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A NONUNIFORM DARK ENERGY FLUID: PERTURBATION EQUATIONS

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ABSTRACT

We propose that galactic dark matter can be described by a nonuniform dark energy fluid. The underlying field is a decaying vector field, which might corresponds to a photon-like but massive particle of 4 degrees of dynamical freedom. We propose a very general Lagrangian for this vector field. The model includes a continuous spectrum of plausible gravity theories, for example, quintessence, $f(R)$, Einstein-Aether, MOND, TeVeS, BSTV, V-Λ theories, and the inflaton scalar field as special cases. We study in detail a special class of models with a fixed norm of the timelike vector field in the physical metric, which includes a nonlinear K_4 term and a Ricci scalar term. We derive the Einstein equations in the perturbed form, which are needed for simulating structure growth in an FRW universe to test such theories. A special case of the model V-Λ shows promise of resembling the ΛCDM cosmology. We show that the vector field has the effect of a nonuniform dark fluid, which resembles dark matter in galaxies and dark energy in the late universe.

Subject headings: cosmology: theory — dark matter — gravitation

1. INTRODUCING A FRAMEWORK FOR VECTOR FIELDS

General relativity (GR) is actually a special case and minimal construction of a range of theories describing the metric of a plausible universe. While completely adequate on small scales, GR by itself predicts a missing mass and missing energy compared to astronomical observations of the metric of the universe on the scale of kpc to Gpc (e.g., Spergel et al. 2007). While the missing mass is arguably explained by dark matter (DM) particle fields in supersymmetry particle physics, the missing energy almost certainly cannot be explained unless the present universe is immersed in an exotic dark energy (DE) field (White 2007). Since both the effects of DM and DE occur when the gravity is weak, one wonders if the underlying fields are tracking the metric field of the gravity (Zhao 2007).

Quantum gravity and string theory often predict a nontrivial coupling of some vector field, which violates CPT symmetry satisfied by standard physics (Kostelecky & Samuel 1989; Kostelecky 2004). It has been considered by Will & Nordvedt (1972) that a vector field can be coupled to the spacetime metric. This creates a "preferred frame" in gravitational physics. A global violation is undesirable, but a local violation is allowed. A four timelike vector field with a nonvanishing time component would select a preferred direction at a given spacetime coordinate. It is an aether-like fluid present everywhere, somewhat like a dark energy with some preferred direction. If such a vector coupling to matter is zero or small, then it can evade current experimental detection (e.g., the CPT violation experiments at Princeton). There has been an increase of interest about such vectors in recent years, especially in works by (to name a few) Kostelecky & Samuel (1989), Kostelecky (2004), Foster & Jacobson (2006), Lim (2005), Bekenstein (2004), Sanders (2005), and Zlosnik et al. (2006, 2007). For example, Foster & Jacobson (2006) noted that a solar system immersed in a unit timelike vector field (called Einstein-Aether, or AE) of small enough mass coupling to the metric is apparently consistent with current measurements of parameterized post-Newtonian (PPN) parameters. Carroll & Lim (2004) noted that such a field can have effects on the Hubble expansion. Inspired by these ideas, several workers, especially Bekenstein (2004), Sanders (2005), and Zlosnik et al. (2007), proposed to extend the application to galaxy scale to use it to explain missing matter (i.e., dark matter). Many have constrained the theory using empirical astronomical data (Famaey & Binney 2005; Zhao & Famaey 2006; Zhao 2006; Famaey et al. 2007), including gravitational lensing (Zhao et al. 2006; Chen & Zhao 2006; Chen 2008; Angus et al. 2007). Most recently Zhao (2007) found a simple Lagrangian within these frameworks to give rise to the DM-DE effects of the right amplitude, offering a possible explanation of coincidence of DE scale and DM scale in ΛCDM cosmology. The model is dubbed Vector-for-Lambda, or V-Λ.

Here we propose a *very general Lagrangian* of the vector field. We show its relation with existing theories. We isolate a simple case and give the full field equations. Most importantly we derive the equations governing perturbation growth in the FRW universe.

To build a covariant theory, one starts with the Einstein-Hilbert action used for GR:

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi G_N}\right) + S_M, \tag{1}$$

where the light speed is $c = 1$, G_N is the gravitational constant, and g is the determinant of the metric $g_{\alpha\beta}$. The signature taken here is $(-, +, +, +)$; we do not distinguish Roman *abcd* and Greek $\alpha\beta\gamma\delta$ for four indices. R is the Ricci scalar, describing the curvature of spacetime. S_M is the matter action that describes the matter distribution. Variation of this action with respect to the metric gives the Einstein equations (EEs):

$$\frac{\delta S}{\delta g^{\alpha\beta}} = 0 \Rightarrow G_{\alpha\beta} = 8\pi G T^{\rm matter}_{\alpha\beta}, \tag{2}$$

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where $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$ is the Einstein tensor and $T^{\rm matter}_{\alpha\beta}$ is the stress-energy tensor of matter defined by $\delta S_M = -\frac{1}{2}\int d^4x(-g)^{1/2} T_{\mu\nu}(x)\delta g^{\mu\nu}(x)$. This tensor describes the matter distribution. In the following we add new fields and terms to the Einstein-Hilbert action.

1.1. *Vector Field with a Dynamic Norm*

Denote a vector field by Z^a, which generally has a variable or dynamic norm. The Lagrangian density of many vector theories can then be cast in the general form[4]

$$L = [1 + f_0(\varphi)]R + f(\varphi, K, J), \tag{3}$$
$$\varphi^2 \equiv Z^a Z_a, \tag{4}$$
$$K \equiv K^{ab}_{cd} \nabla_a Z^b \nabla_c Z^d, \tag{5}$$
$$J \equiv J^{ab}_{cd} \nabla_a Z^b \nabla_c Z^d, \tag{6}$$

where φ^2 is essentially a scalar field made from the norm of the vector field Z^a without introducing new degree of dynamical freedom. The coefficients K^{ab}_{cd} and J^{ab}_{cd} can be lengthy functions of Z^a and the metric g_{ab} with appropriate combinations of upper and lower index. For example, the Lagrangian of the V-Λ model (Zhao 2007) is of the form

$$L = R + f_K(K) + f_J(J), \tag{7}$$

where $K = (Z^a \nabla_a Z^b)(Z^c \nabla_c Z^d)$, $J = \delta^a_b \delta^c_d (\nabla_a Z^b)(\nabla_c Z^d)$. To recover scalar-tensor theories or $f(R)$ theories for dark energy, we set coefficients of K and J to zero, so end up with a Lagrangian

$$L = R + f_0(\varphi)R + f(\varphi), \tag{8}$$

where the vector field Z^a has collapsed into its norm, the scalar φ. All these theories involve only 4 degrees of dynamical freedom at maximum.[5]

As a specific illustration where the coupling coefficients are the simplest, the Lagrangian scalar density can be

$$L(Z, g) = R + L_{012} + L_3 + f_4 L_4 + \sum_{i=5}^{8} a_i L_i + \sum_{i=9}^{\infty} a_i L_i. \tag{9}$$

For simplicity consider setting all coefficients $a_i = 0$ for $i = 9, \ldots, \infty$; we then have

$$L_{012} = a_0 + a_1\varphi^2 + a_2\varphi^4, \quad \varphi^2 \equiv Z^\alpha Z_\alpha, \tag{10}$$
$$L_3 = a_3 \varphi^2 R, \tag{11}$$
$$L_4 = Z^\alpha Z^\beta R_{\alpha\beta}, \tag{12}$$

and

$$L_5 = (\nabla_\alpha Z_\beta)(\nabla^\alpha Z^\beta), \tag{13}$$
$$L_6 = (\nabla_\alpha Z^\alpha)^2, \tag{14}$$
$$L_7 = (\nabla_\alpha Z_\beta)(\nabla^\beta Z^\alpha), \tag{15}$$
$$L_8 = (Z^\beta \nabla_\beta Z^\alpha)(Z^\gamma \nabla_\gamma Z_\alpha), \tag{16}$$

where the coefficients $a_i = \text{const}$.

