}}$ only depends on the indices α_{ij}, β_j such that $\alpha_{ij}, \beta_j \geqslant \ell+1$, while $\widehat{\varphi^E_{\Delta''_{\alpha,\underline{\beta},I}}}$ only depends on the indices α_{ij}, β_j such that $\alpha_{ij}, \beta_j \leqslant \ell$. In particular, if all $\alpha_{ij}, \beta_j \geqslant \ell+1$, then

$$\widehat{\varphi^V_{\Delta_{\alpha,\underline{\beta},I}}}\begin{pmatrix}\xi\\ \mathbf{x}_W\\ 0\end{pmatrix} = \varphi^W_{\Delta'_{\alpha,\underline{\beta},I}}(\mathbf{x}_W)\varphi^E_0(\xi,0).$$

On the other hand, if one of the α_{ij}, β_j is less or equal to ℓ, then

$$\widehat{\varphi}^E_{\Delta''_{\alpha,\underline{\beta},I}}(0,0) = \widehat{\varphi^V_{\Delta_{\alpha,\underline{\beta},I}}}\begin{pmatrix}0\\ \mathbf{x}_W\\ 0\end{pmatrix} = 0.$$

7.2.3. The Fock model. In the Fock model, $\varphi_{nq,\ell'}$ looks particularly simple. We have the following lemma.

Lemma 7.7.

$$\varphi_{nq,\ell'}(\varepsilon_I) = \frac{1}{2^{nq/2+\ell'}}\left(\frac{1}{2\pi i}\right)^{nq+\ell'} \sum_{\substack{\underline{\alpha_1},\ldots,\underline{\alpha_n}\\ \underline{\beta}}} u_1^q \otimes z_{\underline{\alpha_1},1}\cdots z_{\underline{\alpha_n},n}\cdot z_{\underline{\beta},I}\otimes(\omega_{\underline{\alpha_1}}\wedge\cdots\wedge\omega_{\underline{\alpha_q}})\otimes e_{\underline{\beta}}.$$

Here we use the notational conventions of (7.6) and in addition

$$z_{\underline{\alpha_j},j} = z_{\alpha_{j1}j}\cdots z_{\alpha_{jq}j}, \qquad z_{\underline{\beta},I} = z_{\beta_1 i_1}\cdots z_{\beta_{\ell'} i_{\ell'}}. \tag{7.10}$$

7.3. The forms $\varphi_{0,k}$

We now define another class of special forms. We will only do this in the Fock model.

Definition 7.8. We define $\varphi_{0,k} \in \mathrm{Hom}(T^k(\mathbb{C}^n); T^0(U) \otimes \mathcal{F}_{n,V} \otimes T^k(V_\mathbb{C}))$ by

$$\varphi_{0,k}(\varepsilon_I) = \frac{1}{2^k}\left(\frac{1}{2\pi i}\right)^k \sum_{\underline{\beta}} 1 \otimes z_{\underline{\beta},I} \otimes e_{\underline{\beta}}. \tag{7.11}$$

Remark 7.9. The element $\varphi_{0,k}$ is the image of the operator $\mathcal{T}_k$ (see (7.5)) applied to the Gaussian φ_0 under the intertwiner from the Schrödinger to the Fock model. Also note that $\varphi_{0,k}$ is *not* closed, and hence they do not define cocycles.

We also leave the proof of the following lemma to the reader. It follows (in large part) from Remark 7.9 and the corresponding properties of $\varphi_{nq,\ell'}$.

Lemma 7.10.

$$\varphi_{0,k} \in [T^0(U) \otimes T^k(\mathbb{C}^n)^* \otimes \mathcal{F}_{n,V} \otimes T^k(V_\mathbb{C})]^{K' \times K \times S_k},$$

i.e.,

$$\varphi_{0,k} \in C_V^{0,0,k}.$$

From Lemma 7.7, we immediately see the following lemma.

Lemma 7.11.

$$\varphi_{nq,\ell'} = \varphi_{nq,0} \cdot \varphi_{0,\ell'}$$

and

$$\varphi_{0,k_1} \cdot \varphi_{0,k_2} = \varphi_{0,k_1+k_2},$$

where the multiplication is the one in $C_V^\bullet$.

Remark 7.12. This kind of product decomposition for $\varphi_{nq,\ell'}$ and $\varphi_{0,k}$ in Lemma 7.11 only holds in the Fock model. In the Schrödinger model this only makes sense in terms of the operators $\mathcal{D}$ and $\mathcal{T}_{\ell'}$ of Definitions 7.1 and 7.2 respectively.

We apply the projection $\pi_{t(\lambda)}$ (see (5.10)) to define $\varphi_{0,t(\lambda)}$.

Definition 7.13.

$$\varphi_{0,t(\lambda)} := \pi_{t(\lambda)}\varphi_{0,k} \in C_V^{0,0,t(\lambda)}.$$

The following product formula will be important later.

Proposition 7.14. *Let $A = t(\lambda)$ be a filling of the Young diagram associated with λ and let $B = B_{n,\ell}$ be the filling of the $n \times \ell$ rectangular Young diagram introduced in § 3. Then*

$$\varphi_{0,B}^W \cdot \varphi_{0,A}^W = \varphi_{0,B|A}^W.$$

The proposition will follow from the next two lemmas.

Lemma 7.15. *Both* $\varphi^W_{0,B}\cdot\varphi^W_{0,A}$ *and* $\varphi^W_{0,B|A}$ *are elements of*

$$C_W^{0,B|A,0} = [T^0(U)\otimes \mathbb{S}_{B|A}(\mathbb{C}^n)^* \otimes \mathcal{F}_{n,W}\otimes \mathbb{S}_{B|A}(W_{\mathbb{C}})]^{K'\times K_W}.$$

Proof. Since $\mathbb{S}_B(\mathbb{C}^n)^*\otimes\mathbb{S}_A(\mathbb{C}^n)^* = \mathbb{S}_{B|A}(\mathbb{C}^n)^*$ as subspaces of $T^{n\ell+\ell'}(\mathbb{C}^n)$ (see Corollary 3.4), the claim follows in the same way as Lemma 6.16. □

Lemma 7.16.

$$(\varphi^W_{0,B}\cdot\varphi^W_{0,A})(s_B\varepsilon_B\otimes s_A\varepsilon_A) = \varphi^W_{0,B|A}(s_B\varepsilon_B\otimes s_A\varepsilon_A).$$

Proof. This is a short calculation using Lemmas 3.3 and 7.11. Indeed, we have

$$\begin{aligned}(\varphi^W_{0,B}\cdot\varphi^W_{0,A})(s_B\varepsilon_B\otimes s_A\varepsilon_A) &= (\varphi^W_{0,n\ell}\cdot\varphi^W_{0,\ell'})(s_B\varepsilon_B\otimes s_A\varepsilon_A)\\ &= \varphi^W_{0,n\ell+\ell'}(s_B\varepsilon_B\otimes s_A\varepsilon_A) = c(A,B)\varphi^W_{0,n\ell+\ell'}(s_{B|A}\varepsilon_{B|A})\\ &= c(A,B)\varphi^W_{0,B|A}(s_{B|A}\varepsilon_{B|A}) = \varphi^W_{0,B|A}(s_B\varepsilon_B\otimes s_A\varepsilon_A). \quad\square\end{aligned}$$

Now we can prove Proposition 7.14. By Lemma 7.15 we see that $\varphi^W_{0,B}\cdot\varphi^W_{0,A}$ and $\varphi^W_{0,B|A}$ are $U(n)$-equivariant homomorphisms from $\mathbb{S}_{B|A}(\mathbb{C}^n)^*$ to $T^0(U)\otimes\mathcal{F}_{n,W}\otimes\mathbb{S}_{B|A}(W_{\mathbb{C}})$. By Lemma 7.16 they agree on the highest weight vector (see Lemma 3.3), and hence coincide.

8. Local restriction

We retain the notation from the previous sections. In this section, we will give formulas for the restrictions $r_P^{\mathcal{W}}$ and r_P of $\varphi_{nq,\ell'}$. The main result will be then the local restriction formula for $\varphi_{nq,[\lambda]}$.

Proposition 8.1. *We have*

$$(r_{\underline{P}}^{\mathcal{W}}\varphi^V_{nq,\ell'})(\varepsilon_I) = \frac{1}{2^{nq/2+\ell'}}\sum_{\alpha',\underline{\beta}'} u_1^q\otimes\varphi^W_{\Delta'_{\alpha',\underline{\beta}',I}}\otimes\omega_{\alpha'}\otimes e_{\underline{\beta}'}.$$

Here $\varepsilon_I=\varepsilon_{i_1}\otimes\cdots\otimes\varepsilon_{i_\ell}\in T^\ell(\mathbb{C}^n)$, α' *and* $\underline{\beta}'$ *are the same indices as before with*

$$\ell+1\leqslant\alpha'_{ij},\beta'_j\leqslant p.$$

Loosely speaking, $r_P^{\mathcal{W}}(\varphi^V_{nq,\ell'})$ *is obtained from* $\varphi^V_{nq,\ell'}$ *by 'throwing away' all the indices less than or equal to* ℓ*. In particular, if* $n>p-\ell$*, we have*

$$r_{\underline{P}}^{\mathcal{W}}\varphi^V_{nq,\ell'}=0.$$

Proof. This follows from Lemma 7.5, the formula for $\varphi_{nq,\ell}$ in the Schrödinger model, and from Lemma 7.6. For the last statement, we observe that $\omega_{\alpha'}$ is in the nq-exterior power of a $(p-\ell)q$-dimensional space. □

The local restriction looks particularly simple in the Fock model.

Proposition 8.2. *Let $\underline{\alpha_j'}$ and $\underline{\beta'}$ be as before in Proposition 8.1. Then*

$$r_P^{\mathcal{W}}(\varphi_{nq,\ell'}^V(\varepsilon_I)) = \frac{1}{2^{nq/2+\ell'}}\left(\frac{1}{2\pi i}\right)^{nq+\ell'} \sum_{\substack{\underline{\alpha_1'},\dots,\underline{\alpha_n'}\\ \underline{\beta'}}} u_1^q \otimes z_{\underline{\alpha_1'}}\cdots z_{\underline{\alpha_n'}}\cdot z_{\underline{\beta'},I}\otimes(\omega_{\underline{\alpha_1'}}\wedge\cdots\wedge\omega_{\underline{\alpha_q'}})\otimes e_{\underline{\beta'}}.$$

Proof. This follows immediately either from Proposition 8.1 and applying the intertwiner to the Fock model or from Proposition 4.12 and Lemma 7.7. □

Proposition 8.3. *For the restriction of $\varphi_{nq,\ell'}^V$, we have*

$$r_P\varphi_{nq,\ell'}^V = (1_U\otimes 1_{\mathbb{C}^n}\otimes r_P^{\mathcal{W}}\otimes\sigma^*\otimes 1_V)\varphi_{nq,\ell'}^V.$$

Analogous statements hold for $\varphi_{nq,\lambda}^V$ and $\varphi_{nq,[\lambda]}^V$.

