GROWTH RATE OF CLUSTER ALGEBRAS

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Abstract. We complete the computation of growth rate of cluster algebras. In particular, we show that growth of all exceptional non-affine mutation-finite cluster algebras is exponential.

Contents

1. Introduction

This is the fourth paper in the series started in [9, 10, 11].

Cluster algebras were introduced by Fomin and Zelevinsky in the series of papers [14], [15], [2], [17]. Up to isomorphism, each cluster algebra is defined by a skew-symmetrizable $n \times n$ matrix called its *exchange matrix*. Exchange matrices admit *mutations* which can be explicitly described. The cluster algebra itself is a commutative algebra with a distinguished set of generators. All the generators are organized into *clusters*. Each cluster contains exactly n generators (*cluster variables*) for a rank n cluster algebra.

Clusters form a nice combinatorial structure. Namely, clusters can be associated with the vertices of n-regular tree where the collections of generators in neighboring vertices are connected by relations of an especially simple form called *cluster exchange relations*. Exchange relations are governed by the corresponding exchange matrix which in its turn undergoes cluster mutations as described above. The combinatorics of the cluster algebra is encoded by its *exchange graph*, which can be obtained from the n-regular tree by identifying vertices with equal clusters (i.e., the clusters containing the same collection of cluster variables).

This paper is devoted to the computation of the growth rate of exchange graphs of cluster algebras. We say that a cluster algebra is *of exponential growth* if the number of distinct vertices of the exchange graph inside a circle of radius N, i.e., that can be reached from an initial vertex in N mutations, grows exponentially in N. We say that the growth of a cluster algebra is *polynomial* if this number grows at most polynomially depending on N.

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In [12] Fomin, Shapiro and Thurston computed the growth of *cluster algebras originating from surfaces* (or simply *cluster algebras from surfaces* for short). This special class of cluster algebras is characterized by their exchange matrices being signed adjacency matrices of ideal triangulations of marked bordered surfaces. In particular, these matrices are skew-symmetric (we call a cluster algebra *skew-symmetric* if its exchange matrices are skew-symmetric, otherwise we call it *skew-symmetrizable*). Such an algebra has polynomial growth if the corresponding surface is a sphere with at most three holes and marked points in total, and exponential growth otherwise.

Cluster algebras from surfaces have another interesting property: the collections of their exchange matrices (called *mutation classes*) are finite. We call such algebras (and exchange matrices) *mutation-finite*. It was shown in [9] that signed adjacency matrices of ideal triangulations almost exhaust the class of mutation-finite skew-symmetric matrices, namely, there are only eleven (exceptional) finite mutation classes of matrices of size at least 3×3 not coming from triangulations of surfaces. It was also proved in [9] that skew-symmetric algebras that are not mutation-finite (we call them *mutation-infinite*) are of exponential growth.

In [10], we classify skew-symmetrizable mutation-finite cluster algebras. The geometric meaning of this classification is clarified in [11]: all but seven finite mutation classes of skew-symmetrizable (non-skew-symmetric) matrices can be obtained via signed adjacency matrices of ideal triangulations of orbifolds. In the same paper [11] we show that the exchange graph of every cluster algebra originating from an orbifold is quasi-isometric to an exchange graph of a cluster algebra from a certain surface. In this way we compute the growth rate of all cluster algebras from orbifolds.

In [15], Fomin and Zelevinsky classified all finite cluster algebras, i.e., cluster algebras with finitely many clusters. Their ground-breaking result states that any finite cluster algebra corresponds to one of the finite root systems. More precisely, a "symmetrization" of some of the exchange matrices in the corresponding mutation class is a Cartan matrix of the corresponding root system. This observation justifies the following terminology. We say that a cluster algebra is *of finite* (or *affine*) type if a certain sign symmetric version of one of the exchange matrices in the corresponding mutation class is the Cartan matrix of the root system.

Now we are ready to formulate the main result of the current paper. For simplicity reasons we state our result in terms of *diagrams* (see Section 2) rather than in terms of matrices.

Theorem 1.1. *A cluster algebra $\mathfrak{A}$ has polynomial growth if one of the following holds*

(1) $\mathfrak{A}$ *has rank* 2 *(finite or linear growth);*
(2) $\mathfrak{A}$ *is of one of the following types:*
   (a) *finite type A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, or G_2, then $\mathfrak{A}$ is finite;*
   (b) *affine type $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$, or $\widetilde{D}_n$, then $\mathfrak{A}$ has linear growth;*
(3) *the mutation class contains one of the following three diagrams shown in Fig. 1.1:*
   (a) *diagram $\Gamma(n_1, n_2)$, $n_1, n_2 \in \mathbb{Z}_{>0}$, then $\mathfrak{A}$ has quadratic growth;*
   (b) *diagram $\Delta(n_1, n_2)$, $n_1, n_2 \in \mathbb{Z}_{>0}$, then $\mathfrak{A}$ has quadratic growth;*
   (c) *diagram $\Gamma(n_1, n_2, n_3)$, $n_1, n_2, n_3 \in \mathbb{Z}_{>0}$, then $\mathfrak{A}$ has cubic growth;*
(4) $\mathfrak{A}$ *is of one of the following exceptional affine types:*
   (a) *$\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, then $\mathfrak{A}$ is skew-symmetric of linear growth;*
   (b) *$\widetilde{G}_2$, $\widetilde{F}_4$, then $\mathfrak{A}$ is skew-symmetrizable of linear growth.*

Otherwise, $\mathfrak{A}$ has exponential growth.

Remark 1.2. Another independent proof of exponential growth for tubular cluster algebras (namely, $D_4^{(1,1)}, E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}$) is obtained recently in [3].

Remark 1.3. In the paper we consider cluster algebras with connected diagrams only. Nathan Reading mentioned to us that growth of cluster algebras with non-connected diagrams can also be

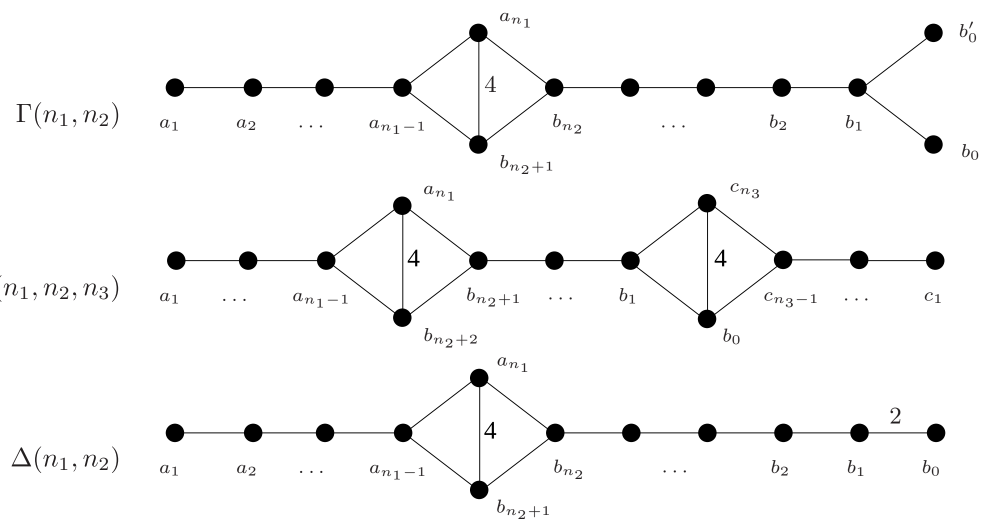

FIGURE 1.1. Diagrams for the cluster algebras of quadratic and cubic growth. All triangles are oriented. Orientations of the remaining edges are of no importance.

derived from Theorem 1.1. Indeed, the number of pairs of vertices in a disjoint union of several rooted graphs at total distance N from the roots is a convolution of the respective functions for the connected components. In particular, this implies that the growth of a cluster algebra is polynomial if and only if for every connected component of its diagram the growth of corresponding cluster algebra is polynomial; the only cluster algebras of linear growth are affine ones with connected diagrams.

The plan of the proof is as follows. Note first that the case (1) of rank two cluster algebras is evident: the exchange graph is either a finite cycle (finite case: A_2, B_2, G_2) or it is an infinite path implying linear growth rate of the cluster algebra. The case (2a) is also clear.

As the next step we mention (Lemma 4.1) that any mutation-infinite cluster algebra has exponential growth (see also [9]). The latter implies that it remains only to determine the growth rate of cluster algebras of finite mutation type.

We collect all already known results on the growth of cluster algebras from surfaces and orbifolds in Section 4. This covers cases (2b) and (3). The polynomial growth of skew-symmetric affine exceptional types (case 4a) is proved using the categorification approach to cluster algebras (see Section 6). The case (4b) follows from (4a) via the *unfolding* construction recalled in Section 2.2. Thus, we are left to prove exponential growth of all the remaining exceptional mutation-finite cluster algebras.

The *mapping class group* of a cluster algebra consists of sequences of mutations that preserve the initial exchange matrix. All nontrivial elements of the mapping class group change the cluster, in particular, different elements of the mapping class group produce different clusters from the initial one. Hence, the exponential growth of the mapping class group implies the exponential growth of the cluster algebra.

To prove exponential growth of remaining exceptional cases we utilize the famous “ping-pong lemma” used in the proof of Tits alternative, that allows us to find a free group with two generators as a subgroup of the mapping class group of a corresponding cluster algebra. This provides an exponential growth of the mapping class group which, in its turn, implies exponential growth of the corresponding cluster algebra.

To apply the ping-pong lemma we consider mutations of g-vectors (see Section 5.2). The strategy consists of finding two elements of the mapping class group such that their actions on the space of g-vectors satisfy conditions of the ping-pong lemma. The proof is accomplished by the detailed case-by-case analysis of g-vector mutations for an appropriate pair of elements of the mapping class group in each exceptional case.

Note that up to this moment we work in coefficient-free settings. In Section 7 we show that growth of cluster algebras does not depend on the coefficients, so the main theorem holds in full generality.

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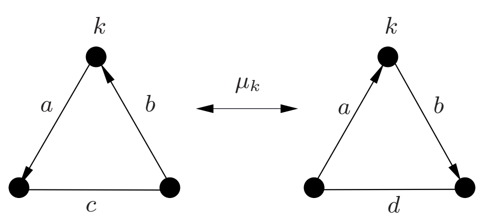
2. Exchange matrices and diagrams

2.1. **Diagram of a skew-symmetrizable matrix.** Following [15], we encode an $n \times n$ skew-symmetrizable integer matrix B by a finite simplicial 1-complex S with oriented weighted edges called a *diagram*. The weights of a diagram are positive integers.

Vertices of S are labeled by $[1, \dots, n]$. If $b_{ij} > 0$, we join vertices i and j by an edge directed from i to j and assign to this edge weight $-b_{ij}b_{ji}$. Not every diagram corresponds to a skew-symmetrizable integer matrix: given a diagram S, there exists a skew-symmetrizable integer matrix B with diagram S if and only if a product of weights along any chordless cycle of S is a perfect square.

Distinct matrices may have the same diagram. At the same time, it is easy to see that only finitely many matrices may correspond to the same diagram. All weights of a diagram of a skew-symmetric matrix are perfect squares. Conversely, if all weights of a diagram S are perfect squares, then there is a skew-symmetric matrix B with diagram S.