This Lagrangian density can be simplified further; note that the L_4 term is related to L_6 and L_7 by a total divergence:

$$f_4 L_4 = [\nabla_a(f_4 W^a) - W^a \nabla_a f_4] - f_4 L_6 - f_4 L_7, \tag{17}$$
$$W^a = (Z^a \nabla_b Z^b - Z^b \nabla_b Z^a). \tag{18}$$

Here we can drop the term proportional to ∇f_4, which is zero if f_4 is a φ-independent constant. The total divergence term, when integrated over volume, can be dropped in the total action S, because the term becomes a surface integration over the boundary according to the Stokes theorem. We can therefore choose not to consider the term L_4, absorbing its contribution in the L_6 and L_7 terms.

[4] It is optional to add a new term $f_4(\varphi) Z^a Z^b R_{ab}$, which is related to $f(\varphi, K, J)$ via a full derivative of no effects.

[5] The Lagrangian of the BSTV theory of Sanders (2005) can be cast into a similar expression but with 5 degrees of freedom in the physical frame, with $L = R + d(\varphi) g^{ab} \nabla_a q \nabla_b q + h(q, \varphi)K - f(q, \varphi)J + 2V(q, \varphi)$, where q is a new dynamical scalar field. In the slow-roll approximation, we can neglect the dynamical term by setting $d(\varphi) \sim 0$, eliminate q altogether by minimizing the action with respect to q, and hence rewrite $L = R + f(\varphi, K, J)$. Essentially the function f is a slow-varying scalar in BSTV with its 5 degrees of dynamical freedom.

1.2. *A General Lagrangian of a Scalar Field Plus a Unit Vector*

So far Z^a is a vector, not required to be unit-norm. It is, however, easier to work with unit vector. Now we decompose the vector Z^a into a scalar field φ representing its norm

$$Z^a = \varphi Æ^a, \tag{19}$$

plus a unit timelike vector $Æ_a = Z_a/\varphi$. Basically,

$$Z^a Z_a \equiv \varphi^2, \qquad Æ^a Æ_a = -1. \tag{20}$$

Note the sign convention for the unit vector.

The timelike unit vector guarantees Lorentz invariance to be broken locally, so that it will always have a nonvanishing timelike component. The additional constraint can be enforced using a nondynamic Lagrange multiplier L^*.

The covariant derivative Z-terms have the following correspondences to covariant derivatives of the Æ field and the scalar φ field:

$$\begin{aligned}
\nabla_a Z_b \nabla^b Z^a &= L_5 = Æ^a Æ^b \nabla_a\varphi\nabla_b\varphi + 2\varphi\nabla_a\varphi Æ^b \nabla_b Æ^a + \varphi^2 K_1,\\
(\nabla_a Z^a)^2 &= L_6 = Æ^a Æ^b \nabla_a\varphi\nabla_b\varphi + 2\varphi Æ^a \nabla_a\varphi\nabla^b Æ_b + \varphi^2 K_2,\\
\nabla_a Z_b \nabla^a Z^b &= L_7 = -\, g^{ab}\nabla_a\varphi\nabla_b\varphi + \varphi^2 K_3,\\
(Z^a\nabla_a Z_c)(Z^b\nabla_b Z^c) &= L_8 = \varphi^2 Æ^a Æ^b \nabla_a\varphi\nabla_b\varphi + \varphi^4 K_4,
\end{aligned} \tag{21}$$

where the K_i values are defined as

$$\begin{aligned}
K_1 &= g^{ab} g_{cd} \nabla_a Æ^c \nabla_b Æ^d = \nabla_a Æ_b \nabla^a Æ^b,\\
K_2 &= \delta^a_c \delta^b_d \nabla_a Æ^c \nabla_b Æ^d = (\nabla_a Æ^a)^2,\\
K_3 &= \delta^a_d \delta^b_c \nabla_a Æ^c \nabla_b Æ^d = \nabla_a Æ_b \nabla^b Æ^a,\\
K_4 &= Æ^a Æ^b g_{cd} \nabla_a Æ^c \nabla_b Æ^d = Æ^a \nabla_a Æ_c Æ^b \nabla_b Æ^c.
\end{aligned} \tag{22}$$

Inspired by the above specific case and redefining the coefficients, we propose a *very general Lagrangian* for a dynamical scalar field φ coupled with a unit-norm vector field $Æ^a$,

$$\begin{aligned}
\mathcal{L}(\varphi, Æ) = {} & [1 + c_0(\varphi)]R + 2V(\varphi) + \sum_{i=1}^{4} c_i(\varphi)K_i + (Æ^a Æ_a - 1)L^*\\
& + \left[d_1(\varphi)g^{ab} + d_2(\varphi)Æ^a Æ^b\right]\nabla_a\varphi\nabla_b\varphi + \left[d_3(\varphi)Æ^a\nabla_a Æ^b + d_4(\varphi)Æ^b\nabla_a Æ^a\right]\nabla_b\varphi,
\end{aligned} \tag{23}$$

where L^* is the Lagrange multiplier and R is the Ricci scalar. The c's and d's are now treated as general functions of the scalar field φ, and these terms are some kind of dynamical dark energy. The scalar field φ is a singlet (a real number). It can be turned nondynamical if $d_1 = d_2 = d_3 = d_4 = 0$. Models with $d_2 = d_3 = d_4 = 0$ are simple quintessence models.

1.3. *Relations with other Proposed Lagrangians*

Our general Lagrangian $L(\varphi, Æ)$ includes the Lagrangian $L(Z, g)$ in equation (9) as a special case. In fact all coefficients a_i in $L(Z, g)$ could be generalized to functions of φ. To see this note that $f_4(\varphi)Z^a Z^b R_{ab}$ in equation (17) would contain a term proportional to

$$W^a\nabla_a f_4 = f_4' W^a \nabla_a\varphi \tag{24}$$

and W^a can be cast into φ and Æ, and the end result are terms all included in $L(\varphi, Æ)$. Likewise, any new terms $a_i L_i$ can be collapsed into φ and Æ representation. For example, Ferreira et al. (2007) proposed to set $a_2 = a_3 = 0$, eliminating the $a_3 R\varphi^2$ coupling term, but including four new terms $a_9 L_9 + a_{10} L_{10} + a_{11} L_{11} + a_{12} L_{12}$ in equation (9); this would not lead to new terms in our $L(\varphi, Æ)$. To see how these new terms are absorbed, we note that

$$\begin{aligned}
Z^b Z_c \nabla_a Z^c \nabla_b Z^a &= L_9 = \varphi^2 Æ^a Æ^b \nabla_a\varphi\nabla_b\varphi + \varphi^3 Æ^b \nabla_b Æ^a \nabla_a\varphi,\\
Z^b Z_c \nabla_a Z^a \nabla_b Z^c &= L_{10} = \varphi^2 Æ^a Æ^b \nabla_a\varphi\nabla_b\varphi + \varphi^3 Æ^a \nabla_a\varphi\nabla^b Æ_b,\\
Z_c Z_d \nabla_a Z^c \nabla^a Z^d &= L_{11} = \varphi^2 g^{ab}\nabla_a\varphi\nabla_b\varphi,\\
Z^a Z^b Z_c Z_d \nabla_a Z^c \nabla_b Z^d &= L_{12} = \varphi^4 Æ^a Æ^b \nabla_a\varphi\nabla_b\varphi.
\end{aligned}$$

Note that the right-hand sides are all *already included* in our Lagrangian equation (23), with a specific assignment of our functions $c_{1,2,3,4}$, $d_{1,2,3,4}$.[6]

The Bekenstein (2004) TeVeS Lagrangian can also be cast (Zlosnik et al. 2006) into that of a pure nonunit norm vector field in the physical metric with a Lagrangian

$$L = R + f_J(J) + f_K(K), \tag{25}$$

where $f_J(J) = J$ and the functional form of $f_K(K)$ is determined the MOND interpolation function. The variables K and J are made of terms

$$K = \sum_{i=5}^{12} k_i(\varphi)L_i, \qquad J = \sum_{i=5}^{12} j_i(\varphi)L_i, \tag{26}$$

where k_i and j_i are functions of the norm φ, and L_i are the eight different combinations of the kinetic terms of the nonunit norm vector Z^a, which then reduces to our Lagrangian $L(\varphi, Æ)$ using scalar field and the unit vector.