Proof. By Definition 5.5, the restriction $r_P : C_V^\bullet \to A_P^\bullet$ is given by $1_U\otimes 1_{\mathbb{C}^n}\otimes r_P^{\mathcal{W}}\otimes(\iota^*\circ\sigma^*)\otimes 1_V$. Then the theorem follows from Proposition 8.2 and Lemma 2.5, and in particular (2.42): the components of $\sigma^*\varphi_{nq,\ell'}^V$ involving $\mathfrak{a}^*$ already become annihilated under $r_P^{\mathcal{W}}$, so ι^* acts trivially on $\sigma^* r_P^{\mathcal{W}}\varphi_{nq,\ell'}^V$. □

We define

$$\varphi_{P,n\ell} = \frac{1}{2^{n\ell}}\left(\frac{1}{2\pi i}\right)^{n\ell}\sum_{\underline{\gamma_1},\dots,\underline{\gamma_n}} u_1^\ell\otimes z_{\underline{\gamma_1},1}\cdots z_{\underline{\gamma_n},n}\otimes(\nu_{\underline{\gamma_1}}\wedge\cdots\wedge\nu_{\underline{\gamma_n}}). \tag{8.1}$$

Here $\underline{\gamma_j} = (\gamma_{jm-\ell+1},\dots,\gamma_{jm})$ is a multi-index of length ℓ such that $\ell+1\leqslant\gamma_{ji}\leqslant p$, and $z_{\underline{\gamma_j},j}$ as in (7.10). Furthermore, we have set

$$\nu_{\underline{\gamma_j}} = \nu_{\gamma_{jm-\ell+1}\ell}\wedge\cdots\wedge\nu_{\gamma_{jm}1}\in\bigwedge\nolimits^{\ell}(\mathfrak{n}_W^*). \tag{8.2}$$

We have the following lemma.

Lemma 8.4.

$$\iota_P(\varphi_{0,B}^W) = \iota_P(\varphi_{0,n\ell}^W) = (-1)^{n\ell(\frac{(q-\ell)(n-1)}{2}+1)}\varphi_{P,n\ell}.$$

Proof. First note that by Proposition 6.15 we have $\iota_P(\varphi_{0,B}^W) = \iota_P(\varphi_{0,n\ell}^W)$. We let $\underline{\beta_1},\dots,\underline{\beta_n}$ be n indices of length ℓ with $\ell+1\leqslant\beta_{ji}\leqslant p$. For the corresponding elements $e_{\underline{\beta_j}}\in T^\ell(W)$, we easily see that

$$\sum_{\underline{\beta_1},\dots,\underline{\beta_n}}(z_{\underline{\beta_1}1}\cdots z_{\underline{\beta_n}n})\otimes\tau_{n\ell}(e_{\underline{\beta_1}}\otimes\cdots\otimes e_{\underline{\beta_n}}) = \sum_{\underline{\beta_1},\dots,\underline{\beta_n}}(z_{\underline{\beta_1}1}\cdots z_{\underline{\beta_n}n})\otimes(\nu_{\underline{\beta_1}}\wedge\cdots\wedge\nu_{\underline{\beta_n}}) \tag{8.3}$$

with $\nu_{\underline{\beta_j}}$ as in (8.2). With that, we conclude

$$\iota_P(\varphi^W_{0,B}) = (-1)^{n\ell(\frac{(q-\ell)(n-1)}{2}+1)} \frac{1}{2^{n\ell}} \left(\frac{1}{2\pi i}\right)^{n\ell} \sum_{\underline{\beta_1},\dots,\underline{\beta_n}} u_1^{\ell} \otimes (z_{\underline{\beta_1}1}\cdots z_{\underline{\beta_n}n}) \otimes (\nu_{\underline{\beta_1}} \wedge \cdots \wedge \nu_{\underline{\beta_n}})$$
$$= (-1)^{n\ell(\frac{(q-\ell)(n-1)}{2}+1)} \varphi_{P,n\ell} \tag{8.4}$$

by (8.1). □

We are now ready for the main result of this section, the local restriction formula for $\varphi_{nq,[\lambda]}$.

Theorem 8.5. *Let A be a standard filling of the Young diagram with ℓ' boxes and let $B_{n,\ell}$ be the standard tableau associated with the n by ℓ rectangle as in § 3. Then*

$$r_P(\varphi^V_{nq,\ell'}) = \iota_P(\varphi^W_{n(q-\ell),n\ell+\ell'}),$$
$$r_P(\varphi^V_{nq,A}) = \iota_P(\varphi^W_{n(q-\ell),B|A}).$$

Furthermore, for the form $\varphi^V_{nq,[A]}$ with harmonic coefficients, we have

$$[r_P(\varphi^V_{nq,[A]})] = [\iota_P(\varphi^W_{n(q-\ell),[B|A]})].$$

Proof. We first note that

$$r_P\varphi^V_{nq,\ell'} = (-1)^{n\ell(\frac{(q-\ell)(n-1)}{2}+1)} \varphi^W_{n(q-\ell),0} \cdot \varphi_{P,n\ell} \cdot \varphi^W_{0,\ell'}.$$

Here we view $\varphi^W_{n(q-\ell),0} \in A_P^{q-\ell,n(q-\ell),0}$ and $\varphi^W_{0,\ell'} \in A_P^{0,0,\ell'}$ in the natural fashion. The analogous statements hold for $\varphi^V_{nq,A}$ and $\varphi^V_{nq,[A]}$. Indeed, this follows immediately from Proposition 8.2 and

$$\sigma^*\omega_{\underline{\alpha'_j}} = (-1)^{\ell} \frac{1}{2^{\ell/2}} \omega_{\alpha'_{j1}p+1} \wedge \cdots \wedge \omega_{\alpha'_{jq-\ell}m-\ell} \wedge \nu_{\alpha'_{jq-\ell+1}\ell} \wedge \cdots \wedge \nu_{\alpha'_{jq}1}, \tag{8.5}$$

which follows from Lemma 2.5. The sign arises from 'sorting' $\sigma^*(\omega_{\underline{\alpha'_1}} \wedge \cdots \wedge \omega_{\underline{\alpha'_q}})$ according to (8.5) into elements $\omega_{\alpha'_\bullet}$ (which lie in $\mathfrak{p}^*_W$) and $\nu_{\alpha'_\bullet}$ (which lie in $\mathfrak{n}^*_W$). From this and Lemma 8.4 we conclude that

$$r_P(\varphi^V_{nq,\ell'}) = \iota_P(\varphi^W_{n(q-\ell),0} \cdot \varphi^W_{0,n\ell} \cdot \varphi^W_{0,\ell'}) = \iota_P(\varphi^W_{n(q-\ell),n\ell+\ell'}).$$

By $S_{\ell'}$-equivariance of ι_P we also obtain

$$r_P(\varphi^V_{nq,A}) = \iota_P(\varphi^W_{n(q-\ell),0} \cdot \varphi^W_{0,B} \cdot \varphi^W_{0,A}) = \varphi^W_{n(q-\ell),B|A}$$

since $\varphi^W_{0,B} \cdot \varphi^W_{0,A} = \varphi^W_{0,B|A}$ (see Proposition 7.14) and $\varphi^W_{n(q-\ell),B|A} = \varphi_{n(q-\ell),0} \cdot \varphi^W_{0,B|A}$ (see Lemma 7.11). The cohomology statement now follows from Proposition 6.15(ii). □

Corollary 8.6. *We have $[r_P(\varphi^V_{nq,[\lambda]})] = 0$ for $n > \min\left(p, \left[\frac{m}{2}\right]\right) - \ell$ (if $\ell \geqslant 2$) and $n > p-1$ or $n > m-2-i(\lambda)$ (if $\ell = 1$).*

Proof. The Schur functor $\mathbb{S}_{[B|A]}(W_{\mathbb{C}})$ vanishes in this range. □

On the other hand, we have the following corollary.

Corollary 8.7. *Let P be a (real) parabolic subgroup as above such that the associated space W is positive definite. Assume that*

$$i(\lambda) \leqslant n \leqslant \begin{cases} \left[\dfrac{p-q}{2}\right] & \text{if } q \geqslant 2 \\ p-1-i(\lambda) & \text{if } q = 1. \end{cases}$$

Then

$$[r_P(\varphi^V_{nq,[\lambda]})] \neq 0.$$

9. Global complexes, theta series, and the global restriction

In this section, we return to the global situation and assume that V, W, E etc are $\mathbb{Q}$-vector spaces. Furthermore, $\underline{P}$ is a standard $\mathbb{Q}$-parabolic subgroup and $P = \underline{P}_0(\mathbb{R})$ for its real points etc. All the 'local' notions (over $\mathbb{R}$) of the previous sections carry over naturally to this situation, and we make use of the already established notation.

Let $L \subset V$ be an even $\mathbb{Z}$-lattice of full rank, i.e., $(x,x) \in 2\mathbb{Z}$ for $x \in L$. In particular, $L \subset L^{\#}$, the dual lattice. We fix $h \in (L^{\#})^n$ once and for all and pick a congruence subgroup $\Gamma \subset \underline{G}(\mathbb{Z})$ of finite index which stabilizes $\mathcal{L} := \mathcal{L}_V = L^n + h$. The associated locally symmetric space $X = \Gamma\backslash D$ is non-compact (since the Witt index of V is positive) but has finite volume.

9.1. Global complexes and theta series

9.1.1. Global complexes. We first define 'global' versions of the 'local' complexes $C^\bullet$ of forms on $X = \Gamma\backslash D$, and $A^\bullet_P$ of forms on $e'(\underline{P}) = \Gamma_P\backslash e(\underline{P})$. We set

$$C^\infty(\Gamma', j, \lambda) := C^\infty\left(\Gamma'\backslash G'; T^j(U) \otimes \mathbb{S}_\lambda(\mathbb{C}^n)^*\right)^{K'} \tag{9.1}$$

for Γ' an (appropriate) arithmetic subgroup of $\mathrm{Sp}(n,\mathbb{Z})$. Note that we can identify $C^\infty(\Gamma', j, \lambda)$ in the usual way with the space of vector-valued C^∞-functions on the Siegel upper half-space of genus n, transforming like a Siegel modular form of type $\det^{j/2} \otimes \mathbb{S}_\lambda(\mathbb{C}^n)$. Furthermore, we denote by $Mod(\Gamma', j, \lambda)$ the space of holomorphic Siegel modular forms of this type. We let

$$\begin{aligned} \widetilde{C}^{j,r,\lambda}_V &= C^\infty(\Gamma', j, \lambda) \otimes [\mathcal{A}^r(D) \otimes \mathbb{S}_\lambda(V_\mathbb{C})]^\Gamma, \\ &\simeq C^\infty(\Gamma', j, \lambda) \otimes \left[\bigwedge\nolimits^r(\mathfrak{p}^*_\mathbb{C}) \otimes \mathbb{S}_\lambda(V_\mathbb{C}) \otimes C^\infty(\Gamma\backslash G)\right]^K \end{aligned} \tag{9.2}$$

and

$$\begin{aligned} \widetilde{A}^{j,r,\lambda}_P &= C^\infty(\Gamma', j, \lambda) \otimes \left[\mathcal{A}^r(e'(P)) \otimes \mathbb{S}_\lambda(V_\mathbb{C})\right]^{\Gamma_P} \\ &\simeq C^\infty(\Gamma', j, \lambda) \otimes \left[\bigwedge\nolimits^r (\mathfrak{n} \oplus \mathfrak{p}_M)^*_\mathbb{C} \otimes \mathbb{S}_\lambda(V_\mathbb{C}) \otimes C^\infty(\Gamma_P\backslash P)\right]^{K_P}. \end{aligned} \tag{9.3}$$

We then define $\widetilde{C}_V^{j,r,[\lambda]}$ and $\widetilde{A}_P^{j,r,[\lambda]}$ as in the local case by harmonic projection onto $\mathbb{S}_{[\lambda]}(V_{\mathbb{C}})$. The local map ι_P induces a global intertwining map of complexes

$$\tilde{\iota}_P : \widetilde{C}_W^{j-\ell,r,\ell\varpi_n+\lambda} \to \widetilde{A}_P^{j,n\ell+r,\lambda}. \tag{9.4}$$

by lifting functions on $\Gamma_W\backslash \mathrm{SO}_0(W_{\mathbb{R}})$ to $\Gamma_M\backslash M$. This induces a map on the cohomology

$$\begin{aligned} &\tilde{\iota}_P : C^{\infty}(\Gamma', j, \lambda) \otimes H^{n(q-\ell)}(X_W, \mathbb{S}_{[\ell\varpi_n+\lambda]}(W_{\mathbb{C}})) \\ &\hookrightarrow C^{\infty}(\Gamma', j, \lambda) \otimes H^{n(q-\ell)}(X_M, H^{n\ell}(\mathfrak{n}, \mathbb{S}_{[\lambda]}(V_{\mathbb{C}}))) \\ &\hookrightarrow C^{\infty}(\Gamma', j, \lambda) \otimes H^{nq}(e'(P), \mathbb{S}_{[\lambda]}(V_{\mathbb{C}})). \end{aligned} \tag{9.5}$$

We also introduce

$$\overline{C}_V^{j,r,\lambda} = C^{\infty}\big(\Gamma'\backslash G'; T^j(U) \otimes \mathbb{S}_{\lambda}(\mathbb{C}^n)^*\big)^{K'} \otimes \mathcal{A}^r(\overline{X}, \mathbb{S}_{\lambda}(\mathcal{V}_{\mathbb{C}})), \tag{9.6}$$

the complex associated with the differential forms on the compactification $\overline{X}$ with values in $\mathbb{S}_{\lambda}(\mathcal{V}_{\mathbb{C}})$, the local system associated with $\mathbb{S}_{\lambda}(V_{\mathbb{C}})$. We then have a restriction map

$$\tilde{r}_P : \overline{C}_V^{\bullet} \to \widetilde{A}_P^{\bullet} \tag{9.7}$$

induced by the inclusion $e'(P) \hookrightarrow \overline{X}$.