As it is shown in [15], mutations of exchange matrices induce *mutations of diagrams*. If S is the diagram corresponding to matrix B, and B' is a mutation of B in direction k, then we call the diagram S' associated to B' a *mutation of S in direction k* and denote it by $\mu_k(S)$. A mutation in direction k changes weights of diagram in the way described in Fig. 2.1 (see e.g. [15]).



$$\pm\sqrt{c} \pm \sqrt{d} = \sqrt{ab}$$

Figure 2.1. Mutations of diagrams. The sign before $\sqrt{c}$ (resp., $\sqrt{d}$) is positive if the three vertices form an oriented cycle, and negative otherwise. Either c or d may vanish. If ab is equal to zero then neither the value of c nor the orientation of the corresponding edge changes.

For a given diagram, the notion of *mutation class* is well-defined. We call a diagram *mutation-finite* if its mutation class is finite.

The following criterion for a diagram to be mutation-finite is well-known (see e.g. [10, Theorem 2.8]).

Lemma 2.1. *A diagram S of order at least 3 is mutation-finite if and only if any diagram in the mutation class of S contains no edges of weight greater than 4.*

2.2. **Unfolding of a skew-symmetrizable matrix.** In this section, we recall the notion of *unfolding* of a skew-symmetrizable matrix.

Let B be an indecomposable $n \times n$ skew-symmetrizable integer matrix, and let BD be a skew-symmetric matrix, where $D = (d_i)$ is diagonal integer matrix with positive diagonal entries. Notice that for any matrix $\mu_i(B)$ the matrix $\mu_i(B)D$ will be skew-symmetric.

We use the following definition of unfolding (communicated to us by A. Zelevinsky) (see [10] and [11] for details).

Suppose that we have chosen disjoint index sets $E_1, \dots, E_n$ with $|E_i| = d_i$. Denote $m = \sum\limits_{i=1}^{n} d_i$. Suppose also that we choose a skew-symmetric integer matrix C of size $m \times m$ with rows and columns indexed by the union of all E_i, such that

(1) the sum of entries in each column of each $E_i \times E_j$ block of C equals b_{ij};

(2) if $b_{ij} \geq 0$ then the $E_i \times E_j$ block of C has all entries non-negative.

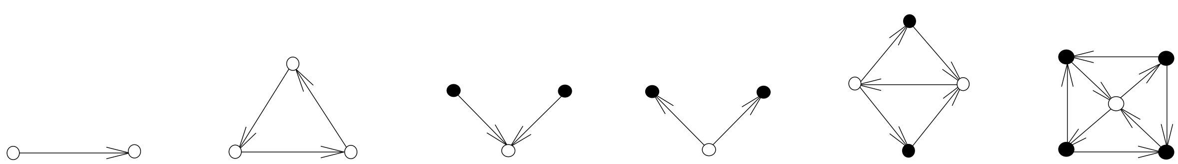
Define a *composite mutation* $\widehat{\mu}_i = \prod_{\hat{\imath} \in E_i} \mu_{\hat{\imath}}$ on C. This mutation is well-defined, since all the mutations $\mu_{\hat{\imath}}$, $\hat{\imath} \in E_i$, for given i commute.

We say that C is an *unfolding* for B if C satisfies assertions (1) and (2) above, and for any sequence of iterated mutations $\mu_{k_1} \dots \mu_{k_m}(B)$ the matrix $C' = \widehat{\mu}_{k_1} \dots \widehat{\mu}_{k_m}(C)$ satisfies assertions (1) and (2) with respect to $B' = \mu_{k_1} \dots \mu_{k_m}(B)$.

3. Block decompositions of diagrams

In [12], Fomin, Shapiro and Thurston gave a combinatorial description of diagrams of signed adjacency matrices of ideal triangulations. Namely, such diagrams are *block-decomposable*, i.e. they are exactly those that can be glued from diagrams shown in Fig. 3.1 (called *blocks*) in the following way.

Call vertices marked in white *outlets*. A connected diagram S is called *block-decomposable* if it can be obtained from a collection of blocks by identifying outlets of different blocks along some partial matching (matching of outlets of the same block is not allowed), where two edges with the same endpoints and opposite directions cancel out, and two edges with the same endpoints and the same directions form an edge of weight 4. A non-connected diagram S is called block-decomposable either if S satisfies the definition above, or if S is a disjoint union of several diagrams satisfying the definition above. If S is not block-decomposable then we call S *non-decomposable*.



I II IIIa IIIb IV V

Figure 3.1. Blocks. Outlets are colored in white.

As it was mentioned above, block-decomposable diagrams are in one-to-one correspondence with adjacency matrices of arcs of ideal (tagged) triangulations of bordered two-dimensional surfaces with marked points (see [12, Section 13] for the detailed explanations). Mutations of block-decomposable diagrams correspond to flips of triangulations. In particular, this implies that mutation class of any block-decomposable diagram is finite.

It was shown in [10, 11] that diagrams of signed adjacency matrices of arcs of ideal triangulations of orbifolds can be described in a similar way. For this, we need to introduce new *s-blocks* shown in Fig. 3.2.

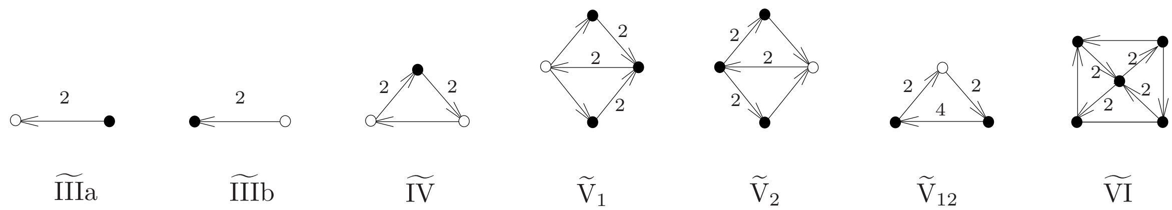


FIGURE 3.2. s-blocks. Outlets are colored in white.

We keep the idea of gluing. A diagram is *s-decomposable* if it can be glued from blocks and s-blocks. We keep the term “block-decomposable” for s-decomposable diagrams corresponding to skew-symmetric matrices.

Like block-decomposable diagrams, s-decomposable diagrams are in one-to-one correspondence with adjacency matrices of arcs of ideal (tagged) triangulations of bordered two-dimensional orbifolds with marked points and orbifold points of degree two (see [11]). As above, mutations of s-decomposable diagrams correspond to flips of triangulations. This implies that mutation class of any s-decomposable diagram is also finite.

Therefore, s-decomposable diagrams form a large class of finite mutation diagrams (and therefore exchange matrices). Moreover, in [10] we proved that together with diagrams of rank 2 they provide almost all diagrams of finite mutation type.

More exactly, the following theorems hold.

Theorem 3.1 (Theorem 6.1 [9])**.** *A connected non-decomposable skew-symmetric mutation-finite diagram of order greater than* 2 *is mutation-equivalent to one of the eleven diagrams* E_6, E_7, E_8, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, X_6, X_7, $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$ *shown in Figure 3.3.*

Theorem 3.2 (Theorem 5.13 [10])**.** *A connected non-decomposable skew-symmetrizable diagram, that is not skew-symmetric, has finite mutation class if and only if either it is of order* 2 *or its diagram is mutation-equivalent to one of the seven types* $\widetilde{G}_2$, F_4, $\widetilde{F}_4$, $G_2^{(*,+)}$, $G_2^{(*,*)}$, $F_4^{(*,+)}$, $F_4^{(*,*)}$ *shown in Fig. 3.4.*

Remark 3.3. In Fig.3.4, we have chosen representatives from the mutation classes of non-decomposable diagrams that are slightly different than the ones from [10, Theorem 5.13]. This is done for simplification of computations in Section 5.

4. GROWTH OF NON-EXCEPTIONAL CLUSTER ALGEBRAS

As was proved in [9], a mutation-infinite skew-symmetric cluster algebra has exponential growth. Very similar considerations lead to the following lemma (it can also be easily derived from the results of Seven [26]).

Lemma 4.1. *Any mutation-infinite skew-symmetrizable cluster algebra has exponential growth.*

Therefore, we are left to describe the growth of mutation-finite cluster algebras.

According to the results of [9] and [10, 11], almost all mutation-finite cluster algebras originate from surfaces or orbifolds. The growth of cluster algebras from surfaces was computed in [12] by investigating mapping class groups of surfaces. In [11], we compute the growth of cluster algebras

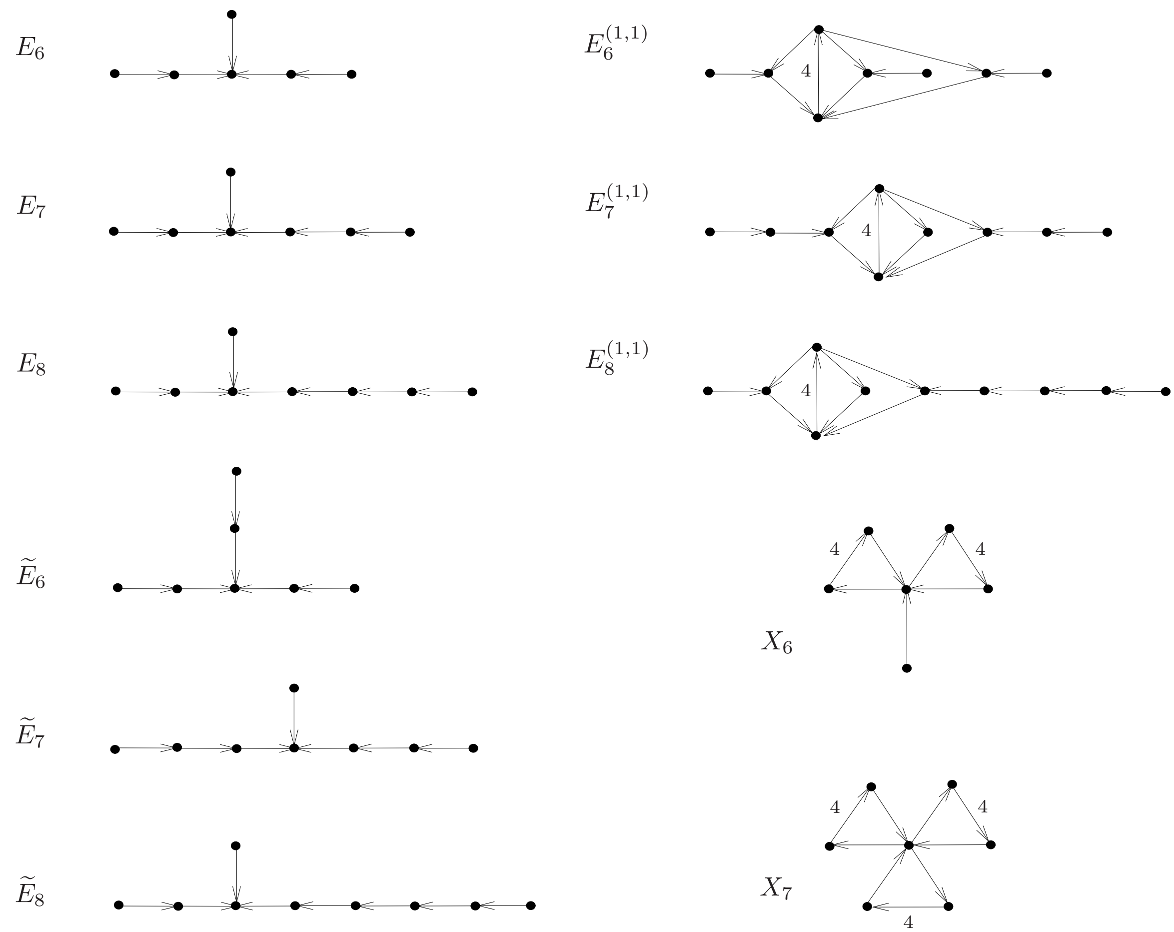


FIGURE 3.3. Non-decomposable skew-symmetric mutation-finite diagrams of order at least 3