In fact we claim that our Lagrangian $L(\varphi, Æ)$ is general enough to include several models in the literature as its special cases:

1. *GR*.—This corresponds to our model with $b_i = 0$, $a_i = 0$ except that a_2 and a_0.
2. *Scalar-tensor gravity*.—When $c_{1,2,3,4}$, $d_{2,3,4} = 0$, and $d_1 = 1$ it reduces to the scalar-tensor gravity. If furthermore $c_0 = 0$, it becomes the standard scalar field theory for inflation and quintessence, etc.
3. *$f(R)$ gravity model*.—When $c_{1,2,3,4}, d_{1,2,3,4} = 0$ we could vary the action with respect to φ (*nondynamical* now) to have $\delta\mathcal{L}/\delta\varphi = 0 \Rightarrow c_0'(\varphi)R + V'(\varphi) = 0$ where a prime means $d/d\varphi$. Solving this equation we get $\varphi(R)$ so that the action becomes that for the $R + f(R)$ theory.
4. *Einstein-Aether model and $f(\mathcal{K})$ model*.—Set c_0, $d_{1,2,3,4} = 0$. When $c_{1,2,3,4} = C_{1,2,3,4}$ are constants and $V(\varphi) = 0$ one obtains Jacobson's linear Æ-theory (Jacobson & Mattingly 2001). More generally assuming $c_{1,2,3,4} = C_{1,2,3,4}\varphi$, we can again vary the action with respect to φ and obtain $\sum_i C_iK_i + V'(\varphi) = 0$. We can solve φ as function of $\mathcal{K} = C_iK_i$, and write the Lagrangian as $f(\mathcal{K})$ (Zlosnik et al. 2007; Zhao 2007).
5. *V-Λ model*.—This requires a nondynamical scalar doublet (λ_K, λ_J) (Zhao 2007). We set $d_{1,2,3,4} = c_{0,1,3} = 0$, and c_2, c_4 are functions of the two independent components (λ_K, λ_J) of the doublet, respectively.
6. *TeVeS*.—This requires the scalar field to be a doublet (φ, μ), that is, it requires two scalar fields. The μ field is nondynamical (e.g., Bekenstein 2004; Zlosnik et al. 2006). The expressions for our functions are lengthy.
7. *BSTV*.—This again requires a scalar doublet (φ, q) (Sander 2005). But we set $d_1 + d_2 = h(q)$ and $d_1 = f(q)$, and $d_3 = d_4 = c_{0,1,2,3,4} = 0$.

There are various other special cases, not in the literature. For example, if we set $d_{1,2,3,4} = 0$, $c_{1,2,3,4} = \varphi C_{1,2,3,4}$, where the C's are constants and $c_0(\phi) \neq 0$, we can vary with respect to φ to get an equation of motion, $\varphi = \varphi(R, K)$, and eliminate the nondynamical φ and its potential $V(\varphi)$, and then cast the Lagrangian as $L = R + F(R, K)$ models. This kind of models are simpler than TeVeS, since the unit vector has only 3 degrees of dynamical freedom and there is no scalar freedom.

2. THE $F(Q)$ MODELS

The dynamics of our general Lagrangian is very rich. To be specific, let us consider the simpler $F(Q)$ where $Q = c_0 M^{-2}R + \mathcal{K}$ models, where we redefine c_i as constants for $i = 1, 2, 3, 4$, and the c_0 term includes a dimensionless (linear) dependence on the Ricci scalar, and redefine variables so that the final total action is

$$S = S_M + \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R + M^2F(Q) + L^*(A^\alpha A_\alpha + 1)\right], \tag{27}$$

$$Q \equiv c_0 M^{-2}R + \mathcal{K}, \tag{28}$$

where L^* is the Lagrangian multiplier, and

$$\mathcal{K} \equiv M^{-2}K^{\alpha\beta}{}_{\gamma\sigma}\nabla_\alpha A^\gamma \nabla_\beta A^\sigma, \tag{29}$$

$$K^{\alpha\beta}{}_{\gamma\sigma} = c_1 g^{\alpha\beta}g_{\gamma\sigma} + c_2\delta^\alpha_\gamma\delta^\beta_\sigma + c_3\delta^\alpha_\sigma\delta^\beta_\gamma + c_4A^\alpha A^\beta g_{\gamma\sigma}. \tag{30}$$

Note that we have replaced the Æ field with A field using the opposite sign convention.

$$A^\alpha A_\alpha = -1. \tag{31}$$

We stick to the A field (instead of Æ field) for the rest of the paper.

In the case that $c_0 = 0$, our action is similar to what was considered by Jacobson and coworkers, except for the nonlinear F-function. Notice that dropping the terms in c_2 and c_4 and considering $c_3 = -c_1$, we find $K^{\alpha\beta}{}_{\gamma\sigma}\nabla_\alpha A^\gamma \nabla_\beta A^\sigma = (c_1/2)F_{\alpha\sigma}F^{\alpha\sigma}$, where $F_{\alpha\sigma}$ is the antisymmetric Maxwell tensor defined by $F_{\alpha\sigma} = \nabla_\alpha A_\sigma - \nabla_\sigma A_\alpha$. This simplification was used by Jacobson and by Bekenstein in TeVeS.

[6] Our Lagrangian for Z^a is a special case if we choose $c_0 = a_3\varphi^2$, $c_{1,2,3} = a_{5,6,7}\varphi^2$, $c_4 = a_8\varphi^4$, $d_1 = a_7 + a_{11}\varphi^2$, $d_2 = a_5 + a_6 + (a_8 + a_9 + a_{10})\varphi^2 + a_{12}\varphi^4$, and $d_{3,4} = 2a_{5,6}\varphi + a_{9,10}\varphi^3$.

Models with $c_{1,2,3,4} = 0$ and $c_0 \neq 0$ are $F(R)$ theories. Models with $c_0 = c_4 = 0$ have been studied by Zlosnik et al. (2007) without giving the full equations. Here we expand on previous results.

2.1. *Field Equations for $F(Q)$ Models*

Now we proceed to obtain the field equations for models, where $Q = c_0 M^{-2} R + \mathcal{K}$. What must be borne in mind when carrying out the variations is that the 2 dynamical degrees of freedom considered are the inverse metric $g^{\mu\nu}$ and the contravariant vector field A^{μ}. The contravariant vector is chosen (and not the covariant one) just because once one has chosen to vary the action with respect to $g^{\mu\nu}$, the result of this variation will be simpler seeing the form of $K^{\alpha\beta}{}_{\gamma\sigma}$, because we have

$$\frac{\delta A^{\mu}}{\delta g^{\alpha\beta}} = 0, \tag{32}$$

where we used the fact that

$$g_{\mu\rho} g^{\rho\sigma} = \delta^{\sigma}_{\mu} \Rightarrow \delta g_{\mu\nu} = -g_{\mu\rho} g_{\nu\sigma} \delta g^{\rho\sigma} \tag{33}$$

$$\rightarrow \frac{\delta A_{\mu}}{\delta g^{\alpha\beta}} = A^{\nu} \frac{\delta g_{\mu\nu}}{\delta g^{\alpha\beta}} = -g_{\mu\alpha} A_{\beta}. \tag{34}$$

The vector equation is obtained by varying the action with respect to A^{μ}:

$$\frac{\delta S}{\delta A^{\alpha}} = 0 \Rightarrow \nabla_{\alpha}\left(F' J^{\alpha}{}_{\beta}\right) - F' y_{\beta} = 2L^{*} A_{\beta}, \tag{35}$$

where we define $F' = dF/d\mathcal{K}$. $J^{\alpha}{}_{\sigma}$ is a tensor current: $J^{\alpha}{}_{\sigma} = (K^{\alpha\beta}{}_{\sigma\gamma} + K^{\beta\alpha}{}_{\gamma\sigma}) \nabla_{\beta} A^{\gamma} = 2K^{\alpha\beta}{}_{\sigma\gamma} \nabla_{\beta} A^{\gamma}$, due to the symmetry here in K, and $y_{\beta} = \nabla_{\sigma} A^{\eta} \nabla_{\gamma} A^{\xi} \left[\delta(K^{\sigma\gamma}{}_{\eta\xi})/\delta A^{\beta}\right] = c_4 M^{-2} A^{\alpha} \nabla_{\alpha} A_{\sigma} \nabla_{\beta} A^{\sigma}$.