9.1.2. The theta series. Using the Schrödinger model $\mathcal{S}(V_{\mathbb{R}}^n)$ of the Weil representation, we now introduce for $\varphi \in C_V^{j,r,\lambda}$ its theta series $\theta(\varphi)$ as follows. For $g' \in G'$, we then define for $z \in D$ the theta series

$$\theta_{\mathcal{L}_V}(g', z, \varphi) = \sum_{\mathbf{x} \in \mathcal{L}_V} \omega(g')\varphi(\mathbf{x}, z). \tag{9.8}$$

We easily see that the series is Γ-invariant as Γ stabilizes $\mathcal{L}_V$. Thus $\theta_{\mathcal{L}_V}$ descends to a closed differential nq-form on the locally symmetric space $X = \Gamma\backslash D$. More precisely, by the standard theta machinery, we have

$$\theta_{\mathcal{L}_V}(\varphi) \in \widetilde{C}_V^{j,r,\lambda} \tag{9.9}$$

for some congruence subgroup $\Gamma' \subseteq \mathrm{Sp}(n, \mathbb{Z})$. Summarizing, the theta distribution $\theta_{\mathcal{L}_V}$ associated with $\mathcal{L}$ gives rise to a $G' \times G$ intertwining map of complexes

$$\theta_{\mathcal{L}_V} : C_V^{\bullet} \longrightarrow \widetilde{C}_V^{\bullet}. \tag{9.10}$$

Remark 9.1. The main point of [11] is that for the Schwartz forms $\varphi_{nq,[\lambda]}$ one has

$$[\theta_{\mathcal{L}_V}(\varphi_{nq,[\lambda]})] \in Mod(\Gamma', j, \lambda) \otimes H^{nq}(X, \mathbb{S}_{\lambda}(\mathcal{V}_{\mathbb{C}})),$$

and the Fourier coefficients are Poincaré dual classes of special cycles with non-trivial local coefficients.

For a similar theta intertwiner for A_P, we note that A_P involves the Weil representation for $W = E^{\perp}/E$. Recall (see Proposition 4.2 and Definition 4.3) that we can extend the action of $\mathrm{O}(W_{\mathbb{R}})$ on $\mathcal{S}(W_{\mathbb{R}}^n)$ to P such that the Weil representation intertwining map $r_P^{\mathcal{W}}$ becomes an MN-intertwiner. In particular, N and M'_P act trivially on $\mathcal{S}(W_{\mathbb{R}}^n)$. We let $\mathcal{L}_W$ be a linear combination of delta functions of (cosets of) lattices in W^n, which is stabilized by Γ_P, that is, by Γ_W. Recall that we defined Γ_W as the image of Γ_P when acting on $E^\perp/E$. It contains $\Gamma \cap \mathrm{SO}_0(W_{\mathbb{R}})$ as a finite subgroup of finite index. Applying the theta distribution associated with $\mathcal{L}_W$ we obtain an intertwining map

$$\theta_{\mathcal{L}_W} : A_P^\bullet \to \widetilde{A}_P^\bullet. \tag{9.11}$$

Furthermore, $\theta_{\mathcal{L}_W}$ commutes with ι_P:

$$\theta_{\mathcal{L}_W} \circ \iota_P = \tilde{\iota}_P \circ \theta_{\mathcal{L}_W}. \tag{9.12}$$

More general, we let

$$A_P^{\bullet,\mathcal{L}_W,\Gamma_W} = \{\varphi \in A_P^\bullet;\, \theta_{\mathcal{L}_W}(\varphi) \text{ is } \Gamma_W\text{-invariant}\}. \tag{9.13}$$

and obtain a map $\theta_{\mathcal{L}_W} : A_P^{\bullet,\mathcal{L}_W,\Gamma_W} \to \widetilde{A}_P^\bullet$ as before.

We will be interested in a particular $\mathcal{L}_W$, which naturally arises from $\mathcal{L}_V$ as follows. Let $\pi_E : E^\perp \to E^\perp/E$ be the natural projection map. We then set

$$\widehat{\mathcal{L}}_W := \pi_E(\mathcal{L}_V \cap E^\perp). \tag{9.14}$$

For this definition, it is crucial to view $W = E^\perp/E$ as a subquotient of V and not as the subspace $E^\perp \cap (E')^\perp$ of V. Namely, $\widehat{\mathcal{L}}_W$ is in general larger than $W \cap \mathcal{L}^n$, which can be empty even when $\widehat{\mathcal{L}}_W$ is not.

Remark 9.2. The notation of $\widehat{\mathcal{L}}_W$ becomes more transparent if one changes to the adelic setting. Adelically, $\mathcal{L}$ corresponds to the characteristic function $\chi_{\mathcal{L}_V}$ of the image of $\mathcal{L}_V$ inside $V(\mathbb{A}_f)$, where $\mathbb{A}_f$ denotes the finite adeles. Then in this setting, $\widehat{\mathcal{L}}_W$ corresponds to the partial Fourier transform of $\chi_{\mathcal{L}_V}$ with respect to $E(\mathbb{A}_f)$ when restricted to W. From this perspective, the assignment $\mathcal{L} \to \widehat{\mathcal{L}}_W$ is the analogue at the finite places of the local restriction map $r_{\underline{P}}$ at the infinite place.

9.2. The global restriction

9.2.1. Smooth forms on smooth manifolds with corners. We begin with a short discussion of the definition of a smooth ℓ-form on a smooth n-manifold with corners M. For more on smooth manifolds with corners we refer the reader to the Appendix of [5] or [24], pp. 363–370. First, for any point $x \in M$ the tangent space $T_x(M)$ is a linear space of dimension n. A differential ℓ-form ω will be a section of $\bigwedge^\ell(T^*(M))$. To say when an ℓ-form ω is smooth on M it suffices to define smooth ℓ-forms on the local models $S_k^n = \mathbb{R}_{\geqslant 0}^k \times \mathbb{R}^{n-k}$.

Definition 9.3. An ℓ-form ω on $\mathbb{R}_{\geqslant 0}^k \times \mathbb{R}^{n-k}$ is smooth if there exists $\epsilon > 0$ and a smooth form $\widetilde{\omega}$ on $\mathbb{R}_{>-\epsilon}^k \times \mathbb{R}^{n-k} \supset \mathbb{R}_{\geqslant 0}^k \times \mathbb{R}^{n-k}$ such that $\widetilde{\omega}$ restricts to ω.

For our purposes we need only two classes of smooth forms. Recall from the appendix of [5] that a point x in a neighbourhood U that maps by a chart φ to the local model S_k^n above with $\varphi(x) = 0$ is said to have index k. The set of points of index greater than or equal to k is denoted as $M^{(k)}$. The subset $M^{(0)}$ is said to be the interior of M, the set $M^{(1)}$ is said to be the boundary of M. The first class of smooth ℓ-forms on M is obtained by extending by zero from $M^{(0)}$ to M the smooth ℓ-forms on $M^{(0)}$ whose coefficients relative to one and hence any system of coordinates vanish to infinite order on $M^{(1)}$. The second class of smooth ℓ-forms on M will consist of the *special* forms. We define an ℓ-form ω in a local model S_k^n to be special if there exists an ℓ-form $\overline{\omega}$ on $\mathbb{R}^{n-k}$ such that $\omega = p_2^*\overline{\omega}$, where $p_2 : S_k^n \to \mathbb{R}^{n-k}$ is the projection on the second factor. We now claim that ω special implies that it is smooth. Indeed if we let $q_2 : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$, then $\widetilde{\omega} := q_2^*\overline{\omega}$ provides the desired extension of ω.

Remark 9.4. This definition of special forms for general smooth manifolds with corners in less restrictive than the definition in [16], Definition 13.2, p. 169, for the case of Borel–Serre compactifications. In this latter definition the form $\overline{\omega}$ is required to have further properties (e.g. local left N_P-invariance) that use the special features of the Borel–Serre compactification.

9.2.2. The restriction formula. We now prove the following theorem.

Theorem 9.5. *Assume that V is different from the $\mathbb{Q}$-split space for signature (p,p). Then (see Remark 9.8 below) the theta series $\theta_{\mathcal{L}_V}(\varphi_{nq,\ell'})$, $\theta_{\mathcal{L}_V}(\varphi_{nq,\lambda})$, $\theta_{\mathcal{L}_V}(\varphi_{nq,[\lambda]})$ extend to smooth forms on the smooth manifold with corners $\overline{X}$.*

Moreover, for a standard rational parabolic $\underline{P}$, the restrictions $\tilde{r}_P$ to the corresponding boundary component $e'(\underline{P})$ of the three series above are given by

$$\tilde{r}_P\left(\theta_{\mathcal{L}_V}(\varphi_{nq,\bullet})\right) = \theta_{\widehat{\mathcal{L}}_W}(r_P\varphi_{nq,\bullet}).$$

Remark 9.6. The statement of the theorem is not correct for the $\mathbb{Q}$-split case for signature (p,p). In that case, one has to replace the Borel–Serre compactification for $\mathrm{SO}(p,p)$ by the big Borel–Serre compactification, as we explain in the final section. With this modification the theorem holds again as stated above.

Combining Theorem 9.5 with Theorem 8.5, we obtain the following corollary.

Corollary 9.7.

$$\tilde{r}_P(\theta_{\mathcal{L}_V}(\varphi^V_{nq,\ell'})) = \widetilde{\iota}_P(\theta_{\widehat{\mathcal{L}}_W}(\varphi^W_{n(q-\ell),n\ell+\ell'})), \quad \tilde{r}_P(\theta_{\mathcal{L}_V}(\varphi^V_{nq,\lambda]})) = \widetilde{\iota}_P(\theta_{\widehat{\mathcal{L}}_W}(\varphi^W_{n(q-\ell),\ell\varphi_n+\lambda})),$$

and

$$[\tilde{r}_P(\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]}))] = [\widetilde{\iota}_P(\theta_{\widehat{\mathcal{L}}_W}(\varphi^W_{n(q-\ell),[\ell\varphi_n+\lambda]}))].$$

Remark 9.8. (i) The proof of Theorem 9.5 also shows that $\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]})$ is 'essentially' a special differential form in the sense of weighted cohomology; see [16]. Namely, $\tilde{r}_P(\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]}))$ is N_P-invariant and while $\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]})$ restricted to a neighbourhood of $e'(P)$ in $\overline{X}$ is not the pullback by the geodesic retraction of its restriction, the difference of $\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]})$ and this pullback has a 'square-exponential' decrease in the coordinates t_i on A_P. In fact, one can distil out of our proof an explicit asymptotic expansion for $\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]})$. In particular, $\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]})$ *extends to a smooth form on the smooth manifold with corners* $\overline{X}$.