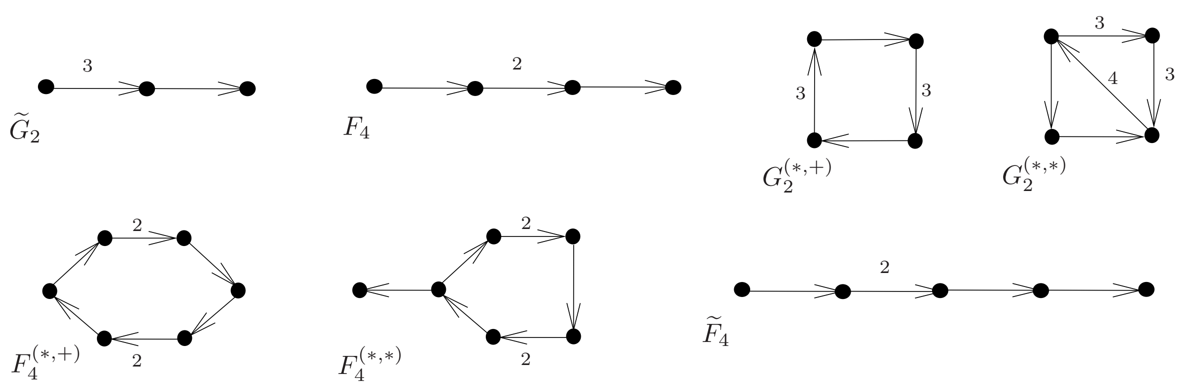


FIGURE 3.4. Non-decomposable mutation-finite non-skew-symmetric diagrams of order at least 3

from orbifolds by making use of unfoldings and proving quasi-isometry of the corresponding exchange graphs (which is a much stronger statement than needed for growth computation), see [11, Section 10]. Below we define a mapping class group of a cluster algebra, and then follow [12, Section 13] to present a uniform explanation for both cases.

Let $\overline{n} = \{1, 2, \dots, n\}$. We denote by $W = \mathbb{Z}_2 * \dots * \mathbb{Z}_2$ the free product of n copies of $\mathbb{Z}_2$ with $i \in \overline{n}$ being a generator of ith copy of $\mathbb{Z}_2$. W is the set of all words without letter repetitions in alphabet $\overline{n}$.

A word $w = i_1\, i_2\, \dots\, i_k \in W$ can be interpreted as a sequence μ_w of mutations of cluster algebra $\mathfrak{A}$, namely, $\mu_w = \mu_{i_k} \circ \dots \circ \mu_{i_2} \circ \mu_{i_1}$.

Definition 4.2. We call a word $w \in W$ *trivial* if $\mu_w(x_i) = x_i$ for any cluster variable x_i, $i \in \overline{n}$, of the initial cluster. Trivial words form a subgroup $W_e \subset W$ that we call *the subgroup of trivial transformations*.

Definition 4.3. A word $w \in W$ is *mutationally trivial* if μ_w preserves the initial exchange matrix B. All mutationally trivial words form *a subgroup of mutationally trivial transformations* denoted by $W_B \subset W$.

Lemma 4.4. *$W_e \subset W_B$ is a normal subgroup.*

Proof. Note first that any word $w \in W_e$ preserves exchange matrix B by [19] and therefore $W_e \subset W_B$. Note also that for all $w \in W_e$, $u \in W_B$ the word $u^{-1}wu$ preserves all initial cluster variables and, hence, belongs to W_e. □

Definition 4.5. The quotient $\mathcal{M} = W_B/W_e$ is a *colored mapping class group* of cluster algebra $\mathfrak{A}$.

Example 4.6. (cluster algebras of rank 2)

(1) The group of trivial transformations W_e of the coefficient-free cluster algebra $\mathfrak{A}$ of type A_2 with the initial exchange matrix $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ consists of all words $(12)^{5k}$ and $(21)^{5k}$. It is generated by word $(12)^5$. (Note that $(21)^5 = (12)^{-5}$.) The group W_B of mutationally trivial transformations is formed by all words $(12)^\ell$ and $(21)^\ell$. It is generated by the word (12) implying that the colored mapping class group $\mathcal{M} = W_B/W_e \simeq \mathbb{Z}_5$.

(2) Similarly, for cluster algebras of types B_2 and C_2 with exchange matrices $B = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$ respectively, the colored mapping class group $\mathcal{M} \simeq \mathbb{Z}_6$. For cluster algebra of type G_2 with exchange matrix $B = \begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix}$ the colored mapping class group $\mathcal{M} \simeq \mathbb{Z}_8$

(3) For cluster algebras of non finite type with exchange matrix $B = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}$, where $ab \geq 4$, the subgroup W_e of trivial transformations is trivial while W_B is still generated by (12) and the mapping class group is an infinite cyclic group, $\mathcal{M} \simeq \mathbb{Z}$.

Example 4.7. Markov cluster algebra. The Markov coefficient-free cluster algebra is a rank 3 cluster algebra with initial exchange matrix $B = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$. Any simple cluster transformation changes the sign of the exchange matrix. Therefore, words (12), (13), (21), (23), (31), (32) generate subgroup W_B. Note that $(21)(12) = (13)(31) = (23)(32) = \mathrm{Id}$. Hence, W_B is generated by three mutationally trivial words (12), (13), and (23). Recall, that the Markov cluster algebra is a cluster algebra of triangulations of once punctured torus whose mapping class group is known to be isomorphic to $SL_2(\mathbb{Z})$, and mutationally trivial words represent all the elements of the mapping class group of the torus. The word (12) corresponds to $\alpha = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$, the word (23) to $\beta = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$ and the word (13) to $\gamma = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \in SL_2(\mathbb{Z})$. Note that $\alpha\beta\gamma^{-1} = -\,\mathrm{Id}$. It is known (see., e.g., [28]), that elements α, β, $-\,\mathrm{Id}$ generate a principle congruence subgroup $\Gamma(2)$ of $SL_2(\mathbb{Z})$, consisting of matrices congruent to Id modulo 2. The quotient $SL_2(\mathbb{Z})/\Gamma(2)$ is isomorphic to the group Σ_3 of permutations of three elements. Therefore, the index $|SL_2(\mathbb{Z}) : \Gamma(2)| = 6$.

Denote by Σ_n the symmetric group of permutations on $\overline{n}$. Any element $\sigma \in \Sigma_n$ acts on any cluster of cluster algebra by a permutation of indices $\sigma(x_i) = x_{\sigma(i)}$. This action conjugates the exchange matrix by the corresponding permutation matrix $M_\sigma \in SL_n(\mathbb{Z})$, i.e. $B \mapsto M_\sigma^{-1} B M_\sigma$. We consider also the action of Σ_n on W by a permutation of the letters of the alphabet $\overline{n}$.

Definition 4.8. We call the elements of the set $\widetilde{W} = W \times \Sigma_n$ *enhanced words.*

Enhanced word $w \times \sigma \in W \times \Sigma_n$ act on cluster algebra by composition $\sigma \circ \mu_w$.

Remark 4.9. It is easy to see that $\widetilde{W}$ is a group. Indeed, the definition of an operation is evident. The composition $(w_1 \times \sigma_1) \circ (w_2 \times \sigma_2)$ can be written again as an enhanced word $w_1\sigma_1^{-1}(w_2) \times \sigma_1\sigma_2$. In particular, $(w \times \sigma)^{-1} = \sigma(w^{-1}) \times \sigma^{-1}$.

Definition 4.10. An enhanced word $w \times \sigma$ is *trivial* if $(w \times \sigma)x_i = x_i \ \forall i \in \overline{n}$. We denote the subgroup of trivial enhanced words by $\widetilde{W}_e$. We also denote the subgroup of mutationally trivial enhanced words by $\widetilde{W}_B$, and *the mapping class group of cluster algebra* $\mathfrak{A}(B)$ by $\widetilde{\mathcal{M}} = \widetilde{W}_B/\widetilde{W}_e$.

Example 4.11. (cluster algebras of rank 2)

(1) case A_2: The group $\widetilde{W}_e$ of trivial enhanced transformations of the coefficient-free cluster algebra $\mathfrak{A}$ of rank 2 with the initial exchange matrix $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ consists of enhanced words of the following four types: $(12121)^{2k} \times \mathrm{Id}$, $(21212)^{2k} \times \mathrm{Id}$, $(12121)^{2k+1} \times \sigma$, $(21212)^{2k+1} \times \sigma$, where $\sigma \in \Sigma_2$ is the permutation $(1 \leftrightarrow 2)$. It is generated by element $(12121) \times \sigma$. The group $\widetilde{W}_B$ is generated by $(1) \times \sigma$ (note that $(2) \times \sigma = ((1) \times \sigma)^{-1}$, $(21212) \times \sigma = ((12121) \times \sigma)^{-1}$, and, finally, $(12121) \times \sigma = ((1) \times \sigma)^5$). Hence, $\widetilde{\mathcal{M}} \simeq \mathbb{Z}_5$.
(2) cases B_2, C_2, and G_2: Similarly, the mapping class groups for B_2, C_2, and G_2 are isomorphic to $\mathbb{Z}_6$, $\mathbb{Z}_6$, and $\mathbb{Z}_8$, respectively.
(3) cluster algebra of non finite type: the mapping class group is $\mathbb{Z}$.

There is a natural embedding $i : W \to \widetilde{W}$, $i(w) = (w \times \mathrm{Id})$. Clearly, $i(W_e) \subset \widetilde{W}_e$ and $i(W_B) \subset \widetilde{W}_B$. Therefore, i induces a homomorphism $\mathbf{i} : \mathcal{M} \to \widetilde{\mathcal{M}}$.

Lemma 4.12. *The map* $\mathbf{i}$ *is an embedding.*

Proof. Indeed, assume that $i(w) \in \widetilde{W}_e$. Then, $w \times \mathrm{Id} \in \widetilde{W}_e$. Hence, $w \times \mathrm{Id}(x_j) = x_j$ implying $\mu_w(x_j) = x_j \ \forall j$. Therefore, $w \in W_e$. □

Remark 4.13. Evidently, $\mathbf{i}(\mathcal{M})$ is a finite index (normal) subgroup of $\widetilde{\mathcal{M}}$. Therefore, the growth rate of $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ is the same.

Example 4.14. (Markov cluster algebra) The mapping class group coincides with the mapping class group of two-dimensional torus with one puncture which is known to be $SL_2(\mathbb{Z})$.