To get the Lagrange multiplier L^{*}, we multiply the equation by A^{β} and contract. Once L^{*} is known, the equation (which has four components $\beta = 0, 1, 2, 3$) yields three constraint equations for the vector. Varying the action with respect to L^{*} will give the constraint on the norm: $A^{\alpha} A_{\alpha} = -1$.

For the variation of the action $S = \int d^4x (-g)^{1/2} L$ with respect to the contravariant metric, one must notice that

$$\frac{\delta S}{\delta g^{\alpha\beta}} = \int d^4x \sqrt{-g} \left(\frac{\delta L}{\delta g^{\alpha\beta}} - \frac{1}{2} g_{\alpha\beta} L \right),$$

where one uses the fact that $\delta g = g g^{\mu\nu} \delta g_{\mu\nu} = -g g_{\mu\nu} \delta g^{\mu\nu}$, g being the determinant of the contravariant metric.

The symmetry of $K^{\alpha\beta}{}_{\sigma\gamma}$ simplifies the equations

$$\frac{\delta(M^2 F)}{\delta g^{\alpha\beta}} = W_{\alpha\beta} + F' \left[Y_{\alpha\beta} + J^{\sigma}{}_{\eta} \frac{\delta(\nabla_{\sigma} A^{\eta})}{\delta g^{\alpha\beta}} \right], \tag{36}$$

with

$$W_{\alpha\beta} = \left(\Xi R_{\alpha\beta} + g_{\alpha\beta} \nabla\nabla\Xi - \nabla_{\alpha}\Xi \nabla_{\beta}\Xi \right), \quad \Xi \equiv \frac{\partial(M^2 F)}{\partial R}, \tag{37}$$

and

$$Y_{\alpha\beta} = \nabla_{\sigma} A^{\eta} \nabla_{\gamma} A^{\xi} \frac{\delta\left(K^{\sigma\gamma}{}_{\eta\xi}\right)}{\delta g^{\alpha\beta}}. \tag{38}$$

The variation of the covariant derivative of the contravariant vector field requires varying the Christoffel symbol (only):

$$\frac{\delta(\nabla_{\sigma} A^{\eta})}{\delta g^{\alpha\beta}} = \frac{\delta\left(\partial_{\sigma} A^{\eta} + \Gamma^{\eta}_{\sigma\rho} A^{\rho}\right)}{\delta g^{\alpha\beta}} = \frac{\delta\left(\Gamma^{\eta}_{\sigma\rho}\right)}{\delta g^{\alpha\beta}} A^{\rho}. \tag{39}$$

And we have $\delta(\Gamma^{\eta}_{\sigma\rho}) = (g^{\eta\tau}/2)(\nabla_{\sigma}\delta g_{\rho\tau} + \nabla_{\rho}\delta g_{\sigma\tau} - \nabla_{\tau}\delta g_{\sigma\rho})$, so one eventually finds

$$F' J^{\sigma}{}_{\eta} \frac{\delta(\nabla_{\sigma} A^{\eta})}{\delta g^{\alpha\beta}} = -\frac{1}{2} \nabla_{\sigma}\left[\mathcal{F}' \left(J_{(\alpha}{}^{\sigma} A_{\beta)} - J^{\sigma}{}_{(\alpha} A_{\beta)} - J_{(\alpha\beta)} A^{\sigma} \right)\right], \tag{40}$$

dropping divergence terms which would once more contribute only by boundary terms. The brackets denote symmetrization, for instance, $J_{(\alpha\beta)} = \frac{1}{2}(J_{\alpha\beta} + J_{\beta\alpha})$.

Putting these together and using

$$\frac{\delta A^\mu A_\mu}{\delta g^{\alpha\beta}} = -A_\alpha A_\beta, \tag{41}$$

we find

$$G_{\alpha\beta} = 8\pi G T^{\text{matter}}_{\alpha\beta} + \hat{T}_{\alpha\beta} - W_{\alpha\beta} \tag{42}$$

and $\hat{T}_{\alpha\beta}$ is the stress-energy tensor of the vector field

$$\begin{aligned}\hat{T}_{\alpha\beta} = {} & \frac{1}{2}\nabla_\sigma\left[\mathcal{F}'\left(J_{(\alpha}{}^{\sigma}A_{\beta)} - J^{\sigma}{}_{(\alpha}A_{\beta)} - J_{(\alpha\beta)}A^{\sigma}\right)\right] \\ & - \mathcal{F}'Y_{(\alpha\beta)} + \frac{1}{2}g_{\alpha\beta}M^2\mathcal{F} + L^* A_\alpha A_\beta.\end{aligned} \tag{43}$$

The above equations are actually true for all $F(R, K)$ models, if F' is interpreted as $\partial F/\partial\mathcal{K}$. It is interesting that the effect of the $c_0 R$ term behaves partly as a rescaling of the gravitational constant by a factor $(1 + \Xi)$. Like in $F(R)$ gravity, the value of $\Xi \propto c_0$ must be very small, to prevent the term $g_{ab}\nabla^2\Xi$ in the correction source term W_{ab} from violating stringent constraints on small scales, for example, the solar system. Unless stated otherwise, we set this source to zero for simplicity.

3. FULL EQUATIONS FOR PERTURBATIONS OF $F(\mathcal{K})$ MODELS IN AN EXPANDING UNIVERSE

3.1. *Metric, Matter, and Einstein Tensor*

With above equations of motion we consider a Friedman-Robertson-Walker (FRW) perturbed metric such that

$$ds^2 = -(1 + 2\epsilon\phi)dt^2 + a(t)^2(1 + 2\epsilon\psi)\left(dx^2 + dy^2 + dz^2\right). \tag{44}$$

A static universe is a special case with all quantities independent of time and $a(t) = 1$. This fairly general metric is a weakly perturbed form of a homogeneous and spatially isotropic universe $ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)$ by setting $\epsilon = 0$; the unperturbed metric is also spatially flat here, that is, with no curvature parameter. The potentials ϕ and ψ are Newtonian gravitational potentials, which are generally nonidentical. This form of perturbed metric neglects both tensor mode (gravitational wave) and vector mode perturbations.

In the following, the equations are developed in orders of ϵ, but ϵ *is not kept for a lighter expression*.

Matter.—For matter fields, we can take

$$T^{\text{matter}}_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu}, \tag{45}$$

which is the stress tensor of a perfect fluid without any anisotropic stress, with a density ρ, a pressure P and with u_μ the fluid four-velocity satisfying $g_{\mu\nu}u^\mu u^\nu = -1$. If we consider a nonrelativistic fluid, hence neglecting the spatial components of u_μ, then in our metric $u_\mu = (-1 - \epsilon\phi, 0, 0, 0)$. We have also

$$T^{\text{matter}}_{00} = (1 + 2\phi)\rho, \tag{46}$$

$$T^{\text{matter}}_{0i} = T^{\text{matter}}_{ij} = 0, \tag{47}$$

$$T^{\text{matter}}_{ii} = a^2(1 + 2\psi)P. \tag{48}$$

Einstein tensor.—Up to linear order in ϵ we find

1. $G_{00} = 3H^2 + 6H\partial_t\psi - \frac{2}{a^2}\partial_i^2\psi$,
2. $G_{0i} = 2(H\partial_i\phi - \partial_t\partial_i\psi)$,
3. $G_{xx} = (\dot{a}^2 + 2a\ddot{a})[-1 + 2(\phi - \psi)] + (\partial_y^2 + \partial_z^2)(\phi + \psi) - 2a^2\partial_t^2\psi + 2a\dot{a}\partial_t(\phi - 3\psi)$,
4. $G_{ij} = -\partial_i\partial_j(\phi + \psi)$ for $i \neq j$.