Moreover, the torus A_P acts on the differential forms in (9.20) with weight

$$\eta_n := \alpha_1^{-n}\alpha_2^{-2n}\cdots\alpha_r^{-rn}$$

(written multiplicatively). Hence (up to the exponentially decreasing part) the forms $\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]})$ represent weighted cohomology classes with weight profile η_n (independent of the coefficient system).

(ii) The restriction of ρ, the half-sum of the (complex) positive roots, to the standard minimal parabolic is given by

$$\prod_{j=1}^{r}\alpha_j^{j(p+q-j-1)/2}.$$

Comparing this with η_n above we easily conclude that the forms $\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]})$ are L^2 if $p+q-r>2n+1$ (since the volume form for the symmetric space D can be expressed in terms of ρ).

(iii) Finally, the proof shows that $\theta_{\mathcal{L}_V}(\varphi^V_{nq,\ell'})$ is exponentially decreasing in the direction of $e'(P)$ if $n>p-\ell$. In particular, $\theta_{\mathcal{L}_V}(\varphi^V_{nq,\ell'})$ is exponentially decreasing for $n=p$.

Proof of Theorem 9.5. It suffices to consider $\varphi^V_{nq,\ell'}$. For $g\in G$ and $g'\in G'$, we let

$$\theta^V_{\alpha,\underline{\beta},I}(g',g)=\sum_{\mathbf{x}\in L^n+h}\omega_V(g')\varphi^V_{\Delta_{\alpha,\underline{\beta},I}}(g^{-1}\mathbf{x})\otimes g^*\omega_\alpha\otimes ge_{\underline{\beta}} \tag{9.15}$$

be the theta series associated with one fixed component of $\varphi^V_{nq,\ell'}$. For the purposes of studying the restriction to $e'(\underline{P})$, we can assume that $g'=1$ (since it intertwines with the restriction) and also $g=a(\mathbf{t})\in A$ (since g varies in a Siegel set and by Lemma 4.1). It also suffices to assume

$$\mathcal{L}_V=\left(L_E^n+h_E\right)\oplus\left(L_W^n+h_w\right)\oplus\left(L_{E'}^n+h_{E'}\right)$$

with $L_E, L_W, L_{E'}$ lattices in E, W, E' respectively.

Lemma 9.9. *Let $a(\mathbf{t})\in A$. Then*

$$\theta^V_{\alpha,\underline{\beta},I}(a(\mathbf{t}))=\det(L_E)^{-n}\sum_{\mathbf{x}_W\in L_W^n+h_W}\ \sum_{\substack{\xi\in(L_E^{\#})^n\\ u'\in L_{E'}^n+h_{E'}}} e\left(2\pi i(\xi,h_E)\right)$$
$$\times|\mathbf{t}|^n\,\widehat{\varphi}^E_{\Delta''_{\alpha,\underline{\beta},I}}(\widetilde{\mathbf{t}}(\xi^t+u'),\widetilde{\mathbf{t}}u')\varphi^W_{\Delta'_{\alpha,\underline{\beta},I}}(\mathbf{x}_W)\otimes a(\mathbf{t})^*\sigma^*\omega_\alpha\otimes a(\mathbf{t})e_{\underline{\beta}}.$$

Proof. This follows directly from Lemma 4.1 and Poisson summation. ☐

Lemma 9.10. *Assume that at least one of the α_{kj} and β_k is less than or equal to ℓ. Then*

$$r_{\underline{P}}\theta^V_{\alpha,\underline{\beta},I}=0.$$

Proof. By the hypothesis we have $W \neq 0$ for all parabolics P. Then $W \otimes \mathbb{R}u_i \subset \mathfrak{n}$ is a weight space for the action of A_P with weight t_i. So in particular, for $a(\mathbf{t}) \in A_P$, we have that all components $t_i \to \infty$ as we approach $e'(P)$. Hence by Lemma 9.9 we clearly see that each term in $\theta^V_{\alpha,\underline{\beta},I}(a(\mathbf{t}))$ is rapidly decreasing as $t_i \to \infty$ for $\underline{P}$ unless both $\xi = u = 0$. But by Lemma 7.6, we have

$$\widehat{\varphi}^E_{\Delta''_{\alpha,\underline{\beta},I}}(0,0) = \widehat{\varphi}^V_{\Delta_{\alpha,\underline{\beta},I}} \begin{pmatrix} 0 \\ \mathbf{x}_W \\ 0 \end{pmatrix} = 0. \quad \square \tag{9.16}$$

Now for the remainder of the proof of Theorem 9.5, assume that

$$\alpha_{kj}, \beta_k \geqslant \ell + 1. \tag{9.17}$$

Again, each term in Lemma 9.9 is rapidly decreasing unless $\xi = u = 0$. So it suffices to consider

$$\widehat{a(\mathbf{t})\varphi^V_{\Delta_{\alpha,\underline{\beta},I}}} \begin{pmatrix} 0 \\ \mathbf{x}_W \\ 0 \end{pmatrix} = |\mathbf{t}| \varphi^W_{\Delta'_{\alpha,\underline{\beta},I}}(\mathbf{x}_W) \otimes a\,(\mathbf{t})^*\, \sigma^* \omega_\alpha \otimes a(\mathbf{t}) e_{\underline{\beta}}. \tag{9.18}$$

Now $a(\mathbf{t})e_{\underline{\beta}} = e_\beta$ by (9.17). We have

$$\sigma^* \omega_{\underline{\alpha_j}} = \frac{(-1)^\ell}{2^{\ell/2}} \omega_{\alpha_{j1} p+1} \wedge \cdots \wedge \omega_{\alpha_{jq-\ell} m-\ell} \wedge \nu_{\alpha_{jq-\ell+1} \ell} \wedge \cdots \wedge \nu_{\alpha_{jq} 1}, \tag{9.19}$$

and A acts trivially on the $\omega_\bullet$, while for the $\nu_\bullet$ we have $a\,(\mathbf{t})^*\, \nu_{ji} = \frac{db_{ji}}{t_i}$, where $1 \leqslant i \leqslant \ell$ and $\ell+1 \leqslant j \leqslant m-\ell$. Here b_{ji} is the coordinate of $W \otimes E$ for $e_j \otimes u_i$ and t_i is the parameter in $a(t_1, \ldots, t_i, \ldots, t_\ell) \in A$. We obtain

$$\begin{aligned}
|\mathbf{t}| a\,(\mathbf{t})^*\, \sigma^* \omega_\alpha &= \frac{(-1)^{n\ell}}{2^{n\ell/2}} |\mathbf{t}| \omega_{\alpha_1\ 1p+1} \wedge \cdots \wedge \omega_{\alpha_{q-\ell}\ 1m-\ell} \wedge \frac{db_{\alpha_{q-\ell+1}\ 1\ell}}{t_\ell} \wedge \cdots \wedge \frac{db_{\alpha_q\ 11}}{t_1} \\
&\quad \wedge \cdots \wedge \omega_{\alpha_1\ np+1} \wedge \cdots \wedge \omega_{\alpha_{q-\ell}\ nm-\ell} \wedge \frac{db_{\alpha_{q-\ell+1}\ n\ell}}{t_\ell} \wedge \cdots \wedge \frac{db_{\alpha_q\ n1}}{t_1} \\
&= \frac{(-1)^{n\ell}}{2^{n\ell/2}} \omega_{\alpha_1\ 1p+1} \wedge \cdots \wedge \omega_{\alpha_{q-\ell}\ 1m-\ell} \wedge db_{\alpha_{q-\ell+1}\ 1\ell} \wedge \cdots \wedge db_{\alpha_q\ 11} \\
&\quad \wedge \cdots \wedge \omega_{\alpha_1\ np+1} \wedge \cdots \wedge \omega_{\alpha_{q-\ell}\ nm-\ell} \wedge db_{\alpha_{q-\ell+1}\ n\ell} \wedge \cdots \wedge db_{\alpha_q\ n1}.
\end{aligned} \tag{9.20}$$

This shows that for (9.18) we have

$$\widehat{a(\mathbf{t})\varphi^V_{\Delta_{\alpha,\underline{\beta},I}}} \begin{pmatrix} 0 \\ \mathbf{x}_W \\ 0 \end{pmatrix} = r_{\underline{P}} \varphi^V_{\Delta_{\alpha,\underline{\beta},I}}(\mathbf{x}_W) \tag{9.21}$$

independently of $\mathbf{t}$. This completes the proof of Theorem 9.5. $\square$

9.3. Non-vanishing

We now prove Theorem 1.3.

By the hypotheses we can find a rational parabolic $\underline{P}$ such that $\dim E = \ell = q$, so W is positive definite and X_W is a point. Then by Theorem 9.5,

$$\begin{aligned}[\tilde{r}_P \theta_{\mathcal{L}_V}(\tau, \varphi^V_{q,[\lambda]})] &= \tilde{\iota}_P[\theta_{\widehat{\mathcal{L}}_W}(\tau, \varphi^W_{0,[\ell\varpi_n+\lambda]})] \\ &\in \mathit{Mod}(\Gamma', m/2, \lambda) \otimes \tilde{\iota}_P(H^0(X_W, \mathbb{S}_{[\ell\varpi_n+\lambda]}(W_{\mathbb{C}}))) \\ &\simeq \mathit{Mod}(\Gamma', m/2, \lambda) \otimes \tau_{nq,\ell'}\left(\mathbb{S}_{[\ell\varpi_n+\lambda]}(W_{\mathbb{C}})\right) \\ &\simeq \mathit{Mod}(\Gamma', m/2, \lambda) \otimes \mathbb{S}_{[\ell\varpi_n+\lambda]}(W_{\mathbb{C}}). \end{aligned} \tag{9.22}$$

So in this case $\tilde{\iota}_P$ is an embedding. Hence the restriction to $e'(P)$ vanishes if and only if the positive definite theta series $\theta_{\widehat{\mathcal{L}}_W}(\tau, \varphi^W_{0,[\ell\varpi_n+\lambda]})$ vanishes. Furthermore, the restriction of the class $[\theta_{\mathcal{L}_V}(\tau, \varphi^V_{q,[\lambda]})]$ cannot arise from an invariant form on D, since in that case one would need to obtain the trivial representation in the coefficients.

To obtain the non-vanishing, we first observe the following lemma.

Lemma 9.11. *Given $\varphi^W_{0,[\ell\varpi_n+\lambda]}$ as above, then there exists a coset of a lattice $\mathcal{L}_W$ which we can take to be contained in $\widehat{\mathcal{L}}_W$ such that*

$$\theta_{\mathcal{L}_W}(\tau, \varphi^W_{0,[\ell\varpi_n+\lambda]}) \neq 0.$$

Proof. We give a very simple argument which we learned from E. Freitag and R. Schulze-Pillot. We can assume that $V = \mathbb{Q}^m$ with the standard inner product. First find a vector $h \in \frac{1}{N_1}(\mathbb{Z}^m)^n$ with $N_1 \in \mathbb{Z}$ such that $\varphi^W_{0,[\ell\varpi_n+\lambda]}(h) \neq 0$. Then pick a lattice $L = N_1 N_2 \mathbb{Z}^m$ such that $\| \sum_{x \in L^n} \varphi^W_{0,[\ell\varpi_n+\lambda]}(x) \| < \|\varphi^W_{0,[\ell\varpi_n+\lambda]}(h)\|$. Such a $N_2 \in \mathbb{Z}$ exists as $\varphi^W_{0,[\ell\varpi_n+\lambda]}$ is a Schwartz function. Then the theta series associated with $\varphi^W_{0,[\ell\varpi_n+\lambda]}$ for $\mathcal{L}_W = L^n + h$ does not vanish. □

From the above, we now can find a $\mathcal{L}'_V$ contained in $\mathcal{L}_V$ such that $\widehat{\mathcal{L}'}_W = \mathcal{L}_W$ with $\theta_{\mathcal{L}_W}(\tau, \varphi^W_{0,[\ell\varpi_n+\lambda]}) \neq 0$. Replace Γ with $\Gamma \cap \operatorname{Stab} L'$. Then $[\tilde{r}_P \theta_{\mathcal{L}_V}(\tau, \varphi^V_{q,[\lambda]})] \neq 0$. This proves Theorem 1.3.