One can note that the mapping class group $\mathcal{M}_S$ of the bordered surface (or bordered orbifold) S is a subgroup of the mapping class group $\widetilde{\mathcal{M}}_{\mathfrak{A}(S)}$ of the corresponding cluster algebra $\mathfrak{A}(S)$. Indeed, fix a triangulation T of the orbifold. Any element g of the mapping class group of the orbifold can be obtained by some sequence of cluster mutations $s_g(T) = \mu_{i_1} \circ \ldots \circ \mu_{i_k}$ which, however, depends on T. At the same time, if the triangulation T' is obtained from T by a mapping class group action, then $s_g(T') = s_g(T) = \mu_{i_1} \circ \ldots \circ \mu_{i_k}$. Therefore, if the mapping class group of the surface (orbifold) contains a free group with at least two generators then the cluster algebra has exponential growth.

Remark 4.15. If the number m of interior marked points on a surface (or orbifold) S is greater than one or $m = 1$ and the surface has a nonempty boundary, then the mapping class group $\mathcal{M}_S$ of the surface is a proper normal subgroup of $\widetilde{\mathcal{M}}_{\mathfrak{A}(S)}$ and the quotient $\widetilde{\mathcal{M}}_{\mathfrak{A}(S)}/\mathcal{M}_S \simeq \mathbb{Z}_2^m$. If $m = 0$ or $m = 1$ and the surface has no boundary, then $\widetilde{\mathcal{M}}_{\mathfrak{A}(S)} \simeq \mathcal{M}_S$. Indeed, this follows easily from [12] for surfaces (and [11] for orbifolds) where it was shown that for $m > 1$ any tagged triangulation can be obtained from any other by a series of flips. Comparing it with the classical result that any two triangulations of the surface are connected by a sequence of flips, and a sequence of flips gives an element of the mapping class group of the surface if and only if the adjacency of the arcs of triangulations are preserved by this sequence, we see that in the first case we can obtain any tagging of marked points, which results in extra $\mathbb{Z}_2$ for every puncture. If $m = 1$ and there is no boundary components or if $m = 0$, then mutations do not change any tagging.

Let us call a *feature* of an orbifold (or surface) a hole, a puncture, or an orbifold point.

The above considerations lead to the following theorem.

Theorem 4.16 ([12], [11])**.** *Cluster algebras corresponding to orbifolds (or surfaces) of genus* 0 *with at most three features have polynomial growth. Cluster algebras corresponding to the other orbifolds (surfaces) grow exponentially.*

Rephrasing this result in terms of diagrams, we obtain the following theorem in [11].

Theorem 4.17. *Let $\mathfrak{A}$ be a cluster algebra with an s-decomposable exchange matrix B. Then $\mathfrak{A}$ has polynomial growth if it corresponds to one of the following diagrams:*

- *finite type A_n, B_n, C_n, or D_n (finite);*
- *affine type $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$, or $\widetilde{D}_n$ (linear growth);*
- *diagram $\Gamma(n_1, n_2)(n_1, n_2 \in \mathbb{Z}_{>0})$ shown in Fig 1.1 (quadratic growth);*
- *diagram $\Delta(n_1, n_2)(n_1, n_2 \in \mathbb{Z}_{>0})$ shown in Fig. 1.1 (quadratic growth);*
- *diagram $\Gamma(n_1, n_2, n_3)(n_1, n_2, n_3 \in \mathbb{Z}_{>0})$ shown in Fig. 1.1 (cubic growth).*

Otherwise $\mathfrak{A}$ has exponential growth.

5. Exceptional cluster algebras of exponential growth

We are left with a short list of exceptional algebras. This section is devoted to the proof of the following theorem.

Theorem 5.1. *Cluster algebras with diagrams of types X_6, X_7, $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$, $G_2^{(*,+)}$, $G_2^{(*,*)}$, $F_4^{(*,+)}$, and $F_4^{(*,*)}$ all have exponential growth.*

The remaining algebras (of affine types $\widetilde{G}_2$, $\widetilde{F}_4$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$) are treated in the next section.

5.1. Ping-pong lemma.

We consider subgroups of the mapping class group G of the corresponding cluster algebra, or the fundamental group of the groupoid of cluster mutations. The elements of the mapping class group are formed by sequences of mutations preserving the chosen initial diagram.

For each exceptional cluster algebra we will find a free subgroup with at least two generators of the mapping class group of the corresponding cluster algebra. Since the free group with two generators grows exponentially, this implies an exponential growth of the mapping class group. Different elements of the mapping class group produce different clusters from the initial one, so exponential growth of the mapping class group implies exponential growth of the cluster algebra.

The proof is based on a case-by-case study of the cluster algebras in question. The main tool is the famous *ping-pong lemma*.

The ping-pong lemma was a key tool used by Jacques Tits in his 1972 paper [27] containing the proof of Tits alternative. Modern versions of the ping-pong lemma can be found in many books, e.g. [23] and others. We will use the following modification of classical ping-pong lemma [24].

Lemma 5.2. *Let G be a group acting on a set X and let $H_1, H_2, \dots, H_k$ be nontrivial subgroups of G where $k \geq 2$, such that at least one of these subgroups has order greater than 2. Suppose there exist disjoint nonempty subsets $X_1, X_2, \dots, X_k$ of X such that the following holds:*

For any $i \neq j$ and for any $h \in H_i$, $h \neq 1$ we have $h(X_j) \subset X_i$.

Then $\langle H_1, \dots, H_k\rangle = H_1 * \dots * H_k$.

Corollary 5.3. *With the assumptions of Lemma 5.2, if we further assume that all H_i are infinite cyclic groups then $\langle H_1, \dots, H_k\rangle$ is a free group with k generators.*

To make the analysis of the mapping class group simpler we will use a tropical degeneration of cluster mutations. Namely, we consider the piecewise linear action of cluster mutations on the space of g-vectors.

5.2. **Mutation of g-vectors.** In this section we recall the definition of g-vectors.

Denote by $\mathbb{T}_n$ the n-regular tree of clusters of the cluster algebra $\mathfrak{A}$ of rank n, and let $t_0 \in \mathbb{T}_n$. Denote by B the exchange matrix at t_0.

Proposition 5.4 ([17], Proposition 3.13, Corollary 6.3)**.** Every pair $(B; t_0)$ gives rise to a family of polynomials $F_{j;t} = F^{B;t_0}_{j;t} \in Z[u_1, ..., u_n]$ and two families of integer vectors $g_{j;t} = g^{B;t_0}_{j;t} = (g_{1j;t}, \dots, g_{nj;t}) \in \mathbb{Z}^n$ (where $j \in \mathbb{N}$ and $t \in \mathbb{T}_n$) with the following properties:

1. Each $F_{j;t}$ is not divisible by any u_i, and can be expressed as a ratio of two polynomials in $u_1, \dots, u_n$ with positive integer coefficients, thus can be evaluated in every semifield $\mathbb{P}$.
2. For any j and t, we have
$$x_{j;t} = x_1^{g_{1j;t}} \cdot \dots \cdot x_n^{g_{nj;t}} \frac{F_{j;t}|_{\mathcal{F}}(\hat{y}_1, \dots, \hat{y}_n)}{F_{j;t}|_{\mathbb{P}}(y_1, \dots, y_n)}, \tag{5.1}$$
where the elements $\hat{y}_j$ are given by $\hat{y}_j = y_j \prod_i x_i^{b_{ij}}$.

Here the tropical semifield $\mathbb{P}$ can be assumed to be trivial for our purposes, and $\mathcal{F}$ can be assumed to be a field of rational functions in $(x_1, \dots, x_n)$ with rational coefficients.

Mutations of g-vectors are described by the following conjecture [17].

Conjecture 5.5 ([17], Conjecture 7.12)**.** *Let $t_0 \longleftrightarrow t_1$ be two adjacent vertices in $\mathbb{T}^n$, and let $B^1 = \mu_k(B^0)$. Then, for any $t \in \mathbb{T}^n$ and $a \in \mathbb{Z}^n_{\geq 0}$, the g-vectors $g^{B^0;t_0}_{a;t} = (g_1, \dots, g_n)$ and $g^{B^1;t_1}_{a;t} = (g'_1, \dots, g'_n)$ are related as follows:*
$$g'_j = \begin{cases} g_k, & \text{if } j = k; \\ g_j + [B^0_{jk}]_+ g_k - B^0_{jk}[g_k]_-, & \text{if } j \neq k, \end{cases} \tag{5.2}$$
where $X_+ = max(X, 0)$ and $X_- = min(X, 0)$ denote the positive and the negative part of the real number X.

For skew-symmetric exchange matrices $B^0 (B^1)$ the conjecture was proved in [7].

In order to prove exponential growth we will use Equation 5.2. Moreover, we will also apply Equation 5.2 to particular skew-symmetrizable exchange matrices. However, all the skew-symmetrizable exchange matrices we consider have skew-symmetric unfoldings, so Equation 5.2 clearly holds.

We will consider all the exceptional types of cluster algebras one by one. Our aim is to find two sequences of mutations acting on the space E_G of g-vectors as in Lemma 5.2.

5.3. X_6 **and** X_7**.** We start with cluster algebras with diagrams X_6 and X_7 shown in Fig. 3.3. Let us label vertices of the diagram X_6 as shown in Fig. 5.1.

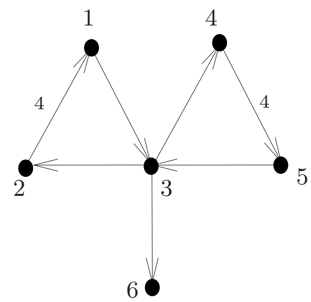


FIGURE 5.1. Diagram for X_6

We consider two following mutation sequences $\mathbf{a} = [3,2,1]^{10}$ and $\mathbf{b} = [3,5,4,2,6]^4$ (by $[i_1,\dots,i_k]$ we mean a sequence of mutations $\mu_{i_k}\dots\mu_{i_1}$). By direct calculation we observe that both $\mathbf{a}$ and $\mathbf{b}$ preserve the diagram shown in Fig. 5.1, or, in other words, they are elements of the mapping class group of X_6.

Note that both $\mathbf{a}$ and $\mathbf{b}$ act on the space E_G of g-vectors of X_6 as described in Section 5.2.