3.2. *Vector Field*

We take a homogenous and spatially isotropic universe for the background, so the vector field must, in the background, respect this isotropy for the modified Einstein equations to have solutions, so only the time component can be nonzero. The constraint on the norm is $g_{\alpha\beta}A^\alpha A^\beta = -1$ so in the background, we take $A^\alpha = \delta^\alpha_0$ and one can then expand it and write

$$A_\mu = g_{\mu\nu}A^\nu = (-1, 0, 0, 0) + \left(-\epsilon\phi, \epsilon B_x, \epsilon B_y, \epsilon B_z\right). \tag{49}$$

The constraint on the total vector $g_{00}A^0A^0 \sim -1$ and $g_{00} = -(1 + 2\epsilon\phi)$ with the perturbed form of the metric fixes $A^0 \sim 1 - \epsilon\phi$ and $A_0 \sim g_{00}A^0 \sim -1 - \epsilon\phi$.

We can also derive ∇A up to linear order in ε, whose nonvanishing components are

$$\nabla_i A_0 = -HB_i, \quad \nabla_0 A_i = \partial_i\phi + \dot{B}_i - HB_i,$$
$$\nabla_i A_j = a^2\left[\dot{\psi} + H(1+2\psi-\phi)\right]\delta_{ij} + \partial_i B_j. \tag{50}$$

Following we carried out the calculations of the Einstein equations up to linear order analytically with this metric. Wherever the expressions become very lengthy, it is helpful to break the expressions into a *nonspatial* part without the B_i terms and a *spatial* part with B_i terms, and denote the two parts by the superscripts A and B. Note also that any term F and its derivatives F' and F'' contain implicitly an unperturbed part and a perturbed part. Finally we define the shorthands $\alpha \equiv c_1 + 3c_2 + c_3$, and define $\partial_i^2 = \partial_i\partial_i$, where $i = 1, 2, 3$ and ∂_i is the comoving spatial derivatives and $\dot{X} = \partial_t X$ is the synchronous cosmic time derivative.

3.3. *Kinetic Scalar and* $F(\mathcal{K})$

We decompose the kinetic scalar $\mathcal{K}$ into the zeroth-, first-, and second-order terms:

$$\delta\mathcal{K} \equiv \mathcal{K} - \frac{3\alpha H^2}{M^2} = \frac{3\alpha H^2}{M^2}\left(-2\phi + 2H^{-1}\dot{\psi} + \frac{2}{3a^2H}\partial_i B_i\right)\epsilon + \epsilon^2\left[\frac{c_4 - c_1}{a^2M^2}\left(\partial_i\phi + \dot{B}_i\right)^2 + \ldots\right], \tag{51}$$

where ellipses include the second-order terms $+(3\alpha/M^2)[-4H\psi\partial_t\psi + 5H^2\phi^2 - 4H\phi\partial_t\psi + (\partial_t\psi)^2] + (6c_2H/M^2)\phi\partial_t\phi$ and other lengthy terms shown elsewhere (Halle 2007). All second-order terms are negligible for the linear perturbation calculations. However, the very first term, $[(c_4 - c_1)/a^2M^2](\partial_i\phi\partial_i\phi)$, should be considered for static galaxies, which are in the nonlinear regime.

The terms F, F', and $\partial_i F'(\mathcal{K})$ are often involved in the Einstein equation. We note they have *different orders of magnitude*, given by

$$\delta F \equiv F - F\left(\frac{3\alpha H^2}{M^2}\right) = F'\delta\mathcal{K}, \quad \delta F' \equiv F' - F'\left(\frac{3\alpha H^2}{M^2}\right) = F''\delta\mathcal{K},$$
$$\partial_i F' - 0 = F''(\mathcal{K})\partial_i\mathcal{K}, \quad \partial_i\mathcal{K} - 0 = \frac{6\alpha}{M^2}\left(-H\partial_i\phi + \partial_t\partial_\psi\right)\epsilon, \tag{52}$$

where we have moved the 0th order terms to the left-hand side. In the special case in which $\alpha = 0$, we find $\delta F'$, δF, $\partial_i F'$, $\delta(\mathcal{K})$ are all zero up to the second order.

3.4. *The Lagrange Multiplier*

The vector equation gives the Lagrange multiplier $L^* = L^{*A} + L^{*B}$, where

$$L^{*A} = \frac{3F'}{1+2\phi}\left[(c_1+c_2+c_3)\left(H^2 + 2H\dot{\psi}\right) + c_2\left(-\frac{\ddot{a}}{a} + H\dot{\phi} - \partial_t^2\psi\right)\right] - \dot{F}'\left(H + 3\dot{\psi}\right) - \frac{c_3\partial_i}{a^2}(F'\partial_i\phi) \text{ and} \tag{53}$$

$$L^{*B} = +2\alpha\frac{\dot{a}}{a^3}F'\partial_i B_i - 3c_2\frac{\dot{a}}{a^3}\partial_i(F'B_i) - \frac{c_2}{a^3}\partial_t(aF'\partial_i B_i) - \frac{c_3}{a^2}\partial_i(F'\partial_t B_i), \tag{54}$$

where we could drop a second-order term $-3c_2(\dot{a}/a^3)B_i\partial_i F' - (c_3/a^2)\partial_i F'\partial_t B_i$. Note that both $\partial_i F'$ and B_i are of first order.

3.5. *Perturbed Equation of Motion of Vector Field*

For the j-component of the equation of motion (EoM) of the vector field, we have

$$0 = \frac{c_4 - c_1}{a}\partial_t\left[aF'\left(\partial_j\phi + \dot{B}_j\right)\right] + \frac{c_1 + c_2 + c_3}{a^2}F'\partial_i\partial_i B_j + \alpha\left[\left(\partial_j\dot{\psi} - H\partial_j\phi\right)F' + H\partial_j F' + B_j\partial_t(HF')\right], \tag{55}$$

where we have dropped second-order terms involving the product of $(\partial_i F')$ and other first-order quantities (Φ, Ψ, B_i). This equation resembles equations (B1) and (B2) in the Appendix of Lim (2005). We consider only the scalar mode here $B_j = \partial_j V$ for $j = 1, 2, 3$.

3.6. *The Spatial Off-Diagonal Terms*

The EE and vector field stress term with $i \neq j$ satisfies up to linear order

$$G_{ij} = -\partial_i\partial_j(\phi + \psi) = \hat{T}_{ij} = -\frac{c_1 + c_3}{2a}\partial_t\left[aF'\left(\partial_i B_j + \partial_j B_i\right)\right]. \tag{56}$$

We can, as in the static case, identify the Newtonian potentials, $\phi + \psi = 0$ in the absence of the anisotropic stress $\hat{T}_{ij}$, that is, in the (magnetic) case $c_1 + c_3 = 0$, (obviously GR is a special case of this).

3.7. *The* 0*i Cross Terms*

The 0i component of the stress tensor

$$\hat{T}_{0j} = \frac{c_4 - c_1}{a}\partial_t\left[aF'\left(\partial_j\phi + \dot{B}_j\right)\right] + \alpha\partial_t(F'H)B_j + \Delta, \tag{57}$$

where $\Delta = +[(c_3 - c_1)/2a^2]\partial_i[F'(\partial_j B_i - \partial_i B_j)]$ is a curl-like term, and could be dropped in case of the scalar mode where $B_i = \partial_i V$; even for vector mode, one can drop the second-order term $[(c_3 - c_1)/2a^2]\partial_i F'(\partial_j B_i - \partial_i B_j)$ safely. So we have the 0x-component EE

$$+2H\partial_j\phi - 2\partial_j\dot{\psi} - \left(\frac{c_4 - c_1}{a}\right)\partial_t\left[aF'\left(\partial_j\phi + \dot{B}_j\right)\right] = -8\pi G\rho u_j \sim 0 \tag{58}$$

for a nonrelativistic matter fluid.

3.8. *The 00th Einstein Equation*

Replacing L^*, we find the 00 component of the vector field stress-energy tensor $\hat{T}_{00}$ satisfies

$$\hat{T}_{00} = \frac{c_4 - c_1}{a^2}\partial_i\left[\left(\partial_i\phi + \dot{B}_i\right)F'\right] + \alpha F'H\left(3H + 6\dot{\psi} + 2a^{-2}\partial_i B_i\right) - \frac{1 + 2\phi}{2}M^2 F, \tag{59}$$

where we could drop a second-order term $[(c_4 - c_1)/a^2]\partial_i F'\partial_t B_i$.