One feature of our method for establishing non-vanishing is that we retain some control over the cover X', since this reduces to the very concrete question of the non-vanishing of positive definite theta series. An easy example for this is the following.

Example 9.12. Consider the integral quadratic form given by

$$y_1 y'_1 + \cdots + y_q y'_q + 2x_1^2 + \cdots + 2x_k^2$$

with $y_i, y'_i, x_j \in \mathbb{Z}$. So $L = \mathbb{Z}^m$ with $m = 2q + k$. Assume that $k \geqslant q$. Note that $L^\# \subset \frac{1}{4}\mathbb{Z}^m$. We let Γ be the subgroup in $\operatorname{Stab}(L)$ which stabilizes $L^\#/L$. Then

$$H^q(\Gamma, \mathbb{Z}) \neq 0.$$

Using our method, this follows from the non-vanishing of the theta series $\sum_{\mathbf{x} \in \mathbb{Z}^k + (\frac{1}{4}, \ldots, \frac{1}{4})} x_1 \cdots x_q e^{4\pi i (\sum x_i^2)\tau}$.

9.4. An L^2-vanishing result for non-trivial coefficients

We now prove Theorem 1.5, that is, that our L^2-non-vanishing result for our theta series is sharp. The proof consists of two parts, the first an L^2-cohomology argument, the second a $(\mathfrak{g}, K)$-cohomology argument.

Assume that $k < i(\lambda)q$, and let $[\eta] \in H^k(X', \mathbb{S}_{[\lambda]}(V_{\mathbb{C}}))^{(2)}$ be a non-zero element represented by a closed square integrable differential k form η. Assume that η is not exact in the L^2-complex. We claim that then the harmonic projection of $\mathcal{H}(\eta)$ of η is non-zero. To prove this it suffices to show that the L^2-cohomology groups of degree $k < i(\lambda)q$ are finite dimensional since such a result of finite dimensionality implies the existence of the usual (i.e., without taking closures in L^2) Hodge decomposition. To this end, note that unless p and q are both odd, then G and K have the same rank over $\mathbb{C}$ and the L^2-cohomology groups of *all* degrees are finite dimensional by the results of [3]. In the exceptional case in which both p and q are odd we apply [6], Chapter III, Theorem 5.1, to deduce that with the exception of the two groups of degrees the two middle dimensions $[\frac{pq-1}{2}], [\frac{pq+1}{2}]$, the L^2-cohomology groups are finite dimensional. But by the hypotheses we have $k < i(\lambda)q < [p/2]q$, so we are below the middle dimensions. Consequently, the usual Hodge decomposition holds, and the harmonic projection $\mathcal{H}(\eta)$ of a non-zero square integrable non-exact closed form η of degree k is a *non-zero* square integrable harmonic form. We find, then, that the pullback of $\mathcal{H}(\eta)$ from X to $\Gamma\backslash G$ would generate a copy of a Vogan–Zuckerman representation $A_{\mathfrak{q}}(\tilde{\lambda})$, occurring in $L^2(\Gamma\backslash G)_d$. But, by a straightforward argument extending that of § 2 in [28], one shows that if λ satisfies $0 < i(\lambda) < [p/2]$, then for any $A_{\mathfrak{q}}(\tilde{\lambda})$ we have $H^k(\mathfrak{g}, K, A_{\mathfrak{q}}(\tilde{\lambda})) = 0$ for all $k < i(\lambda)q$.

10. The big Borel–Serre compactification for rational SO(p, p)

In this section, V is always a $\mathbb{Q}$-split rational quadratic space of signature (p, p) with Witt basis $u_1, \dots, u_{p-1}, u_p, u'_p, u'_{p-1}, \dots, u'_1$.

We will show that our main Theorem 9.5 remains true for the case of rational SO(p, p) but only if we replace the Borel–Serre compactification associated with the usual Tits building of type D_p of (rational) parabolic subgroups of SO(p, p) by the 'big Borel–Serre compactification' of type B_p which will be described below. For this we have to change the underlying root system from type D_p to type B_p by adding reflections (and great subspheres in the Tits building). In terms of groups this is achieved by switching from SO(p, p) to the full orthogonal group O(p, p) (or equivalently, to SO$(p + 1, p)$).

Of course since both compactifications are compactifications of the same locally symmetric space, the two boundaries assigned will be the same as topological spaces but *their structures as manifolds with corners will be different.*

The main issue for us is that the parabolic subgroups of SO(p, p) do *not* correspond bijectively to isotropic flags, but rather to oriflammes; see Lemma 2.3.

By switching to the root system B_p, i.e., considering O(p, p) or SO$(p + 1, p)$, we do obtain a bijection between parabolics and isotropic flags. This is the crucial aspect in constructing the big Borel–Serre compactification.

We first define the big Borel–Serre compactification *extrinsically* by embedding the locally symmetric space $X_{p,p} = \Gamma_{p,p} \backslash D_{p,p}$ for $\mathrm{SO}(p, p)$ into a suitably constructed space $X_{p+1,p} = \Gamma_{p+1,p} \backslash D_{p+1,p}$ for signature $(p+1, p)$ and then considering the closure of $X_{p,p}$ inside the Borel–Serre compactification $\overline{X}_{p+1,p}$. The *intrinsic* big Borel–Serre compactification uses the Tits building for parabolic subgroups for the full orthogonal group $\mathrm{O}(p, p)$.

The extension of $\theta(\varphi_{np,[\lambda]})$ is most easily established by pulling back the usual Borel–Serre compactification and restriction formulas for $(p+1, p)$ using the extrinsic definition. We proceed to give the intrinsic definition and compare the two constructions. It is then most instructive to compare the usual and the big Borel–Serre compactification. Finally, we consider the case of signature (2, 2) in more detail.

10.1. The extrinsic big Borel–Serre compactification

We set $\tilde{V} = V \perp \mathbb{Q}v$ with $(v, v) = 1$. Hence $\tilde{V}$ has signature $(p+1, p)$. We rearrange coordinates such that v becomes the $(p+1)$ st standard basis vector e_{p+1}. We write $\ell_{p+1} = \mathbb{Q}e_{p+1}$ for the line spanned by e_{p+1}. The natural inclusion $V \hookrightarrow \tilde{V}$ defines the inclusion $j_{p+1} : \mathrm{O}(p, p) \to \mathrm{O}(p+1, p)$. We will often identify $\mathrm{O}(p, p)$ with its image under j_{p+1}. The inclusion j_{p+1} induces an inclusion (also denoted as j_{p+1}) of the symmetric spaces $D_{p,p} \hookrightarrow D_{p+1,p}$. We let $\Gamma_{p+1,p}$ denote a congruence subgroup in $\mathrm{SO}(\tilde{V})$ stabilizing $\tilde{\mathcal{L}} = \mathcal{L} \oplus \mathbb{Z}v$ chosen such that it is torsion-free and

$$\Gamma_{p,p} = \mathrm{O}(p, p) \cap \Gamma_{p+1,p}.$$

We may assume, for example if $\Gamma_{p+1,p}$ is neat (the intersection of the subgroup of $\mathbb{C}^*$ generated by the elements of $\Gamma_{p+1,p}$ with the roots of unity is $\{1\}$), that this intersection is contained in $\mathrm{SO}(p, p)$. Let $\sigma \in \mathrm{SO}(\tilde{V})$ be the rational element, that is -1 on V and 1 on ℓ_{p+1}. Then $D_{p,p}$ is the fixed point set of σ acting on $D_{p+1,p}$, that is,

$$j_{p+1} D_{p,p} = D^{\sigma}_{p+1,p}.$$

The inclusion of symmetric spaces induces a map (again denoted as j_{p+1}) of locally symmetric spaces $j_{p+1} : X_{p,p} \to X_{p+1,p}$. Assume now that $\Gamma_{p+1,p}$ is torsion-free. Then it follows from a well-known argument using σ (the 'Jaffe Lemma', Lemma 2.1 of [26]) that j_{p+1} induces an embedding of $X_{p,p}$ into $X_{p+1,p}$.

Definition 10.1 (The extrinsic big Borel–Serre compactification). Assume that $\Gamma_{p,p}$ is torsion-free. The big Borel–Serre compactification $\overline{X}_{p,p}$ is the closure of $X_{p,p}$ in $\overline{X}_{p+1,p}$. We note that the inclusion j_{p+1} induces an embedding $\bar{j}_{p+1} : \overline{X}_{p,p} \to \overline{X}_{p+1,p}$.

We will discuss the properties of the extrinsic big Borel–Serre compactification later in detail. At this point we can already give a quick proof that our theta series extends to the big compactification of $\overline{X}_{p,p}$.

Theorem 10.2. *The forms $\theta(\varphi_{np,[\lambda]})$ on $X_{p,p}$ extend to the big Borel–Serre compactification $\overline{X}_{p,p}$.*

Proof. Let $\widetilde{\varphi}_{np,[\lambda]}$ be the special np-cocycle for $\mathrm{SO}(p+1,p)$ and $\varphi_{np,[\lambda]}$ be that for $\mathrm{SO}(p,p)$. Note that from the explicit formulas for $\widetilde{\varphi}_{np,[\lambda]}$ and $\varphi_{np,[\lambda]}$ we have

$$j_{p+1}^{*}\widetilde{\varphi}_{np,[\lambda]} = \varphi_{np,[\lambda]}\varphi_0^{\ell_{p+1}}. \tag{10.1}$$

Here $\varphi_0^{\ell_{p+1}}$ is the Gaussian associated with the one-dimensional positive definite subspace ℓ_{p+1}. Since the lattice splits, we obtain a corresponding restriction formula

$$j_{p+1}^{*}\theta(\widetilde{\varphi}_{np,[\lambda]}) = \theta(\varphi_{np,[\lambda]})\theta(\varphi_0^{\ell_{p+1}}) \tag{10.2}$$

on the level of theta functions. Note that $\theta(\varphi_0^{\ell_{p+1}})$ is constant on $X_{p,p}$, so the product of the two factors on the right of (10.2) extends to the big Borel–Serre boundary if and only if the first factor extends. Now we have seen above that $\theta(\widetilde{\varphi}_{np,[\lambda]})$ extends over the Borel–Serre boundary of $X_{p+1,p}$. The lemma then follows on considering the following commutative diagram (starting with $\theta(\widetilde{\varphi}_{np})$ in the lower left hand corner).

$$\begin{array}{ccc} A^{\bullet}(\overline{X}_{p+1,p}) & \xrightarrow{\bar{j}_{p+1}^{*}} & A^{\bullet}(\overline{X}_{p,p}) \\ \big\downarrow & & \big\downarrow \\ A^{\bullet}(X_{p+1,p}) & \xrightarrow{j_{p+1}^{*}} & A^{\bullet}(X_{p,p}) \end{array} \qquad \square$$

Remark 10.3. We are required so far to assume that the lattice $\Gamma_{p+1,p}$, and hence $\Gamma_{p,p}$, is torsion-free. However, after we have given the intrinsic description of our compactification, and hence we know that this *intrinsic construction* produces a compactification for the quotient of D by a normal torsion-free subgroup Γ' of $\Gamma \subset \mathrm{SO}(p,p)$, then the extension and the restriction formula will hold for the quotient by the larger lattice Γ because it is invariantly defined. We leave the details to the reader.

10.2. The intrinsic description of the new compactification

We now give an intrinsic description of the big Borel–Serre compactification, that is, it does not use the embedding j_{p+1}.

In what follows if G is any reductive group we will use $\mathcal{P}(G)$ to denote the set of parabolic subgroups of G.