Let us define following subsets of E_G:

$$\begin{aligned}
X^+_{\mathbf{a}}(\varepsilon) &:= \{(T-\nu,-T,0,0,0,\nu),\ \text{where } T>0,\nu>0,\nu<\varepsilon T\},\\
X^-_{\mathbf{a}}(\varepsilon) &:= \{(T,-T+\nu,-\nu,0,0,\nu),\ \text{where } T>0,\nu>0,\nu<\varepsilon T\},\\
X^+_{\mathbf{b}}(\varepsilon) &:= \{(\nu,T-\nu,-T,0,0,T),\ \text{where } T>0,\nu>0,\nu<\varepsilon T\},\\
X^-_{\mathbf{b}}(\varepsilon) &:= \{(-\nu,T-\nu,-T+\nu,0,0,T),\ \text{where } T>0,\nu>0,\nu<\varepsilon T\}.
\end{aligned}$$

We see by inspection for $\varepsilon < 1/15$ that $\mathbf{a},\mathbf{a^{-1}},\mathbf{b},\mathbf{b^{-1}}$ act linearly on $X^\pm_{\mathbf{a}}$ ($X^\pm_{\mathbf{b}}$, correspondingly). In particular,

$$\begin{aligned}
&\mathbf{a}(T-\nu,-T,0,0,0,\nu) &&= (T+14\nu,-T-15\nu,0,0,0,\nu),\\
&\mathbf{a^{-1}}(T,-T+\nu,-\nu,0,0,\nu) &&= (T+15\nu,-T-14\nu,-\nu,0,0,\nu),\\
&\mathbf{b}(\nu,T-\nu,-T,0,0,T) &&= (-\nu,T+2\nu,-T-3\nu,0,0,T+3\nu),\\
&\mathbf{b^{-1}}(-\nu,T-\nu,-T+\nu,0,0,T) &&= (-\nu,T+2\nu,-T-2\nu,0,0,T+3\nu).
\end{aligned}$$

Note also that $X^+_{\mathbf{a}}(\varepsilon)$ is invariant under $\mathbf{a}$. Indeed,

$$\mathbf{a}(T-\nu,-T,0,0,0,\nu) = (T'-\nu,-T',0,0,0,\nu),$$

where $T' = T+15\nu$. Clearly, $\nu < \varepsilon T \le \varepsilon T'$.
Similarly,

$$\mathbf{a^{-1}}(X^-_{\mathbf{a}}(\varepsilon)) \subset X^-_{\mathbf{a}}(\varepsilon),\ \mathbf{b}(X^+_{\mathbf{b}}(\varepsilon)) \subset X^+_{\mathbf{b}}(\varepsilon),\ \mathbf{b^{-1}}(X^-_{\mathbf{b}}(\varepsilon)) \subset X^-_{\mathbf{b}}(\varepsilon).$$

Moreover, for any $v_z \in X^+_{\mathbf{a}}$ we have

$$\lim_{n\to\infty}\frac{\mathbf{a^n}(v_z)}{|\mathbf{a^n}(v_z)|} = (\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0,0,0,0),$$

and, similarly

$$\lim_{n\to\infty}\frac{\mathbf{a^{-n}}(v_z)}{|\mathbf{a^{-n}}(v_z)|} = (\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0,0,0,0,0) \text{ for } v_z \in X^-_{\mathbf{a}},$$

$$\lim_{n\to\infty}\frac{\mathbf{b^n}(v_z)}{|\mathbf{b^n}(v_z)|}=(0,\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},0,0,\frac{1}{\sqrt{3}})\text{ for }v_z\in X^+_{\mathbf{b}},$$
$$\lim_{n\to\infty}\frac{\mathbf{b^{-n}}(v_z)}{|\mathbf{b^{-n}}(v_z)|}=(0,\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},0,0,\frac{1}{\sqrt{3}})\text{ for }v_z\in X^-_{\mathbf{b}}.$$

Computations in Maple show that

$$\mathbf{a}^{10}(0,1,-1,0,0,1)\in X^+_{\mathbf{a}}(\tfrac{1}{15}),\qquad \mathbf{a}^{-10}(0,1,-1,0,0,1)\in X^-_{\mathbf{a}}(\tfrac{1}{15}),$$
$$\mathbf{b}^{10}(1,-1,0,0,0,0)\in X^+_{\mathbf{b}}(\tfrac{1}{15}),\qquad \mathbf{b}^{-10}(1,-1,0,0,0,0)\in X^-_{\mathbf{b}}(\tfrac{1}{15}).$$

Since $\mathbf{a}^{\pm 10}$, $\mathbf{b}^{\pm 10}$ act continuously on E_G, there is a sufficiently small $\epsilon>0$ and an integer $N>0$ such that

$$\mathbf{a}^N\big(X^+_{\mathbf{b}}(\epsilon)\big)\subset X^+_{\mathbf{a}}(\epsilon),\quad \mathbf{a}^{-N}\big(X^+_{\mathbf{b}}(\epsilon)\big)\subset X^-_{\mathbf{a}}(\epsilon),\quad \mathbf{a}^N\big(X^-_{\mathbf{b}}(\epsilon)\big)\subset X^+_{\mathbf{a}}(\epsilon),\quad \mathbf{a}^{-N}\big(X^-_{\mathbf{b}}(\epsilon)\big)\subset X^-_{\mathbf{a}}(\epsilon).$$

Similarly,

$$\mathbf{b}^N\big(X^+_{\mathbf{a}}(\epsilon)\big)\subset X^+_{\mathbf{b}}(\epsilon),\quad \mathbf{b}^{-N}\big(X^+_{\mathbf{a}}(\epsilon)\big)\subset X^-_{\mathbf{b}}(\epsilon),\quad \mathbf{b}^N\big(X^-_{\mathbf{a}}(\epsilon)\big)\subset X^+_{\mathbf{b}}(\epsilon),\quad \mathbf{b}^{-N}\big(X^-_{\mathbf{a}}(\epsilon)\big)\subset X^-_{\mathbf{b}}(\epsilon).$$

Now define

$$X_{\mathbf{a}}=X^-_{\mathbf{a}}(\epsilon)\cup X^+_{\mathbf{a}}(\epsilon),\qquad X_{\mathbf{b}}=X^-_{\mathbf{b}}(\epsilon)\cup X^+_{\mathbf{b}}(\epsilon)$$

Let $H_{\mathbf{a}}=\langle\mathbf{a}^N\rangle$, $H_{\mathbf{b}}=\langle\mathbf{b}^N\rangle$ be two infinite cyclic subgroups of mapping class group. One can easily see that the collection $H_{\mathbf{a}}$, $H_{\mathbf{b}}$ and two subsets $X_{\mathbf{a}}$, $X_{\mathbf{b}}$ satisfy assumptions of Corollary 5.3.

Therefore, we obtain the following.

Lemma 5.6. *The cluster algebra of type X_6 has exponential growth.*

Corollary 5.7. *The cluster algebra of type X_7 has exponential growth.*

Proof. Indeed, the diagram X_7 contains the diagram X_6 as a subdiagram. Hence, exchange graph of X_7 contains exchange graph of X_6 as a subgraph, and therefore also grows exponentially. □

5.4. $G_2^{(*,+)}$ **and its unfolding** $E_6^{(1,1)}$**.** There are two skew-symmetrizable matrices with diagram $G_2^{(*,+)}$. They are denoted by $G_2^{(1,3)}$ and $G_2^{(3,1)}$ according to Saito's notation for extended affine root systems [25]. These two matrices clearly define isomorphic cluster algebras. It was shown in [10] that both exchange matrices with diagram $G_2^{(*,+)}$ have an unfolding with diagram $E_6^{(1,1)}$. We will prove exponential growth of the cluster algebra with diagram $G_2^{(*,+)}$, and then deduce from it exponential growth of $E_6^{(1,1)}$.

The considerations are similar to those of Section 5.3. Let us index the vertices of $G_2^{(*,+)}$ as shown in Fig. 5.2. The choice of labels $(3,1)$ and $(1,3)$ indicates which of the two matrices with this diagram we choose: the entries ± 3 are located in the first and second columns and the third and fourth rows.

We will use the following two mutation sequences: $\mathbf{a}=[1,2,3]^2$ and $\mathbf{b}=[2,3,4]^2$. As above, both $\mathbf{a}$ and $\mathbf{b}$ are elements of the mapping class group of $G_2^{(*,+)}$, i.e. they preserve the diagram shown in Fig. 5.2. Note that $\mathbf{a},\mathbf{b}$ span a subgroup $\langle\mathbf{a},\mathbf{b}\rangle$ in the mapping class group of the diagram.

Consider the actions of $\mathbf{a}$ and $\mathbf{b}$ on the space E_G of g-vectors. Endow E_G with the standard dot product. For a vector $v\in E_G$ or subspace $V\subset E_G$ we denote their orthogonal complements by $v^\perp$ or $V^\perp$. Let $v_{\mathbf{a}}=(-1,-1,3,0)$. For sufficiently small ϵ we define cone

$$C_{\mathbf{a}}(\epsilon)=\left\{\alpha v_{\mathbf{a}}+w,\ \text{where }\alpha>0, w\in v_{\mathbf{a}}^\perp,\frac{|w|}{\alpha|v_{\mathbf{a}}|}<\epsilon\right\}$$

As we have mentioned above, the action of $\mathbf{a}$ on E_G is piecewise linear. However, the action turns out to be linear on $C_{\mathbf{a}}(\epsilon)$ if ϵ is sufficiently small.

More precisely, $\mathbf{a}$ maps the g-vector $v_z=v_{\mathbf{a}}+\bar z$ for sufficiently small $\bar z=(z_1,z_2,z_3,z_4)$ to the vector

$$\mathbf{a}(v_z)=v_{\mathbf{a}}+(2z_1+2z_2+z_3,z_1+3z_2+z_3,-3z_1-6z_2-2z_3,z_4).\tag{5.3}$$

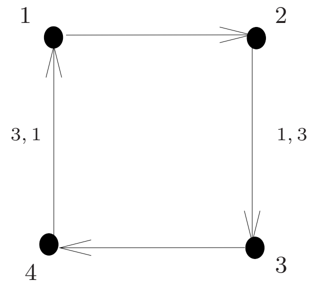


FIGURE 5.2. Diagram for $G_2^{(*,+)}$. The weight "m,n" on the arrow from vertex i to vertex j means that the ratio of $|b_{ij}|$ and $|b_{ji}|$ is equal to m/n.

Direct computation shows that $\mathbf{a} = T^{-1}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} T$, where $T \in GL_4$. Note that the linear operator $\mathbf{a}$ contains (as a direct summand) the Jordan block with eigenvalue one and corresponding eigenvector $v_{\mathbf{a}}$.

Then powers of $\mathbf{a}$ act on $C_{\mathbf{a}}(\epsilon)$ for small epsilon as follows. Denote the second coordinate of the vector $T\bar{z}$ by $\kappa_{\mathbf{a}}$, a simple computation shows $\kappa_{\mathbf{a}} = z_1 + 2z_2 + z_3$.

Then $\mathbf{a}^r(v_z) = v_z + r\kappa_{\mathbf{a}}(v_z)v_{\mathbf{a}}$.

Define $X^+_{\mathbf{a}}(\epsilon) = C_{\mathbf{a}}(\epsilon) \cap \{\kappa_{\mathbf{a}} > 0\}$. Then

$$\lim_{n\to\infty} \frac{\mathbf{a}^n v_z}{|\mathbf{a}^n v_z|} = \frac{v_{\mathbf{a}}}{|v_{\mathbf{a}}|} \text{ if } v_z \in X^+_{\mathbf{a}}(\epsilon).$$

Similarly, define $X^-_{\mathbf{a}}(\epsilon) = C_{\mathbf{a}}(\epsilon) \cap \{\kappa_{\mathbf{a}} < 0\}$. We have

$$\lim_{n\to\infty} \frac{\mathbf{a}^{-n} v_z}{|\mathbf{a}^{-n} v_z|} = \frac{v_{\mathbf{a}}}{|v_{\mathbf{a}}|} \text{ for } v_z \in X^-_{\mathbf{a}}(\epsilon).$$

From Equation 5.3 we see that each $X^\pm_{\mathbf{a}}(\epsilon)$ is invariant under $\mathbf{a}^{\pm 1}$ for ϵ small enough.