Thus, we can write the 00 Einstein equation with $T_{00}^{\mathrm{matter}} = (1 + 2\phi)\rho$ as

$$3(1 - \alpha F')\left[H^2(1 - 2\phi) + 2H\dot{\psi}\right] - \frac{2\alpha F'H}{a^2}\partial_i B_i - \frac{2}{a^2}\partial_i^2\psi - \frac{c_4 - c_1}{a^2}\partial_i\left[F'\left(\partial_i\phi + \dot{B}_i\right)\right] = \left(8\pi G\rho - \frac{M^2 F}{2}\right), \tag{60}$$

where we moved the F term to the right-hand side and divided by the factor $(1 + 2\phi)$.

3.9. *Spatial Diagonal Equations*

The spatial diagonal terms satisfy $\hat{T}^A_{xx} = \hat{T}^A_{yy} = \hat{T}^A_{zz}$ and, for example, $\hat{T}_{xx} = \hat{T}^A_{xx} + \hat{T}^B_{xx}$, where

$$\hat{T}^A_{xx} = a^2(1 + 2\psi)\left[-\alpha\frac{1 - 2\phi}{a^3}\partial_t(F'a^2\dot{a}) + \frac{M^2 F}{2}\right] + \alpha a^2\left[-\dot{F}'\dot{\psi} + F'\left(H\dot{\phi} - 6H\dot{\psi} - \partial_t^2\psi\right)\right]. \tag{61}$$

$$\hat{T}^B_{xx} = -\alpha\left[H\partial_i(F'B_i) + \frac{\partial_t}{3a}(aF'\partial_i B_i)\right] + \Delta, \qquad \Delta = -\frac{c_1 + c_3}{3a}\partial_t[aF'(3\partial_x B_x - \partial_i B_i)], \tag{62}$$

where the summation over $i = 1, 2, 3$ is implicit, and we could drop a second-order term $-\alpha H\partial_i F'B_i$.

Since for matter $T_{ii}^{\mathrm{matter}} = a^2(1 + 2\psi)P$, the modified pressure equation by adding the three spatial diagonal equations becomes

$$\begin{aligned}&\left[-(1 - 2\alpha F')H^2 - (2 - \alpha F')\frac{\ddot{a}}{a} + \alpha\dot{F}'H\right](1 - 2\phi) + \frac{2}{3a^2}\partial_i^2(\phi + \psi)\\ &+ \alpha\left(3HF' + \dot{F}'\right)\dot{\psi} + (2 - \alpha F')\left[H\left(\dot{\phi} - 3\dot{\psi}\right) - \ddot{\psi}\right] + \alpha\left[H\partial_i(F'B_i) + \frac{\partial_t}{3a}(aF'\partial_i B_i)\right] = 8\pi GP + \frac{M^2 F}{2},\end{aligned} \tag{63}$$

where we have moved the vector field term to the left-hand side and divided by the factor $(1 + 2\psi)a^2$ on both sides of the Einstein equation.

4. SPECIAL CASES

We have thus obtained the perturbations of the vector field stress-energy tensor and the Einstein equation for a vector field with a Lagrangian involving a general function of the kinetic term $\mathcal{K}$. As a first check, we recover the linear $F = \mathcal{K}$ model of Lim (2005) and extend it to include a c_4 term; this is given in the Appendix. These perturbation equations are also consistent with Li et al. (2008), which uses a very different formulation. As a summary of equations and further illustrations, let us consider some more special cases in the context of dark matter and the cosmological constant.

Two important quantities for later use are

$$\tilde{\lambda} \equiv \frac{c_4 - c_1}{2}\frac{dF(\mathcal{K})}{d\mathcal{K}}, \qquad \mu \equiv 1 - \tilde{\lambda} = \sqrt{\frac{|\mathcal{K}|}{|\mathcal{K}| + 2}}. \tag{64}$$

As will be evident below, this choice of $F(\mathcal{K})$ recovers MOND in present-day galaxies.

4.1. $F(\mathcal{K}_4)$ *Models with* $c_1 = c_2 = c_3 = 0$

The perturbation equations become much simpler if we concentrate on models with a pure c_4 term. By letting $c_1 = c_2 = c_3 = 0$, we neglect all contributions of other kinematic terms (one can set $c_4 = 2$ with no loss of generality). Up to the linear order the Lagrange multiplier

$$L^* = 0. \tag{65}$$

We find $\psi = -\phi$ from the spatial off-diagonal EE:

$$G_{ij} = -\partial_i \partial_j(\phi + \psi) = \hat{T}_{ij} = 0, \text{ for } i \neq j. \tag{66}$$

Collecting terms in the equations of motion and the EEs, replacing $\psi = -\phi$, and considering only scalar mode $B_i = \partial_i V$ we get

$$G_{0i} = \frac{2}{a}\partial_t \partial_i (a\phi) = -8\pi G \rho u_i \sim 0, \tag{67}$$

$$G_0^0 = 3H^2 - \frac{6H}{a}\partial_t(a\phi) + \frac{2}{a^2}\partial_i^2 \phi = 8\pi G\rho + \hat{T}_0^0, \tag{68}$$

$$\frac{1}{3}G_i^i = \left(-H^2 - 2\frac{\ddot{a}}{a}\right)(1 - 2\phi) = 8\pi G P + \hat{T}_x^x, \tag{69}$$

and

$$\hat{T}_{0i} = 0 = \frac{2\partial_t}{a}X_i, \tag{70}$$

$$\hat{T}_0^0 = \frac{2\partial_i X_i}{a^3} - \frac{M^2 F}{2}, \tag{71}$$

$$\hat{T}_x^x = +\frac{M^2 F}{2}, \tag{72}$$

$$\frac{\tilde{\lambda}^2}{c_4}\mathcal{K} = \left[\frac{X_i}{a(t)^2 M}\right]^2, \tag{73}$$

where we introduce a quantity

$$X_i \equiv \tilde{\lambda}\left[\partial_i(a\phi) + a\dot{B}_i\right], \tag{74}$$

and the terms M^2F and $\tilde{\lambda}$ are fixed functions of $\mathcal{K}$ and hence implicit functions of X_i/Ma^2.

The quantity X_i behaves as a time-independent comoving gravitational force of an effective "dark matter." The time independence is set by the EoM of the vector field or the $\hat{T}_{0i}$ term. Hence, the vector field B_i tracks the spatial variation of the time-independent $\partial_i(a\phi)$ and X_i by the constraint

$$\dot{B}_i = \frac{\tilde{\lambda}}{a}X_i - \frac{1}{a}\partial_i(a\phi). \tag{75}$$

Timewise, as the universe expands, $\mathcal{K}$ and X_i/Ma^2 all approach 0, and $\tilde{\lambda}$ approaches a finite value or zero. Hence, $\dot{B}_i \to 0$.

In general the vector field $A_\mu = (-1 - \phi,\ B_1,\ B_2,\ B_3)$ in $F(\mathcal{K}_4)$ models simply tracks the spacetime metric perturbation ϕ and scale factor $a(t)$, which tracks the dominant source, be it radiation or baryonic matter. Metric perturbation can be printed in the B_1, B_2, and B_3 fields even in the absence of baryonic matter. Note that the effects of the vector field are more complex than a change of gravitational constant of a baryon-radiation fluid, where Silk damping can erase perturbations. The vector field is not coupled to photons or baryons directly, so its perturbations can be passed onto baryons after last scattering. The $\hat{T}_{00}$ stress contains a DM-like source term, which decays with the redshift as fast as the baryonic density $\bar{\rho} = \bar{\rho}_{\rm com}a^{-3}$, but keeping the effective DM-to-baryon contrast time-independent. That is,

$$\frac{a^{-3}\rho_{\rm DM,com}}{a^{-3}\bar{\rho}_{\rm com}} = \text{independent of time}, \qquad \rho_{\rm DM,com} \equiv \partial_i X_i. \tag{76}$$

We can further introduce another parameter for the equation of state parameter defined by

$$w \equiv \frac{\hat{T}_x^x}{\hat{T}_0^0} = \left(1 - \frac{2\partial_i X_i a^{-3}}{M^2 F/2}\right)^{-1}. \tag{77}$$

Clearly in the case of the early universe and CMB, a^{-3} is large, so

$$w = 0. \tag{78}$$

That is, the equation of the state of the vector field is almost exactly dark matter like. This is important to understand why the vector field can replace dark matter (DM) in galaxies. We show next that the $F(\mathcal{K}_4)$ model is essentially a nonuniform dark energy. One difference with real DM is that DM density perturbations can grow in comoving coordinates while the "dark" source term $2\partial_i X_i/a^3$ in $F(\mathcal{K}_4)$ corresponds to a static nonuniform density in comoving coordinates.