There are four key ingredients of a Borel–Serre compactification; see [4], III.9 (and § 2.4 above).

(1) The Tits building $\mathcal{B}(G)$ (or rather its quotient by the arithmetic group $\Gamma \subset G$ under consideration).
(2) For each rational parabolic P of G there is the split torus A_P which is the connected component of the identity of the centre of P/N.
(3) For each rational parabolic subgroup P there is the associated 'Borel–Serre face' $e(P) := P/A_P K_P$. Here $K_P = P \cap K$ is as before the subgroup of P that stabilizes the base point z_0 of the associated symmetric space.

(4) The set $\Phi(P, A_P)$ of restrictions of the set of positive roots to A_P, which governs the topology around the boundary faces, and in particular, convergence to a point in the boundary. The reader will note that the definition of convergence will not be changed if the elements of $\Phi(P, A_P)$ are replaced by positive scalar multiples. Furthermore, one obtains the same set of convergent sequences if in the rule of [4], p. 328, one replaces $\Phi(P, A_P)$ by $\Delta(P, A_P)$, the set restrictions to A_P of the *simple* roots in the root system associated with the maximal torus A_{P_0} for a chosen minimal parabolic P_0.

Definition 10.4 (The intrinsic big Borel–Serre compactification). The intrinsic big Borel–Serre compactification $\overline{X}_{p,p}$ is obtained by applying the 'uniform construction of Borel and Ji' ([4], § III.9) to the Tits building $\mathcal{B}(\mathrm{O}(p, p))$ for the full orthogonal group together with the root system B_p.

The term 'intrinsic compactification' is a bit premature since one still needs to show that the construction really gives a compact space. At this point it is only a formal procedure. Moreover, it is *a priori* not clear that we can freely change the root system from D_p to B_p. Only once we have established the equivalence to the extrinsic description will this be justified. Note however, that the full orthogonal group $\mathrm{O}(p, p)$ gives rise to the same symmetric space as $\mathrm{SO}(p, p)$.

We now describe some of the features of the new construction.

10.2.1. The new building $\mathcal{B}(\mathrm{O}(p, p))$ and the map of parabolic subgroups. Recall that we defined the standard totally isotropic subspaces $E_k = \mathrm{span}(u_1, \ldots, u_k)$ in V and the spaces $E_+ = E_p = \mathrm{span}(u_1, \ldots, u_{p-1}, u_p)$ and $E_- = \mathrm{span}(u_1, \ldots u_{p-1}, u'_p)$.

We first note (see e.g. [1, 14]) the following lemma.

Lemma 10.5. *The (standard) parabolic subgroups of* $\mathrm{O}(p, p)$ *are the stabilizers of the (standard) isotropic flags (in* E_p*), and every isotropic flag determines a parabolic. Thus the associated Tits building* $\mathcal{B}(\mathrm{O}(p, p))$ *is the spherical building associated with the partially ordered set of isotropic flags in* V *and the parabolic subgroups of* $\mathrm{O}(p, p)$ *are the stabilizers of the faces of the building.*

Example 10.6. We illustrate this fundamental difference from the special orthogonal group $\mathrm{SO}(p, p)$. Let $P \subset \mathrm{O}(p, p)$ be the stabilizer of the isotropic subspace E_{p-1}. Then

$$P = \left\{ \begin{pmatrix} g & c_2\, c_3 & \ldots \\ 0 & h & \ldots \\ 0 & 0 & g^* \end{pmatrix} \right\}$$

with $g \in \mathrm{GL}_{p-1}(\mathbb{R})$, $h \in \mathrm{O}(1, 1)$, $c_i \in \mathbb{R}^{p-1}$ (column vectors) and g^* as in (2.14). Note that $\mathrm{O}(1, 1) = \mathrm{SO}(1, 1) \cup w\mathrm{SO}(1, 1)$ and $\mathrm{SO}(1, 1) = \left\{ \begin{pmatrix} b & \\ & b^{-1} \end{pmatrix} \right\}$. Here $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence P is a maximal parabolic subgroup of $\mathrm{O}(p, p)$.

Now consider $P' = P \cap \mathrm{SO}(p,p)$, the stabilizer of E_{p-1} in $\mathrm{SO}(p,p)$. Now we have

$$P' = \left\{ \begin{pmatrix} g & c_2 & c_3 & \dots \\ 0 & b & 0 & \dots \\ 0 & 0 & b^{-1} & \dots \\ 0 & 0 & 0 & g^* \end{pmatrix} \right\}.$$

Thus P' is strictly contained in the stabilizer of both isotropic p-planes E_+ and E_-. Hence *is not a maximal parabolic* and we can associate P' with *two* isotropic flags, namely (E_{k-1}, E_+) and (E_{k-1}, E_-), i.e., the oriflamme (E_+, E_-).

The situation in general is as follows.

Definition 10.7. We say an isotropic flag $\mathbf{F}$ in V is bad if an isotropic subspace of dimension $p-1$ occurs in $\mathbf{F}$. We say a parabolic in $\mathrm{O}(p,p)$ is bad if it stabilizes a bad flag. Otherwise we call $\mathbf{F}$ and $P_{\mathbf{F}}$ good.

We then have the following lemma.

Lemma 10.8. *Let $P \subset \mathrm{O}(p,p)$ be a parabolic subgroup stabilizing the flag $\mathbf{F}$. Set $P' = P \cap \mathrm{SO}(p,p)$.*

(i) *Assume that P is good. Then P' is the stabilizer of the flag $\mathbf{F}$(see also Lemma 2.3(2)).*

(ii) *Assume that P is bad, stabilizing a flag $F_1 \subset \cdots F_k \subset F_{p-1} (\subset F_p)$ with $\dim F_{p-1} = p-1$ and $\dim F_p = p$. (F_p might or might not be there). Let $F_{p,1}, F_{p,2}$ be as in Lemma 2.3 (3). Then P' is the stabilizer of the oriflamme $(F_1, \dots, F_k, F_{p,1}, F_{p,2})$.*

We now describe how each top dimensional simplex of the Tits building $\mathcal{B}(\mathrm{SO}(p,p))$ of type D_p will be bisected to obtain $\mathcal{B}(\mathrm{O}(p,p))$. Each spherical chamber (a top dimensional, i.e. $p-1$-dimensional, simplex) contains a distinguished edge e (the edge joining the two vertices corresponding to highest dimensional isotropic subspaces). Let f be the $p-3$ face that is opposite to e. Hence the chamber is the join $e * f$. Let b be the barycentre of e. Then we bisect each spherical chamber by the codimension 1 interior simplex $b * f$. We make a choice of one of the two halves of the original spherical fundamental chamber $\Delta_{D_p} = \Delta'$ and call it the fundamental spherical chamber $\Delta_{B_p} = \Delta$ of $\mathcal{B}(\mathrm{O}(p,p))$. The resulting non-thick building is the building of type B_p on which the big Borel–Serre compactification will be modelled. Note that if F is a face of $\mathcal{B}(\mathrm{O}(p,p))$, then there will be a unique face F' of $\mathcal{B}(\mathrm{SO}(p,p))$ such that the interior F^0 is contained in F'.

Since the parabolic subgroups are exactly the subgroups that fix faces of the buildings, the map $F \mapsto F'$ induces a map $\mathcal{P}(\mathrm{O}(p,p)) \to \mathcal{P}(\mathrm{SO}(p,p))$ of parabolic subgroups. In fact, it is exactly the assignment $P \mapsto P' = P \cap \mathrm{SO}(p,p)$ in Lemma 10.8. For good flags the claim is obvious, since in that case by definition P' is the subgroup of $\mathrm{SO}(p,p)$ that fixes the same face F. Thus the only difficulty is when the face F corresponds to a bad flag. In this case the face F fixed by the original parabolic P has dimension 1 less than $\mathcal{F}(F)$. But in this case F^0 is contained in the interior of F' and if $g \in \mathrm{SO}(p,p)$ fixes an interior point to the face F', then it fixes all of F'. The claim follows. Note that $F \mapsto F'$ is a bijection on faces of dimension less than or equal to $p-1$, but it is two-to-one on top faces.

10.2.2. The new split central split torus A_P. We define the subtorus A_P of A_{P_0} to be the centre of $L = P \cap P^{\theta_0}$ where θ_0 is the Cartan involution corresponding to our chosen base point z_0. Note that we cannot define it as the annihilator of an appropriate subset I of the simple roots of $\mathrm{SO}(p,p)$. However we can define it as the annihilator of an appropriate subset I of the simple roots of the *new root system of type* B_p; see below, in particular Lemma 10.18. These roots are defined intrinsically only up to positive multiples but this is enough to unambiguously define A_P. We will denote the new torus by A_P.

10.2.3. The new face $e(P)$. Given A_P, we define the associated face $e(P)$ of the Borel–Serre enlargement by $e(P) = P/A_P K_P$. Hence the cells $e(P)$ are assembled using the simplicial complex associated with the partially ordered set of isotropic flags in V. The point is that the split torus A_P can be *strictly smaller* for certain parabolics in the new compactification (because P and its Levi subgroup L will have extra connected components causing its centre to be smaller; see Example 10.6) and consequently the face $e(P)$ will be strictly larger. In Theorem 10.11 we will record this in detail.

10.2.4. The new system of roots of type B_p and the set $\Phi(P, A_P)$. There is a subset of the positive roots restricted to A_P to be denoted as $\Phi(P, A_P)$ and the corresponding system of simple roots restricted to A_P to be denoted as $\Delta(P, A_P)$. This is the most complicated change to describe intrinsically. We define the Weyl group W of the maximal torus A_{P_0} as usual as the normalizer in $\mathrm{O}(p,p)$. But now the element

$$w = \begin{pmatrix} I_{p-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{p-1} \end{pmatrix}$$

is in W. Hence the Weyl group for $\mathrm{O}(p,p)$ is strictly larger than the one for $\mathrm{SO}(p,p)$. In fact, with this additional reflection (which interchanges u_p and u_p') one obtains the Weyl group for the root system B_p. While this does not define directly the new roots it defines the root hyperplanes. The choice of the fundamental chamber in the new Tits building defines a positive Weyl chamber, or equivalently the correct orientation of the hyperplanes. (This corresponds to the choice of defining the standard parabolics in $\mathrm{O}(p,p)$ to be the stabilizers of flags in E_+ or E_-.) For each root hyperplane we choose a linear functional which vanishes on the hyperplane and is positive on the cone on Δ_{B_p}. This new collection of linear functionals we will call the (new) positive roots to be denoted as Φ. In terms of the Tits building this amounts to the following. We have already added the new walls to the spherical building at infinity and chosen the fundamental spherical chamber Δ_{B_p}. We now extend them inside A_{P_0} to obtain the standard linear action of the Weyl group of type B_p as a reflection group. In more detail, given the split torus A_{P_0} which we identify with its Lie algebra $\mathfrak{a}$, we consider the corresponding apartment $\mathcal{A}$ in $\mathcal{B}(\mathrm{O}(p,p))$ (the boundary of A_{P_0}). The building structure on $\mathcal{B}(\mathrm{O}(p,p))$ gives us a collection of great spheres in the apartment $\mathcal{A}$. If we regard the apartment $\mathcal{A}$ as the sphere at infinity of A_{p_0} (each ray leaving the origin of A_{P_0} corresponds to a unique point of $\mathcal{A}$), then the collection of great spheres corresponds to (the boundaries of) a collection of hyperplanes in A_{P_0}. Reflections in these hyperplanes give rise to the standard representation of the Coxeter group of type B_p. The chosen spherical chamber Δ_{B_p} corresponds to a Weyl chamber in A_{P_0} which we will also denote as Δ_{B_p}.

Definition 10.9. $\Phi(P, A_P)$ is the set of restrictions to A_P of the roots in Φ.

Remark 10.10. We did not use the Lie algebra $\mathfrak{n}$ of P in this definition. We will see later that what we are doing is pulling back the usual A_P and $\Phi(P, A_P)$ from $\mathrm{SO}(p+1,p)$ using the embedding j_{p+1}.