Let $v_{\mathbf{b}} = (0, -1, 1, 1)$. We consider the action of $\mathbf{b}$ on E_G in a neighborhood of the ray $\{\alpha \cdot v_{\mathbf{b}} \,\big|\, \alpha > 0\}$.

Define the cone $C_{\mathbf{b}}(\epsilon) = \Big\{\alpha v_{\mathbf{b}} + w, \text{ where } \alpha > 0, w \in v_{\mathbf{b}}^\perp, \frac{|w|}{\alpha|v_{\mathbf{b}}|} < \epsilon\Big\}$.

For sufficiently small $\bar{z} = (z_1, z_2, z_3, z_4)$, $\mathbf{b}$ maps the g-vector $v_z = v_{\mathbf{b}} + \bar{z}$ to the vector

$$\mathbf{b}(v_z) = v_{\mathbf{b}} + (z_1, 4z_2 + 2z_3 + z_4, -3z_2 - z_3 - z_4, -3z_2 - 2z_3) \tag{5.4}$$

The action is linear on $C_{\mathbf{b}}(\epsilon)$ and the corresponding linear operator is a direct sum of the identity operator on 2-dimensional space and a 2×2 Jordan block with eigenvalue 1 with corresponding eigenvector $v_{\mathbf{b}}$.

Let $\kappa_{\mathbf{b}}(v_z) = z_2 + (2/3)z_3 + (1/3)z_4$, then we have $\mathbf{b}^r(v_z) = v_z + r\kappa_{\mathbf{b}}(v_z)v_{\mathbf{b}}$ for any positive integer r. Denote $X^+_{\mathbf{b}}(\epsilon) = C_{\mathbf{b}}(\epsilon) \cap \{\kappa_b > 0\}$. Then Equation 5.4 implies that

$$\lim_{n\to\infty} \frac{\mathbf{b}^n v_z}{|\mathbf{b}^n v_z|} = \frac{v_{\mathbf{b}}}{|v_{\mathbf{b}}|} \text{ for } v_z \in X^+_{\mathbf{b}}(\epsilon).$$

Similarly, we can define $X^-_{\mathbf{b}}(\epsilon) = C_{\mathbf{b}}(\epsilon) \cap \{\kappa_{\mathbf{b}} < 0\}$. Then

$$\lim_{n\to\infty} \frac{\mathbf{b}^{-n} v_z}{|\mathbf{b}^{-n} v_z|} = \frac{v_{\mathbf{b}}}{|v_{\mathbf{b}}|} \text{ for } v_z \in X^-_{\mathbf{b}}(\epsilon).$$

One can also note that $X^+_{\mathbf{b}}(\epsilon)$ is invariant under $\mathbf{b}$ and $X^-_{\mathbf{b}}(\epsilon)$ is invariant under $\mathbf{b}^{-1}$ for sufficiently small ϵ. Straightforward computations using Maple show that $\mathbf{b}^{\pm 10}(v_{\mathbf{a}}) \in C_{\mathbf{b}}(\epsilon)$, where ϵ is small enough for Equation 5.4 to hold. Moreover, $\kappa_{\mathbf{b}}(\mathbf{b}^{10}(v_{\mathbf{a}})) > 0$, while $\kappa_{\mathbf{b}}(\mathbf{b}^{-10}(v_{\mathbf{a}})) < 0$.

Vice versa, $\mathbf{a}^{\pm 10}(v_{\mathbf{b}}) \in C_{\mathbf{a}}(\epsilon)$, where ϵ is small enough for Equation 5.3 to hold. Also, $\kappa_{\mathbf{a}}(\mathbf{a}^{10}(v_{\mathbf{b}})) > 0$, while $\kappa_{\mathbf{a}}(\mathbf{a}^{-10}(v_{\mathbf{b}})) < 0$.

Hence, for any $\epsilon > 0$ small enough we can find a sufficiently large positive integer N_ϵ such that $\mathbf{b}^{N_\epsilon}(X_{\mathbf{a}}(\epsilon)) \subset X^+_{\mathbf{b}}(\epsilon)$, $\mathbf{b}^{-N_\epsilon}(X_a(\epsilon)) \subset X^-_{\mathbf{b}}(\epsilon)$, $\mathbf{a}^{N_\epsilon}(X_{\mathbf{b}}(\epsilon)) \subset X^+_{\mathbf{a}}(\epsilon)$, $\mathbf{a}^{-N_\epsilon}(X_b(\epsilon)) \subset X^-_{\mathbf{a}}(\epsilon)$.

Note that the collection of two infinite cyclic groups $H_{\mathbf{a}} = \langle \mathbf{a}^{N_\epsilon} \rangle$, $H_{\mathbf{b}} = \langle \mathbf{b}^{N_\epsilon} \rangle$, and two sets $C_{\mathbf{a}}(\epsilon)$, $C_{\mathbf{b}}(\epsilon)$ satisfy the assumptions of Corollary 5.3. Thus,

Lemma 5.8. *A cluster algebra with diagram of type $G_2^{(*,+)}$ has exponential growth.*

Corollary 5.9. *The cluster algebra of type $E_6^{(1,1)}$ has exponential growth.*

Proof. The exchange matrix with diagram $E_6^{(1,1)}$ is an unfolding of the exchange matrix with diagram $G_2^{(*,+)}$. In particular, any mutation in $G_2^{(*,+)}$ is lifted to a sequence of mutations of $E_6^{(1,1)}$, and the mapping class group of $E_6^{(1,1)}$ contains the mapping class group of $G_2^{(*,+)}$ as a subgroup. Hence, the growth of $E_6^{(1,1)}$ is exponential. □

5.5. $G_2^{(*,*)}$ **and its unfolding** $E_8^{(1,1)}$**.** There are two distinct skew-symmetrizable matrices with diagram $G_2^{(*,*)}$, which are denoted by $G_2^{(3,3)}$ and $G_2^{(1,1)}$ (see [10, Table 6.3]). We will prove that cluster algebras corresponding to matrices $G_2^{(1,1)}$ and $G_2^{(3,3)}$ have exponential growth. The considerations are almost identical: one needs to take the same sequences of mutations, but different vectors $v_{\mathbf{a}}, v_{\mathbf{b}}, \kappa_{\mathbf{a}}, \kappa_{\mathbf{b}}$. We will indicate below details that differ. Then exponential growth of the unfolding $E_8^{(1,1)}$ of $G_2^{(1,1)}$ follows.

The reasoning is similar to that from Section 5.4. The following diagram represents $G_2^{(1,1)}$, see Fig. 5.3. The diagram of $G_2^{(3,3)}$ is obtained by reversing the orientation of all the arrows.

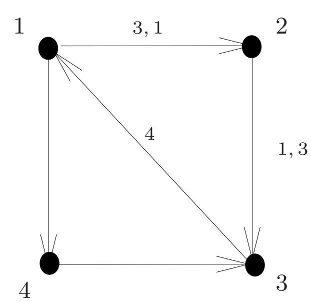


FIGURE 5.3. Diagram for $G_2^{(1,1)}$

Set $\mathbf{a} = [4,1,2]^4$, $v_{\mathbf{a}} = (-2,1,0,1)$ ($v_{\mathbf{a}} = (2,-3,0,-1)$ for $G_2^{(3,3)}$), and $\mathbf{b} = [4,3,2]^4$, $v_{\mathbf{b}} = (0,-1,2,-1)$ ($v_{\mathbf{b}} = (0,3,-2,1)$ for $G_2^{(3,3)}$).

Define cones $C_{\mathbf{a}}(\epsilon)$ and $C_{\mathbf{b}}(\epsilon)$ as above.

For small $\bar{z} = (z_1, z_2, z_3, z_4)$ define the g-vector $v_z = v_{\mathbf{a}} + \bar{z}$. Then

$$\mathbf{a}(v_z) = v_{\mathbf{a}} + (13z_1 + 18z_2 + 6z_4, -6z_1 - 8z_2 - 3z_4, z_3, -6z_1 - 9z_2 - 2z_4). \tag{5.5}$$

For $G_2^{(3,3)}$ we have

$$\mathbf{a}(v_z) = v_{\mathbf{a}} + (13z_1 + 6z_2 + 6z_4, -18z_1 - 8z_2 - 9z_4, z_3, -6z_1 - 3z_2 - 2z_4). \tag{5.6}$$

As above, the action of $\mathbf{a}$ on E_G is a linear transformation, which is a direct sum of the identity operator on 2-dimensional space and a 2×2 Jordan block with unit eigenvalue and eigenvector $v_{\mathbf{a}}$.

Define $\kappa_{\mathbf{a}}(v_z) = 6z_1 + 9z_2 + 3z_4$ ($\kappa_{\mathbf{a}}(v_z) = 6z_1 + 3z_2 + 3z_4$ for $G_2^{(3,3)}$). Then for any positive integer r we have $\mathbf{a}^r(v_z) = v_z - r\kappa_{\mathbf{a}}(v_z)v_{\mathbf{a}}$ ($\mathbf{a}^r(v_z) = v_z + r\kappa_{\mathbf{a}}(v_z)v_{\mathbf{a}}$ in the case of $G_2^{(3,3)}$).

Define $X^+_{\mathbf{a}}(\epsilon)$, $X^-_{\mathbf{a}}(\epsilon)$, $X^+_{\mathbf{b}}(\epsilon)$, and $X^-_{\mathbf{b}}(\epsilon)$ as above.

From Equation 5.5 (5.6) we see that each $X^{\mp}_{\mathbf{a}}(\epsilon)$ is invariant under $\mathbf{a}^{\pm\mathbf{1}}$ for ϵ small enough. If $v_z \in X^-_{\mathbf{a}}(\epsilon)$ then $\lim_{n\to\infty} \dfrac{\mathbf{a}^n v_z}{|\mathbf{a}^n v_z|} = \dfrac{v_{\mathbf{a}}}{|v_{\mathbf{a}}|}$. If $v_z \in X^+_{\mathbf{a}}(\epsilon)$ then $\lim_{n\to\infty} \dfrac{\mathbf{a}^{-n} v_z}{|\mathbf{a}^{-n} v_z|} = \dfrac{v_{\mathbf{a}}}{|v_{\mathbf{a}}|}$.

For sufficiently small $\bar z = (z_1, z_2, z_3, z_4)$ define the g-vector $v_z = v_{\mathbf{b}} + \bar z$. Then

$$\mathbf{b}(v_z) = v_{\mathbf{b}} + (z_1, -8z_2 - 6z_3 - 3z_4, 18z_2 + 13z_3 + 6z_4, -9z_2 - 6z_3 - 2z_4) \tag{5.7}$$

For $G_2^{(3,3)}$ we have

$$\mathbf{b}(v_z) = v_{\mathbf{b}} + (z_1, -8z_2 - 18z_3 - 9z_4, 6z_2 + 13z_3 + 6z_4, -3z_2 - 6z_3 - 2z_4) \tag{5.8}$$

The corresponding linear transformation is a direct sum of an identity operator on 2-dimensional space and a Jordan block of size 2×2 with eigenvalue one and the corresponding eigenvector $v_{\mathbf{b}}$. If $\kappa_{\mathbf{b}} = 9z_2 + 6z_3 + 3z_4$ ($\kappa_{\mathbf{b}} = 3z_2 + 6z_3 + 3z_4$ for $G_2^{(3,3)}$) then $\mathbf{b}^r(v_z) = v_z + r\kappa_{\mathbf{b}}(v_z)v_{\mathbf{b}}$ ($\mathbf{b}^r(v_z) = v_z - r\kappa_{\mathbf{b}}(v_z)v_{\mathbf{b}}$ for $G_2^{(3,3)}$).