In summary $F(\mathcal{K}_4)$ gravity gives particularly simple equations. The meaning of these equations has been explored in part in the VΛ model of Zhao (2007). The model has the effect of a nonuniform dark energy, which mimics galactic dark matter, can seed cosmic perturbations, but does not boost structure formation.

4.2. *Homogenous and Isotropic Universe*

As a second check, we consider the general case of $F' \neq cst$, but in the simple case of the expanding uniform universe. The only nonzero components of the Einstein tensor are $G_{00} = 3H^2$ and $G_{xx} = G_{yy} = G_{zz} = -(\dot{a}^2 + 2a\ddot{a})$.

We therefore have the 00 modified Einstein equation

$$3(1 - \alpha F')H^2 + \frac{1}{2}M^2 F = 8\pi G\rho \tag{79}$$

and the modified pressure equation

$$-(1 - 2\alpha F')H^2 - 2\left(1 - \frac{1}{2}\alpha F'\right)\frac{\ddot{a}}{a} + \alpha\dot{F}'H - \frac{M^2 F}{2} = 8\pi GP. \tag{80}$$

These results are identical to that of Zlosnik et al. (2007). This means simply that the c_4 term does *not* contribute to the expansion except for providing a zero point of pressure.

4.3. *A Possible Origin of Cosmological Constant*

To see how the c_4 term can contribute as cosmological constant, let us consider Hubble expansion in the simple case in which we set $\alpha = 0$. For such models $\mathcal{K} = 0$. The equations for expansion become very simple:

$$3H^2 + \frac{M^2 F}{2} = 8\pi G\rho, \tag{81}$$

$$-H^2 - 2\frac{\ddot{a}}{a} - \frac{M^2 F}{2} = 8\pi GP, \tag{82}$$

so the equation of state of the vector field is

$$w = -1 \tag{83}$$

for the Hubble expansion at all redshift.

Following Zhao (2007) we set the zero point $F(\mathcal{K}_{\rm solar}) = 0$ in the solar-system-like strong gravity regime, where $\mathcal{K}_{\rm solar} \sim 10^{16}$, since the gravity near the Earth's orbit is about $10^8 M$ (where $M \sim 10^{-10}$ m s^{-2}). Thus,

$$\frac{M^2 F}{2} = \int_{\mathcal{K}_{\rm solar}}^{\mathcal{K}} 2\tilde{\lambda}\, d\mathcal{K}. \tag{84}$$

Taylor expanding in the limit of weak gravity $\mathcal{K} \sim 0$, we have $M^2F/2 \approx -\Lambda_0 + (M^2/c_4)\tilde{\lambda}\mathcal{K} \approx -\Lambda_0 + (\tilde{\lambda}/a^2)\partial_i\phi\partial_i\phi$, which has no first-order term but can have a zero-point constant Λ_0, given by

$$\Lambda_0 = -M^2 c_4^{-1} \int_{\mathcal{K}_{\rm solar}}^{0} 2\tilde{\lambda}\, d\mathcal{K} \sim M^2 \ln \mathcal{K}_{\rm solar}, \tag{85}$$

where for reasons evident later we take $\mu = 1 - \tilde{\lambda} = [|\mathcal{K}|/(|\mathcal{K}| + 2)]^{1/2}$. As we show below, Λ_0 plays the role of the cosmological constant. Interestingly, $\Lambda_0 \sim M^2 \ln 10^{16} \sim 36M^2 \sim H_0^2$, which is the observed amplitude of the cosmological constant. The logarithm factor explains why the observed Λ is significantly greater than M^2.

4.4. *Static Limit*

As another application, we apply our equations to the regime of quasi-static galaxies. We set the background expansion factor $a = 1$. In the static limit, the spatial terms of the vector appear only at second order in all the equations. $\hat{T}_{\alpha\beta}$ has no cross terms (up to linear order), so we find $\psi = -\phi$ from $G_{ij} = 0$ equation. And the only nonzero component of the Einstein tensor is

$$-G_0^0 = G_{00} = 2a^{-2}\partial_i^2\phi. \tag{86}$$

For the vector field we have

$$-\hat{T}_0^0 = a^{-2}\partial_i\left(2\tilde{\lambda}\partial_i\phi\right) - \hat{T}_x^x, \tag{87}$$

$$\hat{T}_x^x = \frac{1}{2}FM^2 \sim -\Lambda_0, \tag{88}$$

where the pressure term $\hat{T}_x^x$ is generally much smaller than $\hat{T}_0^0 \sim 8\pi G\rho$. Thus, the equation of state of the vector field is

$$w \sim 0 \tag{89}$$

in static galaxies where $|k|^2\phi \gg \Lambda_0$.

From the Einstein 00th equation, and neglecting the pressure term, we find the modified Poisson equation

$$a^{-2}\partial_i(2\mu\partial_i\phi) = 8\pi G\rho, \qquad \mu \equiv 1 - \tilde{\lambda}. \tag{90}$$

We hence recover equation (9) of Zlosnik et al. (2007), except that we do not require $c_4 = 0$.

The above equation resembles the MOND Poisson equation in the static limit. However, MOND also requires for a present-day galaxy $\mu \to \sqrt{y}$ when $y \equiv \partial_i\phi\partial_i\phi/(M^2a^2) \ll 1$ and $\mu \to 1$ when $y \gg 1$, where we identify M with the MOND critical acceleration a_0; i.e., $(M)^{1/2} \equiv a_0 \sim 10^{-10}$ m s^{-2}. With no loss of generality we set $c_4 = 2$, $c_1 = 0$. The easiest way to *match the MOND function* with F' together is to require $\alpha = 0$, $y = \mathcal{K}/2$, and

$$\mu = 1 - \frac{c_4 - c_1}{2}F'(\mathcal{K}) = \sqrt{\frac{|\mathcal{K}|}{|\mathcal{K}| + 2}}. \tag{91}$$

The latter corresponds to the standard μ function of classical MOND, which fits rotation curves of hundreds of nearby spiral galaxies extremely well.

5. POSSIBLE COVARIANT DEPENDENCE OF THE MONDIAN BEHAVIOR ON REDSHIFT, ENVIRONMENT, AND HISTORY

As a final application, we note that it is possible to deviate from MOND when we consider galaxy models with $\alpha = c_1 + 3c_2 + c_3 \neq 0$ in a nonstatic universe. As before, we set $\mu = [|\mathcal{K}|/(|\mathcal{K}| + 2)]^{1/2}$. However, the kinetic scalar $\mathcal{K}$ is up to second order

$$\mathcal{K} \sim 100\alpha\frac{H(z)^2}{H_0^2} + 2y, \qquad y \equiv \frac{1}{M^2a^2}(\partial_i\phi\partial_i\phi), \tag{92}$$

where $\mathcal{K}_0 \equiv 3\alpha H_0^2/M^2 \sim 100\alpha$ for $M \sim H_0/6$,

Hence we find $\mu = \left\{|y + 50\alpha H(z)^2/H_0^2|/\left[1 + |y + 50\alpha H(z)^2/H_0^2|\right]\right\}^{1/2}$ to depend on redshift.