10.3. The intrinsic and the extrinsic big Borel–Serre compactifications coincide

Theorem 10.11. *The intrinsic and the extrinsic big Borel–Serre compactifications of $X_{p,p}$ coincide. In particular, the cells $e'(P)$ are assembled using the simplicial complex associated with the partially ordered set of isotropic flags in V.*

From this we now easily check that all results from § 9 carry over with no change to the big Borel–Serre compactification for the split (p,p)-case. In particular, we have the following theorem.

Theorem 10.12. *The restriction theorems, Theorem 9.5 and Corollary 9.7, hold in the big Borel–Serre compactification of $X_{p,p}$.*

Remark 10.13. In fact, the restriction in the small Borel–Serre compactification to faces associated with good parabolics goes through as before as well, with no change. It is the restriction to bad faces which causes problems.

To prove Theorem 10.11 we will first prove the analogue of the theorem for the partial compactifications (Borel–Serre enlargements) of the symmetric spaces $D_{p,p}$ and $D_{p+1,p}$. We will denote the corresponding enlargements by $\overline{D}_{p,p}$ (constructed using $\mathcal{P}(\mathrm{O}(p,p))$) and $\overline{D}_{p+1,p}$. Recall that earlier we already saw that $D_{p,p} = D^{\sigma}_{p+1,p}$. We claim that the corresponding equation also holds for the enlargements. We have the following proposition.

Proposition 10.14. (i) $\overline{D}_{p,p} = \overline{D}^{\sigma}_{p+1,p}$.

(ii) *Let $\widetilde{P}$ be the stabilizer of an isotropic flag $\widetilde{\mathbf{F}}$ in $\widetilde{V}$ and suppose that $\widetilde{P}$ is normalized by σ. Then the subspaces of the flag $\widetilde{\mathbf{F}}$ are in fact contained in V. We let $\mathbf{F}$ be the* *associated isotropic flag in V and P be the stabilizer of* $\mathbf{F}$*, whence*

$$P = \widetilde{P}^{\sigma}.$$

(iii) *Suppose that* $e(\widetilde{P})^{\sigma}$ *is non-empty. Then* $\widetilde{P}$ *is normalized by* σ *and*

$$e(\widetilde{P})^{\sigma} = e(P).$$

(iv) $\overline{D}^{\sigma}_{p+1,p} = D_{p,p} \coprod \coprod_{P \in \mathcal{P}(\mathrm{O}(p,p))} e(P)$.

On the building level this means that the map j_{p+1} induces a simplicial embedding of $\mathcal{B}(\mathrm{O}(p,p))$ onto $\mathcal{B}(\mathrm{SO}(p+1,p))^{\sigma}$ carrying apartments isomorphically onto apartments. The image is the fixed subbuilding $\mathcal{B}(\mathrm{SO}(p+1,p))^{\sigma}$.

The proposition will be a consequence of the following discussion.

We note that the inclusion $\overline{D}_{p,p} \subset \overline{D}^{\sigma}_{p+1,p}$ is obvious. The reverse inclusion will follow once we have proved (iv). We immediately see that

$$\overline{D}^{\sigma}_{p+1,p} = D^{\sigma}_{p+1,p} \coprod \coprod_{P \in \mathcal{P}(\mathrm{SO}(p+1,p))} e(\widetilde{P})^{\sigma}.$$

Clearly (iv) will follow from (iii). (ii) and (iii) will be a consequence of the next three lemmas. In order to prove (iii) we need to first prove (ii). We first note the following lemma.

Lemma 10.15. *Suppose that* $\widetilde{E}$ *is a totally isotropic subspace of* $\widetilde{V}$ *such that* $\sigma(\widetilde{E}) = \widetilde{E}$*. Then* $\widetilde{E} \subset V$.

We now show that Lemma 10.15 implies (ii). Indeed, $\widetilde{P}$ is the stabilizer of a unique isotropic flag $\widetilde{\mathbf{F}}$. Now since $\widetilde{P}$ is its own normalizer and we are assuming that σ normalizes $\widetilde{P}$, we find $\sigma \in \widetilde{P}$ and consequently σ carries each of the subspaces in $\widetilde{\mathbf{F}}$ into itself. Hence by Lemma 10.15, each of these subspaces is contained in V. We let $\mathbf{F}$ denote the associated isotropic flag in V and let P be its stabilizer in $\mathrm{O}(p,p)$. We now prove that $\widetilde{P}^{\sigma} = P$. First we claim that $\widetilde{P}^{\sigma}$ is contained in $\mathrm{O}(p,p)$. Indeed, since $g \in \widetilde{P}^{\sigma}$, we have $g^{-1}\sigma g = \sigma$, whence g carries the line through e_{p+1} into itself, whence $g \in \mathrm{O}(p,p)$. But also by definition, $\widetilde{P}^{\sigma}$ fixes $\mathbf{F}$, whence we have

$$\widetilde{P}^{\sigma} = P.$$

Thus it remains to prove (iii). This we do in the next two lemmas.

Lemma 10.16. *If* $e(\widetilde{P})$ *contains a fixed point of* σ*, then* $\sigma \in \widetilde{P}$ *and hence* $\sigma(e(\widetilde{P})) = e(\widetilde{P})$*. In fact, we have*

$$\sigma(e(\widetilde{P})) = e(\widetilde{P}) \iff \sigma \in \widetilde{P}. \tag{10.3}$$

Proof. It follows from the basic result of [5], Corollary 7.7(1) (with $P = Q$), that

$$\sigma(e(\widetilde{P})) \cap e(\widetilde{P}) \neq \emptyset \iff \sigma \in \widetilde{P}. \quad \square$$

Lemma 10.17. *Suppose $P = \widetilde{P}^\sigma$. Then we have*

$$e(P) = e\,(\widetilde{P})^\sigma. \tag{10.4}$$

Proof. We only need to show that $e(P) \subset e\,(\widetilde{P})^\sigma$. So suppose that $x \in e(\widetilde{P})$ is fixed by σ. Let y be the diagonal matrix with $p+1$ entries 1 followed by p entries -1. Then conjugation with y induces Cartan involutions of $\mathrm{SO}(p+1,p)$ and $\mathrm{O}(p,p)$. It is standard that we may construct a Levi decomposition $\widetilde{P} = \widetilde{M} \cdot \widetilde{N}$ with $\widetilde{M} = \widetilde{P} \cap y\widetilde{P}y^{-1}$, whence $\sigma \in \widetilde{M}$. Note that

$$e(\widetilde{P}) = (\widetilde{M}\widetilde{N})/\widetilde{K} \cap \widetilde{M}.$$

Choose a lift $x' = \widetilde{m}\widetilde{n}$ of x to $\widetilde{P}$. Then x being fixed under σ implies that $\widetilde{n}$ is fixed under σ, which implies that $\widetilde{n}$ is in the unipotent radical N of P. Also $\widetilde{m}$ is fixed modulo $\widetilde{K} \cap \widetilde{M}$. Thus it remains to show that the group $M = \widetilde{M}^\sigma$ acts transitively on the fixed point set of σ on its associated symmetric space $\widetilde{M}/(\widetilde{K} \cap \widetilde{M})$. But the fixed point set is connected (because the unique geodesic joining any two fixed points must also be fixed). Hence we may obtain the fixed point set by exponentiating the fixed subspace of $\widetilde{\mathfrak{p}}$, the tangent space of $D_{p+1,p}$ at the point z_0 fixed by the above Cartan involution. But this fixed subspace is $\mathfrak{p}$, the tangent space of $D_{p,p}$ at z_0. $\square$

We have now completed the proof of Proposition 10.14.

We also need to show that the convergence criterion applied to the topology of $\overline{D}_{p,p}$ is induced from the topology of $\overline{D}_{p+1,p}$ (and hence using the root system of type B_p). This follows from the following lemma which the reader can verify.

Lemma 10.18. *$\Phi(P, A_P)$ is the set of weights of A_P acting on the nilradical $\widetilde{\mathfrak{n}}$ of the parabolic subalgebra of the corresponding parabolic $\widetilde{P}$ ($\widetilde{P}^\sigma = P$) via the inclusion $j_{p+1} : L \to \widetilde{L}$.*

Theorem 10.11 will follow from the next lemma.

Lemma 10.19. *Suppose that $\Gamma_{p+1,p}$ is torsion-free and there exists $\gamma \in \Gamma_{p+1,p}$ such that $\gamma(e(P)) \cap e(P) \neq \emptyset$. Then $\gamma \in P \cap \Gamma_{p,p}$. In particular, the image of $e(P)$ in $\overline{X}_{p+1,p}$ is the quotient of $e(P)$ by $P \cap \Gamma_{p,p}$.*

Proof. Suppose $x \in e(P)$ satisfies that $y = \gamma(x) \in e(P)$. Then $\sigma\gamma^{-1}\sigma\gamma(x) = x$ since σ fixes x and y. But the action of $\Gamma_{p+1,p}$ on the Borel–Serre enlargement of $D_{p+1,p}$ is fixed-point-free since by [5], Theorem 9.3, it acts properly and we have assumed that it is torsion-free. Hence $\sigma\gamma\sigma = \gamma$ and consequently $\gamma \in \Gamma_{p,p}$. The lemma now follows from Corollary 7.7(1) of [5]. $\square$

This concludes the proof of Theorem 10.11.

10.4. Relating the small and the big Borel–Serre compactifications of $X_{p,p}$

We now have two compactifications of $X_{p,p}$: the usual Borel–Serre compactification and the new 'big' Borel–Serre compactification that we have just described.

For P a parabolic in $\mathrm{O}(p,p)$, we will write $P' = P \cap \mathrm{SO}(p,p)$ as before. We will denote the corresponding face in the small Borel–Serre enlargement by $e(P')$.

Proposition 10.20. *Suppose that P is a good parabolic in $\mathrm{O}(p,p)$. Then:*

(1) $e(P) = e(P')$.

(2) $A_P = A_{P'}$.

(3) *If the last subspace in the flag has dimension strictly less than p (and hence strictly less than $p-1$) then*

$$\Phi^{B_p}(P, A_P) = \Phi^{D_p}(P', A_{P'}).$$

If the last element in the flag has dimension p, then $\Phi^{B_p}(P, A_P)$ and $\Phi^{D_p}(P', A_{P'})$ will coincide except for the last entry, which in the first case will be the restriction of t_p and in the second case will be the restriction of t_p^2 (the squaring makes no difference in terms of the convergence criterion).

We will leave the proof of this proposition to the reader.

We now state what happens if P is bad. We may assume that the associated flag is standard, contained in the totally isotropic subspace $E_p = E_+$.

Proposition 10.21. *Suppose that P is a bad parabolic in $\mathrm{O}(p,p)$. There are two cases.*

(i) *Suppose first that the last subspace in the flag has dimension $p-1$. Then:*

(1) $e(P) \cong e(P') \times \mathbb{R}_+$.

(2) $A_P \times \mathbb{R}_+ = A_{P'}$. *Note that there is a projection map $\pi_p : A_{P'} \to A_P$ which omits the last coordinate t_p. This map is split by the map $i_p : A_P \to A_{P'}$ which puts a 1 in the last component.*

(3) *Then $\Delta^{B_p}(P, A_P)$ is the set of restrictions of the old simple roots of type D_p to A_P and $\Delta^{D_p}(P', A_{P'})$ is the set of restrictions of the old simple roots of type D_p to the larger torus $A_{P'}$. This may be restated as follows. We may identify A_P and $A_{P'}$ with quotient tori of A and hence we may identify their character groups with subgroups of the character group of A. Suppose that $A_{P'}$ has dimension $r+1$, whence A_P hence dimension r. Then $|\Delta^{D_p}(P', A_{P'})| = r+1$ and $|\Delta^{B_p}(P, A_P)| = r$. Then the first $r-1$ elements of the two sets of restricted simple roots 'coincide' in the sense that as characters of A they are the pullbacks of the restrictions of the roots $t_i/t_{i+1}, 1 \leqslant i \leqslant p-2$, to A_P and $A_{P'}$ (and so some of these may be trivial), the last element of $\Delta^{B_p}(P, A_P)$ is t_{p-1} and the last two elements of $\Delta^{D_p}(P', A_{P'})$ are t_{p-1}/t_p and $t_{p-1}t_p$.*

(ii) *Now suppose the last element in the flag has dimension p, and so the last two elements are E_{p-1} and E_p; then:*

(1) $e(P) = e(P')$.