Note now, that $\mathbf{b}(X^+_{\mathbf{b}}(\epsilon)) \subset X^+_{\mathbf{b}}(\epsilon)$, and $\mathbf{b}^{-\mathbf{1}}(X^-_{\mathbf{b}}(\epsilon)) \subset X^-_{\mathbf{b}}(\epsilon)$. Furthermore, as in the previous case, we have

$$\lim_{n\to\infty} \frac{\mathbf{b}^n v_z}{|\mathbf{b}^n v_z|} = \frac{v_{\mathbf{b}}}{|v_{\mathbf{b}}|} \text{ for } v_z \in X^+_{\mathbf{b}}(\epsilon),$$
$$\lim_{n\to\infty} \frac{\mathbf{b}^{-n} v_z}{|\mathbf{b}^{-n} v_z|} = \frac{v_{\mathbf{b}}}{|v_{\mathbf{b}}|} \text{ for } v_z \in X^-_{\mathbf{b}}(\epsilon).$$

Again, $\mathbf{b}^{\pm 10}(v_{\mathbf{a}}) \in X^{\pm}_{\mathbf{b}}(\epsilon)$, $\mathbf{a}^{\pm 10}(v_{\mathbf{b}}) \in X^{\mp}_{\mathbf{a}}(\epsilon)$, where ϵ is small enough for Equations 5.5 and 5.7 to hold.

Therefore, we can conclude that for any $\epsilon > 0$ small enough we can find a sufficiently large positive integer N_ϵ such that $\mathbf{b}^{N_\epsilon}(X_{\mathbf{a}}(\epsilon)) \subset X^+_{\mathbf{b}}(\epsilon)$, $\mathbf{b}^{-N_\epsilon}(X_a(\epsilon)) \subset X^-_{\mathbf{b}}(\epsilon)$, $\mathbf{a}^{N_\epsilon}(X_{\mathbf{b}}(\epsilon)) \subset X^-_{\mathbf{a}}(\epsilon)$, $\mathbf{a}^{-N_\epsilon}(X_b(\epsilon)) \subset X^+_{\mathbf{a}}(\epsilon)$.

Now we apply Corollary 5.3 to get the result.

Lemma 5.10. *Cluster algebras of type $G_2^{(1,1)}$ and $G_2^{(3,3)}$ have exponential growth.*

Equivalently,

Corollary 5.11. *Cluster algebras with diagram of type $G_2^{(*,*)}$ have exponential growth.*

The fact that $E_8^{(1,1)}$ is an unfolding of $G_2^{(1,1)}$ implies the following corollary.

Corollary 5.12. *The cluster algebra of type $E_8^{(1,1)}$ has exponential growth.*

5.6. **$F_4^{(*,+)}$, its unfolding $E_7^{(1,1)}$, and $F_4^{(*,*)}$.** In this section we will prove exponential growth for $F_4^{(*,+)}$, its unfolding $E_7^{(1,1)}$, and $F_4^{(*,*)}$. The arguments follow almost literally the arguments of Sections 5.4 and 5.5. Therefore we describe below only the differences between the cases in question and the cases $G_2^{(*,*)}$, $G_2^{(*,+)}$.

The diagram representing $F_4^{(*,+)}$ is shown in Fig. 5.4 (again, labels show the choice of one of the two matrices, which differ by permutations of rows and columns only).

Let $\mathbf{a} = [5,4,3,2,1]$ and $\mathbf{b} = [2,1,6,5,4]$, $v_{\mathbf{a}} = (-1,0,0,0,1,0)$, $v_{\mathbf{b}} = (0,1,0,-1,0,0)$.

In a neighborhood of $v_{\mathbf{a}}$ set $v_z = v_{\mathbf{a}} + (z_1, z_2, z_3, z_4, z_5, z_6)$. Then

$$\mathbf{a}(v_z) = v_{\mathbf{a}} + (-z_1 - z_2 - z_3 - z_4 - 2z_5, 2z_1 + z_2 + z_3 + z_4 + 2z_5, z_2, z_3, z_4 + z_5, z_6) \tag{5.9}$$

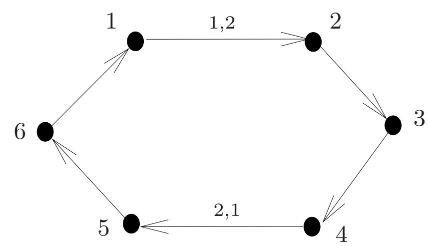


FIGURE 5.4. Diagram for $F_4^{(*,+)}$

for sufficiently small $(z_1, z_2, z_3, z_4, z_5, z_6)$.

Note that the Jordan form of the linear operator $\mathbf{a}$ is a direct sum of an identity operator, negative one times an identity operator, a rotation operator of order four (with eigenvalues of magnitude one) and a 2×2 Jordan block with eigenvalue one whose eigenvector is $v_{\mathbf{a}}$.

Computing coordinates of the corresponding transformation matrix we set $\kappa_{\mathbf{a}} = \frac{1}{2}(z_1 + z_2 + z_3 + z_4 + z_5)$. Then $\mathbf{a}^r(v_z) = v_z + r\kappa_{\mathbf{a}}(v_z)v_{\mathbf{a}}$ whenever r is a multiple of four.

In a neighborhood of $v_{\mathbf{b}}$ we denote $v_z = v_{\mathbf{b}} + (z_1, z_2, z_3, z_4, z_5, z_6)$. Then

$$\mathbf{b}(v_z) = v_z + (z_6, 2z_1 + z_2, z_3, -2z_1 - 2z_2 - z_4 - 2z_5 - 2z_6, z_1 + z_2 + z_4 + z_5 + z_6, z_5) \tag{5.10}$$

for sufficiently small $(z_1, z_2, z_3, z_4, z_5, z_6)$.

Similarly, the linear transformation $\mathbf{b}$ is a direct sum of the 2×2 Jordan block with eigenvalue one and an identity operator, negative one times an identity operator, and a rotation of order four. The eigenvector corresponding to the Jordan block is $v_{\mathbf{b}}$.

Set $\kappa_{\mathbf{b}} = z_1 + (1/2)z_2 + (1/2)z_4 + z_5 + z_6$. Then $\mathbf{b}^r(v_z) = v_z + r\kappa_{\mathbf{b}}(v_z)v_{\mathbf{b}}$ whenever r is a multiple of four.

As above, using Corollary 5.3 we conclude:

Lemma 5.13. *The cluster algebra of type $F_4^{(*,+)}$ has exponential growth.*

Corollary 5.14. *The cluster algebra of type $E_7^{(1,1)}$ has exponential growth.*

Proof. $E_7^{(1,1)}$ is an unfolding of $F_4^{(*,+)}$. □

Finally, we show that the growth of cluster algebras with diagram $F_4^{(*,*)}$ is exponential. As in the case of $G_2^{(*,*)}$, there are two distinct skew-symmetrizable matrices with this diagram, which correspond to extended affine root systems $F_4^{(1,1)}$ and $F_4^{(2,2)}$ (see [10]). Below we consider $F_4^{(1,1)}$. The considerations for $F_4^{(2,2)}$ are almost identical; we give the differing details in parentheses.

The diagram representing $F_4^{(1,1)}$ is shown in Fig. 5.5. The diagram representing $F_4^{(2,2)}$ is obtained by reversing the orientations of all the arrows.

In both cases we considered the same pair of elements of the mapping class group $\mathbf{a} = [1, 2, 3, 4, 5]^2$ and $\mathbf{b} = [4, 5, 6, 1, 2]^2$.

Then choose $v_{\mathbf{a}} = (-1, -1, -1, 1, 1, 0)$, $v_{\mathbf{b}} = (-1/2, 1, 0, -1/2, -1/2, 1/2)$ for $F_4^{(1,1)}$ (for $F_4^{(2,2)}$ $v_{\mathbf{a}} = (1, 1, 1, -2, -2, 0)$, $v_{\mathbf{b}} = (1, -2, 0, 2, 2, -1)$).

In a neighborhood of $v_{\mathbf{a}}$, denote $v_z = v_{\mathbf{a}} + (z_1, z_2, z_3, z_4, z_5, z_6)$. Then

$$\mathbf{a}(v_z) = v_{\mathbf{a}} + (-z_3 - 2z_4, -z_1 - z_2 - z_3 - 2z_4 - 2z_5, z_1, z_2 + z_3 + 2z_4 + z_5, z_3 + z_4 + z_5, z_6) \tag{5.11}$$

for sufficiently small $(z_1, z_2, z_3, z_4, z_5, z_6)$ (for $F_4^{(1,1)}$).

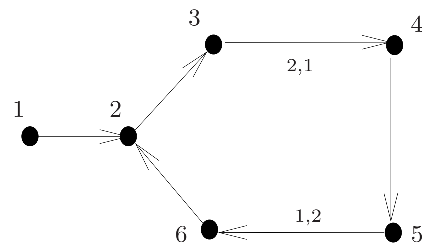


Figure 5.5. Diagram for $F_4^{(1,1)}$

For $F_4^{(2,2)}$ we have

$$\mathbf{a}(v_z) = v_{\mathbf{a}} + \\ + (z_3, z_1 + z_2 + z_3 + z_4, z_1 + 2z_2 + 2z_3 + z_4 + z_5, -2z_1 - 2z_2 - 2z_3 - z_4 - z_5, -2z_2 - 2z_3 - z_4, z_6). \tag{5.12}$$

The linear operator $\mathbf{a}$ is a direct sum of a Jordan block of size 2 with eigenvalue one and eigenvector $v_{\mathbf{a}}$, an identity operator, and a rotation of order three.

Define $\kappa_{\mathbf{a}} = \frac{1}{3}z_1 + \frac{2}{3}z_2 + z_3 + \frac{4}{3}z_4 + \frac{2}{3}z_5$ ($\kappa_{\mathbf{a}} = \frac{1}{3}z_1 + \frac{2}{3}z_2 + z_3 + \frac{2}{3}z_4 + \frac{1}{3}z_5$ for $F_4^{(2,2)}$), then $\mathbf{a}^r(v_z) = v_z + r\kappa_{\mathbf{a}}(v_z)v_{\mathbf{a}}$ whenever r is a multiple of three.

In a neighborhood of $v_{\mathbf{b}}$, denote $v_z = v_{\mathbf{b}} + (z_1, z_2, z_3, z_4, z_5, z_6)$. Then

$$\mathbf{b}(v_z) = v_{\mathbf{b}} + \\ + (2z_5 + z_6, -z_2 - 2z_4 - 4z_5 - 2z_6, z_3, z_4 + z_5 + z_6, z_1 + z_2 + z_4 + 2z_5 + z_6, -z_1) \tag{5.13}$$

for sufficiently small $(z_1, z_2, z_3, z_4, z_5, z_6)$.