Finally, coming back to $F(Q)$ models, the free function now depends on $Q = c_0M^{-2}R + \mathcal{K}$, which depends on the Ricci scalar, which is crudely speaking the density of the system. For galaxies in an expanding universe,

$$Q = c_0M^{-2}R + \mathcal{K} \sim (6c_0 + 3\alpha)\frac{H^2}{M^2} + Q_0, \quad Q_0 = \frac{2c_0\partial_i\partial_i\phi + 2\partial_i\phi\partial_i\phi}{M^2a^2}. \tag{93}$$

Setting $\alpha = -2c_0$, we can also opt out the H^2 term or the redshift dependence and make $Q = Q_0$. For example, if the MOND function $\mu = 1 - [(c_4 - c_1)/2]dF/dQ = [Q/(Q + 2)]^{1/2}$, then MONDian behavior will depend on density. The zero-gravity $\partial_i\phi = 0$ region has $\mu \sim Q = Q_0 = (2c_0\partial_i\partial_i\phi)/(M^2a^2) \sim (8c_0\pi G\rho)/M^2 \sim 100c_0\delta_\rho \ll 1$, where δ_ρ is the overdensity over the cosmic mean, and we assume $c_0 \ll 1$. So the dark matter effect μ^{-1} could be bigger in a fluffy galaxy cluster than in a dense galaxy in these models. In the solar system Q is large due to high density and strong gravity. Hence $\mu = 1$, and we recover GR-like behavior. The $F(Q)$ models also contain a correction to the Einstein equation due to a source proportional to $-c_0W_{\alpha\beta} \sim -c_0F'R_{\alpha\beta} \sim 0$, where the free function $F' \sim 0$–1. This correction can be neglected in the case $c_0 \ll 1$, as in most $F(R)$ gravity models.

Let us come back to our general Lagrangian $L(\varphi, A)$ with a dynamical φ freedom if $0 = d_1 = d_3 = d_4$ and $d_2 = 1$. The term $A^aA^b\nabla_a\varphi\nabla_b\varphi$ creates a quintessence-like source term in cosmology but does not contribute to static galaxies. However, in time-dependent systems, this coupling of A^a and φ means that the MOND $\mu = \varphi$ in these models has not reached its steady state prediction; for example, $\mu = [Q/(Q + 2)]^{1/2}$. Instead it must be solved from its own equation of motion in an unrelaxed system under merging.

In short, the covariant version offers new possibilities of tailoring the MOND behavior as a function of environment, redshift, and history. These possibilities of covariant dependence of the MOND μ-function are generally welcome, since some of the MOND's worst outliers are with gravitationally lensed galaxy clusters under merging at modest redshift, for example, the Bullet Clusters at $z = 0.3$; clusters have generally lower density than spiral galaxies, where the empirical formula of MOND applies well. In this sense, the empirical MOND formula is *not* a universal rule, and there are a range of possible fundamental rules giving the effects of dark matter and dark energy.

6. CONCLUSION

We have outlined a framework for studying the dark matter and dark energy effects of a vector field. We have isolated a few simple cases in which the perturbation equations for structure formation are the simplest. Our equations reduce to the nonlinear Hubble equation and the nonlinear Poisson equations in the literature. Our simplest model with $c_4 \neq 0 = c_1 = c_2 = c_3 = c_0 = d_1 = d_2 = d_3 = d_4$ gives particularly simple Einstein equations. Including other coefficients leads to a range of new behaviors in structure formation. We itemize our main results as follows.

1. The rotation curves of most spiral galaxies can be explained if we adopt the MOND dielectric parameter $\mu(\mathcal{K}) = 1 - [(c_4 - c_1)/2]F' = (|\mathcal{K}|/|\mathcal{K} + 2|)^{1/2}$, where $\mathcal{K} \sim 2y$ and where $\sqrt{y}$ is the gravity measured in units of the acceleration scale M.

2. The metric-tracking vector-field is described by a four-vector $A_a = (-1 - \phi, B_1, B_2, B_3)$. It acts as a dark fluid of certain four-velocity. This fluid is able to store up perturbations in vacuum in a cold dark matter fashion without being dissipated by photons, hence giving the seed for formation of baryonic structures after the epoch of last scattering (Dodelson & Liguori 2006; Zhao 2007).
3. This dark fluid has a nonconstant equation of state parameter w. In the pure $c_4\mathcal{K}$ case the fluid behaves as a $w = -1$ cosmological constant Λ_0 in Hubble expansion, and $w = 0$ dark matter in static galaxies.
4. The small amplitude of vacuum pressure $\Lambda_0 \sim H_0^2$ is explained by the vector field's pressure in galaxies, if the zero point of the pressure is set at the solar system. Here Λ_0 is the maximum pressure difference between very strong and very weak gravity.
5. There are covariant $F(R + \mathcal{K})$ models with $\alpha = c_1 + 3c_2 + c_3 \neq 0$, and/or $c_0 \neq 0$ which allows the MOND dielectric function to depend on redshift and density; hence, MOND is no longer a universal rule.

Our perturbation equations can be fairly straightforwardly generalized by superimposing two $F(Q_1)$ and $F(Q_2)$ terms together. For example, in the V-Λ model (Zhao 2007), one replaces $F \to F(c_4\mathcal{K}_4) + F_2(J)$, where $F_2 \propto J \propto \mathcal{K}_2$ in the matter-dominated regime. This J-term has effects orthogonal to that of the $\mathcal{K}_4$ term. It can mimic the effects of dark matter in the Hubble equation, but does not contribute to galaxy rotation curves.

The generality of the equations presented here gives the opportunity of exploring various realistic cases. With these it is in principle possible to numerically simulate structure formation and cosmic microwave background to falsify this $F(R + \mathcal{K})$ class of models in the style of Skordis et al. (2006) and Li et al. (2008).

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APPENDIX

LINEAR MODELS WITH $F(\mathcal{K}) = \mathcal{K}$

As a first application of our results, we generalize the linear model of Lim (2005) to include a $c_4\mathcal{K}_4$ term.

We let $F(\mathcal{K}) = \mathcal{K}$, hence $F' = 1$. We have

$$M^2F(\mathcal{K}) = M^2\mathcal{K} = 3\alpha H^2 + 3\alpha H^2\left(-2\phi + 2H^{-1}\partial_t\psi + \frac{2}{3a^2H}\partial_i B_i\right)\epsilon + \left[\frac{c_4 - c_1}{a^2}\left(\partial_i\phi + \dot{B}_i\right)^2 + \ldots\right]O\left(\epsilon^2\right). \tag{A1}$$

The vector equation gives the Lagrange multiplier

$$L^*(1 + 2\phi) = 3(c_1 + c_2 + c_3)\left(H^2 + 2H\dot{\psi}\right) + 3c_2\left(-\frac{\ddot{a}}{a} + H\dot{\phi} - \partial_t^2\psi\right) - \frac{c_3}{a^2}\partial_i^2\phi + 2(c_1 + c_2 + c_3)\frac{\dot{a}}{a^3}\partial_i B_i - \frac{c_2 + c_3}{a^2}\partial_t(\partial_i B_i), \tag{A2}$$

the 00 component of the stress-energy tensor

$$\hat{T}_{00} = \frac{c_4 - c_1}{a^2}\partial_i^2\phi + 3\alpha H^2 + 6\alpha H\partial_t\psi - \frac{1 + 2\phi}{2}M^2\mathcal{K} + 2\alpha\frac{\dot{a}}{a^3}\partial_i B_i + \frac{c_4 - c_1}{a^2}\partial_t(\partial_i B_i), \tag{A3}$$

and the spatial diagonal term

$$\hat{T}_{xx} = -\alpha\frac{1 + 2\psi - 2\phi}{a}\partial_t\left(a^2\dot{a}\right) + \frac{1}{2}a^2(1 + 2\psi)M^2\mathcal{K} + \alpha a^2\left(-6H\dot{\psi} + H\dot{\phi} - \partial_t^2\psi\right) - \frac{\alpha}{3}(4H + \partial_t)\partial_i B_i - \frac{c_1 + c_3}{3a}\partial_t[a(3\partial_x B_x - \partial_i B_i)]. \tag{A4}$$

The 0x component of the stress tensor

$$\hat{T}_{0x} = (c_4 - c_1)(\partial_t\partial_i\phi + H\partial_i\phi) + \frac{c_4 - c_1}{a}\partial_t(a\partial_t B_x) + \alpha\partial_t(H)B_x + \frac{c_3 - c_1}{2a^2}\partial_i(\partial_x B_i - \partial_i B_x), \tag{A5}$$

and the spatial off-diagonal terms

$$\hat{T}_{ij} = -\frac{c_1 + c_3}{2}(H + \partial_t)\partial_{(i}B_{j)}, \tag{A6}$$

where the parentheses mean symmetric permutation of i and j.

It is reassuring that the above equations agree with those of Lim (2005) if we set $c_4 = 0$. This confirms our results up to the linear order in the case that $F' = 1$.

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