(2) $A_P = A_{P'}$.

(3) $\Delta^{D_p}(P', A_{P'})$ *and* $\Delta^{B_p}(P, A_P)$ *have the same cardinality* r*, and their first* $r-1$ *elements coincide. The last two non-trivial elements of* $\Delta^{B_p}(P, A^{B_p})$ *are* t_{p-1}/t_p *and* t_p *and the last two non-trivial elements of* $\Delta^{D_p}(P', A_{P'})$ *are the restrictions of* t_{p-1}/t_p *and* $t_{p-1}t_p$.

Proof. We prove (i) for the special case in which P is the stabilizer of the isotropic subspace E_{p-1}; see Example 10.6. For $P' = P \cap \mathrm{SO}(V_{\mathbb{R}})$ we easily see that

$$A_{P'} = \left\{ \begin{pmatrix} aI_{p-1} & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b^{-1} & 0 \\ 0 & 0 & 0 & a^{-1}I_{p-1} \end{pmatrix} ;\ a, b \in \mathbb{R}_+ \right\}$$

and

$$\Delta^{D_p}(P', A_{P'}) = \{a/b, ab\}.$$

Consequently if Y_{p-1} denotes the symmetric space associated with $\mathrm{SL}(E_{p-1})$, we have a diffeomorphism (ignoring the fibre bundle structure)

$$e(P') \cong Y_{p-1} \times (W \otimes E_{p-1}) \times \bigwedge^2 E_{p-1}$$

with $W = \mathrm{span}(u_p, u_p')$. But for the Levi of P in the full group $\mathrm{O}(p,p)$, we have

$$Z(L) = Z(L \cap \mathrm{SO}(p,p)) \cap Z(w)$$

with w as in Example 10.6, whence we have

$$A_P = \left\{ \begin{pmatrix} aI_{p-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a^{-1}I_{p-1} \end{pmatrix} ;\ a \in \mathbb{R}_+ \right\}$$

and

$$\Delta^{B_p}(P, A_P) = \{a\}.$$

Hence we have a diffeomorphism

$$e(P) \cong Y_{p-1} \times \mathbb{R}_+ \times (W \otimes E_{p-1}) \times \bigwedge^2 E_{p-1}.$$

For (ii), suppose that the last subspace has dimension p. For convenience we assume that P is the stabilizer of the flag (E_{p-1}, E_+). Then

$$P = \left\{ \begin{pmatrix} g & c & \dots & \dots \\ 0 & b & \dots & \dots \\ 0 & 0 & b^{-1} & \dots \\ 0 & 0 & 0 & g^* \end{pmatrix} \right\} \quad \text{and} \quad L = \left\{ \begin{pmatrix} g & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b^{-1} & 0 \\ 0 & 0 & 0 & g^* \end{pmatrix} \right\}$$

with $g \in \mathrm{GL}_{p-1}(\mathbb{R})$, $c \in \mathbb{R}^{p-1}$, $b \in \mathbb{R}^*$. Hence

$$A_P = A_{P'} = \left\{ \begin{pmatrix} aI_{p-1} & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b^{-1} & 0 \\ 0 & 0 & 0 & a^{-1}I_{p-1} \end{pmatrix} ;\ a, b \in \mathbb{R}_+ \right\},$$

but

$$\Delta^{B_p}(P, A_P) = \{a/b, b\} \quad \text{and} \quad \Delta^{D_p}(P', A_{P'}) = \{a/b, ab\}. \quad \square$$

10.5. Signature (2, 2)

We now consider the case of signature $(2, 2)$ in detail. In particular, we illustrate in this case the failure of the restriction formula for the small Borel–Serre compactification.

10.5.1. Comparison of the two compactifications. For $\mathrm{SO}(2, 2)$, each apartment of the underlying Tits building (the building of parabolic subgroups of $\mathrm{SO}(2, 2)$) is a square: the building of type $D_2 = A_1 \times A_1$. In the usual Borel–Serre compactification each of the four vertices is blown up to a circle bundle over a quotient of the upper half-plane by a subgroup of finite index in $\mathrm{SL}(2, \mathbb{Z})$, i.e., modular curves. Each edge is blown up to a 2-torus, and the two circle bundles over the modular curves corresponding to the two vertices of the edge are glued along this torus.

We now describe the big Borel–Serre compactification. In this case the underlying building (the non-thick Tits building associated with the complex of isotropic flags in $\mathbb{Q}^{2,2}$) has apartments which are octagons. We will regard these octagons as the barycentric subdivisions of the above squares. We blow up the original vertices to the same circle bundles over modular curves as before. We blow up the four new vertices (the barycentres of the original edges) to trivial 2-torus bundles over $\mathbb{R}_+$ compactified by adding two points 0 and ∞. We can glue the four new 3-manifolds to the four old ones because each has boundary components homeomorphic to the 2-torus. There is one such glueing for each of the eight edges of the octagon. It is critical to observe that not only do we use a new glueing scheme—the non-thick building of type $B_2 = C_2$ associated with the isotropic flag complex—but also there are some new cells $e(P)$ that do not occur in the usual Borel–Serre compactification.

In detail, we consider one fixed edge of the apartment of the Tits building for $\mathrm{SO}(2, 2)$ corresponding to the basis $\{u_1, u_2, u_2', u_1'\}$. Namely, we let $Q'_\pm$ be the maximal parabolic in $\mathrm{SO}(2, 2)$ which is the stabilizer of the isotropic plane $E_\pm$ spanned by u_1, u_2 and u_1, u_2' respectively. The intersection $P' = Q'_+ \cap Q'_-$ stabilizes the oriflamme (E_+, E_-). Recall that in this situation the maximal split torus A is given by $\{a(t_1, t_2) = \mathrm{diag}(t_1, t_2, t_2^{-1}, t_1^{-1});\ t_i > 0\}$. We set $W := \mathrm{span}(u_2, u_2')$. Then:

(i) $e(Q'_+) \simeq \mathbb{H} \times \mathbb{R}$ with trivial bundle structure. The collar neighbourhood in $\overline{D}$ is given by $e(Q'_+) \times \{a(t, t);\ t^2 > T\}$.

(ii) $e(P') = N_{P'} \simeq W \simeq \mathbb{R}^2$. The collar neighbourhood in $\overline{D}$ is given by $e(P') \times \{a(t_1, t_2);\ t_1 t_2 > T, t_1/t_2 > T\}$.

(iii) $e(Q'_-) \simeq \mathbb{H} \times \mathbb{R}$ with trivial bundle structure. The collar neighbourhood in $\overline{D}$ is given by $e(Q'_+) \times \{a(t, t^{-1}); t^2 > T\}$.

Furthermore, $e(Q'_+)$ and $e(Q'_-)$ are glued in $e(P')$ with the respective $\mathbb{R}$-fibres glued to the 'x-direction' of $\mathbb{H}$.

Now we consider the analogous picture for $\mathrm{O}(2,2)$. The faces $e(Q_\pm)$ for the stabilizers $Q_\pm$ of the planes $E_\pm$ stay the same (with slightly different neighbourhoods). But now there are three parabolics P, P_+, P_- whose restriction to $\mathrm{SO}(2,2)$ is P', and we blow up $e(P')$ by using $e(P) \simeq e(P') \times \mathbb{R}_+$ and glue $e(P)$ to $e(Q_\pm)$ along $e(P_\pm)$. The blow-up variable in $\mathbb{R}_+$ in the neighbourhood of $e(P')$ is given by t_1/t_2. We have:

(i) P is the stabilizer of the line $E_1 = \mathbb{R}u_1$. Then $e(P) = \{a(1, t_2)\} \times W$ with collar neighbourhood $e(P) \times \{a(t, 1); t_2 > T\}$.

(ii) $P_\pm$ are the stabilizers of the flag $\mathbb{R}u_1 \subset E_\pm$. Then $e(P_\pm) \simeq W$. Collar neighbourhoods are given by $e(P_+) \times \{a(t_1, t_2); t_1t_2 > T, t_2 > T\}$ and $e(P_-) \times \{a(t_1, t_2); t_1t_2^{-1} > T, t_2^{-1} > T\}$ respectively.

Inside $e(P) \simeq \{a(1, t_2)\} \times W$, one approaches $e(P_\pm)$ by letting $t_2 \to \infty$ and $t_2 \to 0$ respectively.

10.5.2. Non-existence and existence of the restriction for the case of $\mathrm{SO}(2,2)$. In this subsection we will explain why $\theta(\varphi_{2,0})$ does not extend to $\overline{X}_{p,p}$ if $\overline{X}_{p,p}$ is the *small* Borel compactification of $\mathrm{SO}(p,p)$.

Namely, $\theta(\varphi_{2,0})$ does not extend to the 2-torus $e'(P')$, where P' is the stabilizer of the oriflamme (E_+, E_-). We will see below that the limit as we approach $e'(P')$ is undefined (it depends on the way in which we approach the corner). We have just seen that the corner $e'(P')$ is the intersection of the two maximal faces $e'(Q'_\pm)$, trivial circle bundles over quotients of the upper half-plane.

It suffices to study $\theta(\varphi_{2,0})(a(t_1,t_2)) = \sum_{y_1,y_2,y'_2,y'_1} \varphi_{2,0}(t_1^{-1}y_1, t_2^{-1}y_2, t_2y_2, t_1y'_1)$ as we go to the corner. Here y_i, y'_i are the Witt coordinates of V. In this case the 2-form $\theta(\varphi_{2,0})$ has four components. Three of the components go to zero as $\alpha_1 = t_1t_2$ and $\alpha_2 = t_1/t_2$ go to infinity; essentially because $t_1 = \sqrt{\alpha_1\alpha_2} \to \infty$, we can apply the partial Fourier transform and the Poisson summation argument from §9 to the sum on y_1. We find that the limit as $t_1 \to \infty$ coincides (up to a constant) with $\sum_{(y_2,y'_2)} \widetilde{H}_2(t_2^{-1}y_2 + t_2y'_2)e^{-\pi(t_2^{-2}y_2^2+t_2^2(y'_2)^2)}\frac{dt_2}{t_2} \wedge (\frac{dw_2}{t_2} + t_2dw'_2)$. Here w_2, w'_2 are the variables for the 2-torus $e'(P')$ realized as a quotient of $W = \mathbb{R}u_2 \oplus \mathbb{R}u'_2$. Now the resulting limit is supposed to be a 2-form on the corner $e'(P')$, that is, a form in the coordinates w_2, w'_2 on the torus. However note that the limit depends on t_2 (and also involves the coordinate differential dt_2). Thus it depends on how we approach the boundary and consequently is not well-defined. In particular, as claimed, the form $\theta(\varphi_{2,0})$ does not extend to a well-defined 2-form on the manifold with corners $\overline{X}$.

In the big Borel–Serre compactification the problems go away. For the face $e'(P)$, t_2 is the extra variable for $e'(P) = e'(P') \times \mathbb{R}_+$, and we obtain the above form as the limit as $t_1 \to \infty$. The other faces $e'(P_\pm)$, as sets, are again the 2-torus $e'(P')$ but now approached by taking $t_1/t_2, t_2 \to \infty$ and $t_1t_2, t_2^{-1} \to \infty$ respectively. Then the Poisson summation argument on the sum on y_1, y_2 and y_1, y_2', respectively, gives vanishing.

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