For $F_4^{(2,2)}$ we have

$$\mathbf{b}(v_z) = v_{\mathbf{b}} + \\ + (z_5 + z_6, -z_2 - z_4 - 2z_5 - 2z_6, z_3, z_4 + z_5 + 2z_6, 2z_1 + 2z_2 + z_4 + 2z_5 + 2z_6, -z_1). \tag{5.14}$$

The linear operator $\mathbf{b}$ is a direct sum of an identity operator, a rotation of order three and a 2×2 Jordan block with eigenvalue one and eigenvector $v_{\mathbf{b}}$.

Define $\kappa_{\mathbf{b}} = \frac{2}{3}z_1 + \frac{4}{3}z_2 + \frac{4}{3}z_4 + \frac{8}{3}z_5 + 2z_6$ ($\kappa_{\mathbf{b}} = \frac{1}{3}z_1 + \frac{2}{3}z_2 + \frac{1}{3}z_4 + \frac{2}{3}z_5 + z_6$ for $F_4^{(2,2)}$). Then $\mathbf{b}^r(v_z) = v_z - r\kappa_{\mathbf{b}}(v_z)v_{\mathbf{b}}$ whenever r is a multiple of three.

As above, using Corollary 5.3 we conclude

Lemma 5.15. *Cluster algebras of type $F_4^{(1,1)}$ and $F_4^{(2,2)}$ have exponential growth.*

Equivalently,

Corollary 5.16. *Cluster algebras with diagram of type $F_4^{(*,*)}$ have exponential growth.*

6. Growth rates of affine cluster algebras

We are left with exceptional cluster algebras of affine type. This section is devoted to the proof of linear growth of affine cluster algebras. We start with skew-symmetric (simply-laced) affine cluster algebras (whose diagrams can be understood as quivers), and then use unfoldings to complete the proof of Theorem 1.1 in the coefficient-free case.

Let Q be a quiver without oriented cycles, and with n vertices. Let A_Q be the cluster algebra associated to Q.

Let k be an algebraically closed field. We write kQ for the path algebra of Q, and kQ-mod for its module category. The bounded derived category of this abelian category is denoted $D^b(kQ)$. This category is triangulated, and therefore equipped with a shift autoequivalence [1]; it also has an Auslander-Reiten autoequivalence τ.

Given a triangulated category and an auto-equivalence, there is an orbit category, in which objects in the same orbit with respect to the autoequivalence are isomorphic. By definition, the cluster category is the orbit category $\mathcal{C}_Q = D^b(kQ)/[1]\tau^{-1}$, which is again triangulated by a result of Keller [20]. This category is called the cluster category associated to Q, and was introduced in [4].

Thanks to the embedding of objects of kQ-mod as stalk complexes in degree zero inside $D^b(kQ)$, there is a functor from kQ-mod to $\mathcal{C}_Q$, which embeds kQ-mod as a (non-full) subcategory of $\mathcal{C}_Q$. We write P_i for the indecomposable projective kQ module with simple top at vertex i; we also write P_i for the corresponding object of $\mathcal{C}_Q$ via the above embedding.

An object E in $\mathcal{C}_Q$ is called rigid if it satisfies $\mathrm{Ext}^1_{\mathcal{C}_Q}(E,E) = 0$. The crucial result relating the cluster algebra to the cluster category is the following [8, 5]: the cluster variables in the cluster algebra A_Q are naturally in one-one correspondence with the rigid indecomposables of $\mathcal{C}_Q$. We will make the (slightly non-standard) choice of identifying the cluster variable u_i from the initial seed with P_i. A (basic) cluster tilting object in $\mathcal{C}_Q$ is the direct sum of the collection of rigid indecomposables objects corresponding to the cluster variables of some cluster.

Say that two rigid indecomposable objects E, F in $\mathcal{C}_Q$ are compatible if $\mathrm{Ext}^1_{\mathcal{C}_Q}(E,F) = 0$. (By the 2-Calabi-Yau property of cluster categories, this is equivalent to the condition that $\mathrm{Ext}^1_{\mathcal{C}_Q}(F,E) = 0$.) Two rigid indecomposables are compatible if and only if the corresponding cluster variables are both contained in some cluster. Cluster tilting objects can be given a representation-theoretic description: T is a cluster tilting object if T is the direct sum of a maximal collection of pairwise-compatible distinct rigid indecomposable objects in $\mathcal{C}_Q$.

The autoequivalence τ of $D^b(kQ)$ descends to an autoequivalence of $\mathcal{C}_Q$. It therefore induces an action on the indecomposable objects of C_Q. Write X^p_i for the indecomposable object $\tau^p P_i$, where $p \in \mathbb{Z}$ and $1 \leq i \leq n$. These objects are pairwise non-isomorphic, and each is rigid. We refer to these indecomposables as *transjective*. It will be convenient to define a function q on the transjective indecomposable modules by setting $q(X^p_i) = p$.

There are also a finite number of other rigid indecomposable objects in $\mathcal{C}_Q$. They are referred to as the regular rigid indecomposable objects. They lie in finite τ-orbits.

We now prove a sequence of lemmas:

Lemma 6.1. *Any cluster tilting object contains at least two (non-isomorphic) indecomposable transjective summands.*

Proof. We first show that no cluster tilting object has exactly one transjective summand. Suppose that $X \oplus R$ were a cluster tilting object, with X indecomposable tranjective, and R regular. Since all the regular indecomposable summands of R lie in finite τ-orbits, there is some non-zero $m \in \mathbb{Z}$ such that $\tau^m R \simeq R$. Since τ is an auto-equivalence of $\mathcal{C}_Q$, it follows that $\tau^{tm} P$ is compatible with $\tau^{tm} R \simeq R$ for any $t \in \mathbb{Z}$. This would mean that the cluster algebra A_Q has a collection of $n-1$ cluster variables contained in an infinite number of clusters, which is impossible. (It is always the case that $n-1$ cluster variables are contained in either 0 or 2 clusters.)

It now follows that no cluster tilting object in $\mathcal{C}_Q$ contains zero tranjective summands either, since any cluster tilting object can be obtained by a finite number of mutations from the cluster tilting object $\bigoplus_i P_i$ [4]. Since each mutation changes exactly one summand of the cluster tilting object, a sequence of mutations leading to a cluster tilting object with no transjective summands would have to pass through a cluster tilting object with exactly one transjective summand, which we have already shown is impossible. This proves the lemma. □

Lemma 6.2. *There is a bound N such that any tranjective rigid indecomposable compatible with X_i^p is of the form X_j^r with for some $1 \leq j \leq n$ and $p-N \leq r \leq p+N$.*

Proof. Since τ is an autoequivalence, it suffices to check the statement for one element in each transjective τ-orbit. We will check it for each P_i.

Fix i with $1 \leq i \leq n$. Consider the cluster algebra A_i obtained by freezing the vertex i. The principal part of the exchange matrix of A_i corresponds to the quiver Q with the vertex i removed. This is a collection of Dynkin quivers, and thus corresponds to a cluster algebra of finite type. It follows that there are only finitely many cluster variables in A_i. These cluster variables correspond to the cluster variables of A_Q which are compatible with P_i. Since there are only finitely many of them, we can pick a bound N_i so that all the indecomposable transjective objects compatible with P_i are of the form X_j^r with $-N_i \leq r \leq N_i$. Now set N to be the maximum of all the N_i. □

Let T be a cluster tilting object. Take the mean value of $q(E)$ as E runs through the transjective indecomposable summands of T (a non-empty set by Lemma 6.1), and denote that mean value by $q(T)$.

Corollary 6.3. *If M is a transjective summand of a cluster tilting object T, then $|q(T)-q(M)| \leq N$.*

Lemma 6.4. *If T and T' are cluster tilting objects related by a single mutation, then $|q(T')-q(T)| \leq N$.*

Proof. Let M be the summand of T which does not appear in T', and let M' be the summand of T' which does not appear in T. If neither M not M' is transjective, then $q(T') = q(T)$, and we are done. Otherwise, without loss of generality, suppose that M is transjective.

If E is any other transjective summand of T (and there is at least one such E by Lemma 6.1), then $|q(E) - q(M)| \leq N$ by Lemma 6.2. This implies the desired result if M' is not transjective.

If M' is transjective, $|q(M')-q(E)| \leq N$ by Lemma 6.2 again. It follows that the difference between the sum of the q-values of T and T' is at most $2N$, and thus their mean values differ by at most N. □

Theorem 6.5. *The growth rate of any affine simply-laced cluster algebra is linear.*

Proof. Pick a starting cluster T. Let M be a transjective summand of T. Let T' be obtained by applying k mutations to T. Let M' be any transjective summand of T'. Applying Corollary 6.3 twice and Lemma 6.4 once, it follows that $|q(M') - q(M)| \leq N(k+2)$. The number of transjective indecomposable objects within this range is $2nN(k+2)$, while the number of regular rigid indecomposable objects is finite. It follows that the number of cluster variables which can be obtained by k mutations starting from a given cluster is linearly bounded in k, as desired. □

Corollary 6.6. *The growth rate of any affine cluster algebra is linear.*

Proof. The diagrams $\widetilde{B}_n$ and $\widetilde{C}_n$ are s-decomposable, therefore, the vertices of the exchange graph of any cluster algebra with one of these diagrams are indexed by the triangulations of the corresponding orbifold (depending only on the diagram). This implies that the growth is the same for any skew-symmetrizable matrix with diagram $\widetilde{B}_n$ (or $\widetilde{C}_n$). Further, for any of these diagrams there is a matrix with an affine unfolding ($\widetilde{D}_{n+1}$ for $\widetilde{B}_n$, and $\widetilde{D}_{n+2}$ for $\widetilde{C}_n$). Since the growth rate of any cluster algebra is not faster than the growth rate of any its unfolding, we obtain linear growth for any cluster algebra with diagram $\widetilde{B}_n$ or $\widetilde{C}_n$.

For either of the diagrams $\widetilde{F}_4$ and $\widetilde{G}_2$ there are two skew-symmetrizable matrices with these diagrams, see [10, Table 6.3]. All four matrices have affine unfoldings, namely, $\widetilde{E}_6$ and $\widetilde{E}_7$ for $\widetilde{F}_4$, and $\widetilde{D}_4$, $\widetilde{E}_6$ for $\widetilde{G}_2$. Again, this implies linear growth. □

The latter Corollary accomplishes the proof of Theorem 1.1 in coefficient-free case.

7. Coefficients

In this section we prove the following lemma.

Lemma 7.1. *The growth rate of a cluster algebra does not depend on its coefficients.*

Proof. It is easy to see from the definition that the exchange graph of a cluster algebra covers the exchange graph of the coefficient-free cluster algebra with the same exchange matrix. In particular, we have nothing to prove for algebras with exponential growth, so we only need to explore cases (1)–(4) from Theorem 1.1.

In [16], Fomin and Zelevinsky conjectured [16, Conjecture 4.14] that the exchange graph of a cluster algebra depends only on the exchange matrix. This conjecture is known to be true in many cases, including:

- for cluster algebras of finite type [15], which covers case (2a);
- for cluster algebras of rank 2 (immediately following from the finite type case), which covers case (1);
- for cluster algebras from surfaces [13] and orbifolds [11], which covers cases (2b) and (3).
- for skew-symmetric cluster algebras [6], which covers case (4a) (and parts of the previous cases).

Further, the unfolding argument does not depend on coefficients, so case (4b) is implied by (4a). □

This completes the proof of Lemma 7.1 and thus also the proof of Theorem 1.1.

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